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Absolutely continuous spectrum of Stark operators

Michael Christ(1) and Alexander Kiselev(2)

Abstract. We prove several new results on the absolutely continuous spectra of perturbed one-dimensional Stark operators. First, we find new classes of perturbations, characterized mainly by smoothness conditions, which preserve purely absolutely continuous spectrum. Then we establish stability of the absolutely continuous spectrum in more general situations, where imbedded singular spectrum may occur. We present two kinds of optimal conditions for the stability of absolutely continuous spectrum: decay and smoothness. In the decay direction, we show that a sufficient (in the power scale) condition is $|q(x)| \le C(1+|x|)^{-1/4-\varepsilon}$; in the smoothness direction, a sufficient condition in Hölder classes is $q \in C^{1/2+\varepsilon}(\mathbf{R})$. On the other hand, we show that there exist potentials which both satisfy $|q(x)| \le C(1+|x|)^{-1/4}$ and belong to $C^{1/2}(\mathbf{R})$ for which the spectrum becomes purely singular on the whole real axis, so that the above results are optimal within the scales considered.

1. Introduction

In this paper we consider the Stark operator

$$H_q = -\frac{d^2}{dx^2} - x + q(x) \tag{1.1}$$

defined on the whole real line $\mathbf{R}$. This operator describes a charged quantum particle in a constant electric field subject to an additional electric potential $q(x)$. There exists an extensive physical and mathematical literature on Stark operators; for a review, see e.g. [11]. When $q(x)=0$, the operator has purely absolutely continuous spectrum. The question we wish to address is which classes of perturbations

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q preserve this property. We will consider two classes of conditions that ensure preservation of the absolutely continuous spectrum: smoothness and decay. The first result on the smoothness condition was proven by Walter [39], who showed that if the potential is bounded and has two bounded derivatives, the spectrum remains purely absolutely continuous. Similar results were obtained by Bentosela, Carmona, Duclos, Simon, Souillard and Weder in [4] using Mourre's method. A corollary noted in [4] is a drastic change in the spectral properties of Schrödinger operators of Anderson model type, say

$$H_\omega = -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} a_n(\omega)V(x-n),$$

where a_n are independent identically distributed random variables and the potential $V \in C_0^2(0,1)$, when a constant electric field is switched on. The spectrum changes from almost surely pure point to purely absolutely continuous. Recently, Sahbani [32], [33] relaxed the smoothness conditions of [39] and [4] (see the remark after Theorem 1.6). On the opposite side of the smoothness scale, Delyon, Simon and Souillard [13] showed that for a periodic array of δ function potentials with random couplings in a constant electric field, the spectrum is purely singular. Avron, Exner and Last [3] realized that the spectrum may be purely singular even for a deterministic periodic array of very singular interactions, such as δ'. Generalizations of these results, as well as other models with singular potentials, were considered in [24], [14], [2], [23], [5] and [1]. There remained a gap, however, between the classes of potentials for which localization was known to occur, and those for which the spectrum was known to remain absolutely continuous.

As far as decay conditions are concerned, it is well known that if $q(x)$ satisfies $|q(x)| \le C(1+|x|)^{-\alpha}$, $\alpha > \frac{1}{2}$, then the spectrum remains purely absolutely continuous [38]. Moreover, there are examples where $|q(x)x^{1/2}| \le C$ and isolated imbedded eigenvalues appear. If $|q(x)|x^{1/2} \to \infty$, it was shown by Naboko and Pushnitski [25] that dense (imbedded) point spectrum may appear on all of $\mathbf{R}$. We remark that for the operator without electric field, the decay threshold where imbedded eigenvalues may appear is the power -1; of course, it is physically natural that it is more difficult to get an imbedded eigenvalue in the presence of the constant electric field. However, if we do not wish to rule out imbedded singular spectrum, it has been shown in [18] that the absolutely continuous spectrum of a perturbed Stark operator still fills the whole real axis when $|q(x)| \le C(1+|x|)^{-\alpha}$, $\alpha > \frac{1}{3}$. The question what is the critical rate of decay for which the spectrum may become purely singular remained open.

Our main goal in this paper is to prove two sharp results on the preservation of the absolutely continuous spectrum of Stark operators. Recall that $f(x)$ is called

Hölder continuous with exponent α $(f\in C^\alpha(\mathbf{R}))$ if

$$\|f\|_{C^\alpha}=\sup_{x\in\mathbf{R}}|f(x)|+\sup_{x,y\in\mathbf{R}}\frac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty.$$

We will analyze solutions of the generalized eigenfunction equation

$$-u''-xu+q(x)u=Eu. \tag{1.2}$$

Here $-x$ represents a background potential due to a constant electrical field, while q is some perturbation.

Theorem 1.1. *Assume that the potential $q(x)$ is Hölder continuous with exponent $\alpha>\frac{1}{2}$. Then an essential support of the absolutely continuous part of the spectral measure coincides with the whole real axis. Moreover, for a.e. E, all solutions $u(x,E)$ of equation* (1.2) *satisfy $u(x,E)=O(x^{-1/4})$ and $u'(x,E)=O(x^{1/4})$, as $x\to+\infty$.*

Remarks. 1. An essential support of μ is a set S such that $\mu(\mathbf{R}\backslash S)=0$ and $\mu(S_1)>0$ for any $S_1\subset S$ of positive Lebesgue measure.

2. In this and subsequent theorems, only the behavior of $q(x)$ for $|x|$ large matters. We will always implicitly assume q to be locally integrable, and will state only additional hypotheses which concern its behavior for large x. On the negative part of the real axis, it is sufficient for all our conclusions to require that $q(x)-x\to+\infty$, as $x\to-\infty$. We prefer to state the results in a slightly weaker form to avoid making statements too cumbersome.

Theorem 1.2. *Assume that the potential $q(x)$ is locally integrable, and that $q(x^2)\in L^p(\mathbf{R})$ for some $1\le p<2$. Then an essential support of the absolutely continuous part of the spectral measure coincides with the whole real axis. Moreover, for a.e. E, all solutions $u(x,E)$ of equation* (1.2) *satisfy $u(x,E)=O(x^{-1/4})$ and $u'(x,E)=O(x^{1/4})$, as $x\to+\infty$.*

Remarks. 1. In particular, the assumption of Theorem 1.2 is satisfied if $|q(x)|\le C(1+|x|)^{-\alpha}$ for some $\alpha>\frac{1}{4}$.

2. An explicit expression for the leading term in an asymptotic expansion of $u(x,E)$, as $x\to+\infty$, can also be derived under the hypotheses of Theorems 1.1 and 1.2; see Theorem 1.3 and Section 4.

Both Theorems 1.1 and 1.2 are direct corollaries of the following more general result.

Theorem 1.3. *Consider a Stark operator H_q on $\mathbf{R}$. Assume that the potential $q(x)$ admits a decomposition $q=q_1+q_2$, where both $q_1(x^2)$ and $x^{-1}q_2'(x^2)$ belong to $(L^1+L^p)(\mathbf{R})$, $1\le p<2$ and that there exists $\zeta<1$ such that $|q_2(x)|\le\zeta|x|$ for sufficiently large $|x|$. Then for almost every energy E there exists a solution $u_+(x,E)$ of equation* (1.2) *with the asymptotic behavior*

$$u_+(x,E)=\frac{e^{i\phi(x,E)}}{(x-q_2(x)+E)^{1/4}}(1+o(1)), \quad \textit{as } x\to+\infty, \tag{1.3}$$

where

$$\phi(x,E)=\int_0^x\left(\sqrt{t-q_2(t)+E}-\frac{q_1(t)}{2\sqrt{x-q_2(t)+E}}\right)dt.$$

Remark. A sufficient condition on the derivative is that $|q'(x)|\le C(1+|x|)^{\alpha}$ for some $\alpha<\frac{1}{4}$. A surprising aspect of this theorem is that the perturbation q_2 is allowed virtually as much growth as the constant electric field potential, and more flexibility on the derivative.

Theorem 1.2 follows immediately; Theorem 1.1 requires a simple argument showing that any C^{α} potential with $\alpha>\frac{1}{2}$ can be represented as in Theorem 1.3. We sketch this argument in Section 4.

We recall that in the case of a Schrödinger operator without constant electric field, the absolutely continuous spectrum is preserved for potentials with power decay rate $\alpha>\frac{1}{2}$ [6], [29] and [12]. There exist potentials V satisfying $|V(x)x^{1/2}|\le C$ for which the absolutely continuous spectrum is destroyed [20], [21], so that $\alpha=\frac{1}{2}$ is a sharp threshold. It is natural that in the presence of a constant electrical field, the absolutely continuous spectrum is preserved under more slowly decaying perturbations of the potential.

The next result shows optimality of Theorems 1.1 and 1.2. Fix $f\in C_0^{\infty}(0,1)$, not identically zero, and let $a_n(\omega)$ be independent random variables with uniform distribution in $[0,2\pi]$. Set $c=\left(\frac{3}{2}\right)^{2/3}$. Let us define

$$q(x)=c\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}f\left(\sqrt{cx}-n\right)\sin\left(\tfrac{4}{3}x^{3/2}+a_n(\omega)\right). \tag{1.4}$$

(There is nothing magic in the choice of c; however, this choice will simplify computations later.) We have the following result.

Theorem 1.4. *Let $q(x)$ be a random potential given by* (1.4). *Then for a.e. ω, the spectrum of the corresponding perturbed Stark operator is purely singular on the whole real line.*

In particular, any realization of q defined by (1.4) satisfies $|q(x)| \le Cx^{-1/4}$ and belongs to $C^{1/2}(\mathbf{R})$, so Theorem 1.4 assures sharpness of Theorems 1.1 and 1.2 in Hölder spaces and in the power scale, respectively.

On a more detailed level, we may want to distinguish between perturbations for which the absolutely continuous spectrum is preserved, but imbedded singular spectrum may appear, and perturbations which preserve purely absolutely continuous spectrum. Our final results provide two criteria ensuring pure absolute continuity of the spectrum. Recall that a function $g(x)$ defined on the real line is called *smooth in Zygmund's sense* if

$$\int_0^1 \sup_{x\in\mathbf{R}} |g(x+\varepsilon)-2g(x)+g(x-\varepsilon)| \frac{d\varepsilon}{\varepsilon^2} < \infty.$$

Theorem 1.5. *Assume that the potential $q(x)$ is bounded, has bounded continuous first derivative and is smooth in Zygmund's sense. Then the Stark operator* (1.1) *has purely absolutely continuous spectrum on the whole real axis.*

We can also allow the potential to grow at a rate arbitrarily close to that of the constant electric field, and still have purely absolutely continuous spectrum, provided that we impose a slightly different condition on smoothness.

Theorem 1.6. *Assume that the potential $q(x)$ satisfies the condition $q(x) = O(x^\alpha)$, as $|x| \to \infty$, for some $\alpha < 1$, is differentiable, and that its derivative $q'(x)$ is Dini continuous:*

$$\int_0^1 \sup_{x\in\mathbf{R}} |q'(x+\varepsilon)-q'(x-\varepsilon)| \frac{d\varepsilon}{\varepsilon} < \infty.$$

Then the spectrum of the perturbed Stark operator (1.1) *is purely absolutely continuous on the whole real axis.*

Remark. The classes of potentials that are smooth in Zygmund's sense or have Dini continuous derivative were first considered in this context by Sahbani in [32] and [33]. Using the conjugate operator approach, he proved that under conditions similar to Theorem 1.5 or Theorem 1.6 (but with stronger growth restrictions) the spectrum is absolutely continuous with perhaps some imbedded eigenvalues. His results extend (in a slightly weaker form) to the higher dimensional setting.

We will later discuss examples demonstrating that the criteria given by Theorems 1.5 and 1.6 are fairly sharp.

We employ two different approaches to prove the stated results. First, to prove Theorems 1.4, 1.5, and 1.6, we apply a Liouville transformation to reduce the Stark operator to a form reminiscent of the Schrödinger operator without electric field, but with the energy entering in a non-standard way. We then use a Prüfer transformation to analyze the asymptotic behavior of solutions. In the proofs of the other results, it is more convenient to represent (1.2) as a first-order system and to employ estimates for the solution series. Some of the tools for these estimates come from our recent work [6], [7], [8] and [9]. As soon as we have control over the asymptotic behavior of solutions, we can apply the whole axis version of subordinacy theory due to Gilbert [15], or the approximate eigenvectors criterion of [10], to draw spectral conclusions.

After this paper had been submitted, Killip [17] proved that if $q(x^2) \in L^2(\mathbf{R})$, the absolutely continuous spectrum fills the whole real axis. His method, related to that of [12], is quite different from ours and yields less specific information concerning the generalized eigenfunctions.

2. Preservation of purely absolutely continuous spectrum

We begin by proving Theorems 1.5 and 1.6 as a warm-up. All the proofs of spectral properties in this paper rely on the study of solutions of the equation (1.2), $-u''-xu+q(x)u=Eu$. The link between the behavior of solutions and spectral results is provided by Gilbert–Pearson subordinacy theory, more particularly, by the whole-line version of this theory due to Gilbert [15]. Recall that a real solution $u_1(x,E)$ of (1.2) is called *subordinate on the right* if for any other real linearly independent solution $u_2(x,E)$ we have

$$\lim_{N\to\infty} \frac{\int_0^N |u_1(x,E)|^2\,dx}{\int_0^N |u_2(x,E)|^2\,dx} = 0.$$

Subordinacy on the left is defined similarly. Note that it is easy to see that for equation (1.2), under the assumptions of any of our theorems, there is always a solution subordinate (in fact, L^2) on the left since the potential goes to $+\infty$ there. The main result of Gilbert implies that singular spectrum may only be supported on the set of energies where there exists a solution subordinate on both sides. Moreover, the set of the energies where there exists a solution subordinate on one side, but there is no subordinate solution on the other side, is an essential support of the absolutely continuous spectrum, of multiplicity one. Therefore, our goal is to prove that for all energies (if we want to show pure absolute continuity), or for a.e. energy (if we allow imbedded singular spectrum), there is no solution of (1.2) subordinate on the right.

In the equation (1.2), let us perform a Liouville transformation given by (see, e.g. [27])

$$\xi(x) = \int_0^x \sqrt{t}\,dt = \tfrac{2}{3}x^{3/2}, \quad \phi(\xi) = x(\xi)^{1/4}u(x(\xi)). \tag{2.1}$$

This transformation introduces an irrelevant singularity at the origin; henceforth we always work outside some neighborhood of 0.

The resulting function ϕ satisfies the Schrödinger equation

$$-\phi''+\left(\frac{5}{36\xi^2}+\frac{q(c\xi^{2/3})-E}{c\xi^{2/3}}\right)\phi=\phi, \tag{2.2}$$

where $c=\left(\frac{3}{2}\right)^{2/3}$. Let us introduce a short-hand notation $V(\xi,E)$ for the expression in brackets in (2.2). Let us further apply a Prüfer transformation to the equation for ϕ, setting for each E,

$$\begin{aligned}\phi(\xi,E)&=R(\xi,E)\sin\theta(\xi,E),\\ \phi'(\xi,E)&=R(\xi,E)\cos\theta(\xi,E).\end{aligned} \tag{2.3}$$

The equations for R and θ are as follows:

$$(\log R(\xi,E))'=\tfrac{1}{2}V(\xi,E)\sin 2\theta(\xi,E), \tag{2.4}$$

$$\theta'(\xi,E)=1-\tfrac{1}{2}V(\xi,E)(1-\cos 2\theta(\xi,E)). \tag{2.5}$$

Our main goal in the proof of Theorem 1.5 will be to show the convergence of the integral

$$\int_1^N\left(\frac{5}{36\xi^2}+\frac{-E+q(c\xi^{2/3})}{c\xi^{2/3}}\right)\sin 2\theta(\xi,E)\,d\xi, \quad \text{as } N\to\infty, \tag{2.6}$$

for every E. This goal is motivated by the following proposition.

Proposition 2.1. *Suppose that*

$$\limsup_{x\to\infty}\frac{|q(x)|}{x}=\zeta<1,$$

and for a given E, the integral (2.6) *converges for all initial values of $\theta(0,E)$. Then for this value of E, there is no subordinate on the right solution of the equation* (1.2).

Proof. If for a given value of E the integral (2.6) converges, it follows from (2.4) and (2.5) that all solutions of the equation (2.2) are bounded and, moreover, any solution ϕ_β (where β parametrizes the boundary condition) has the asymptotic behavior as $\xi\to+\infty$,

$$\phi_\beta(\xi,E)=C_\beta\sin(\xi+g_\beta(\xi,E))(1+o(1)).$$

where $|g'_\beta(\xi,E)|<\zeta<\zeta_1<1$ for ξ sufficiently large. Going back to the original equation (1.2), we infer that every solution $u_\beta(x,E)$ has the asymptotic behavior as $x\to\infty$,

$$u_\beta(x,E)=\frac{C_\beta}{x^{1/4}}\sin\left(\tfrac{2}{3}x^{3/2}+f_\beta(x,E)\right)(1+o(1)),$$

where $|f'_\beta(x,E)|<\zeta_1 x^{1/2}$ for sufficiently large x. For any $u_\beta(x,E)$ we find

$$\begin{aligned}\int_1^N |u_\beta(x,E)|^2\,dx &= C_\beta^2\int_1^N \frac{1}{x^{1/2}}\sin^2\left(\tfrac{2}{3}x^{3/2}+f_\beta(x,E)\right)dx(1+o(1))\\ &= C_\beta^2\left(N^{1/2}-\frac{1}{2}\int_1^N\frac{1}{x^{1/2}}\cos\left(\tfrac{4}{3}x^{3/2}+2f_\beta(x,E)\right)dx\right)(1+o(1)).\end{aligned}\tag{2.7}$$

Consider the integral

$$I_\beta(N)=\int_1^N\frac{1}{x^{1/2}}e^{4ix^{3/2}/3+2if_\beta(x,E)}\,dx.\tag{2.8}$$

Integrating by parts, with $e^{2if_\beta(x,E)}$ being differentiated, we obtain that $I_\beta(N)$ is equal to

$$-\int_N^\infty\frac{1}{x^{1/2}}e^{4ix^{3/2}/3}e^{2if_\beta(N,E)}\,dx+2i\int_1^N f'_\beta(x,E)e^{2if_\beta(x,E)}\int_x^\infty\frac{1}{t^{1/2}}e^{4it^{3/2}/3}\,dt\,dx.$$

Since

$$\int_x^\infty\frac{1}{t^{1/2}}e^{4it^{3/2}/3}\,dt=\frac{1}{2x}e^{4ix^{3/2}/3}(1+o(1)),$$

we obtain that

$$|I_\beta(N)|\le 2\zeta_1 N^{1/2}$$

for N sufficiently large. Returning to (2.7), it is straightforward to conclude that any solution $u_\beta(x,E)$ satisfies, for sufficiently large N,

$$C_\beta^2(1-\zeta_1)N^{1/2}\le\int_1^N|u_\beta(x,E)|^2\,dx\le C_\beta^2(1+\zeta_1)N^{1/2}.$$

Therefore, all solutions have the same rate of L^2 norm growth, as $N\to\infty$, and there is no subordinate solution. $\square$

Now we establish convergence of (2.6) for every energy under the assumptions of Theorem 1.5, thus completing the proof of this result.

Proof of Theorem 1.5. Let us write

$$\theta_\beta(\xi,E)=\xi+g_\beta(\xi,E),$$

with β parametrizing the initial condition at 0 and

$$|g'_\beta(\xi,E)|\le\frac{C}{\xi^{2/3}} \tag{2.9}$$

uniformly in β. Clearly we can ignore the short-range quadratic decay term and consider only

$$\int_1^N\frac{E-q(c\xi^{2/3})}{c\xi^{2/3}}e^{2i(\xi+g_\alpha(\xi,E))}\,d\xi.$$

Moreover, the integral

$$\begin{aligned}\int_1^N\frac{1}{\xi^{2/3}}e^{2i(\xi+g_\alpha(\xi,E))}\,d\xi=&-\biggl(\int_\xi^\infty\frac{e^{2i\eta}}{\eta^{2/3}}\,d\eta\biggr)e^{2ig_\alpha(\xi,E)}\biggr|_0^N\\&+\int_1^N e^{2ig_\alpha(\xi,E)}g'_\alpha(\xi,E)\int_\xi^\infty\frac{e^{2i\eta}}{\eta^{2/3}}\,d\eta\,d\xi\end{aligned}$$

is clearly convergent due to (2.9). It remains to estimate

$$\int_3^N\frac{q(c\xi^{2/3})}{c\xi^{2/3}}e^{2i(\xi+g_\alpha(\xi,E))}\,d\xi$$

uniformly as $N\to\infty$ (we shifted the region of integration for convenience). Fix h, $\frac{1}{2}\pi>h>0$, and consider the equality

$$\begin{aligned}(e^{2ih}&+e^{-2ih}-2)\int_3^N\frac{q(c\xi^{2/3})}{c\xi^{2/3}}e^{2i(\xi+g_\alpha(\xi,E))}\,d\xi\\&=O(1)+\int_3^N e^{2i\xi}\biggl(\frac{q(c(\xi+h)^{2/3})e^{2ig_\alpha(\xi+h,E)}}{(\xi+h)^{2/3}}\\&\quad+\frac{q(c(\xi-h)^{2/3})e^{2ig_\alpha(\xi-h,E)}}{(\xi-h)^{2/3}}-\frac{2q(c\xi^{2/3})e^{2ig_\alpha(\xi,E)}}{\xi^{2/3}}\biggr)\,d\xi\\&=O(1)+\int_3^N\frac{e^{2i(\xi+g_\alpha(\xi,E))}}{\xi^{2/3}}(q(c(\xi+h)^{2/3})+q(c(\xi-h)^{2/3})-2q(c\xi^{2/3}))\,d\xi.\end{aligned}$$

Since we assumed that $q\in C^1$, it suffices to control

$$\int_3^N\frac{1}{\xi^{2/3}}\bigl|q\bigl(c\bigl(\xi^{2/3}+\tfrac{2}{3}\xi^{-1/3}h\bigr)\bigr)+q\bigl(c\bigl(\xi^{2/3}+\tfrac{2}{3}\xi^{-1/3}h\bigr)\bigr)-2q(c\xi^{2/3})\bigr|\,d\xi. \tag{2.10}$$

Set $\varepsilon=\frac{2}{3}c\xi^{-1/3}h$, then uniformly in N, the integral (2.10) is bounded by

$$C\int_0^1 |q(c_1\varepsilon^{-2}+\varepsilon)+q(c_1\varepsilon^{-2}-\varepsilon)-2q(c_1\varepsilon^{-2})|\frac{d\varepsilon}{\varepsilon^2}$$

which is finite by assumption. $\square$

Remark. Without change, the proof goes through even with the weaker growth assumption $q(x)=O(x^{1/2-\varepsilon})$ for some $\varepsilon>0$.

Proof of Theorem 1.6. The proof of this theorem is very similar to the preceding proof. However, it is convenient to employ a slight variation of the Prüfer transformation. Namely, we let

$$\begin{aligned}\sqrt{1-V(\xi,E)}\,\phi(\xi,E) &= \tilde{R}\sin\tilde{\theta}(\xi,E),\\ \phi'(\xi,E) &= \tilde{R}\cos\tilde{\theta}(\xi,E).\end{aligned}$$

This transformation is well-defined for large ξ where $V(\xi,E)<1$, and this suffices for our purpose since we are interested in the asymptotic behavior at $+\infty$. The equations for $\tilde{R}$ and $\tilde{\theta}$ are

$$(\log\tilde{R})'(\xi,E) = -\frac{V'(\xi,E)}{4(1-V(\xi,E))}(1-\cos 2\tilde{\theta}(\xi,E)), \tag{2.11}$$

$$\tilde{\theta}'(\xi,E) = \sqrt{1-V(\xi,E)} - \frac{V'(\xi,E)}{4(1-V(\xi,E))}\cos 2\tilde{\theta}(\xi,E). \tag{2.12}$$

The role analogous to the integral (2.6) is played by

$$\int_1^N \frac{V'(\xi,E)}{1-V(\xi,E)}\cos 2\tilde{\theta}(x,E)\,d\xi; \tag{2.13}$$

the other term on the right-hand side of (2.11) can be integrated explicitly. If the integral (2.13) converges for a given energy E, then for this energy there is no solution subordinate on the right. This can be shown in a direct analogy to the proof of Proposition 2.1; the details are left to the reader.

Expressing $V'/(1-V)$ in terms of q, we see that it is enough to show the convergence of $\int_a^N q'(c\xi^{2/3})\xi^{-1}e^{i\tilde{\theta}(\xi,E)}\,d\xi$, as $N\to\infty$. From (2.12) and the assumption of the theorem, it follows that

$$\tilde{\theta}(\xi,E) = \xi+\tilde{g}_\alpha(\xi,E),$$

where $|\tilde{g}'_\alpha(\xi,E)|\le C\xi^{-\delta}$ for some $\delta>0$. Now fix h, $0<h<\frac{1}{2}\pi$, and consider

$$\begin{aligned}
&(e^{ih}-e^{-ih})\int_a^N q'(c\xi^{2/3})\frac{1}{\xi}e^{i\tilde{\theta}(\xi,E)}\,d\xi\\
&\quad=O(1)+\int_a^N e^{i\xi}\left(e^{i\tilde{g}_\alpha(\xi+h)}\frac{q'(c(\xi+h)^{2/3})}{\xi+h}-e^{i\tilde{g}_\alpha(\xi-h)}\frac{q'(c(\xi-h)^{2/3})}{\xi-h}\right)d\xi\\
&\quad=O(1)+\int_a^N e^{i\xi+i\tilde{g}_\alpha(\xi)}\frac{1}{\xi}(q'(c(\xi+h)^{2/3})-q'(c(\xi-h)^{2/3}))\,d\xi\\
&\quad\le C\left(1+\int_a^N \frac{1}{\xi}|q'(c(\xi+h)^{2/3})-q'(c(\xi-h)^{2/3})|\,d\xi\right).
\end{aligned}$$

Setting

$$\varepsilon=\frac{c(\xi+h)^{2/3}-c(\xi-h)^{2/3}}{2},$$

and making a change of variable in the last integral, we find that for a sufficiently large a, the controlling integral (2.13) is bounded by

$$C\left(1+\int_0^1 |q'(f(\varepsilon)+\varepsilon)-q'(f(\varepsilon)-\varepsilon)|\,\frac{d\varepsilon}{\varepsilon}\right)$$

which is finite by the assumption of Dini continuity (here $f(\varepsilon)=\frac{1}{2}[c(\xi+h)^{2/3}+c(\xi-h)^{2/3}]$). $\square$

We remark that the results of Theorems 1.5 and 1.6 are rather sharp. For example, let $E=0$, then (1.2) reduces to

$$-\phi''+\left(\frac{5}{36\xi^2}+\frac{q(c\xi^{2/3})}{c\xi^{2/3}}\right)\phi=\phi.$$

Let us denote by $wn(\xi)$ the classical Wigner–von Neumann potential [26], [28]. Choose q so that the expression in the brackets coincides with $wn(\xi)$ for $\xi\ge1$. It is not difficult to show that we can take $q=0$ on $(-\infty,0)$ and q smooth and bounded on $(0,1)$ so that the whole equation (1.2) has an eigenvalue at $E=0$. (The issue is gluing together the L^2 solution on $-\infty$ and the L^2 solution produced by the Wigner–von Neumann potential on ∞. It can always be achieved by choosing q appropriately on $(0,1)$: see, e.g. [36] for a similar argument.) Notice that in this case

$$q(x)=x\left(wn\left(\frac{3x^{3/2}}{2}\right)-\frac{5}{81x^3}\right).$$

The Wigner–von Neumann potential has asymptotic behavior [28]

$$wn(x) = -\frac{8\sin 2x}{x} + O\left(\frac{1}{x^2}\right),$$

with the $O(x^{-2})$ term also smooth with derivatives decaying at the same rate. We see that

$$q(x) = \frac{C_1}{x^{1/2}} \sin C_2 x^{3/2} + q_1(x),$$

where $q_1(x)$ is better behaved in all respects. This function $q(x)$ narrowly misses the class of functions smooth in Zygmund's sense. Indeed, $|q(x+\varepsilon)+q(x-\varepsilon)-2q(x)| = \varepsilon q_2(x,\varepsilon)$, where q_2 is uniformly bounded. Also, q has a bounded, continuous derivative, which however fails to be Dini continuous.

3. Main theorem

Here we prove Theorem 1.3. As before, we are going to study the asymptotic behavior of the solutions to equation (1.2),

$$-u'' - xu(x) + q(x)u(x) = Eu(x),$$

as $x \to \infty$. The L^1 part of the perturbation can be treated by standard means (such as, for example, Levinson's theorem), so we will assume that $q_1(x^2) \in L^p(0,\infty)$ and $x^{-1} q_2'(x^2) \in L^p(0,\infty)$. Write (1.2) as a system

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -x+q(x)-E & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}. \tag{3.1}$$

We are going to perform a series of transformations with this system, similarly to [7] and [9]. Applying first a variation of parameters-type transformation

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} e^{i\psi(x,E)} & e^{-i\psi(x,E)} \\ i\psi'(x,E)e^{i\psi(x,E)} & -i\psi'(x,E)e^{-i\psi(x,E)} \end{pmatrix} z \tag{3.2}$$

we arrive at

$$z' = \begin{pmatrix} -i\mathcal{E} & -i\bar{\mathcal{E}}e^{-2i\psi} \\ i\mathcal{E}e^{2i\psi} & i\bar{\mathcal{E}} \end{pmatrix} z,$$

where

$$\mathcal{E}(x,E) = \frac{1}{2\psi'(x,E)}(-i\psi''(x,E) + \psi'(x,E)^2 - x + q(x) - E). \tag{3.3}$$

Letting

$$z=\begin{pmatrix} e^{-i\int_0^x \mathcal{E}(t,E)\,dt} & 0 \\ 0 & e^{i\int_0^x \bar{\mathcal{E}}(t,E)\,dt} \end{pmatrix} y \tag{3.4}$$

leads to

$$y'=\begin{pmatrix} 0 & -i\bar{\mathcal{E}}e^{-2i\psi+i\int_0^x \operatorname{Re}\mathcal{E}(t,E)\,dt} \\ i\mathcal{E}e^{2i\psi-i\int_0^x \operatorname{Re}\mathcal{E}(t,E)\,dt} & 0 \end{pmatrix} y. \tag{3.5}$$

We are going to choose

$$\psi'(x,E)=\sqrt{x-q_2(x)+E}\,,$$

so that

$$\mathcal{E}(x,E)=\frac{q_1(x)}{2\sqrt{x-q_2(x)+E}}-i\frac{1-q_2'(x)}{2(x-q_2(x)+E)}. \tag{3.6}$$

Let $Q(x)=(x^{-1}q_1(x)^2+x^{-2}q_2'(x)^2+x^{-2})^{1/2}$, and

$$a(x,E)=\frac{i\mathcal{E}(x,E)}{Q(x)}. \tag{3.7}$$

Let us introduce the multilinear operators

$$\begin{aligned} &S_n(f_1,\dots,f_n)(x,E)\\ &=\int_x^\infty\int_{t_1}^\infty\dots\int_{t_{n-1}}^\infty \prod_{j=1}^n\Big[e^{2i(-1)^{n-j}(\psi(t_j,E)-\int_0^{t_j}\operatorname{Re}\mathcal{E}(t,E)\,dt)}a_j(t_j,E)f_j(t_j)\,dt_j\Big] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\widetilde{S}_n(f_1,\dots,f_n)(\xi,E)\\ &=\int_x^\infty\int_{t_1}^\infty\dots\int_{t_{n-1}}^\infty \prod_{j=1}^n\Big[e^{2i(-1)^{n-j}(\psi(t_j^2,E)-\int_0^{t_j^2}\operatorname{Re}\mathcal{E}(t,E)\,dt)}a_j(t_j^2,E)f_j(t_j)\,dt_j\Big], \end{aligned} \tag{3.9}$$

where a_j is equal to a for $n-j$ even and $\bar{a}$ for $n-j$ odd. Set $\tilde{f}(t)=2f(t^2)t$. Notice that making the change of variable $t_j=s_j^2$ in (3.8) leads to

$$S_n(f_1,\dots,f_n)(x,E)=\widetilde{S}(\tilde{f}_1,\dots,\tilde{f}_n)(x^{1/2},E). \tag{3.10}$$

Notice that by assumption,

$$\widetilde{Q}(x)=2\left(q_1(x^2)^2+\frac{1}{x^2}q_2'(x^2)^2+\frac{1}{x^2}\right)^{1/2}$$

belongs to $L^p(\mathbf{R})$ with $p<2$. Iterating system (3.5) starting from the vector $(1,0)$ and using (3.10), we obtain the formal series expansion for one of the solutions

$$
\begin{aligned}
y_+(x,E)&=\begin{pmatrix} \sum_{n=0}^{\infty} S_{2n}(Q,\dots,Q)(x,E) \\ -\sum_{n=1}^{\infty} S_{2n-1}(Q,\dots,Q)(x,E)\end{pmatrix}\\
&=\begin{pmatrix} \sum_{n=0}^{\infty} \widetilde{S}_{2n}(\widetilde{Q},\dots,\widetilde{Q})(x^{1/2},E) \\ -\sum_{n=1}^{\infty} \widetilde{S}_{2n-1}(\widetilde{Q},\dots,\widetilde{Q})(x^{1/2},E)\end{pmatrix}
\end{aligned}
\tag{3.11}
$$

(we stipulate $S_0(Q)(\xi,E)=\widetilde{S}_0(\widetilde{Q})(\xi,E)\equiv 1$ in the above formula). Introduce the operator

$$\widetilde{S}f(E)=\int_0^{\infty}\widetilde{S}(t,E)f(t)\,dt,\tag{3.12}$$

where

$$\widetilde{S}(t,E)=a(t^2,E)e^{2i(\psi(t^2,E)-\int_0^{t^2}\operatorname{Re}\mathcal{E}(t,E)\,dt)}.$$

Proposition 3.1. *Fix a compact interval $J\subset\mathbf{R}$. Assume that the operator $\widetilde{S}$ maps $L^p(\mathbf{R})$ boundedly to $L^r(J)$, for some $p<2<r$. Then for any $\widetilde{Q}\in L^p(\mathbf{R})$, the series* (3.11) *converges for a.e. $E\in J$. Moreover, for a.e. $E\in J$, the solution $y_+(x,E)$ of the system* (3.5) *given by* (3.11) *has the asymptotic behavior*

$$y_+(x,E)=\begin{pmatrix}1\\0\end{pmatrix}+o(1).\tag{3.13}$$

Proof. The proof is based on results of [7], [8] and [9]. Introduce a multilinear operator M_n, acting on n functions $g_k(x,E)$, by

$$M_n(g_1,\dots,g_n)(x,x',E)=\int_{x\le x_1\le\dots\le x_n\le x'}\prod_{k=1}^{n}[g_k(x_k,E)\,dx_k].$$

In the special case when there is a single function g such that each g_k is either g or $\bar{g}$, we write simply $M_n(g)(x,x',E)$. In particular, the multilinear transforms $\widetilde{S}$

in the series (3.11) have a structure identical to M_n, with $g(x,E)=\widetilde{S}(x,E)\widetilde{Q}(x)$. Define

$$M_n^*(g_1,\dots,g_n)(E)=\sup_{x\le x'\in\mathbf{R}}|M_n(g_1,\dots,g_n)(x,x',E)|,$$

$$M_n^*(g)(E)=\sup_{x\le x'\in\mathbf{R}}|M_n(g)(x,x',E)|,$$

$$G(g,E)=\sum_{r=0}^{1}\sum_{m=1}^{\infty}m\biggl(\sum_{j=1}^{2^m}\biggl|\int_{E_j^m}g(x,E)\,dx\biggr|^2\biggr)^{1/2}.$$

In Proposition 4.2 and in the proof of Theorem 1.3 of [8] it is shown that

$$M_n^*(g_1,\dots,g_n)(E)\le C^n\prod_{k=1}^{n}G(g_k,E), \tag{3.14}$$

$$M_n^*(g)(E)\le C^n\frac{G(g,E)^n}{\sqrt{n!}} \tag{3.15}$$

for some universal constant $C<\infty$. If the operator $\widetilde{S}$ satisfies the required $L^p(\mathbf{R})$-$L^r(J)$ bound, then it is not hard to see that

$$\|\widetilde{G}(\widetilde{S}(Q),E)\|_{L^r(J)}\le C\|Q\|_{L^p(\mathbf{R})} \tag{3.16}$$

(see Proposition 3.3 of [7]). The estimates (3.14), (3.15) and (3.16) allow to show a.e. E convergence of the series (3.11), and to prove Proposition 3.1. For details we refer to [7], Section 4, where a similar argument is given. □

Remark. An alternative route to the same result is to consider an energy dependent potential $\mathcal{E}(x,E)$, rather than to introduce $a(x,E)$. The paper [9] follows this approach.

It remains to show that the operator $\widetilde{S}$ satisfies an $L^p(\mathbf{R})$-$L^r(J)$ bound for any compact J and any $1\le p<2$, $r=p/(p-1)$, provided that the potential q satisfies the assumptions of Theorem 1.2. Such a result would follow by complex interpolation if we establish the $L^1(\mathbf{R})$-$L^\infty(J)$ and $L^2(\mathbf{R})$-$L^2(J)$ bounds. The first bound is evident since $a(x^2,E)$ is bounded and the oscillatory exponential

$$e^{2i(\psi(x,E)-\int_0^x \operatorname{Re}\mathcal{E}(t,E)\,dt)}$$

is bounded too. Next we establish the key L^2-L^2 bound for the operator $\widetilde{S}$.

Proposition 3.2. *Let $J \subset \mathbf{R}$ be a compact interval. Assume that*

$$|\partial_E^\beta a(x,E)| \le C \tag{3.17}$$

for $\beta=0,1,2$ and every $E \in J$, and that the potential q satisfies the assumptions of Theorem 1.3. Then for any $f \in L^2(\mathbf{R})$ we have

$$\|\widetilde{S}f\|_{L^2(J)} \le C\|f\|_{L^2(\mathbf{R})}$$

for the operator $\widetilde{S}$ defined by (3.12).

Proof. Notice that

$$\|\widetilde{S}f\|^2_{L^2(J)} \le \int_{\mathbf{R}}\int_{\mathbf{R}} f(x)\bar{f}(y) \times \left(\int_J \eta(E)a(x^2,E)\bar{a}(y^2,E)e^{2i\int_{x^2}^{y^2}(\psi'(t,E)-\operatorname{Re}\mathcal{E}(t,E))\,dt}\,dE\right)dx\,dy,$$

where $\eta(E)$ is a positive $C_0^\infty(\mathbf{R})$ function satisfying $\eta(E) \ge 1$ for $E \in J$. We can rewrite the kernel in the brackets in the above formula by making the change of variable $t=s^2$,

$$K(x,y) = \int_J \eta(E)a(x^2,E)\bar{a}(y^2,E)e^{4i\int_y^x(s\psi'(E,s^2)-s\operatorname{Re}\mathcal{E}(s^2,E))\,ds}\,dE. \tag{3.18}$$

Let us write for simplicity

$$\sigma(s,E) = (\psi'(s^2,E)-\operatorname{Re}\mathcal{E}(s^2,E))s.$$

Since

$$\partial_E\left(s\sqrt{s^2-q_2(s^2)+E}\,\right) = \frac{s}{2\sqrt{s^2-q_2(s^2)+E}} = 1+o(1)$$

for large s, a direct computation using the assumption on q shows that there exists N such that for any $x,y>0$ such that $|x-y| \ge N$, we have uniformly in $E \in J$,

$$\begin{aligned} \left|\int_x^y \partial_E\sigma(s,E)\,ds\right| &\ge C_1|x-y|, \\ \left|\int_x^y \partial_E^\beta\sigma(s,E)\,ds\right| &\le C_2|x-y|, \quad \beta=2,3 \end{aligned} \tag{3.19}$$

with positive constants C_1 and C_2. Integrate by parts two times in (3.18), with

$$e^{4i\int_x^y \sigma(s,E)\,ds}\int_x^y \partial_E\sigma(s,E)\,ds$$

being integrated. Using (3.19) and (3.17) we obtain

$$|K(x,y)| \leq \frac{C}{1+|x-y|^2} \quad \text{for all } x \text{ and } y,$$

which implies the desired L^2-L^2 bound. □

The fact that $a(x,E)$ satisfies (3.17) can be checked directly from the definitions (3.3) and (3.7).

Now we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Proposition 3.2 ensures that Proposition 3.1 applies under the conditions of the theorem. Notice that

$$e^{i\psi(x,E)-i\int_0^x \mathcal{E}(t,E)\,dt} = \frac{1}{(x-q_2+E)^{1/4}} e^{i\int_0^x (\sqrt{t-q_2(t)+E}-q_1(t)/2\sqrt{t-q_2(t)+E})\,dt}.$$

Applying transformations (3.4), (3.2) and (2.1) to the solution $y_+(\xi,E)$ and using (3.13), we obtain for a.e. $E\in J$ a solution $u_+(x,E)$ of the equation (1.2) with the asymptotic behavior

$$u_+(x,E) = \frac{1}{(x-q_2+E)^{1/4}} e^{i\int_0^x (\sqrt{t-q_2(t)+E}-q_1(t)/2\sqrt{t-q_2(t)+E})\,dt}(1+o(1)),$$

coinciding with (1.3), and

$$u_+'(x,E) = \frac{i}{(x-q_2+E)^{1/4}} e^{i\int_0^x (\sqrt{t-q_2(t)+E}-q_1(t)/2\sqrt{t-q_2(t)+E})\,dt}(1+o(1)).$$

There also exists a solution $u_-(x,E)$ which is just the complex conjugate of u_+. In particular, for a.e. E, any solution $u(x,E)$ of (1.2) satisfies

$$|u(x,E)| \leq \frac{C}{x^{1/4}} \quad \text{and} \quad |u'(x,E)| \leq Cx^{1/4}. \tag{3.20}$$

By the results of [10], to show that the essential support of the absolutely continuous part of the spectral measure fills all of the real line, it suffices to show that for a.e. E there exists a sequence $\psi_n(x,E)$ such that

$$\limsup_{n\to\infty} \frac{|\psi_n(0,E)|+|\psi_n'(0,E)|}{\|\psi_n\|\,\|(H_q-E)\psi_n\|} > 0. \tag{3.21}$$

We can obtain an appropriate sequence $\psi_n(E)$ by taking a smooth cutoff function $\eta(x)$ satisfying $0\leq\eta(x)\leq 1$, $\eta(x)=1$ if $x<\frac{1}{2}$ and $\eta(x)=0$ if $x>1$, and letting

$\psi_n(x,E)=\tilde{u}(x,E)\eta(2^{-n}x)$, where $\tilde{u}$ is the solution of (1.2) which is L^2 at $-\infty$. Then $|\psi_n(0,E)|+|\psi'(0,E)|>c(E)>0$ since $\tilde{u}$ is nonzero. Moreover, using (3.20) gives $\|\psi_n\|\sim 2^{n/4}$ and $\|(H_q-E)\psi_n\|\sim 2^{-n/4}$, so that

$$\|\psi_n\|\,\|(H_q-E)\psi_n\|\leq C(E).$$

Therefore, (3.21) is satisfied at a.e. E.

An alternative (but more technical) way to finish the proof is to apply subordinacy theory using more detailed information on the asymptotic behavior, similarly to Proposition 2.1. □

4. Potentials in $C^{1/2+\varepsilon}$

In this section we are going to prove a result from which Theorem 1.1 will follow. Define

$$\overline{D}^{\alpha} f(x)=\sup_{y:|x-y|<1}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}},\quad 0<\alpha\leq 1.$$

Theorem 4.1. *Assume that the potential $q(x)$ satisfies $|q(x)|\leq\zeta|x|$ with $\zeta<1$ for all sufficiently large $|x|$, and $x^{-\alpha}\overline{D}^{\alpha}q(x^2)\in L^p(1,\infty)$ for some $p<2$ and $1\geq\alpha>0$. Then an essential support of the absolutely continuous part of the spectral measure coincides with the whole real axis.*

Remark. In particular, if $q\in C^{\alpha}(\mathbf{R})$ with $\alpha>\frac{1}{2}$, then $\overline{D}^{\alpha}q\in L^{\infty}(\mathbf{R})$ and the assumption of Theorem 4.1 is satisfied.

Proof. We will show that if q satisfies the conditions of the theorem, then it can be represented as a sum $q=q_1+q_2$ with $q_1(x^2)\in L^p(\mathbf{R})$ and $x^{-1}q_2'(x^2)\in L^p(\mathbf{R})$, $|q_2(x)|\leq\zeta_1 x$ with $\zeta_1<1$ for large x. Then the result follows from Theorem 1.3. Fix a function $\eta(x)\in C_0^{\infty}(0,1)$ such that $\int_{\mathbf{R}}\eta(x)\,dx=1$. Set

$$q_2(x)=x^{1/2}\int_{\mathbf{R}}\eta(x^{1/2}(x-y))q(y)\,dy,$$
$$q_1(x)=x^{1/2}\int_{\mathbf{R}}\eta(x^{1/2}(x-y))(q(x)-q(y))\,dy.$$

Then

$$|q_1(x)|\leq x^{(1-\alpha)/2}\int_{\mathbf{R}}\eta(x^{1/2}(x-y))\overline{D}^{\alpha}q(x)\,dy=\frac{1}{x^{\alpha/2}}\overline{D}^{\alpha}q(x).$$

Therefore, by assumption, $q_1(x^2)\in L^p(\mathbf{R})$. The property $|q_2(x)|\le\zeta_1 x$ with $\zeta_1<1$ for all large x is clear from the definition. Also

$$\begin{aligned} q_2'(x) = {} & \frac{1}{2x^{1/2}}\int_{\mathbf{R}}\eta(x^{1/2}(x-y))q(y)\,dy + x\int_{\mathbf{R}}\eta'(x^{1/2}(x-y))(q(y)-q(x))\,dy \\ & + \frac{1}{2}\int_{\mathbf{R}}\eta'(x^{1/2}(x-y))(x-y)q(y)\,dy. \end{aligned} \tag{4.1}$$

Due to the bound $|q(x)|\le\zeta x$, the first term on the right-hand side of (4.1) is bounded, and so harmless. In the third term, $|x-y|\le x^{-1/2}$ where the integrand is nonzero, hence it is also bounded (using $x^{1/2}\int_{\mathbf{R}}|\eta'(x^{1/2}(x-y)|\,dy\le C$). Finally, in the second term we estimate

$$|q(x)-q(y)|\le\frac{C}{x^{\alpha/2}}\overline{D}^{\alpha}q(x)$$

where η is nonzero, and so this term is bounded by $Cx^{(1-\alpha)/2}\overline{D}^{\alpha}q(x)$. Hence, in all, $x^{-1}q_2'(x^2)\in L^p(\mathbf{R})$. $\square$

5. A counterexample

Our main goal in this section is to prove Theorem 1.4, showing that the results of Theorems 1.1 and 1.2 are sharp. Namely, we will prove that there exist potentials which both belong to $C^{1/2}(\mathbf{R})$, and decay at the rate $x^{-1/4}$, yet which lead to singular spectrum on the whole real axis. Hence one example will show optimality of both theorems. Apart from establishing optimality, the example we are going to construct is interesting in its own right, suggesting the mechanism to get singular spectrum in a background constant electric field "at the lowest cost", and giving potentials with interesting dynamical properties. Our potentials will be random, and the construction is inspired by [20], although there are some notable differences. In [20], random perturbations of the free operator of the form

$$V(x)=\sum_{n=1}^{\infty}\frac{a_n(\omega)}{n^{\alpha}}f(x-n) \tag{5.1}$$

were considered (here f and a_n can be taken as in (1.4)). This model has a transition of spectral properties from absolutely continuous to singular at $\alpha=\frac{1}{2}$ (see also [21] for a similar earlier model).

By subordinacy theory, it is enough to construct the potential on the positive half-axis with the above decay and smoothness properties such that for a.e. $E\in\mathbf{R}$

there exists a solution subordinate on the right. We are going to analyze the equation (2.2),

$$-\phi''+\left(\frac{5}{36\xi^2}+\frac{q(c\xi^{2/3})-E}{c\xi^{2/3}}\right)\phi=\phi,$$

and its Prüfer variables representation (2.4), (2.5),

$$(\log R(\xi,E))'=\frac{1}{2}\left(\frac{5}{36\xi^2}+\frac{q(c\xi^{2/3})-E}{c\xi^{2/3}}\right)\sin 2\theta(\xi,E),$$

$$\theta'(\xi,E)=1-\frac{1}{2}\left(\frac{5}{36\xi^2}+\frac{q(c\xi^{2/3})-E}{c\xi^{2/3}}\right)(1-\cos 2\theta(\xi,E)).$$

Let us denote by β the boundary condition for θ at the origin. Let $a_n(\omega)$ be independent random variables with uniform distribution in $[0,2\pi]$. Fix a function $f\in C_0^\infty(0,1)$, f not identically zero. Consider a family of random potentials $q(x)$ chosen so that

$$\frac{q(c\xi^{2/3})}{c\xi^{2/3}}=\frac{1}{\xi^{2/3}}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}f(\xi^{1/3}-n)\sin(2\xi+a_n(\omega)). \tag{5.2}$$

Notice that in the coordinate x, according to the Liouville transformation, our potential $q(x)$ looks exactly like (1.4),

$$q(x)=c\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}f\left(\sqrt{cx}-n\right)\sin\left(\tfrac{4}{3}x^{3/2}+a_n(\omega)\right).$$

In particular, any such q satisfies $|q(x)|\le Cx^{-1/4}$ and belongs to $C^{1/2}(\mathbf{R})$. We begin the proof of Theorem 1.4 with a series of auxiliary statements. The key, as before, will be the analysis of the asymptotic behavior of solutions. First, we are going to show that in order to apply subordinacy theory, it is sufficient to study the asymptotic behavior of $R(\xi,E)$.

Lemma 5.1. *For any $q(x)$ satisfying $|q(x)|\le Cx$, any energy E and any boundary condition β, we have*

$$\int_1^L \phi^2(\xi,E,\beta)\,d\xi\le\int_1^L R^2(\xi,E,\beta)\,d\xi\le C_q(E)\int_1^L\phi^2(\xi,E,\beta)\,d\xi.$$

Proof. Notice that the condition on q ensures that $q(c\xi^{2/3})\xi^{-2/3}$ is bounded. By the definition of R, it is enough to show that the L^2 norm of the derivative of ϕ on $[1,L]$ is controlled by some constant $C_q(E)$ times the L^2 norm of ϕ. The proof of the latter fact is a simple elliptic regularity type argument, and is well known. We refer, for example, to [22], the proof of Theorem 2.1C, for a proof of a completely parallel result. □

Theorem 5.2. *Consider the Prüfer variables equations* (2.4) *and* (2.5), *with potential q defined by* (5.2). *Then for any $E\in\mathbf{R}$ and any boundary condition β we have for a.e. ω,*

$$\lim_{\xi\to\infty}\frac{\log R(\xi,E,\beta)}{\log\xi}=\Lambda(E), \tag{5.3}$$

where

$$\Lambda(E)=\frac{3\pi}{8}\left|\hat f\left(\frac{3E}{C}\right)\right|^2 \tag{5.4}$$

(*here $\hat f$ denotes the Fourier transform $\int_{\mathbf{R}}\exp(i\xi x)f(x)\,dx$ of f*).

Remark. Notice that the "Lyapunov exponent" $\Lambda(E)$ is positive everywhere except for perhaps a discrete set of points since the Fourier transform of f is analytic.

We begin with a sequence of auxiliary lemmas. Consider the unique solution of (2.5) satisfying some fixed boundary condition β. First, notice that the integral

$$\int_1^N\left(\frac{5}{36\xi^2}-\frac{E}{c\xi^{2/3}}\right)\sin 2\theta(\xi,E)\,d\xi$$

stays bounded, as $N\to\infty$, for any ω, by the argument in the proof of Theorem 1.5, and so in the equation (2.4) for $R(\xi,E)$ it is enough to consider the integral

$$\frac{1}{2}\int_1^N\frac{q(c\xi^{2/3})}{c\xi^{2/3}}\sin 2\theta(\xi,E)\,d\xi. \tag{5.5}$$

First we are going to analyze $R(\xi,E)$ along a sequence $\xi=n^3$, as suggested by the form of the potential (5.2), since the parts with independent phases are supported on $[n^3,(n+1)^3]$. Notice that

$$\begin{aligned}&\left|\int_{n^3}^{\xi}\frac{q(c\xi^{2/3})}{c\xi^{2/3}}\sin 2\theta(\xi,E)\,d\xi\right|\\ &=\left|\frac{1}{n^{1/2}}\int_{n^3}^{\xi}\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\sin(2\xi+a_n(\omega))\sin 2\theta(\xi,E)\,d\xi\right|\le\frac{C}{n^{1/2}}\end{aligned} \tag{5.6}$$

for $\xi\in[n^3,(n+1)^3]$. Therefore, if we control $R(n^3,E)$, we will have sufficient control for all ξ, in particular

$$\lim_{x\to\infty}\frac{R(x,E)}{\log x}=\frac{1}{3}\lim_{n\to\infty}\frac{R(n^3,E)}{\log n}. \tag{5.7}$$

Denote by θ_n the value of θ at the beginning of the nth interval, $\theta_n=\theta(n^3,E)=\theta(n^3,E,\beta,\omega)$.

Define

$$F(\xi,E)=\frac{1}{2}\int_{n^3}^{\xi}\frac{E}{c\eta^{2/3}}\,d\eta=\frac{3}{2c}E(\xi^{1/3}-n). \tag{5.8}$$

Lemma 5.3. *We can represent $\theta(\xi,E)$ on $[n^3,(n+1)^3]$ in the form*

$$\theta(\xi,E)=\theta_n+\xi+F(\xi,E)+\Theta(\xi,E)$$

$$+\frac{1}{2n^{1/2}}\int_{n^3}^{\xi}\frac{1}{\eta^{2/3}}f(\eta^{1/3}-n)\sin(2\eta+a_n(\omega))\cos(2\eta+2F(\eta,E)+2\theta_n)\,d\eta, \tag{5.9}$$

where

$$|\Theta(\xi,E)|\le\frac{C(E)}{n}.$$

Proof. Let us analyze the terms arising in the equation (2.5) for the angle θ. The proof reduces to showing that

$$\Theta(\xi,E)=\frac{1}{2}\int_{n^3}^{\xi}\Bigg(\left(\frac{5}{36\eta^2}+\frac{E}{c\eta^{2/3}}\right)\cos 2\theta(\eta,E)-\left(\frac{5}{36\eta^2}+\frac{q(c\eta^{2/3})}{c\eta^{2/3}}\right)$$
$$+\frac{q(c\eta^{2/3})}{c\eta^{2/3}}(\cos 2\theta(\eta,E)-\cos(2\eta+2F(\eta,E)+2\theta_n))\Bigg)\,d\eta$$

satisfies the required estimate. The first term satisfies

$$\left|\int_{n^3}^{\xi}\left(\frac{5}{36\eta^2}+\frac{E}{c\eta^{2/3}}\right)\cos 2\theta(\eta,E)\,d\eta\right|\le\frac{C}{n^2} \tag{5.10}$$

by an integration by parts argument identical to that in the proof of Theorem 1.5. The second term satisfies

$$\left|\int_{n^3}^{\xi}\left(\frac{5}{36\eta^2}+\frac{q(c\eta^{2/3})}{c\eta^{2/3}}\right)d\eta\right|\le\frac{C}{n^3},$$

also by a simple integration by parts estimate for the second summand. For the last term, we have

$$\int_{n^3}^{\xi}\frac{q(c\eta^{2/3})}{c\eta^{2/3}}(\cos 2\theta(\eta,E)-\cos(2\eta+2F(\eta,E)+2\theta_n))\,d\eta$$

$$=2\int_{n^3}^{\xi}\frac{f(\eta^{1/3}-n)}{n^{1/2}\eta^{2/3}}\sin(2\eta+a_n(\omega))\sin(\theta(\eta,E)+\eta+F(\eta,E)+\theta_n) \tag{5.11}$$
$$\times\sin(\theta(\eta,E)-\eta-F(\eta,E)-\theta_n)\,d\eta.$$

Now, we have by (2.5),

$$|\theta(\eta,E)-\eta-F(\eta,E)-\theta_n|$$
$$=\left|\frac{1}{2}\int_{n^3}^{\eta}\left[\left(\frac{q(ct^{2/3})}{ct^{2/3}}+\frac{5}{36t^2}\right)(1-\cos 2\theta(t,E))-\frac{E}{ct^{2/3}}\cos 2\theta(t,E)\right]dt\right|$$
$$\le\frac{C}{n^{1/2}}$$

for $\eta\in[n^3,(n+1)^3]$ by the decay of q and the estimate (5.10). Putting this into (5.11), we bound the last term in the expression for $\Theta(\xi,E)$ by Cn^{-1}. □

Next we need Lemma 8.4 from [20], concerning series of random variables. We will denote by $\mathrm{Exp}(X)$ the expectation of the random variable X. Assume that $X_j(\omega)$ are independent random variables with zero mean $\mathrm{Exp}(X_j)=0$, and let

$$Z_n = X_n f_n(X_1, \dots, X_{n-1}), \tag{5.12}$$

where f_n are some measurable functions. We have the following result.

Lemma 5.4. ([20]) *Suppose that* $\mathrm{Exp}(Z_n^2)\leq Cn^{-2\alpha}$. *Then for a.e.* ω,
(1) *if* $\alpha<\frac{1}{2}$ *and* $\beta>\frac{1}{2}(1-2\alpha)$, *then*

$$\lim_{n\to\infty}\left|\sum_{j=1}^{n} Z_j\right|\frac{1}{n^\beta}=0;$$

(2) *if* $\alpha=\frac{1}{2}$ *and* $\beta>\frac{1}{2}$, *then*

$$\lim_{n\to\infty}\left|\sum_{j=1}^{n} Z_j\right|\frac{1}{(\log n)^\beta}=0;$$

(3) *if* $\alpha>\frac{1}{2}$,

$$\lim_{n\to\infty}\left|\sum_{j=1}^{n} Z_j\right|=Y_\infty$$

exists, and for any $\beta<\alpha-\frac{1}{2}$,

$$\lim_{n\to\infty}\left|\sum_{j=n}^{\infty} Z_j\right|n^\beta=0.$$

Consider

$$\log\frac{R((n+1)^3,E)}{R(n^3,E)}=I_n(E,\omega)$$
$$=\frac{1}{2n^{1/2}}\int_{n^3}^{(n+1)^3}\frac{f(\xi^{1/3}-n)}{\xi^{2/3}}\sin(2\xi+a_n(\omega))\sin 2\theta(\xi,E)\,d\xi. \tag{5.13}$$

We have the following lemma.

Lemma 5.5. *The expression $I_n(E,\omega)$ may be represented as*

$$I_n(E,\omega)=\frac{9\pi}{8n}\left|\hat{f}\left(\frac{3E}{c}\right)\right|^2+Z_n^{(1)}+Z_n^{(2)}, \tag{5.14}$$

where $|Z_n^{(1)}(\omega)|\le Cn^{-3/2}$, $\{Z_n^{(2)}(\omega)\}_{n=1}^{\infty}$ *is a sequence of random variables of type* (5.12), *and* $(Z_n^{(2)})^2$ *has expectation* $\le Cn^{-1}$.

Proof. The first observation is that we can replace $2\theta(\xi,E)$ in (5.13) by

$$2\xi+2F(\xi,E)+\theta_n+\int_{n^3}^{\xi}\frac{q(c\eta^{2/3})}{c\eta^{2/3}}\cos(2\eta+2F(\eta,E)+2\theta_n)\,d\eta.$$

By Lemma 5.3, the value in (5.13) will change by at most $Cn^{-3/2}$, and we can put this difference into $Z_n^{(1)}$. Also by an estimate similar to (5.6),

$$\begin{aligned}\sin&\left(2\xi+2F(\xi,E)+2\theta_n+\int_{n^3}^{\xi}\frac{q(c\eta^{2/3})}{c\eta^{2/3}}\cos(2\eta+2F(\eta,E)+2\theta_n)\,d\eta\right)\\ &=\cos(2\xi+2F(\xi,E)+2\theta_n)\int_{n^3}^{\xi}\frac{q(c\eta^{2/3})}{c\eta^{2/3}}\cos(2\eta+2F(\eta,E)+2\theta_n)\,d\eta\\ &\quad+\sin(2\xi+2F(\xi,E)+2\theta_n)+O\left(\frac{1}{n}\right),\end{aligned} \tag{5.15}$$

and so it suffices to consider the contributions from the first two terms on the right-hand side, putting the $O(n^{-1})$ terms (multiplied by $n^{-1/2}$) into $Z_n^{(1)}$. We will write $Z_n^{(2)}$ as a sum of four parts, each satisfying the conclusion of the lemma. The first of these four is

$$\frac{1}{2n^{1/2}}\int_{n^3}^{(n+1)^3}\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\sin(2\xi+a_n(\omega))\sin(2\xi+2F(\xi,E)+2\theta_n)\,d\xi.$$

Clearly its mean is zero, and the expectation of its square is $O(n^{-2})$. It remains to discuss the contribution arising from the first term on the right-hand side in (5.15),

$$\begin{aligned}\frac{1}{2n}\int_{n^3}^{(n+1)^3}&\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\sin(2\xi+a_n(\omega))\cos(2\xi+2F(\xi,E)+2\theta_n)\\ &\times\int_{n^3}^{\xi}\frac{1}{\eta^{2/3}}f(\eta^{1/3}-n)\sin(2\eta+a_n(\omega))\cos(2\eta+2F(\eta,E)+2\theta_n)\,d\eta\,d\xi\\ &=\frac{1}{4n}\left(\int_{n^3}^{(n+1)^3}\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\sin(2\xi+a_n(\omega))\cos(2\xi+2F(\xi,E)+2\theta_n)\right)^2.\end{aligned} \tag{5.16}$$

Write the product of trigonometric functions as

$$\sin(2\xi+a_n(\omega))\cos(2\xi+2F(\xi,E)+2\theta_n)=\tfrac{1}{2}\sin(4\xi+2F(\xi,E)+2\theta_n+a_n(\omega))\\ +\tfrac{1}{2}\sin(2F(\xi,E)+2\theta_n-a_n(\omega)).$$

The first term, by integration by parts, gives a contribution of the order n^{-3}. Introducing the notation

$$I_n^{\sin}=\int_{n^3}^{(n+1)^3}\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\sin(2F(\xi,E)+2\theta_n)\,d\xi,$$
$$I_n^{\cos}=\int_{n^3}^{(n+1)^3}\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\cos(2F(\xi,E)+2\theta_n)\,d\xi$$

we obtain that, modulo small corrections that can be put into $Z_n^{(1)}$, the right-hand side of (5.16) is equal to

$$\frac{1}{8n}((I_n^{\sin})^2\cos^2 a_n(\omega)+(I_n^{\cos})^2\sin^2 a_n(\omega)+I_n^{\sin}I_n^{\cos}\sin 2a_n(\omega)). \tag{5.17}$$

The contribution of the last term in the brackets is one of the random variables of type (5.12) constituting $Z_n^{(2)}$. Subtracting from the first two terms their expectations, we get the other two constituents of $Z_n^{(2)}$. Finally, note that $\operatorname{Exp}(\cos^2 a_n(\omega))=\operatorname{Exp}(\sin^2 a_n(\omega))=\pi$, and so after subtracting $Z_n^{(2)}$ from (5.17) and taking expectations we deduce that the right-hand side of (5.16), modulo the sum of the various terms Z_n, equals

$$\frac{\pi}{8n}((I_n^{\sin})^2+(I_n^{\cos})^2)=\frac{\pi}{8n}\left|\int_{n^3}^{(n+1)^3}\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)e^{2iF(\xi,E)}\,d\xi\right|^2.$$

Recalling that $F(\xi,E)=3E(\xi^{1/3}-n)/2c$, and making the change of variable $t=\xi^{1/3}$, we obtain

$$\frac{9\pi}{8n}\left|\hat{f}\left(\frac{3E}{c}\right)\right|^2,$$

proving the lemma. □

Proof of Theorem 5.2. The proof follows directly from combining the results of Lemma 5.5, Lemma 5.4, and the relation (5.7), choosing $\beta<1$ in conclusion (2) of Lemma 5.4. □

The final step in our construction is showing that for a.e. ω and a.e. E there exists a subordinate solution.

Proposition 5.6. *Define the potential q by (5.2). Consider any energy E such that $\hat{f}(3E/c)\neq 0$. Then for a.e. ω there exists a boundary condition $\beta(\omega)=\beta(\omega,E)$ such that*

$$\lim_{\xi\to\infty}\frac{\log R(\xi,E,\beta(\omega))}{\log\xi}=-\Lambda(E). \tag{5.18}$$

where $\Lambda(E)=(3\pi/8)|\hat{f}(3E/c)|^2$.

Proof. By Lemma 8.7 of [20] (which is formulated there in a discrete setting but extends trivially to our situation), it is sufficient to show that for a.e. ω,

$$\varrho(\xi,E)=\log\frac{R(\xi,E,0)}{R(\xi,E,\frac{1}{2}\pi)}$$

has a limit $\varrho_\infty(E)$, as $\xi\to\infty$, and that

$$\limsup_{\xi\to\infty}\frac{\log|\varrho(\xi,E)-\varrho_\infty(E)|}{\log\xi}\leq -2\Lambda(E). \tag{5.19}$$

Notice that by constancy of the Wronskian,

$$R(\xi,E,0)R(\xi,E,\tfrac{1}{2}\pi)\sin\big(\theta(\xi,E,0)-\theta(\xi,E,\tfrac{1}{2}\pi)\big)=\text{constant}.$$

Therefore, by Theorem 5.2 for a.e. ω,

$$\lim_{\xi\to\infty}\frac{\log\big|\theta(\xi,E,0)-\theta(\xi,E,\frac{1}{2}\pi)\big|}{\log\xi}=-2\Lambda(E). \tag{5.20}$$

On the other hand, by Lemma 5.3,

$$\begin{aligned}\varrho((n+1)^3,E)=\sum_{j=1}^{n+1}\frac{1}{n^{1/2}}\int_{n^3}^{(n+1)^3}&\frac{1}{\xi^{2/3}}f(\xi^{1/3}-n)\sin(2\xi+a_n(\omega))\\ &\times\big(\sin(2\xi+2F(\xi,E)+2\theta_n(\xi,E,0))\\ &-\sin\big(2\xi+2F(\xi,E)+2\theta_n(\xi,E,\tfrac{1}{2}\pi)\big)\big)\,d\xi+O\Big(\frac{1}{n}\Big).\end{aligned}$$

Expanding $\sin(2\xi+a_n(\omega))$ into a sum of products, and using (5.20), we can apply Lemma 5.4 to prove convergence to ϱ_∞ and estimate the rate of convergence obtaining (5.19). The estimate (5.6) as before allows us to pass from estimates over the sequence n^3 to estimates over all ξ. $\square$

Our main Theorem 1.4 follows immediately from Proposition 5.6, Theorem 5.2 and Lemma 5.1.

Proof of Theorem 1.4. Going back to the x variable representation, we find that for a.e. ω and for a.e. E, there exist both a decaying solution $u_d(x,E)$ and a growing solution $u_g(x,E)$, and the asymptotic behavior of their L^2 norms taken over $[0,L]$ is given by

$$\log\int_0^L |u_d(x,E)|^2\,dx = \max\{0, \tfrac{1}{2}-3\Lambda(E)\}\log L(1+o(1)), \tag{5.21}$$

$$\log\int_0^L |u_g(x,E)|^2\,dx = \left(\tfrac{1}{2}+3\Lambda(E)\right)\log L(1+o(1)). \tag{5.22}$$

Hence, given a generic ω, there is a solution subordinate on the right for a.e. E. □

We would like to conclude by making several remarks about some curious properties of the random potential (1.4). First of all, the structure of the potential (1.4) indicates that to get singular or point spectrum "at minimal cost" in terms of decay or smoothness of a random potential, it is important to have correlations over large (increasing) distances. In particular, in our example correlation distance grows as a square root of the distance from the origin. This contrasts with the free Schrödinger operator case [21], [20], where long distance correlation would have been ineffectual. Another characteristic feature is the presence of increasingly fast oscillations at infinity.

The half-line Stark operator with potential (1.4) exhibits a number of interesting spectral and dynamical properties which are similar to the properties of potentials with the borderline decay rate in a model of [20] in the free case. Introduce X for the coordinate operator acting by $Xf(x)=|x|f(x)$. Denote also the inner product of two vectors by $\langle\psi_1,\psi_2\rangle$. Given an initial state ψ, $\psi(t)$ stands for its unitary evolution $e^{-iH_qt}\psi$. One reasonable measure of propagation properties is the averaged moments of the coordinate operator

$$\langle\langle X^m\psi(t),\psi(t)\rangle\rangle_T = \frac{1}{T}\int_0^T \langle\psi(t), X^m\psi(t)\rangle\,dt.$$

Scaling

$$\langle\langle X^m\psi(t),\psi(t)\rangle\rangle_T \approx CT^m$$

corresponds to the constant velocity ballistic rate, while the behavior

$$\langle\langle X^m\psi(t),\psi(t)\rangle\rangle_T \approx CT^{2m}$$

corresponds to the superballistic constant acceleration rate, as for the free Stark operator. We will denote by P_c the spectral projector on the continuous part of the spectrum of the operator H_q. Also, given a Borel measure μ, we will say that μ has local Hausdorff dimension $d(E)$ at the point E, if for every $\varepsilon>0$ there exists $\delta>0$ such that the restriction of μ to $[E-\delta, E+\delta]$ gives zero weight to any set of Hausdorff dimension $d(E)-\varepsilon$, and is supported on a set of Hausdorff dimension $d(E)+\varepsilon$. Let us denote by $H_{q,\beta}$ the Schrödinger operator (1.1) defined on the positive half-line with the boundary condition $\beta u(0)-u'(0)=0$. Finally, let μ_β be the spectral measure associated with $H_{q,\beta}$ in a canonical way, namely

$$\langle (H_{q,\beta}-z)^{-1}\delta_0, \delta_0\rangle = \int \frac{d\mu_\beta(t)}{t-z},$$

where δ_0 is the delta-function distribution at 0 (see, e.g. [35], Sections I.1–I.6). The following proposition holds.

Proposition 5.7. *Assume that $q_\omega(x)$ is given by (1.4). Then for a.e. β and ω, the spectral measure of $H_{q,\beta}$ is pure point on the set where $\Lambda(E)>\frac{1}{6}$, and is singular continuous with local Hausdorff dimension*

$$d(E)=1-6\Lambda(E)$$

on the set where $\Lambda(E)<\frac{1}{6}$. Moreover, for a.e. β and ω, for any ψ such that $P_c(\beta,\omega)\psi\neq 0$, we have

$$\langle\langle X^m\psi(t), \psi(t)\rangle\rangle_T \geq C_{\varepsilon,m,\beta,\omega}T^{2m(1-\varepsilon)} \quad \textit{for any } \varepsilon>0.$$

Proof. The proof of this proposition is analogous to the proof of Theorem 8.10 of [20] and Theorem 5.1 of [19], where similar properties were established for a model (5.1) of [20] with $\alpha=\frac{1}{2}$ in a discrete setting. Let us sketch the arguments for the sake of completeness. First, let

$$A=\left\{E \mid \Lambda(E)>\tfrac{1}{6}\right\}.$$

Notice that by (5.4) and the properties of f, A is a finite union of disjoint bounded intervals. By Proposition 5.6, for a.e. ω and for a.e. $E\in A$, there exists a solution $u_{\beta(\omega)}(x,E,\omega)\in L^2(\mathbf{R})$. Also, the set A belongs to the spectrum of $H_{q,\beta}$ for any ω and β since q is decaying. The general theory of rank one perturbations then implies that for a.e. ω and for a.e. β, the spectrum on A is dense pure point (see, e.g. [30], Theorem 5.1, or [34], [35]).

Now let E be such that $\Lambda(E)<\frac{1}{6}$, and fix $\varepsilon>0$. Consider an interval $I_\delta=[E-\delta, E+\delta]$ such that the values of the Hausdorff dimension function $d(E)=1-6\Lambda(E)$ on I_δ belong to the interval $(d(E)-\varepsilon, d(E)+\varepsilon)$. We can choose $\delta>0$ since $\Lambda(E)$ is continuous. Let

$$\|u\|_L^2=\int_0^L |u(x)|^2\,dx,$$

and recall that the upper α derivative of the measure μ is defined by

$$D^\alpha\mu(E)=\limsup_{\delta\to 0}\frac{\mu(E-\delta, E+\delta)}{(2\delta)^\alpha}.$$

By the Jitomirskaya–Last extension of the subordinacy theory [16], the spectral measure μ_β satisfies

$$D^\alpha\mu_\beta(E)=\infty \quad \text{if and only if} \quad \liminf_{L\to\infty}\frac{\|u_\beta\|_L^{2-\alpha}}{\|u_{\beta\perp}\|_L^\alpha}=0, \tag{5.23}$$

where u_β is a nonzero solution satisfying the boundary condition at zero, and $u_{\beta\perp}$ is the solution satisfying an orthogonal boundary condition $\beta^{-1}u(0)+u'(0)=0$ (in fact, any linearly independent solution will do in place of $u_{\beta\perp}$). From the estimates (5.21) and (5.22) it follows that for a.e. ω and a.e. $E\in I_\delta$, there exists a boundary condition $\beta(\omega)$ such that

$$\inf\{\alpha \mid D^\alpha\mu_{\beta(\omega)}(E)=\infty\}=d(E). \tag{5.24}$$

Recall two basic facts from the general rank one perturbation theory: first,

$$\int_{\mathbf{R}}\mu_\beta(S)\,d\beta=m(S)$$

for any Borel measurable set S, where m is the Lebesgue measure. Second, we can have $D^1\mu_{\beta_1}(E)=D^1\mu_{\beta_2}(E)=\infty$ for some $\beta_1\neq\beta_2$ only on a fixed (for a given ω) exceptional set of energies of Lebesgue measure zero. The first fact, in the context of Schrödinger operators, is due to Simon and Wolff [37]. The second is a simple consequence of an explicit formula relating the Cauchy transforms of μ_{β_1} and μ_{β_2}. See, for example, equation (I.13) of [35] for more details. Using these properties, we see that (5.24) implies that for a.e. ω and and a.e. β, we have

$$\inf\{\alpha \mid D^\alpha\mu_\beta(E)=\infty\}=d(E)$$

for every $E\in I_\delta$ in the support of μ_β. Similarly, we also obtain that for a.e. ω and β, the solution $u_\beta(x,E)$ satisfies the bound (5.21) for the energies in the support of μ_β.

By the choice of δ and well-known relations between D^α derivatives and dimensional properties of Radon measures (see [31]) it follows that for a.e. ω and β, the restriction of the spectral measure μ_β to I_δ is supported on a set of Hausdorff dimension $\leq d(E)+\varepsilon$ and gives zero weight to any set of Hausdorff dimension $\leq d(E)-\varepsilon$.

To show the dynamical bound, notice that Theorem 1.2 of [19] says that if the spectral measure μ is α-continuous on a set S, and for every $E\in S$, the generalized eigenfunction $u(x,E)$ satisfies

$$\limsup_{L\to\infty} \frac{1}{L^\gamma}\|u\|_L^2 < \infty,$$

then for any vector ψ with nonzero projection on S, that is, with $P_S(\psi)\neq 0$,

$$\langle\langle |X|^m\psi(t),\psi(t)\rangle\rangle_T \geq CT^{m\alpha/\gamma}.$$

The proof is now completed as in Theorem 5.1 in [19], noticing that in our context, $\alpha\sim 1-6\Lambda(E)$, while by (5.21), $\gamma\sim\frac{1}{2}-3\Lambda(E)$. $\square$

Remarks. 1. The methods of [19] also give a stronger dynamical estimate that for a.e. β and ω and for every $\varepsilon>0$ and $\varrho>0$, there exists a constant $C_{\beta,\omega,\varrho,\varepsilon}$ such that if $R_T=C_{\beta,\omega,\varrho,\varepsilon}T^{2-\varepsilon}$, then

$$\langle\|\psi(t)\|_{R_T}^2\rangle_T \leq \|\psi-P_c(\beta,\omega)\psi\|^2+\varrho\|\psi\|^2. \tag{5.25}$$

From (5.25) it follows that the whole part of the wavepacket lying in the continuous spectral subspace travels at a rate $\gtrsim T^{2(1-\varepsilon)}$, even though we can choose f in (1.4) so that the spectral dimension $d(E)$ is arbitrarily close to zero in some parts of the support of the spectral measure.

2. Proposition 5.7 is not true for a fixed boundary condition. In fact, for a.e. ω and for a dense G_δ set of boundary conditions $\beta\in\mathbf{R}$, the spectrum of $H_{q,\beta}$ is going to be purely singular continuous [30].

A final remark, which is more of academic interest, is that our method can be modified in a straightforward way to treat perturbations of background operators

$$-\frac{d^2}{dx^2}-x^\lambda,$$

where $0<\lambda<2$ (the operator becomes non-selfadjoint for larger λ). In particular, one obtains that the absolutely continuous spectrum of such operators is preserved for perturbations q satisfying $|q(x)|\leq C(1+|x|)^{-(2-\lambda)/4-\varepsilon}$ or $q\in C^{(2-\lambda)/2\lambda+\varepsilon}(\mathbf{R})$, where ε is an arbitrary small positive number. These results are optimal in the power scale and in Hölder spaces, respectively.

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Michael Christ
Department of Mathematics
University of California
Berkeley, CA 94720-3840
U.S.A.
email: mchrist@math.berkeley.edu

Alexander Kiselev
Department of Mathematics
University of Chicago
Chicago, IL 60637
U.S.A.
email: kiselev@math.uchicago.edu