

Hybrids between hyperbolic and elliptic differential operators with constant coefficients

Johan Fehrman

Introduction

A partial differential operator with constant coefficients is said to be elliptic (hypoelliptic) if it has a fundamental solution which is real analytic (infinitely differentiable) outside the origin. It is said to be hyperbolic if it has a fundamental solution which vanishes outside some proper cone with vertex at the origin. These classes of operators are well known and their properties have been studied in some detail (see e.g. Hörmander [5] and Atiyah—Bott—Gårding [2]). We shall here make a detailed study of a class of operators called hybrid or hyperbolic-elliptic operators, defined by having fundamental solutions that are analytic outside proper cones with vertices at the origin. Like the elliptic and hyperbolic operators these hybrids have principal parts in the same class and weaker lower order terms are the only lower order terms that can be added to a hybrid operator without destroying its character. The hybrids between the hypoelliptic and hyperbolic operators, characterized algebraically by Shirota [9], do not share these simple properties. For them, as for the hypoelliptic operators, the principal part is not a relevant concept.

I shall now list the main results. First some notation and definitions. Points in \mathbf{R}^n will usually be denoted by ξ, η while points in \mathbf{R} will be denoted by s, t . When $P(\xi)$ is a polynomial, $P(D)$, $D=(\partial/\partial x_1, \dots, \partial/\partial x_n)$, denotes the associated differential operator. Homogeneous polynomials will be denoted by a . The class of hybrid or hyperbolic-elliptic operators having fundamental solutions analytic outside proper cones with vertices at the origin, on which $x=0$ or $\langle x, \theta \rangle > 0$ will be denoted by $he(\theta)$. The subclass of homogeneous elements in $he(\theta)$ is called $He(\theta)$. Both these classes can be characterized algebraically. The results, parallel to those of the hyperbolic classes $Hyp(\theta)$ and $hyp(\theta)$, run as follows:

(i) $a \in \text{He}(\theta)$ if and only if there is a number $c > 0$ such that

$$\xi, t \text{ real, } 0 < |t| < c|\xi| \Rightarrow a(\xi - it\theta) \neq 0$$

(ii) $P \in \text{he}(\theta)$ if and only if $P_m(\theta) \neq 0$, P_m being the principal part of P , and there are numbers $c_1, c_2 > 0$ such that

$$\xi, t \text{ real, } c_1 < t < c_2|\xi| \Rightarrow P(\xi - it\theta) \neq 0.$$

The connection between these two classes is given by the following statement that generalizes a theorem by S. Leif Svensson [8] for hyperbolic operators:

$$(iii) \quad P \in \text{he}(\theta) \Leftrightarrow P_m \in \text{He}(\theta) \quad \text{and} \quad P < P_m.$$

Here P_m is the principal part of P and $P < P_m$ means that P is weaker than P_m in the sense of Hörmander [5].

As a corollary we get

$$(iv) \quad \text{he}(\theta) = \text{he}(-\theta).$$

Replacing c in (i) and c_2 in (ii) by ∞ we get the definitions of the classes $\text{Hyp}(\theta)$ and $\text{hyp}(\theta)$. The statements corresponding to (iii) and (iv) are, for these classes, well known.

To describe the singular support of the fundamental solution of a hybrid operator we start by recalling some fundamental facts from the hyperbolic case.

Let $a \in \text{Hyp}(\theta)$ and let A denote the real hypersurface $a(\xi) = 0$. Further let $\Gamma(A, \theta)$ be the component of $\mathbf{R}^n - A$ containing θ . Then $\Gamma(A, \theta)$ is an open convex cone and $a \in \text{Hyp}(\eta)$ for every $\eta \in \Gamma(A, \theta)$. The dual

$$K(A, \theta) = \{x; \langle x, \eta \rangle \geq 0 \forall \eta \in \Gamma(A, \theta)\} \quad \text{of} \quad \Gamma(A, \theta)$$

is called the propagation cone.

Let $P \in \text{hyp}(\theta)$ have principal part a . Then $P(D)$ has a fundamental solution $E(P, \theta, x)$ with support in $K(A, \theta)$. Furthermore if $P = a + b$ one has the formula

$$E(P, \theta, x) = \sum_0^\infty (-1)^k b(D)^k E(a^{k+1}, \theta, x),$$

connecting the fundamental solution of P with those of the powers of a .

The singular support of $E(P, \theta, x)$ can be described by introducing the localizations a_ξ of a . They are the first nonvanishing coefficients of the expansions

$$a(\xi + t\zeta) = t^p a_\xi(\zeta) + O(t^{p+1}), \quad t \rightarrow 0.$$

It turns out that $a_\xi \in \text{Hyp}(\theta)$ and $\Gamma(A_\xi, \theta) \supset \Gamma(A, \theta)$ for all $\xi \in \mathbf{R}^n$. Also

$$\Gamma(A, \theta) = \bigcap_{0 \neq \xi \in \mathbf{R}^n} \Gamma(A_\xi, \theta).$$

The corresponding union

$$W(A, \theta) = \bigcup_{0 \neq \xi \in \mathbb{R}^n} K(A_\xi, \theta)$$

is called the wave front surface. This is a closed subset of $K(A, \theta)$ containing the boundary of $K(A, \theta)$, and the fundamental solution $E(P, \theta, x)$ is analytic outside $W(A, \theta)$.

In the hybrid case there are corresponding statements. If a is the principal part of P then

$$P \in \text{he}(\theta), \quad \xi \text{ real} \neq 0 \Rightarrow a_\xi \in \text{Hyp}(\theta)$$

and $P \in \text{he}(\eta)$ for each η in the intersection

$$\Gamma(A, \theta) = \bigcap_{0 \neq \xi \in \mathbb{R}^n} \Gamma(A_\xi, \theta).$$

The corresponding wave front surface is

$$W(A, \theta) = \bigcup_{0 \neq \xi \in \mathbb{R}^n} K(A_\xi, \theta).$$

The difference with the hyperbolic case is that the cone $\Gamma(A, \theta)$ is in general smaller than the component of the complement of A containing θ . Correspondingly, the wave front surface does not bound its convex hull. If, e.g., $a(\xi) = \xi_1^m + \dots + \xi_n^m$, $\theta = (1, 1, \dots, 1)$ and $m > 1$ is odd, then $a \in \text{He}(\theta)$ and $\Gamma(A, \theta)$ turns out to be the intersection of all the half-spaces $\eta_j + \eta_k > 0$, $j \neq k$. The closure of $\Gamma(A, \theta)$ then intersects A only at the origin. The wave front surface consists of all $x = (x_1, \dots, x_n)$ of the form $x_j = \xi_j^{m-1}$ where $a(\xi) = 0$ and has only the points $(0, \dots, 1, 0, \dots, 1, 0, \dots, 0)$ in common with the boundary of its convex hull. The special case $m = n = 3$ was treated by Zeilon [9] and we give some other examples of hybrid operators in section 2.

Precisely as for hyperbolic operators, the interest in the wave front surface comes from the following statement.

(v) If $P \in \text{he}(\theta)$, $P(D)$ has a fundamental solution which is analytic outside the wave front surface.

For second order hyperbolic operators, this is of course classical. Zeilon [9] had it in the case he considered, Atiyah—Bott—Gårding [2] proved it for hyperbolic operators and K. G. Andersson [1] proved it in a general case that comprises ours.

Finally I would like to thank prof. Lars Gårding who introduced me to the subject which led to this paper and who has given me much help and valuable advice as the work progressed.

1. Algebraic properties of hybrid polynomials

We start by deriving necessary conditions for an operator $P(D)$ to have a fundamental solution that is analytic outside a proper closed cone. By proper we mean that it does not contain any straight lines.

Lemma 1.1. *Suppose that $P(D)$ has a fundamental solution which is analytic outside a proper closed cone K with vertex at the origin. Then there are positive locally bounded functions $c_0(\eta)$, $c_1(\eta)$ defined in the dual cone $\Gamma = \{\eta; \langle x, \eta \rangle > 0 \forall x \in K - \{0\}\}$ of K such that:*

$$\xi, t \text{ real}, \eta \in \Gamma, c_0(\eta) \log(2 + |\xi|) < t < c_1(\eta) |\xi| \Rightarrow P(\xi - i t \eta) \neq 0.$$

Proof. Denote the fundamental solution by E and let $u \in C^\infty$ be a solution of $P(D)u = 0$.

Choose a cut-off function $\varphi \in C_0^\infty$ such that $\varphi(x) = 1$ if $|x| < R$ and $\varphi(x) = 0$ if $|x| > R + 1$, where R is a fixed constant.

We have

$$\begin{aligned} u(x) &= P(D)E * u(x) = P(D)(\varphi E) * u(x) + \\ &+ P(D)((1 - \varphi)E) * u(x) = P(D)((1 - \varphi)E) * u(x), \end{aligned}$$

since $P(D)u = 0$. Further, $\text{supp } P(D)((1 - \varphi)E) \subset \{x; R < |x| < R + 1\}$.

Put $K_\epsilon = \{x \in \mathbb{R}^n; d(x, K) < \epsilon\}$ where d denotes the Euclidean distance and let ψ be a C^∞ function equal to 1 in K_ϵ and 0 outside $K_{2\epsilon}$.

Introduce for simplicity the following notation

$$F_1 = \psi P(D)((1 - \varphi)E), \quad F_2 = (1 - \psi) P(D)((1 - \varphi)E).$$

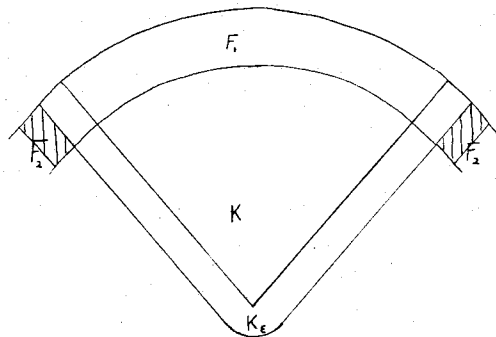


Figure 1.

The supports of F_1 and F_2 intersect in the shaded areas. According to the above we have

$$u(x) = F_1 * u(x) + F_2 * u(x)$$

which implies

$$D^\alpha u(0) = F_1 * D^\alpha u(0) + D^\alpha F_2 * u(0). \quad (1.1)$$

To obtain the condition in the lemma we shall choose u as an exponential solution $e^{i\langle x, \zeta \rangle}$, $P(\zeta)=0$ and estimate the right hand side of (1.1).

To estimate the first term let A be a compact subset of Γ . By choosing ε small, we can find a compact neighbourhood ω of $\text{supp } F_1$ such that

$$h(\eta) = \sup_{x \in \omega} \langle x, \eta \rangle$$

has a negative upper bound in $-A$.

Let $\chi \in C^\infty$ be equal to 1 in a neighbourhood of $\text{supp } F_1$ and 0 outside ω . Then, if m denotes the degree of P and M the order of E on ω , we have

$$F_1(v) = F_1(\chi v) = E((1-\varphi)P(-D)(\psi\chi v)),$$

implying

$$|F_1(v)| \leq C \sum_{|\beta| \leq M+m} (\sup |D^\beta \varphi| + \sup |D^\beta \psi|) \cdot \sum_{|\beta| \leq M+m} \sup_{\omega} |D^\beta v|.$$

Here C is independent of φ and ψ . Using this we get the following estimate for the first term in (1.1):

$$\begin{aligned} |F_1 * D^\alpha u(0)| &= |F_1(D^\alpha u(-y))| \leq \\ &\leq C \sum_{|\beta| \leq M+m} (\sup |D^\beta \varphi| + \sup |D^\beta \psi|) \cdot |\zeta^\alpha| (1 + |\zeta|)^{M+m} e^{h(\text{Im } \zeta)}. \end{aligned} \quad (1.2)$$

Here and in the sequel C denotes various constants independent of α . To estimate the second term in (1.1) we shall use the analyticity of E outside K . We then need to control the derivatives of φ and ψ so we choose $\varphi = \varphi_N$, $\psi = \psi_N$ depending on N , such that

$$\sup |D^\alpha \varphi_N| + \sup |D^\alpha \psi_N| \leq C^{|\alpha|+1} N^{|\alpha|} \quad (1.3)$$

when $|\alpha| \leq N$. That this is possible will be proved later. On compact parts of $\mathbf{R}^n - K$, E satisfies

$$|D^\alpha E(y)| \leq C^{|\alpha|+1} |\alpha|^{|\alpha|},$$

and this combined with (1.3), with N replaced by $N+m$, and Leibniz' formula gives

$$|D^\alpha F_2(y)| \leq C^{|\alpha|+m+1} (N+m)^{|\alpha|+m} \quad \text{if } |\alpha| \leq N,$$

and from this we get

$$|D^\alpha F_2 * u(0)| \leq C^{|\alpha|+m+1} (N+m)^{|\alpha|+m} e^{c|\text{Im } \zeta|} \quad (1.4)$$

if $|\alpha| \leq N$. Combining (1.1), (1.2), (1.3) and (1.4) for suitable α with $|\alpha| = N$, we get

$$|\zeta|^N \leq C(N+m)^{M+m} |\zeta|^N (1 + |\zeta|)^{M+m} e^{h(\text{Im } \zeta)} + C_0^{N+m+1} (N+m)^{N+m} e^{c|\text{Im } \zeta|}.$$

We now choose $N = [\zeta/C_0 e] - m$ and get when $|\zeta|$ is large

$$1 \leq C(1 + |\zeta|)^{2M+2m} e^{h(\operatorname{Im} \zeta)} + C(1 + |\zeta|)^{m+1} e^{c|\operatorname{Im} \zeta| - c'|\zeta|}.$$

Putting $\zeta = \xi - i\eta$, $|\xi|$ large, $\eta \in A$ (to make $h(-\eta) < 0$), we see that this inequality is violated if

$$c_0 \log(2 + |\xi|) < t < c_1 |\xi|$$

provided that c_0 and c_1 are properly chosen.

This last inequality contains, after a further choice of c_0 , the condition that $|\xi|$ be large so the lemma is proved apart from the verification of (1.3). We formulate this as a separate lemma.

Lemma 1.2. *Let A be a subset of \mathbf{R}^n . Put $A_\delta = \{x \in \mathbf{R}^n; d(x, A) < \delta\}$ and let $0 < \delta_1 < \delta_2$ be given numbers. Then for each N there exists a function $\varphi_N \in C_0^\infty(A_{\delta_2})$ such that*

1. $\varphi_N(x) = 1$ when $x \in A_{\delta_1}$,
2. $|\alpha| \leq N \Rightarrow \sup |D^\alpha \varphi_N| \leq C^{|\alpha|+1} N^{|\alpha|}$,

where C is a number independent of α and N .

Proof. Choose $\varphi \in C_0^\infty(|x| < 1)$ with $\varphi(x) \geq 0$, $\int \varphi(x) dx = 1$. With $\varphi_j(x) = j^n \varphi(jx)$ we put

$$\psi_N = \varphi_2 * \underbrace{\varphi_{2N} * \dots * \varphi_{2N}}_{N \text{ factors}}.$$

Then $\psi_N \in C_0^\infty(|x| < 1)$, $\int \psi_N(x) dx = 1$ and ψ_N is non-negative. Letting at most one derivative fall on each of the N last factors and no derivative on φ_2 , we get

$$|D^\alpha \psi_N(x)| \leq \sup \varphi_2 \cdot (2N)^{|\alpha|} \left(\sup_{|\beta|=1} \int |\varphi^{(\beta)}(x)| dx \right)^{|\alpha|}$$

i.e. $|D^\alpha \psi_N(x)| \leq C^{|\alpha|+1} N^{|\alpha|}$ when $|\alpha| \leq N$.

Now let δ be a number satisfying $\delta_1 < \delta < \delta_2$ and let χ be the function equal to one on A_δ and equal to zero elsewhere. With $\psi_N^\varepsilon(x) = \varepsilon^{-n} \psi_N(x/\varepsilon)$ we put

$$\varphi_N^\varepsilon = \chi * \psi_N^\varepsilon.$$

Then one easily checks that if ε is fixed, $\varepsilon < \delta - \delta_1$, $\varepsilon < \delta_2 - \delta$, $\varphi_N = \varphi_N^\varepsilon$ has the required properties.

Next we shall eliminate the logarithm occurring in lemma 1.1.

Lemma 1.3. *Let Γ be an open cone and let $c_0(\eta)$, $c_1(\eta)$ be two positive locally bounded functions defined in Γ . Assume that the polynomial P satisfies the condition*

$$\xi, t \text{ real}, \eta \in \Gamma, c_0(\eta) \log(2 + |\xi|) < t < c_1(\eta) |\xi| \Rightarrow P(\xi - i\eta) \neq 0.$$

Then there exist two positive locally bounded functions $c'_0(\eta)$ and $c'_1(\eta)$ defined in Γ such that

$$\xi, t \text{ real}, \eta \in \Gamma, c'_0(\eta) < t < c'_1(\eta)|\xi| \Rightarrow P(\xi - it\eta) \neq 0.$$

Proof. A subset of \mathbf{R}^n is said to be algebraic if it is defined by a finite number of polynomial equalities or inequalities. Finite unions of algebraic sets are called semi-algebraic. Let M be a semi-algebraic subset of Γ and let c be a real number. According to a theorem of Seidenberg and Tarski, (see Hörmander [5] appendix) we are allowed to use the symbols \forall and \exists in the defining relations of a set without destroying its semi-algebraic character. Thus it follows that the set

$$B_r = \{s; P(\xi - it\eta) \neq 0 \forall \xi, t, \eta; |\xi| = r, \eta \in M, s < t < cr\}$$

is semi-algebraic.

As in Hörmander [5] appendix, we conclude that there are numbers a and b such that

$$\inf_{s \in B_r} s = ar^b(1 + o(1)) \quad \text{as } r \rightarrow \infty, \quad (1.5)$$

provided that B_r is non-empty and we allow a to be $-\infty$.

The assumptions imply that $B_r \neq \emptyset$ if c is chosen properly. Since $s = C \log r$ is in B_r for large r and suitable C we must have $a \leq 0$ or $b \leq 0$ in (1.5) which implies the lemma.

Lemma 1.4. Suppose that the polynomial P satisfies the following condition:

There is an open cone Γ and two positive locally bounded functions c_0 and c_1 defined in Γ such that

$$\xi, t \text{ real}, \eta \in \Gamma, c_0(\eta) < t < c_1(\eta)|\xi| \Rightarrow P(\xi - it\eta) \neq 0. \quad (1.6)$$

Then it follows that $P_m(\eta) \neq 0$ for all $\eta \in \Gamma$, where P_m denotes the principal part of P .

Proof. Assume that $P_m(\eta) = 0$ for some $\eta \in \Gamma$. We exclude the trivial case when P is a constant and consider the function $t \rightarrow P(\xi - t\eta)$ (ξ not necessarily real).

Case 1. $P(\xi - t\eta) = P(\xi) \forall \xi, t$. Choose ξ_0 with $P(\xi_0) = 0$. We have

$$0 = P(\xi_0 - t\eta) = P(\operatorname{Re} \xi_0 - \operatorname{Re} t\eta - i \operatorname{Im} t(\eta - \operatorname{Im} \xi_0 / \operatorname{Im} t)).$$

This clearly contradicts (1.6) if $\operatorname{Re} t$, $\operatorname{Im} t$ and $\operatorname{Re} t / \operatorname{Im} t$ are large enough.

Case 2. $P(\xi - t\eta) = Q(\xi)t^v + R(\xi, t)$ where $v < m$, $Q \neq 0$ and $R(\xi, t)$ has degree less than v in t .

a) Q not constant.

Then take ξ_0, ξ_1 with $Q(\xi_0) = 0$ but $Q(\xi_0 + s\xi_1) \neq 0$. This implies that there is a root $t = \mu(s)$ of $P(\xi_0 + s\xi_1 - t\eta)$ such that $|\mu(s)| \rightarrow \infty$ as $s \rightarrow 0$. A Puiseux expansion of $\mu(s)$ then shows that $\mu(s) = as^{-b}(1 + o(1))$ when $s \rightarrow 0$, where $a \neq 0$ and $b > 0$.

From this we see that for given $\varepsilon > 0$, we can find a straight line through the origin in the complex plane such that when s is restricted to this line we have $\operatorname{Im} \mu(s) < \varepsilon |\operatorname{Re} \mu(s)|$ and $\operatorname{Im} \mu(s) \rightarrow \infty$ as $s \rightarrow 0$.

Combining this with $0 = P(\xi_0 + s\xi_1 - \mu(s)\eta) = P(\operatorname{Re} \xi_0 + \operatorname{Re} s\xi_1 - \operatorname{Re} \mu(s)\eta - i \operatorname{Im} \mu(s)(\eta - \operatorname{Im} \xi_0 / \operatorname{Im} \mu(s) - \operatorname{Im} s\xi_0 / \operatorname{Im} \mu(s)))$, we get a contradiction to (1.6).

b) $Q(\xi) = c$ is constant.

Choose ξ_0 with $P_m(\xi_0) \neq 0$ and let $\mu_1(s), \dots, \mu_v(s)$ denote the roots of $P(s\xi_0 - \eta) = 0$.

We have $\prod_{i=1}^v \mu_i(s) = (-1)^v / c \cdot P(s\xi_0)$ and since by assumption $v < m$ and $P(s\xi_0)$ has degree m in s , at least one of the roots, say $\mu_1(s)$, must have a Puiseux expansion around infinity with leading term $a \cdot s^b$ where $a \neq 0$, $b > 1$, i.e. $\mu_1(s) = as^b(1 + o(1))$ as $s \rightarrow \infty$.

Again from

$$0 = P(s\xi_0 - \mu_1(s)\eta) = P(\operatorname{Re} s\xi_0 - \operatorname{Re} \mu_1(s)\eta - i \operatorname{Im} \mu_1(s)(\eta - \operatorname{Im} s\xi_0 / \operatorname{Im} \mu_1(s)))$$

we obtain a contradiction letting s tend to infinity along a suitably chosen straight line in the complex plane.

Definition 1.5. We say that a polynomial P is hyperbolic elliptic or a hybrid if there is an open cone Γ , two positive locally bounded functions $c_0(\eta)$, $c_1(\eta)$ defined in Γ such that P satisfies (1.6).

This class of polynomials will be denoted $\operatorname{he}(\Gamma)$.

We can now collect some of the previous results in the following.

Theorem 1.6. Suppose that the differential operator $P(D)$ has a fundamental solution which is analytic outside a proper closed cone K with vertex at the origin. Then it follows that $P \in \operatorname{he}(\Gamma)$, where $\Gamma = \{\eta \in \mathbb{R}^n; \langle x, \eta \rangle > 0 \forall x \in K - \{0\}\}$.

2. Homogeneous hybrid polynomials

Let a be a homogeneous polynomial belonging to $\operatorname{he}(\Gamma)$. Then it follows immediately that a satisfies

$$\xi, t \text{ real}, \quad \eta \in \Gamma, \quad 0 < |t| < c(\eta)|\xi| \Rightarrow a(\xi + it\eta) \neq 0, \quad (2.1)$$

where $c(\eta)$ is a positive locally bounded function in Γ .

Definition 2.1. The class of homogeneous polynomials belonging to $\operatorname{he}(\Gamma)$ will be denoted by $\operatorname{He}(\Gamma)$.

We have

$$a \in \operatorname{He}(\Gamma) \text{ if and only if } a \text{ satisfies (2.1).}$$

Lemma 2.2. *If P_m denotes the principal part of P we have*

$$P \in \text{he}(\Gamma) \Rightarrow P_m \in \text{He}(\Gamma).$$

Proof. $\tau^m P(\tau^{-1}(\xi - i\eta)) \rightarrow P_m(\xi - i\eta)$ when $\tau \rightarrow 0$, and the convergence is locally uniform in t when $1/2 < |\xi| < 3/2$ and $\eta \in M$, where M is a compact part of Γ .

By lemma 1.4 the limit is not identically zero, so it follows from the argument principle that the zeros of $t \rightarrow P_m(\xi - i\eta)$ are limit points of the zeros of $t \rightarrow \tau^m P(\tau^{-1}(\xi - i\eta))$. This and the fact that there are constants $c_0, c_1 > 0$ such that

$$\xi, t \text{ real}, \quad 1/2 < |\xi| < 3/2, \quad \tau > 0, \quad \eta \in M,$$

$$c_0 \tau < t < c_1 \Rightarrow \tau^m P(\tau^{-1}(\xi - i\eta)) \neq 0$$

implies that

$$P_m(\xi - i\eta) \neq 0 \quad \text{if} \quad \xi, t \text{ real}, \quad |\xi| = 1, \quad \eta \in M, \quad 0 < t < c_1.$$

Now the lemma follows from the homogeneity of P_m .

We shall now prove that condition (2.1) for a single vector η is in fact sufficient to ensure that (2.1) is valid for some cone Γ containing η . So we introduce the following definition.

Definition 2.3. A homogeneous polynomial a will be called hyperbolic elliptic or a hybrid with respect to θ if there exists a constant $c > 0$ such that a satisfies the following condition

$$\xi, t \text{ real}, \quad 0 < |t| < c|\xi| \Rightarrow a(\xi + it\theta) \neq 0. \quad (2.2)$$

This class of polynomials will be denoted by $\text{He}(\theta)$.

Examples. Let $a \in \text{He}(\theta)$ and let $\lambda_1(\xi), \dots, \lambda_m(\xi)$ denote the zeros of the polynomial $t \rightarrow a(\xi + t\theta)$ so that (note that $a(\theta) \neq 0$ is a consequence of (2.2))

$$a(\xi + t\theta) = a(\theta) \prod_1^m (t - \lambda_k(\xi)).$$

The condition (2.2) can then be expressed as a condition on the zeros $\lambda_k(\xi)$ namely

$$\xi \text{ real} \Rightarrow \text{Im } \lambda_k(\xi) = 0 \quad \text{or} \quad |\lambda_k(\xi)| \geq c|\xi| \quad \text{for each } k. \quad (2.3)$$

To see this, we note that from $a(\xi + \text{Re } \lambda_k \theta + i \text{Im } \lambda_k \theta) = 0$ it follows that either $\text{Im } \lambda_k = 0$ or $|\text{Im } \lambda_k| \geq c|\xi + \text{Re } \lambda_k \theta|$. In the latter case either $|\text{Re } \lambda_k| \geq 1/(2|\theta|)|\xi|$ or else $|\text{Im } \lambda_k| \geq c/2 \cdot |\xi|$ so (2.3) follows.

This is true for elliptic and hyperbolic polynomials where none of respectively all of the $\lambda_k(\xi)$ are real. Since obviously products remain in $\text{He}(\theta)$, products of elliptic and hyperbolic operators in $\text{He}(\theta)$ are in $\text{He}(\theta)$.

The lineality $L(P)$ of any polynomial P is the set of η for which $P(\xi + t\eta) = P(\xi) \forall \xi, t$. $L(P)$ is a linear space and $L(P) = 0$ means that P is complete in the sense that it is not a polynomial in less than n linear combinations of the variables ξ .

It follows from (2.2) that a non-hyperbolic polynomial in $\text{He}(\theta)$ must be complete. In fact, if $\eta \in L(a)$ and $a(\xi + i\theta) = 0$ for some real ξ and t it follows that $a(\xi + s\eta + it\theta) = 0$ for all s , which implies that $t = 0$ or $|t| \geq c|\xi + s\eta|$ for all s by (2.2). Now the latter condition is absurd so the statement follows.

The polynomials in $\text{He}(\theta)$ of degree 0 are the nonvanishing constants. When the degree is one we can after a linear change of variables write $a(\xi)$ as a multiple of ξ_1 or $\xi_1 + i\xi_2$. Since $\xi_1 \in \text{Hyp}(\theta, 1)$ when $\theta_1 \neq 0$ it follows that $\text{He}(\theta) = \text{Hyp}(\theta, 1)$ except when $n = 2$, in which case $\text{He}(\theta)$ also contains elliptic elements. The only real non-elliptic elements in $\text{He}(\theta)$ of degree two are the hyperbolic ones. In fact by a change of variables and after multiplying by a constant we can assume that $\theta = (1, 0, \dots, 0)$ and $a(\xi) = \xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \dots - \xi_q^2$. An easy argument shows that then necessarily $p = 1$.

Let A be the real hypersurface $a(\xi) = 0$. If a is real and A is non singular outside the origin, then $a \in \text{He}(\theta)$ provided that no real straight line with direction θ is tangent to A outside the origin. In fact, then $a(\theta) \neq 0$ and the real zeros of $t \rightarrow a(\xi + t\theta)$ are distinct for all real $\xi \neq 0$. Since a is real the complex roots come in pairs. Using the continuity of the zeros, this would imply the existence of a multiple zero if the complex roots were not bounded away from zero on the unit sphere.

This criterion shows that the polynomials

$$a(\xi) = \xi_1^m + \dots + \xi_n^m \quad m \geq 3, \quad m \text{ odd}$$

are in $\text{He}(\theta)$ when

$$j \neq k \Rightarrow \theta_j + \theta_k > 0, \quad (2.4)$$

(when m is even they are elliptic and hence in $\text{He}(\theta)$ for all $\theta \neq 0$). In fact, it suffices to verify that then

$$\xi_1^m + \dots + \xi_n^m = 0 \Rightarrow \theta_1 \xi_1^{m-1} + \dots + \theta_n \xi_n^{m-1} > 0$$

for all real $\xi \neq 0$. When all θ_k are > 0 this is clear. Otherwise it follows from (2.4) that at most one θ_k , say θ_n , is ≤ 0 . By Jensen's inequality,

$$|\xi_n| \leq (|\xi_1|^m + \dots + |\xi_{n-1}|^m)^{\frac{1}{m}} \leq (\xi_1^{m-1} + \dots + \xi_{n-1}^{m-1})^{\frac{1}{m-1}}$$

so that then

$$\theta_1 \xi_1^{m-1} + \dots + \theta_n \xi_n^{m-1} \geq (\theta_1 + \theta_n) \xi_1^{m-1} + \dots + (\theta_1 + \theta_n) \xi_{n-1}^{m-1} > 0.$$

It is easy to see that if $m > 0$ is even the polynomials $a(\xi) = \xi_1^m - \xi_2^m - \dots - \xi_n^m$ are in $\text{He}(\theta)$ when $\theta = (1, 0, \dots, 0)$.

In the following theorem we will use the notions of localization and local hyperbolicity. For convenience we will state the definitions here. The reader is referred to Atiyah—Bott—Gårding [2], Gårding [4] and Hörmander [6] for details.

Definition 2.4. Let $a(\xi)$ be a homogeneous polynomial and expand $a(\xi+t\zeta)$ in ascending powers of t

$$a(\xi+t\zeta) = t^p a_\zeta(\xi) + O(t^{p+1}) \quad \text{as } t \rightarrow 0,$$

where $a_\zeta(\xi)$ is the first coefficient that does not vanish identically in ξ . The polynomial $\zeta \rightarrow a_\zeta(\xi)$ is called the localization of a at ζ and p is called the multiplicity of a at ζ .

Definition 2.5. A function $f(\xi)$, analytic in a neighbourhood of the origin in \mathbf{C}^n , is said to be locally hyperbolic with respect to $\theta \in \mathbf{R}^n$ if

$$\xi \in \mathbf{R}^n, \quad \text{Im } t \neq 0 \Rightarrow f(\xi + t\theta) \neq 0 \quad (2.5)$$

when ξ, t are small enough. The class of these functions is denoted by $\text{Hyp}_{\text{loc}}(\theta)$.

Definition 2.6. A homogeneous polynomial $a(\xi)$ is said to be hyperbolic with respect to $\theta \in \mathbf{R}^n$ if (2.5) holds for a . This class of polynomials is denoted by $\text{Hyp}(\theta)$ ($\text{Hyp}(\theta, m)$ if the degree of a is m .)

Definition 2.7. When a is a homogeneous polynomial, let A denote the real surface $a(\xi)=0, \xi \in \mathbf{R}^n$. If $a \in \text{Hyp}(\theta)$, let $\Gamma(a, \theta) = \Gamma(A, \theta)$ be the component of $\mathbf{R}^n - A$ that contains θ .

Let $f(\xi)$ be analytic in a neighbourhood of the origin in \mathbf{C}^n . Expand f in homogeneous terms f_v of degree $v, f = \sum_0^\infty f_v$. We denote the first nonvanishing term f_m by Pf . The number m is called the degree of f . The subclass of functions f in $\text{Hyp}_{\text{loc}}(\theta)$ of degree m will be denoted by $\text{Hyp}_{\text{loc}}(\theta, m)$. We have

$$f \in \text{Hyp}_{\text{loc}}(\theta, m) \Rightarrow Pf \in \text{Hyp}(\theta, m).$$

This follows, as in the proof of lemma 2.2, from $r^{-m}f(r(\xi+t\theta)) \rightarrow Pf(\xi+t\theta)$, if we know that $Pf(\theta) \neq 0$. In fact this was included in the original definition of local hyperbolicity, but was later found to be a consequence of (2.5) (see Gårding [3]).

When $f \in \text{Hyp}_{\text{loc}}(\theta)$ we put $\Gamma_0(f, \theta) = \Gamma(Pf, \theta)$. We will use the notation T_η to denote the translation operator, i.e. $(T_\eta f)(\xi) = f(\eta + \xi)$. When the localization a_ζ of a at ζ is hyperbolic with respect to θ , we put $\Gamma(a_\zeta, \theta) = \Gamma(A_\zeta, \theta)$, where A_ζ denotes the real surface $a_\zeta(\xi)=0, \xi \in \mathbf{R}^n$.

Definition 2.8. A map $\tau \rightarrow c_\tau$, from a topological space to open sets in some \mathbf{R}^n , is said to be inner continuous if any compact set contained in c_{τ_0} is also contained in c_τ when τ is close enough to τ_0 .

It is proved in Gårding [4] that if $h \in \text{Hyp}_{\text{loc}}(\theta)$, the function

$$\mathbf{R}^n - \{0\} \ni \xi \rightarrow \Gamma_0(T_\xi h, \theta)$$

is inner continuous when ξ is small enough.

Definition 2.9. A map $\tau \rightarrow M_\tau$, from a topological space to compact sets in some \mathbf{R}^n is said to be outer continuous if any compact neighbourhood of M_{τ_0} contains M_τ when τ is close enough to τ_0 .

For reference we quote the Main Lemma of Gårding [4].

Main Lemma. Let $f \in \text{Hyp}_{\text{loc}}(\theta)$ and let η belong to a compact part of $\Gamma_0(f, \theta)$. Then

$$\xi \text{ real, } \text{Im } s \text{ Im } t \geq 0, \text{ Im } (s+t) \neq 0 \Rightarrow f(\xi + s\theta + t\eta) \neq 0,$$

provided that ξ, s, t are small enough.

Now comes one of the main results:

Theorem 2.10. A homogeneous polynomial a belongs to $\text{He}(\theta)$ if and only if all the functions $\xi \rightarrow T_\zeta a(\xi)$ are in $\text{Hyp}_{\text{loc}}(\theta)$ when $\zeta \neq 0$ is real. If these conditions hold then $a \in \text{He}(\Gamma)$ where

$$\Gamma = \bigcap_{0 \neq \zeta \in \mathbf{R}^n} \Gamma(A_\zeta, \theta).$$

Proof. Let $a \in \text{He}(\theta)$ and let $0 \neq \zeta \in \mathbf{R}^n$ be fixed. Then $a(\zeta + \xi + t\theta) = a(\zeta + \xi + \text{Re } t\theta + i \text{Im } t\theta) \neq 0$ provided that $0 < |\text{Im } t| < c|\zeta + \xi + \text{Re } t\theta|$.

This is clearly satisfied if ξ and t are small and $\text{Im } t \neq 0$, which means that $T_\zeta a \in \text{Hyp}_{\text{loc}}(\theta)$. Conversely assume that $T_\zeta a \in \text{Hyp}_{\text{loc}}(\theta)$ for all real $\zeta \neq 0$. Then by definition $a(\zeta + \xi + it\theta) \neq 0$ if ξ, t real, $|\xi| < \delta_\zeta$, $0 < |t| < \delta_\zeta$. A covering of the unit sphere then gives

$$\xi, t \text{ real, } |\xi| = 1, 0 < |t| < \delta \Rightarrow a(\xi + it\theta) \neq 0,$$

which by homogeneity implies that $a \in \text{He}(\theta)$.

From $T_\zeta a \in \text{Hyp}_{\text{loc}}(\theta)$ we conclude that the cones $\zeta \rightarrow \Gamma_0(T_\zeta a, \theta) = \Gamma(A_\zeta, \theta)$ (note that $PT_\zeta a = a_\zeta$) form an inner continuous family. Since $\bigcap_{\zeta \neq 0} \Gamma(A_\zeta, \theta)$ contains θ , it follows from this, that Γ is an open cone containing θ .

Let K a compact part of Γ . Then the Main Lemma of Gårding [4] gives

$$a(\zeta + \xi + it\eta) \neq 0$$

provided that ξ and t are small real, $t \neq 0$ and $\eta \in K$. As before a covering of the sphere $|\zeta| = 1$ proves that $a \in \text{He}(\Gamma)$.

We will call a polynomial $a \in \text{Hyp}(\theta)$ strongly hyperbolic with respect to θ , if the roots t of $a(\xi + t\theta)$ are real and different for all real ξ which are not proportional to θ .

Proposition 2.11. Let a be a homogeneous polynomial with real coefficients. Suppose that the localizations a_ζ are strongly hyperbolic with respect to θ for all real $\zeta \neq 0$. Then it follows that $a \in \text{He}(\theta)$.

Proof. Let p denote the multiplicity of a at ζ and m the degree of a . Expanding $a(\zeta + \xi + t\theta)$ in powers of t we get

$$a(\zeta + \xi + t\theta) = \sum_0^m t^v f_v(\xi).$$

For $\xi=0$ we have $f_v(0)=0$ when $0 \leq v < p$ and $f_p(0)=a_\zeta(\theta) \neq 0$ by assumption. From the continuity of the roots it follows that for small ξ , p of the roots of $t \rightarrow a(\zeta + \xi + t\theta)$ are small, while the other roots are bounded away from zero (depending on ζ). We shall prove that the small roots are real when ξ and ζ are real and $\zeta \neq 0$.

Factorizing we have

$$a(\zeta + \xi + t\theta) = a(\theta) \prod_1^m (t + \mu_k(\xi)) \quad (2.6)$$

where $-\mu_j(\xi)$, $1 \leq j \leq m$, are the zeros of the polynomial. We arrange the labelling so that $-\mu_1, \dots, -\mu_p$ are the small roots. Also, factorizing $a_\zeta(\xi + t\theta)$ we have

$$a_\zeta(\xi + t\theta) = a_\zeta(\theta) \prod_1^p (t + \lambda_k(\xi)),$$

which defines the roots $-\lambda_k(\xi)$, $1 \leq k \leq p$. From (2.6) we get

$$\tau^{-p} a(\zeta + \tau(\xi + t\theta)) = a(\theta) \prod_1^p (t + \mu_k(\tau\xi)/\tau) \cdot \prod_{p+1}^m (\tau t + \mu_k(\tau\xi))$$

and this, combined with the continuity of the roots and the fact that $\tau^{-p} a(\zeta + \tau(\xi + t\theta)) \rightarrow a_\zeta(\xi + t\theta)$ when $\tau \rightarrow 0$, implies that

$$\mu_k(\tau\xi)/\tau \rightarrow \lambda_k(\xi) \quad \text{as } \tau \rightarrow 0, \quad (2.7)$$

if the roots are labelled properly. Also, the convergence is uniform on compact sets.

Now assume that for arbitrarily small τ some $\mu_i(\tau\xi)$, $1 \leq i \leq p$, is non-real for some ξ with $|\xi| \leq 1$. Then there are sequences $\tau_v \rightarrow 0$ and $\xi_v \rightarrow \xi^0$, $|\xi_v| \leq 1$, such that say $\text{Im } \mu_1(\tau_v \xi_v) > 0$ for all v . The set

$$V = \{(\tau, \xi, \text{Re } t, \text{Im } t) \in \mathbb{R}^{n+3}; \tau^{-p} a(\zeta + \tau(\xi + t\theta)) = 0\}$$

is algebraic and from what has just been said it follows that the point $(0, \xi^0, 0, 0)$ belongs to the closure of $V \cap \{(\tau, \xi, \text{Re } t, \text{Im } t) \in \mathbb{R}^{n+3}; \text{Im } t > 0\}$. The curve selection lemma (see Milnor [7]) shows that there is a real analytic curve $s \rightarrow (\tau(s), \xi(s)) \in \mathbb{R}^{n+1}$, defined for small s , such that $(\tau(0), \xi(0)) = (0, \xi^0)$, $\tau(s) \neq 0$ and such that some root $\mu_i(\tau(s) \cdot \xi(s))$, $1 \leq i \leq p$, say $\mu_1(\tau(s) \cdot \xi(s))$ is non-real when $s > 0$ is small. Since $\mu_1(\xi)$ is real if ξ is proportional to θ , a Taylor expansion shows that for some $k \geq 0$ we have

$$\xi(s) = p(s)\theta + s^k \eta + O(s^{k+1})$$

where p is a polynomial of degree less than k and η is not proportional to θ . Since $\mu_1(\zeta + s\theta) = \mu_1(\zeta) + s$ we conclude that $\mu_1(\tau(s) \cdot (s^k \eta + O(s^{k+1})))$ is non-real and from (2.7) we get

$$\mu_1(\tau(s)(s^k \eta + O(s^{k+1}))) / \tau(s) s^k \rightarrow \lambda_1(\eta) \quad \text{when } s \rightarrow 0, \quad s > 0. \quad (2.8)$$

Now, since the coefficients of a are real $\bar{\mu}_1$ is also a root, say $\bar{\mu}_1 = \mu_2$ and since by assumption λ_1 is real, we conclude, taking complex conjugates in (2.8), that $\lambda_1(\eta) = \lambda_2(\eta)$. This contradicts the strong hyperbolicity of a_ζ and thus proves that $\mu_i(\tau\zeta)$, $1 \leq i \leq p$, are real when ζ, τ are real, $|\zeta| \leq 1$ and τ is small. This, combined with (2.6) and the fact $\mu_k(\zeta)$ are bounded away from zero when $p < k \leq m$, shows that when ζ is a fixed real vector different from zero, we have

$$\zeta, t \text{ small real, } t \neq 0 \Rightarrow a(\zeta + \zeta + it\theta) \neq 0,$$

and as before a covering gives the desired result.

We shall give an example showing the necessity of the assumptions in this proposition.

Example. The polynomial

$$a(\zeta) = \zeta_1^2 + \zeta_2^2 + 2i\zeta_1\zeta_3$$

is not in $\text{He}(\theta)$, $\theta = (1, 0, 0)$. In fact, $a(it, 1, \zeta_3) = -(t + \zeta_3)^2 + 1 + \zeta_3^2$ has a root $t \sim 1/(2\zeta_3)$ for large ζ_3 . On the other hand the localizations a_ζ of a are either constants (when $a(\zeta) \neq 0$) or we have $a_\zeta(\xi) = 2i\zeta_3\xi_1$ (when $a(\zeta) = 0 \Leftrightarrow \zeta_1 = \zeta_2 = 0$) and these are all strongly hyperbolic with respect to θ . This shows that the condition that the coefficients are real in the previous proposition is necessary. That the localizations have to be strongly hyperbolic for the conclusion to be valid follows from the example

$$a(\zeta) = (\zeta_1^2 + \zeta_2^2)^2 + 4\zeta_1^2\zeta_3^2$$

where the localizations a_ζ are either constants or multiples of ζ_1^2 . Since this polynomial has the previous one as a factor, it is not in $\text{He}(\theta)$, $\theta = (1, 0, 0)$.

The wave front surface. We shall end this section by studying the wave front surface

$$W(A, \theta) = \bigcup_{0 \neq \xi \in \mathbb{R}^n} K(A_\xi, \theta)$$

of a polynomial $a \in \text{He}(\theta)$. Here the cones

$$K(A_\xi, \theta) = \{x; \langle x, \eta \rangle \geq 0 \forall \eta \in \Gamma(A_\xi, \theta)\}$$

are called the local propagation cones. Putting as before

$$\Gamma(A, \theta) = \bigcap_{0 \neq \xi \in \mathbb{R}^n} \Gamma(A_\xi, \theta),$$

we shall call its dual

$$K(A, \theta) = \{x; \langle x, \eta \rangle \geq 0 \forall \eta \in \Gamma(A, \theta)\}$$

the singular propagation cone of a . We have

Lemma 2.12. *Let $a \in \text{He}(\theta)$. Then the wave front surface is a closed set whose convex hull is equal to the singular propagation cone of a .*

Proof. From the inner continuity of the map $\xi \rightarrow \Gamma(A_\xi, \theta)$ we conclude that if $x \notin K(A_{\xi_0}, \theta)$, then $y \notin K(A_\xi, \theta)$, provided that y, ξ are close to x and ξ_0 respectively. This and a covering argument shows that if $x \notin W(A, \theta)$ there is a neighbourhood of x which is disjoint from $W(A, \theta)$.

We know from theorem 2.10 that $a \in \text{He}(\theta)$ implies that $a \in \text{He}(\Gamma)$, where $\Gamma = \bigcap_{\xi \neq 0} \Gamma(A_\xi, \theta)$. Since $a \in \text{He}(\tilde{\Gamma})$ implies that $a_\xi \in \text{Hyp}(\eta)$ for all real $\xi \neq 0$ and all $\eta \in \tilde{\Gamma}$, it follows that Γ is the maximal cone for which $a \in \text{He}(\Gamma)$.

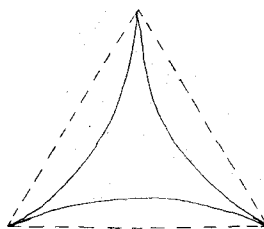
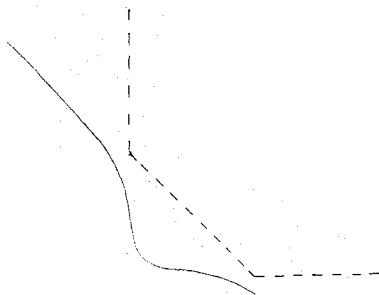
Assume that $\text{ch } W(A, \theta) \not\subseteq K(A, \theta)$ where ch denotes the convex hull. Then according to section 4. and theorem 1.6, $a \in \text{He}(\Gamma_1)$ where Γ_1 is the dual of $\text{ch } W(A, \theta)$. But Γ , being the dual of $K(A, \theta)$, is strictly contained in Γ_1 , contradicting the maximality of Γ .

The following lemma is proved in the hyperbolic case in [2]. The same proof works in the hybrid case so we have:

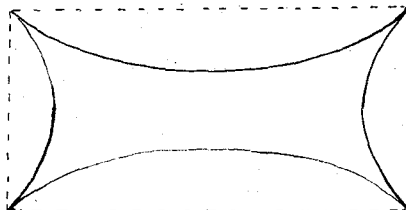
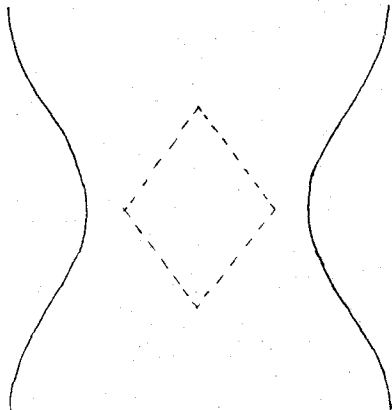
Lemma 2.13. *The wave front surface is contained in a proper conical subvariety.*

Examples. We have sketched the image in real projective two-space of some A and $W(A, G)$. The figures to the left show A and the ones to the right the corresponding $W(A, \theta)$. Dotted lines indicate the boundaries of $\Gamma(A, \theta)$ and $K(A, \theta)$ respectively.

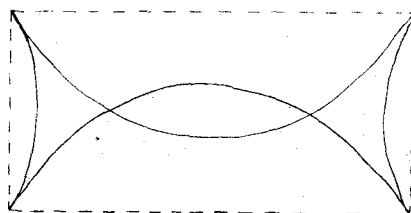
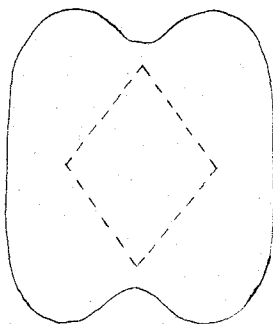
$$1. a(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3, \quad \theta = (1, 1, 1)$$



$$2. a(\xi) = \xi_1^4 + \xi_1^2(\xi_2^2 + \xi_3^2) - \xi_3^4, \quad \theta = (1, 0, 0)$$

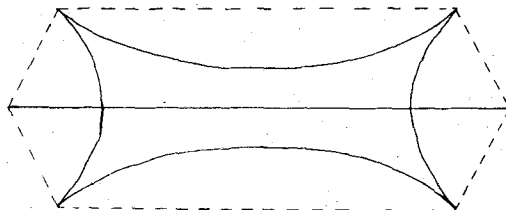
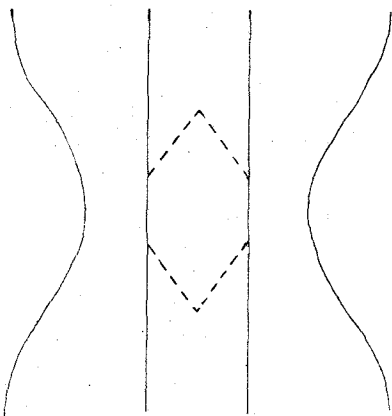


$$3. a(\xi) = \xi_1^2 \xi_2^2 - \xi_1^3 \xi_2 - \xi_1^2 \xi_3^2 + \xi_3^4, \quad \theta = (2, 1, 0)$$



$$4. a(\xi) = \varepsilon \xi_1^6 + \xi_1^4(\varepsilon \xi_2^2 - (1 - \varepsilon) \xi_3^2) - \xi_1^2(\xi_2^2 \xi_3^2 + (1 + \varepsilon) \xi_3^4) + \xi_3^6,$$

where $\varepsilon > 0$ is small, $\theta = (1, 0, 0)$



3. The non-homogeneous case

Precisely as for homogeneous polynomials it is possible to simplify the definition of $\text{he}(\theta)$ by requiring the crucial condition (1.6) for only one vector. The definition runs as follows:

Definition 3.1. A polynomial P is said to be a hybrid or hyperbolic-elliptic with respect to θ if

- (i) $P_m(\theta) \neq 0$ where P_m denotes the principal part of P .
- (ii) There are constants $c_1, c_2 > 0$ such that

$$\xi, t \text{ real, } c_1 < t < c_2 |\xi| \Rightarrow P(\xi - it\theta) \neq 0.$$

The class of these operators will be denoted by $\text{he}(\theta)$.

Note that (i) is not a consequence of (ii) as follows from the example $P(\xi) = i\xi_1 + i\xi_2^2$, $\theta = (1, 0)$.

Lemma 2.2 still holds and the proof is the same. We have

Lemma 3.2. $P \in \text{he}(\theta) \Rightarrow P_m \in \text{He}(\theta)$, where P_m denotes the principal part of P .

The class $\text{Hyp}_{\text{loc}}(\theta)$ of locally hyperbolic functions was an important companion to $\text{He}(\theta)$. The corresponding companion to $\text{he}(\theta)$ is the class $\text{hyp}_{\text{loc}}(\theta)$ of locally impurely hyperbolic functions. The definition is as follows (Gårding [4]).

Definition 3.3. A function $f(\xi, \tau)$, analytic in a neighbourhood of the origin, is said to be locally impurely hyperbolic with respect to $\theta \in \mathbb{R}^n$, if there is a number $c > 0$ such that

$$\xi, t \text{ real, } \text{Im } t/\tau > c \Rightarrow f(\xi - t\theta, \tau) \neq 0 \quad (3.1)$$

when ξ, t, τ are small enough. (When $\tau = 0$ the inequality should read $\text{Im } t \neq 0$.) The class of functions satisfying (3.1) will be denoted by $\text{hyp}_{\text{loc}}(\theta)$ ($\text{hyp}_{\text{loc}}(\theta, m)$ if m is the degree of $f(\cdot, 0)$).

Note. Gårding has $|\text{Im } t| > c|\tau|$ in (3.1) and he also requires that $Pf(\theta, 0) \neq 0$ where P denotes the principal part. The last requirement is superfluous because $f(\xi, \tau) \in \text{hyp}_{\text{loc}}(\theta)$ obviously implies that $f(\xi, 0) \in \text{Hyp}_{\text{loc}}(\theta)$. As explained below the first requirement is also superfluous.

We shall need some of Gårding's key results, but since we cannot quote them directly, we shall state the modified versions here. First, Gårding's version of Svensson's theorem.

Svensson's theorem. Let $f(\xi, \tau) \in \text{hyp}_{\text{loc}}(\theta, m)$ and expand in powers of τ

$$f(\xi, \tau) = \sum_{v=0}^{m-1} \tau^v f_{m,v}(\xi) + \tau^m f_m(\xi, \tau). \quad (3.2)$$

Then $\deg f_{m,v} \equiv m-v$ when $0 < v < m$ and the quotients

$$t^v f_{m,v}(\xi - it\theta) / f_{m,0}(\xi - it\theta), \quad 0 < v < m,$$

are bounded for small $(\xi, t) \neq (0, 0)$. Conversely, if $f_{m,0} \in \text{Hyp}_{\text{loc}}(\theta, m)$ and if $f_{m,v}$, $0 < v \leq m$, are analytic, the quotients bounded and f defined by (3.2), then f belongs to $\text{hyp}_{\text{loc}}(\theta)$ and there is a constant $c > 0$ such that

$$\xi \text{ real}, \quad |\text{Im } t| > c|\tau| \Rightarrow f(\xi - it\theta, \tau) \neq 0, \quad (3.3)$$

when ξ, τ, t are small enough and τ is allowed to be complex.

The proof is the same as in Gårding [4], only change the formula on page 77 to $s \text{ real} \Rightarrow \text{Im } \mu_k(s)/s \leq c$ and on page 79, $\text{Im } \lambda_k(r) = O(r^{p+1})$ shall be changed to $\text{Im } \lambda_k(r)/r^{p+1} \leq c$.

For the reader's convenience we also state Gårding's Main Lemma*, which we shall use later.

Main lemma*. When $f \in \text{hyp}_{\text{loc}}(\theta)$ satisfies (3.3) for complex τ , and η belongs to a compact part of $\Gamma_0(f(\cdot, 0), \theta)$, then

$$\xi \text{ real}, \quad \text{Im } s \text{ Im } t \geq 0, \quad |\text{Im } s| > c|\tau| \Rightarrow f(\xi + s\theta + t\eta, \tau) \neq 0$$

when ξ, τ, s, t are small enough.

The following theorem connects the definitions 3.1 and 3.2.

Theorem 3.4. Let P be a polynomial of degree m with principal part a . Then $P \in \text{he}(\theta)$ if and only if the functions $f_\zeta(\xi, \tau) = \tau^m P(\tau^{-1}(\xi + \zeta))$ are in $\text{hyp}_{\text{loc}}(\theta)$ for all real $\zeta \neq 0$. If these conditions hold then $P \in \text{he}(\Gamma)$ where

$$\Gamma = \bigcap_{0 \neq \xi \in \mathbb{R}^n} \Gamma(A_\xi, \theta).$$

Proof. Let $P \in \text{he}(\theta)$. Then from the definition $P(\tau^{-1}(\zeta + \xi - t\theta)) \neq 0$ provided ξ, ζ, τ are real, $\tau \neq 0$ and

$$c_1 < \text{Im } t/\tau < c_2 |\zeta + \xi - \text{Re } t\theta|/|\tau|.$$

This condition certainly holds if $\text{Im } t/\tau > c_1$, $\zeta \neq 0$ is fixed and ζ, t are small. when $\tau = 0$, $f_\zeta(\xi, 0) = a(\zeta + \xi)$, and by lemma 3.2 and theorem 2.10, this belongs to $\text{Hyp}_{\text{loc}}(\theta)$. This shows that $f_\zeta \in \text{hyp}_{\text{loc}}(\theta)$ when $\zeta \neq 0$ is real. Conversely suppose that this holds and let K be a compact part of Γ . Then, by Svensson's theorem and Main Lemma*, to every real ζ with $|\zeta| = 1$ there are numbers c_ζ and d_ζ such that

$$\xi \text{ real}, \quad \eta \in K, \quad |\text{Im } t| > c_\zeta |\tau| \Rightarrow \tau^m P(\tau^{-1}(\zeta + \xi + t\eta)) \neq 0$$

when $|\zeta| < d_\zeta$, $|t| < d_\zeta$, $|\tau| < d_\zeta$. Hence, by a covering argument, there are numbers c_1 and d_1 such that $\tau^m P(\tau^{-1}(\zeta + t\eta)) \neq 0$ when ζ is real, $|\zeta| = 1$, $|\text{Im } t| > c_1 |\tau|$ and $|t| < d_1$, $|\tau| < d_1$. Letting $\tau = 0$, we see that $a(\zeta + t\eta) \neq 0$ if ζ real $|\zeta| = 1$, $\text{Im } t \neq 0$, and

$|t| < d_1$, which implies in particular that $a(\eta) \neq 0$. Letting instead $\tau \neq 0$ real, $t = i\tau s$ purely imaginary and $\zeta = \tau\xi$, we get that $P(\xi + is\eta) \neq 0$ when $c < |s| < d_1 |\xi|$. This proves the theorem and also gives;

Corollary 3.5. $\text{he}(\theta) = \text{he}(-\theta)$.

A polynomial P is said to be weaker than another polynomial Q , and we write $P < Q$, if there is a constant c such that

$$\tilde{P}(\xi) \leq c\tilde{Q}(\xi)$$

for all real ξ , where

$$\tilde{P}(\xi) = \left(\sum |P^{(\alpha)}(\xi)|^2 \right)^{\frac{1}{2}}$$

(Hörmander [5]). The following theorem extends Svensson's original result.

Theorem 3.6. *Let P be a polynomial of degree m with principal part P_m . Then $P \in \text{he}(\theta)$ if and only if $P_m \in \text{He}(\theta)$ and P is weaker than P_m .*

Proof. Put $f_\zeta(\xi, \tau) = \tau^m P(\tau^{-1}(\zeta + \xi))$. Suppose that $P \in \text{he}(\theta)$. Then by theorem 3.4 we have $f_\zeta \in \text{hyp}_{\text{loc}}(\theta)$ when $\zeta \neq 0$ is real. We have $f_\zeta(\xi, 0) = P_m(\zeta + \xi) = P_{m\zeta}(\xi) + R(\xi, \zeta)$. Denoting the multiplicity of P_m at ζ by ν , we thus have $f_\zeta \in \text{hyp}_{\text{loc}}(\theta, \nu)$ where ν depends on ζ . Expanding $f_\zeta(\xi, \tau)$ in powers of τ , we get

$$f_\zeta(\xi, \tau) = \sum_{k=0}^m \tau^k P_{m-k}(\zeta + \xi)$$

where P_μ is the part of P that is homogeneous of degree μ . From Svensson's theorem we can now conclude that there is a constant c_ζ , depending on ζ , such that

$$|t^k P_{m-k}(\xi + \zeta + it\theta)| \leq c_\zeta |P_m(\xi + \zeta + it\theta)| \quad (3.4)$$

if $k < \nu$ and ξ, t are small real. We fix k and put

$$A_k = \{\zeta \in \mathbb{R}^n; |\zeta| = 1 \text{ and } P_m^{(\alpha)}(\zeta) = 0 \forall \alpha; |\alpha| = k\}.$$

When $\zeta \in A_k$ we have $\nu = \deg P_{m\zeta} \geq k+1$, so (3.4) is valid in A_k . Since A_k is compact, a covering argument proves that there is a constant c such that

$$|t^k P_{m-k}(\xi + it\theta)| \leq c |P_m(\xi + it\theta)|,$$

provided that t is small and ξ is in a neighbourhood of A_k . By homogeneity we get

$$|P_{m-k}(\xi + i\theta)| \leq c |P_m(\xi + i\theta)|$$

if $|\xi|$ is large and ξ belongs to a conical neighbourhood of A_k . Lemma 3.1.5 in Hörmander [5] then implies

$$\tilde{P}_{m-k}(\xi + i\theta) \leq c\tilde{P}_m(\xi + i\theta)$$

for large and hence for all ξ in a conical neighbourhood of A_k . Finally Taylor's formula implies that

$$\tilde{P}_{m-k}(\xi) \leq c\tilde{P}_m(\xi)$$

in this neighbourhood.

Outside a conical neighbourhood of A_k we have trivially $\tilde{P}_m(\xi) \leq c|\xi|^{m-k}$ and this combined with the above gives

$$\tilde{P}_{m-k}(\xi) \leq c\tilde{P}_m(\xi)$$

for all ξ , i.e. P_{m-k} is weaker than P_m .

Conversely, if $P_m \in \text{He}(\theta)$, theorem 2.10 implies that $P_m \in \text{He}(\Gamma)$, with Γ as in that theorem. Thus we have

$$P_m(\xi + \zeta + i\theta) = P_m(\xi + \text{Re } \zeta + i(\theta + \text{Im } \zeta)) \neq 0$$

if ξ is real and large and ζ is small. From lemma 4.1.1 in Hörmander [5] it then follows that

$$|P_m^{(a)}(\xi + i\theta)| \leq c|P_m(\xi + i\theta)|$$

if $|\xi| \geq c_1$, say, and thus

$$\tilde{P}_m(\xi + i\theta) \leq c|P_m(\xi + i\theta)| \quad \text{if } |\xi| \geq c_1.$$

Since $P < P_m$ implies $P_k < P_m$ according to lemma 5.5.1 in Hörmander [5], we get (c denotes different constants)

$$|P_k(\xi + i\theta)| \leq \tilde{P}_k(\xi + i\theta) \leq c\tilde{P}_k(\xi) \leq c\tilde{P}_m(\xi) \leq c\tilde{P}_m(\xi + i\theta) \leq c|P_m(\xi + i\theta)|$$

if $|\xi| \geq c_1$.

By homogeneity this implies

$$|P_k(\xi + it\theta)| \leq c|t|^{k-m}|P_m(\xi + it\theta)| \quad \text{if } |\xi| \geq c_1|t|,$$

and thus

$$|P(\xi + it\theta) - P_m(\xi + it\theta)| \leq c|t|^{-1}|P_m(\xi + it\theta)|$$

if $|\xi| \geq c_1|t|$. This implies that $P(\xi + it\theta) \neq 0$ if ξ and t are real $2c < |t| < 1/c_1|\xi|$ and $P_m(\xi + it\theta) \neq 0$, from which we conclude that $P \in \text{he}(\theta)$.

We end this section by proving a result that will be needed later.

Theorem 3.7. *Let $P = a + b \in \text{he}(\theta)$ have principal part a and degree m . Further let $\xi \rightarrow M_\xi$ be an outer continuous function with M_ξ a compact part of $\Gamma(A_\xi, \theta)$ satisfying $M_{t\xi} = M_\xi$ when $t \neq 0$ is real. Then there are positive constants c_0 , c_1 and c_2 such that*

$$\xi, t \text{ real}, \quad \eta \in M_\xi, \quad c_1 < |t| < c_2|\xi| \Rightarrow |P(\xi - it\eta)| \geq c_0|t|^m.$$

Proof. We have from theorem 2.10 that $T_\xi a \in \text{Hyp}_{\text{loc}}(\theta, p)$ for all real ξ different from zero, where p denotes the multiplicity of a at ξ . Factorizing we can write

$$T_\xi a(\zeta + t\eta) = F(\zeta, t, \eta) \prod_1^p (t + \lambda_k(\zeta, \eta))$$

where the λ_k , $1 \leq k \leq p$, are real and equal to zero when $\zeta = 0$. Further F is continuous when ζ and t are small and $\eta \in M_\xi$. Since $F(0, 0, \eta) = a_\xi(\eta) \neq 0$, when $\eta \in M_\xi$, we get, with some $c_\xi > 0$,

$$|a(\xi + \zeta + it\eta)| \geq c_\xi |t|^p$$

provided that ζ , t are real and small. A covering of the unit sphere then proves the theorem if $P = a$ is homogeneous.

Now we can as in the previous proof use lemma 4.1.1 in Hörmander [5] to conclude that

$$b(\xi + it\eta) \leq c|t|^{-1}|a(\xi + it\eta)|$$

when ξ , t are real, $\eta \in M_\xi$, $1 < |t| < c_1|\xi|$, and from $P = a + b = a(1 + b/a)$ the theorem follows easily.

4. Construction of fundamental solutions

Let $P \in \text{he}(\theta)$ have principal part a . We are going to show that P has a fundamental solution which is analytic outside the wave front surface $W(P, \theta)$ of P , defined as

$$W(P, \theta) = \bigcup_{\xi \neq 0} K(A_\xi, \theta)$$

where

$$K(A_\xi, \theta) = \{x; \langle x, \eta \rangle \geq 0 \forall \eta \in \Gamma(A_\xi, \theta)\}$$

is the so called local propagation cone belonging to the localization a_ξ of a at ξ . This result is a special case of a theorem by K. G. Andersson (see [1]). The proof in our case is simpler and is similar to the presentation in Atiyah—Bott—Gårding [2].

Let $P \in \text{he}(\theta)$ have principal part a . From theorem 3.7 with $M_\xi = \{\theta\}$ we see that t_0 and γ can be chosen so that $|P(\xi - it_0\theta)| \geq \text{const} > 0$ if $|\xi| \geq \gamma$. In fact, with the notation in that theorem, we just choose t_0 with $t_0 > c_1$ and γ satisfying $\gamma > t_0/c_2$.

We can thus define a distribution E by

$$E(\varphi) = (2\pi)^{-n} \int_{|\xi| \geq \gamma} \frac{\hat{\varphi}(-\xi + it_0\theta)}{P(\xi - it_0\theta)} d\xi, \quad (4.1)$$

where $\varphi \in C_0^\infty$, $\hat{\varphi}(\xi) = \int e^{-i\langle x, \xi \rangle} \varphi(x) dx$ and $d\xi = d\xi_1 \wedge \dots \wedge d\xi_n$. Here $E = E(P, \theta, t_0, \gamma)$ depends on t_0 and γ but we have:

Lemma 4.1. *Let $P \in \text{he}(\theta)$ and define E as above. Then, modulo entire analytic functions, E is a fundamental solution of $P(D)$ and is independent of t_0 and γ as long*

as t_0 and γ are chosen so as to satisfy the condition $c_1 < t_0 < c_2 |\xi|$ in theorem 3.7, when $|\xi| > \gamma$.

Proof. We have

$$\begin{aligned} P(D)E(\varphi) &= E(P(-D)\varphi) = (2\pi)^{-n} \int_{|\xi| \geq \gamma} \hat{\varphi}(-\xi + it_0\theta) d\xi = \\ &= (2\pi)^{-n} \int \hat{\varphi}(-\xi + it_0\theta) d\xi - (2\pi)^{-n} \int_{|\xi| < \gamma} \hat{\varphi}(-\xi + it_0\theta) d\xi. \end{aligned}$$

Put $\Omega_R = \{\zeta = -\xi + is\theta; |\xi| \leq R, 0 \leq s \leq t_0\}$. Since $\hat{\varphi}$ is analytic it follows that the form $\hat{\varphi}(\zeta) d\zeta$, $d\zeta = d\zeta_1 \wedge \dots \wedge d\zeta_n$, is closed which according to Stokes' theorem implies that

$$\int_{\partial\Omega_R} \hat{\varphi}(\zeta) d\zeta = 0.$$

On the part of the boundary $\partial\Omega_R$ where $|\xi| = R$ we have by the Paley—Wiener theorem that

$$|\hat{\varphi}(-\xi + is\theta)| \leq C_N (1 + |-\xi + is\theta|)^{-N} e^{c_0|\theta|} \leq C'_N R^{-N}.$$

We can thus let R tend to infinity in the above integral and get

$$\int \hat{\varphi}(-\xi + it_0\theta) d\xi = \int \hat{\varphi}(-\xi) d\xi = (2\pi)^n \varphi(0),$$

where the last equality follows from Fourier's inversion formula. We thus get using the definition of the Fourier transform and Fubini's theorem that $P(D)E = \delta + h$, where $h(x) = -(2\pi)^{-n} \int_{|\xi| \geq \gamma} \exp(-i\langle x, -\xi + it_0\theta \rangle) d\xi$ is entire analytic. Thus choosing f entire analytic with $P(D)f = h$ proves the first part of the lemma. To show that $E(P, \theta, t_0, \gamma_0) - E(P, \theta, t_1, \gamma_1)$ is entire analytic we first note that if $\gamma_0 < \gamma$ we have

$$(E(P, \theta, t_0, \gamma_0) - E(P, \theta, t_0, \gamma))(\varphi) = (2\pi)^{-n} \int_{\gamma_0 < |\xi| < \gamma} \frac{\hat{\varphi}(-\xi + it_0\theta)}{P(\xi - it_0\theta)} d\xi.$$

It follows from the definition of the Fourier transform and Fubini's theorem that this defines an entire analytic function, so it suffices to show that $E(P, \theta, t_0, \gamma) - E(P, \theta, t_1, \gamma)$ is entire analytic when γ is large.

We then choose γ so large that the condition $c_1 < st_0 + (1-s)t_1 < c_2\gamma$ in theorem 3.7 is satisfied for all $0 \leq s \leq 1$. Then $|P(\xi - i(st_0 + (1-s)t_1)\theta)| \geq \text{const} > 0$ if $0 \leq s \leq 1$ and $|\xi| \geq \gamma$, so the form $\hat{\varphi}(-\xi)/P(\xi) d\xi$ is holomorphic when $\zeta = \xi - i(st_0 + (1-s)t_1)\theta$, $0 \leq s \leq 1$ and $|\xi| \geq \gamma$. The same reasoning as above with Ω_R now equal to $\{\zeta = \xi - i(st_0 + (1-s)t_1)\theta; 0 \leq s \leq 1, \gamma \leq |\xi| \leq R\}$ shows that

$$(E(P, \theta, t_0, \gamma) - E(P, \theta, t_1, \gamma))(\varphi) = (2\pi)^{-n} \int_K \frac{\hat{\varphi}(-\zeta)}{P(\zeta)} d\zeta,$$

where $K = \{\zeta = \xi - i(st_0 + (1-s)t_1)\theta; |\xi| = \gamma, 0 \leq s \leq 1\}$ with proper orientation. Since K is compact it follows that the last integral defines an entire analytic function.

To prove that E is analytic outside the wave front surface we shall replace the constant vectorfield $\xi \rightarrow t_0 \theta$ in (4.1) by a smooth variable vectorfield $\xi \rightarrow v(\xi)$. We collect the facts that we need about v in the following:

Lemma 4.2. *Let $P \in \text{he}(\theta)$ have principal part a . Then*

1. *There exists a C^∞ vectorfield v , homogeneous of degree one, such that $v(\xi) \in \Gamma(A_\xi, \theta)$ if $\xi \neq 0$, and a piecewise smooth homotopy $w(s, \xi)$ that for suitable t_0 connects $w(0, \xi) = t_0 \theta$ and $v(\xi)$. Further the derivative of w is bounded by a constant times $|\xi|$ and $|P(\xi - iw(s, \xi))| \geq \text{const} > 0$ when ξ is large and as $0 \leq s \leq 1$.*
2. *If $x \notin W(P, \theta)$ we can choose the vectorfield $v(\xi)$ and the homotopy $w(s, \xi)$ in 1. so as to satisfy $\langle x, v(\xi) \rangle < 0$ respectively $\langle x, w(s, \xi) \rangle \leq \text{const}$ for all $\xi \neq 0$ and for all $0 \leq s \leq 1$.*

Proof. Let $x \notin W(P, \theta)$. Then $x \notin K(A_\xi, \theta)$ when $\xi \neq 0$, and hence, by definition we can find an $\eta_\xi \in \Gamma(A_\xi, \theta)$ such that $\langle x, \eta_\xi \rangle < 0$. From the inner continuity of the map $\xi \rightarrow \Gamma(A_\xi, \theta)$ we see that $\eta_\xi \in \Gamma(A_{\xi+\zeta}, \theta)$ then ζ is small. A covering of the unit sphere gives points η_i and neighbourhoods U_{ξ_i} , $1 \leq i \leq N$, such that $\eta_i \in \Gamma(A_{\xi_i}, \theta)$ when $\xi \in U_{\xi_i}$ and such that $\langle x, \eta_i \rangle < 0$ for all $1 \leq i \leq N$. Let φ_i be a partition of unity on the unit sphere subordinate to the cover U_{ξ_i} and put

$$v(\xi) = |\xi| \sum_{i=1}^N \varphi_i(\xi/|\xi|) \eta_i.$$

Then $v(\xi) \in \Gamma(A_\xi, \theta)$ and $\langle x, v(\xi) \rangle < 0$.

By choosing M_ξ in theorem 3.7 as $\{s\theta + (1-s)v(\xi)/|\xi|; 0 \leq s \leq 1\}$ we see that

$$|P(\xi - it(s\theta + (1-s)v(\xi)/|\xi|))| \geq c_0 t^m$$

if ξ, t real, $0 \leq s \leq 1$ and $c_1 < t < c_2 |\xi|$. Choosing a $t = t_0$ in this interval and varying s , we obtain a homotopy between $t_0 \theta$ and $t_0 v(\xi)/|\xi|$ and then by varying t_0 we obtain a homotopy connecting $t_0 v(\xi)/|\xi|$ and $cv(\xi)$ if $c_1 < cv(\xi) < c_2 |\xi|$, which is satisfied for large ξ and suitable c . Since $\langle x, v(\xi) \rangle < 0$ if $\xi \neq 0$ it is trivial that $\langle x, w(s, \xi) \rangle \leq \text{const}$ when $\xi \neq 0$, $0 \leq s \leq 1$, and that proves the lemma with $v = cv$.

Theorem 4.3. *Let $P \in \text{he}(\theta)$. Then P has a fundamental solution that is real analytic outside the wave front surface.*

Proof. Let $x_0 \notin W(P, \theta)$ and choose v according to the previous lemma such that $\langle x_0, v(\xi) \rangle < 0$ for all $\xi \neq 0$. If ω is a small neighbourhood of x_0 we thus have with a positive constant c ,

$$\langle x, v(\xi) \rangle \leq -c|\xi| \quad \text{when } x \in \omega \quad \text{and } \xi \neq 0. \quad (4.2)$$

Let $\varphi \in C_0^\infty(\omega)$ and put

$$\Omega_R = \{\zeta = \xi - iw(s, \xi); \gamma \leq |\xi| \leq R, 0 \leq s \leq 1\},$$

where $w(s, \xi)$ is the homotopy in lemma 4.2. Then $|P(\zeta)| \geq \text{const} > 0$ in Ω_R if γ is

properly chosen. Since $\varphi(-\zeta)/P(\zeta)$ is analytic in Ω_R the form $\varphi(-\zeta)/P(\zeta) d\zeta$, $d\zeta = d\zeta_1 \wedge \dots \wedge d\zeta_n$, is closed, which according to Stokes' theorem implies

$$\int_{\partial\Omega_R} \frac{\varphi(-\zeta)}{P(\zeta)} d\zeta = 0. \quad (4.3)$$

Since $\langle x, \operatorname{Im} \zeta \rangle$ is bounded from above when $x \in \omega$, $\zeta \in \Omega_R$ we have $|\varphi(-\zeta)| = O(|\zeta|^{-N})$ for all N if $\zeta \in \Omega_R$. This implies that we can let $R \rightarrow \infty$ in (4.3), giving

$$E(P, t_0, \eta, \gamma)(\varphi) = (2\pi)^{-n} \int_K \varphi(-\zeta)/P(\zeta) d\zeta + (2\pi)^{-n} \int_{\substack{\zeta = \xi - i\nu(\xi) \\ |\xi| \geq \gamma}} \varphi(-\zeta)/P(\zeta) d\zeta \quad (4.4)$$

with $K = \{\zeta \in \Omega_R : |\operatorname{Re} \zeta| = \gamma\}$ properly oriented. Since K is compact the first integral defines an analytic function in \mathbb{C}^n and since (4.2) holds, the second integral defines a real analytic function in ω . This proves the theorem.

To abbreviate we will not write out the dependence of t_0 and γ in $E(P, t_0, \eta, \gamma)$ in the sequel. We will always assume that they are properly chosen and write $E(P, \eta)$ or $E(P, \eta, x)$ when $x \notin W(A, \theta)$.

It is now easy to prove the following result mentioned in the introduction.

Theorem 4.4. *Let $P = a + b \in \operatorname{he}(\theta)$ have principal part a . Then, with convergence in the distribution sense, we have*

$$E(P, \theta) = \sum_{k=0}^{\infty} (-1)^k b(D)^k E(a^{k+1}, \theta).$$

Proof. From (4.1) we have

$$E(P, \theta)(\varphi) = (2\pi)^{-n} \int_{|\xi| \geq \gamma} \frac{\varphi(-\xi + i t \theta)}{P(\xi - i t \theta)} d\xi.$$

From the proof of theorem 3.6 we have that $|b(\xi - i t \theta)| \leq c \cdot t^{-1} |a(\xi - i t \theta)|$ for large ξ , implying that

$$\frac{1}{P(\xi - i t \theta)} = \frac{1}{a(\xi - i t \theta)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{b(\xi - i t \theta)}{a(\xi - i t \theta)} \right)^k$$

with uniform convergence if $t > 2c$. Thus choosing γ and t properly we can integrate term by term which proves the result.

When $P = a$ is homogeneous we can simplify our formulas by introducing polar coordinates. As a preparation, put

$$\chi_p(z) = (2\pi)^{-n} \int_1^{\infty} e^{irz} r^{p-1} dr \quad \text{when } \operatorname{Im} z > 0.$$

From $\chi'_p(z) = i\chi_{p+1}(z)$ and $\chi_1(z) = (2\pi)^{-n} i e^{iz}/z$ we get

$$\chi_p(z) = \frac{(2\pi)^{-n} i^p (p-1)!}{z^p} + f_p(z), \quad p \geq 1,$$

$$\chi_p(z) = \frac{(2\pi)^{-n} i^{2-p} z^{-p} \log z}{(-p)!} + f_p(z), \quad p \leq 0,$$

where $f_p(z)$ are entire analytic functions for all p . We will denote the singular part of χ_p by χ_p^0 i.e. $\chi_p^0(z) = \chi_p(z) - f_p(z)$. To simplify the calculations we will also assume that the constant γ appearing earlier can be chosen equal to one. This can always be achieved by a change of scale if necessary.

Theorem 4.5. *Let $a \in \text{He}(0)$ be a polynomial in \mathbb{C}^n of degree m . If $x_0 \notin W(A, \theta)$ there exists a neighbourhood ω of x_0 , a vector field $v(\xi)$ such that, modulo an entire analytic function, $a(D)$ has a fundamental solution of the form*

$$F(a, \theta, x) = \int_{\Omega} \chi_a(\langle x, \xi \rangle) a(\xi)^{-1} \omega(\xi) \quad \text{when } x \in \omega. \quad (4.5)$$

Here $q = n - m$, $\omega(\xi) = \sum (-1)^i \xi_i d\xi_1 \wedge \dots \wedge \widehat{d\xi_i} \wedge \dots \wedge d\xi_n$ and

$$\Omega = \{\xi = \xi - iv(\xi); |\xi| = 1\}.$$

Proof. Formula (4.4) shows that when $\varphi \in C_0^\infty(\omega)$ and $\langle x, v(\xi) \rangle \leq -c|\xi|$ when $x \in \omega$, the integral

$$F(a, \theta)(\varphi) = (2\pi)^{-n} \int_{\substack{\xi = \xi - iv(\xi) \\ |\xi| > 1}} \frac{\widehat{\varphi}(-\xi)}{a(\xi)} d\xi$$

defines a fundamental solution of $a(D)$ modulo an entire analytic function. Inserting the definition of the Fourier transform and switching to polar coordinates gives

$$\begin{aligned} F(a, \theta, x) &= (2\pi)^{-n} \int_{\substack{\xi = \xi - iv(\xi) \\ |\xi| = 1}} \omega(\xi) \int_1^\infty \frac{e^{ir\langle x, \xi \rangle}}{a(\xi)} r^{n-1-m} dr = \\ &= \int_{\Omega} \chi_a(\langle x, \xi \rangle) a(\xi)^{-1} \omega(\xi), \end{aligned}$$

and since obviously $\int_{\Omega} \chi_a(\langle x, \xi \rangle) a(\xi)^{-1} \omega(\xi)$ is entire analytic the result follows.

The Herglotz—Petrovsky—Leray formulas. One main point of [2] was to prove that, outside the wave front surface, the derivatives of the fundamental solution of hyperbolic operators are periods of rational closed differential forms in projective space. The corresponding formulas were named after Herglotz, Petrovsky and Leray who found various special cases. We shall sketch the corresponding result in the hybrid case.

To state it we shall need the Petrovsky homology classes, defined as follows. When x is outside the wave front surface $W(A, \theta)$, there are real C^∞ vector fields $v(\xi)$, absolutely homogeneous of degree one, i.e. $v(\lambda\xi) = |\lambda|v(\xi)$, such that $v(\xi) \in \text{Re } X \cap \Gamma(A_\xi, \theta)$ for every $\xi \neq 0$. Here X is the complex hyperplane $\langle x, \xi \rangle = 0$, and in the sequel we let A denote the complex surface $a(\xi) = 0$. Let Z be \mathbb{C}^n and let Z^* be the corresponding complex projective space, i.e. $Z^* = \dot{Z}/\dot{C}$. When $t > 0$ is small enough, the map $\xi \rightarrow \xi - itv(\xi)$ from the real $(n-1)$ -sphere oriented by $\langle x, \xi \rangle \omega(\xi) > 0$, where $\omega(\xi) = \sum (-1)^{i-1} \xi_i d\xi_1 \wedge \dots \wedge \widehat{d\xi_i} \wedge \dots \wedge d\xi_n$, into Z^* defines a $(n-1)$ -cycle of the pair $(Z^* - A^*, X^*)$. Here the star denotes images in projective space. All these

cycles are homologous and define a homology class in $H_{n-1}(Z^* - A^*, X^*)$ denoted by $2\alpha(A, x, \theta)^*$. The boundary $\beta(A, x, \theta)^* = \partial\alpha(A, x, \theta)^* \in H_{n-2}(X^* - A^* \cap X^*)$ is called the Petrovsky cycle (class).

The difference $F(a, \theta, x) - F(a, -\theta, x)$, where F is given by (4.5), can be analyzed precisely as in [2] p. 175—176. Keeping track of the entire function that appears in the hybrid case, one gets the following result, where $t_x: H_{n-2}(X^* - X^* \cap A^*) \rightarrow H_{n-1}(Z^* - A^* \cup X^*)$ is a tube operation described in [2] p. 173.

Theorem 4.6. *Let $a \in \text{He}(\theta)$ be of degree m and let $E(a, \theta, x)$ be the fundamental solution of $a(D)$ with singular support in $W(A, \theta)$ given by (4.5). If x is not in $\pm W(A, \theta)$ then, modulo entire functions, we have*

$$D^v(E(a, \theta, x) - E(a, -\theta, x)) = c_v \int_{\alpha^*} \langle x, \xi \rangle^q a(\xi)^{-1} \omega(\xi) \quad (4.6)$$

if $q \geq 0$ and

$$D^v(E(a, \theta, x) - E(a, -\theta, x)) = c_v \int_{t_x \beta^*} \langle x, \xi \rangle^q a(\xi)^{-1} \omega(\xi) \quad (4.7)$$

when $q < 0$. Here $q = m - n - |v|$, $c_v \neq 0$ is a constant, $\alpha^* = \alpha(A, x, \theta)^* \in H_{n-1}(Z^* - A^*, X^*)$ and $\beta^* = \partial\alpha(A, x, \theta)^* \in H_{n-2}(X^* - X^* \cap A^*)$ is the Petrovsky class.

In the hyperbolic case the term $E(a, -\theta, x)$ vanishes when x is not in $-K(A, \theta)$; in the hybrid case it is real analytic there.

Lacunae, sharp fronts, the Petrovsky conditions. A component L of the complement of the wave front surface $W(A, \theta)$ is said to be a lacuna for $P \in \text{he}(\theta)$ (with principal part a) if there is an entire function f such that $f(x)$ is equal to the fundamental solution $E(P, \theta, x)$ when x is restricted to L . The fundamental solution E is said to be sharp from L at a point $y \in \partial L$ if E has an analytic extension from L to $L \cup N$, where N is a neighbourhood of y . In the hyperbolic case, these notions were studied in detail in [2] and [3]. We shall touch briefly on some results for hybrid operators that follow from these two papers.

The question of lacunae is tied to the Petrovsky condition,

$$\beta(A, x, \theta)^* = 0 \quad \text{in } H_{n-2}(X^* - X^* \cap A^*).$$

It follows from (4.7) that if $a \in \text{Hyp}(\theta, m)$ and this condition holds for one x in L , then $E(a, \theta, x)$ is a polynomial of degree $m - n$ in L . More generally, by theorem 4.4 the fundamental solution of any $P \in \text{hyp}(\theta)$ whose principal part is a power of a is an entire function in L . Hence the Petrovsky condition implies that L is a lacuna for every such P ([2] Theorem 10.3). In the hybrid case the Petrovsky condition only implies that the fundamental solutions $E = E(P)$ have sharp fronts at all points of ∂L except the origin. In fact, then the term $E(a, -\theta, x)$ of (4.6) does not necessarily vanish and although it is holomorphic at $W(A, \theta)$ outside the origin, it need not be holomorphic at the origin.

For many hybrid operators a , the Petrovsky condition holds in some component L of the complement of the wave front surface. If, e.g. $n=3$, then according to [2] formula (6.26), an equivalent condition is that $A^* \cap X^*$ be real. In example 1 page 223 this is true when x is inside the curved triangle that constitutes $W(A, \theta)$. Hence in this case the fundamental solution has sharp fronts from inside the wave front surface except at the origin. This is also a consequence of the fact that the local Petrovsky condition ([3] Chapter III, formula (10.2)) applies equally well to hyperbolic and hybrid operators. It states that

$$\beta(A, x, \theta)^* \in H_{n-2}(Y^* - Y^* \cap A^*)$$

where x is in some component L of the complement of wave front surface and close to a point $0 \neq y \in \partial L$. The formula should be taken in the sense that the Petrovsky class belongs to the image of the right hand side induced by projections $Y^* - Y^* \cap \cap A^* \rightarrow X^* - X^* \cap A^*$. When the local Petrovsky condition holds, then all fundamental solutions of hybrid operators whose principal part is a power of a have sharp fronts at y from L . The proof is as in [3] p. 183 and example 10.3 of [3] shows that outside the origin all the fundamental solutions of our examples 1, 2, 3, 4 page 223 have sharp fronts from inside the regions bounded by lines that curve inwards

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Johan Fehrman
Institut Mittag-Leffler
Aurav. 17
S-182 62 Djursholm
Sweden