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Indices, characteristic numbers and essential commutants of Toeplitz operators

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Abstract. For an essentially normal operator T, it is shown that there exists a unilateral shift of multiplicity m in $C^*(T)$ if and only if $\gamma(T)\neq 0$ and $\gamma(T)|m$. As application, we prove that the essential commutant of a unilateral shift and that of a bilateral shift are not isomorphic as C^*-algebras. Finally, we construct a natural C^*-algebra $\mathcal{E}+\mathcal{E}_*$ on the Bergman space $L^2_a(B_n)$, and show that its essential commutant is generated by Toeplitz operators with symmetric continuous symbols and all compact operators.

1. Introduction

It is well known that the index formula of Gohberg and Krein [9] gives indices of Toeplitz operators on the unit circle as minus the winding numbers of their symbols. This may be the simplest and the most enlightening case in the entire index theory. Let D be the unit disk of the complex plane, S^1 its boundary and $C(S^1)$ be the space of all continuous functions on S^1. Write $H^2(D)$ for the classical Hardy space on the unit disk, and $\mathcal{K}$ for all compact operators on $H^2(D)$. Denote by $C^*(T_z)$ the C^*-algebra generated by the Hardy shift T_z and the identity operator, which equals to the C^*-algebra generated by all Toeplitz operators with symbols in $C(S^1)$. One thus has the short exact sequence, [5],

$$(1.1)\qquad 0\longrightarrow\mathcal{K}\longrightarrow C^*(T_z)\xrightarrow{\pi} C(S^1)\longrightarrow 0.$$

In the language of homological algebra, [4], the above sequence is an extension of $\mathcal{K}$ by $C(S^1)$. The importance of this extension is that the index theory of Toeplitz operators is implicit. Let $\pi^1(S^1)$ be the group of homotopy classes of continuous

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maps from S^1 to $\mathbf{C}\setminus\{0\}$. Then the index ind gives an isomorphism from $\pi^1(S^1)$ to the integer group $\mathbf{Z}$ by

$$\operatorname{ind}([f])=\operatorname{ind}(\pi^{-1}(f)).$$

Let T be an essentially normal operator on a separable Hilbert space H, and $\sigma_e(T)$ be the essential spectrum of T. Write $C^*(T)$ for the C^*-algebra generated by T, the identity operator I and all compact operators. Then a natural extension of $\mathcal{K}$ by $C(\sigma_e(T))$ arises at this point

$$0\longrightarrow\mathcal{K}\longrightarrow C^*(T)\xrightarrow{\pi}C(\sigma_e(T))\longrightarrow 0. \tag{1.2}$$

As usual, let $\pi^1(\sigma_e(T))$ denote the group of homotopy classes of continuous maps from $\sigma_e(T)$ to $\mathbf{C}\setminus\{0\}$. Fredholm index derives thus a homomorphism

$$\operatorname{ind}\colon\pi^1(\sigma_e(T))\longrightarrow Z,\quad \operatorname{ind}([f])=\operatorname{ind}(\pi^{-1}(f)).$$

Although the homomorphism ind is not an isomorphism in general, its range, range(ind), is a subgroup of $\mathbf{Z}$, that is, there exists a non-negative integer m such that range(ind)$=m\mathbf{Z}$. This integer m which is said to be the characteristic number of T, and is denoted by $\gamma(T)$, reveals many intrinsic properties of T. For example, from [4, Corollary 11.2] we see that $\gamma(T)=0$ if and only if T is a compact perturbation of a normal operator. The explicit description of $\gamma(T)$ will be given in Section 3.

For the use of subsequent sections, in Section 2, we prove several results on the index of Toeplitz operator with composition symbol. These results may be known by many people, although we do not find the related references.

The study in Section 3 is closely related to the work of Englis [7], [8], and Barria and Halmos [2]. Let T be an essentially normal operator, and its essential spectrum $\sigma_e(T)$ be non-countable. We prove that there exists a bilateral shift W in $C^*(T)$, and there exists a unilateral shift S of multiplicity m in $C^*(T)$ if and only if $\gamma(T)\neq 0$ and $\gamma(T)|m$. In particular, it is shown that the essential commutant of a unilateral shift and that of a bilateral shift are not isomorphic as C^*-algebras, which is posed in [7].

In [7], Englis proved that the set of all Toeplitz operators on the Bergman space $L_a^2(\Omega)$ is dense in all bounded operators in the strong operator topology, and its norm closure contains all compact operators. In view of the result of Englis, one is convinced that the C^*-algebra $\mathcal{T}^\infty(\Omega)$ generated by all Toeplitz operators on the Bergman space is "quite huge". Let B_n be the unit ball of $\mathbf{C}^n$, and $L_a^2(B_n)$ be the Bergman space on B_n. Set

$$\mathcal{E}=\{T\,|\,T_{z_j}T-TT_{z_j}\in\mathcal{K},\ j=1,\ 2,\dots,\ n\}$$

and

$$\mathcal{E}_* = \{T \mid T_{z_j}^* T - T T_{z_j} \in \mathcal{K},\ j = 1,\ 2, \ldots,\ n\}.$$

In Section 4, it is shown that $\mathcal{E}+\mathcal{E}_*$ is a C^*-algebra, and its essential commutant is $\mathcal{T}(SC(B_n))$, that is, an operator A essentially commutes with any operator in $\mathcal{E}+\mathcal{E}_*$ if and only if A has the form

$$A = T_f + \text{compact},$$

with f continuous on $\overline{B}_n$, and $f(z)=f(\bar{z})$, $z \in S^{2n-1}$ (the boundary of $\overline{B}_n$). This implies that $\mathcal{E}+\mathcal{E}_*$ is a proper subset of all bounded operators, though there is no unilateral shift which essentially commutes with each operator in $\mathcal{E}+\mathcal{E}_*$.

For a related study, in the version of the Hardy space $H^2(\mathbf{D})$, see [10].

2. Indices of Toeplitz operators with composition symbols

In this section, we shall be concerned with the question of how to compute Fredholm indices of Toeplitz operators with composition symbols on the Hardy space $H^2(D)$ or the Bergman space $L_a^2(D)$ of the unit disk D. These results will be used in subsequent sections.

From Coburn [5] or McDonald [11], one sees that a Toeplitz operator T_ϕ with continuous symbol on the Hardy space $H^2(D)$ (or on the Bergman space $L_a^2(D)$) is Fredholm if and only if ϕ is nonzero on the unit circle S^1 (i.e. $\phi|_{S^1}$ is nonzero). For such a Fredholm Toeplitz operator T_ϕ on $H^2(D)$, Gohberg and Krein [9] proved that the Fredholm index of T_ϕ equals to $-\gamma(\phi)$, where $\gamma(\phi)$ is the winding number of $\phi(z)$ about the origin as z describes the unit circle once in the positive sense (i.e., D is to be on the left as a point moves on the unit circle in the positive direction). Let $\mathcal{F}$ denote all invertible functions in $C(S^1)$. For ϕ_1 and ϕ_2 in $\mathcal{F}$, we say that ϕ_1 is homotopic to ϕ_2 if there exists a continuous function

$$F\colon S^1 \times [0,1] \longrightarrow \mathbf{C}_* (= \mathbf{C} \setminus \{0\})$$

such that $F(z,0)=\phi_1(z)$ and $F(z,1)=\phi_2(z)$ for $z \in S^1$. It is well known that the group $\pi^1(S^1)$ of homotopy classes of $\mathcal{F}$ is the integer group $\mathbf{Z}$, and each ϕ is homotopic to some z^n, [6]. This fact yields that $\operatorname{ind}(T_\phi)=n$ if and only if ϕ is homotopic to z^{-n}. The following result illustrates how to characterize the index of a Toeplitz operator with composition symbol.

Theorem 2.1. *Let $\phi\in\mathcal{F}$, and let $f\colon \mathbf{C}_*\to\mathbf{C}_*$ be a continuous function. Then*

$$\operatorname{ind}(T_{f\circ\phi})=\gamma(f)\operatorname{ind}(T_\phi).$$

Proof. Let $\operatorname{ind}(T_\phi)=n$. Then ϕ is homotopic to z^{-n}, that is, there exists a continuous function

$$F\colon S^1\times[0,1]\longrightarrow\mathbf{C}_*$$

such that $F(z,0)=\phi(z)$ and $F(z,1)=z^{-n}$ for $z\in S^1$. Consider the continuous composition function

$$f\circ F\colon S^1\times[0,1]\longrightarrow\mathbf{C}_*.$$

This means that $f\circ\phi$ is homotopic to $f\circ z^{-n}$. So,

$$\operatorname{ind}(T_{f\circ\phi})=\operatorname{ind}(T_{f\circ z^{-n}}).$$

Let $\gamma(f)=m$, that is, the winding number of $f(z)$ about the origin is m as z describes the unit circle. This thus forces f (restricted to S^1) to be homotopic to z^m, that is, there exists a continuous function

$$G\colon S^1\times[0,1]\longrightarrow\mathbf{C}_*$$

such that $G(z,0)=f(z)$ and $G(z,1)=z^m$ for $z\in S^1$. Define a function

$$\widetilde{G}\colon S^1\times[0,1]\longrightarrow\mathbf{C}_*$$

by $\widetilde{G}(z,t)=G(z^{-n},t)$. One then easily checks

$$\widetilde{G}(z,0)=G(z^{-n},0)=f(z^{-n}),\quad \widetilde{G}(z,1)=G(z^{-n},1)=z^{-mn},$$

that is, $f(z^{-n})$ is homotopic to z^{-mn}. It follows that

$$\operatorname{ind}(T_{f\circ\phi})=\operatorname{ind}(T_{f\circ z^{-n}})=\operatorname{ind}(T_{z^{-mn}})=mn=\gamma(f)\operatorname{ind}(T_\phi).$$

This completes the proof.

Corollary 2.2. *Let f and ϕ be invertible functions in $C(S^1)$, and $|\phi|=1$. Then*

$$\operatorname{ind}(T_{f\circ\phi})=\gamma(f)\operatorname{ind}(T_\phi).$$

Proof. Set $\tilde{f}(z)=f(z/|z|)$ on $\mathbf{C}_*$. Theorem 2.1 implies that

$$\operatorname{ind}(T_{f\circ\phi})=\operatorname{ind}(T_{\tilde{f}\circ\phi})=\gamma(f)\operatorname{ind}(T_\phi)$$

which immediately yields the desired conclusion.

For an invertible function ϕ in $C(S^1)$, let f be an invertible and continuous function defined on $\phi(S^1)$. If f can be continuously extended into a function $\tilde{f}$ from $\mathbf{C}_*$ to $\mathbf{C}_*$, this then implies that $\operatorname{ind}(T_\phi)$ is a divisor of $\operatorname{ind}(T_{f\circ\phi})$. However this is not true in general. For example, take $\phi=z+2$ and $f=z-2$, then $\operatorname{ind}(T_{f\circ\phi})=\operatorname{ind}(T_z)=-1$, but $\operatorname{ind}(T_\phi)=0$.

Now return to the Bergman space $L^2_a(D)$. Suppose that ϕ is continuous on the closure $\bar{D}$ of D , then T_ϕ acting on $L^2_a(D)$ is unitarily equivalent to a compact perturbation of $T_{\phi|_{S^1}}$ acting on $H^2(D)$, [5]. Thus we can restate Theorem 2.1 on the Bergman space $L^2_a(D)$.

Theorem 2.3. *Let ϕ be bounded function on the closure $\bar{D}$ of D. Suppose that ϕ is continuous and bounded away from zero on some neighborhood of the unit circle. Then for any continuous function f on the complex plane $\mathbf{C}$ whose only possible zero point is zero, we have*

$$\operatorname{ind}(T_{f\circ\phi}) = \gamma(f)\operatorname{ind}(T_\phi).$$

In view of Theorem 2.3, we require that f is a continuous function with the only possible zero point being zero. Otherwise, the composition function $f\circ\phi$ might be out of $L^2_a(D)$. For example, take $\phi=z$, $f=1/z$. However, if we assume that f is continuous from $\mathbf{C}_*$ to $\mathbf{C}_*$ and is bounded on $D\backslash\{0\}$, then the same conclusion is true. We thus have following corollary.

Corollary 2.4. *Let ϕ satisfy the assumption of Theorem 2.3. Then for $f\in\mathcal{F}$ we have*

$$\operatorname{ind}(T_{\tilde{f}\circ\phi}) = \gamma(f)\operatorname{ind}(T_\phi),$$

where $\tilde{f}(z)=f(z/|z|)$ for $z\in\mathbf{C}_*$.

Let T be an essentially normal operator, and $\widetilde{T}$ the image of T in the corresponding Calkin algebra. In nature, if T is Fredholm, one can define the index of $\widetilde{T}$, $\operatorname{ind}(\widetilde{T})$, by $\operatorname{ind}(T)$. Assume that the essential spectrum $\sigma_e(T)$ of T lies on a Jordan curve Γ, then there exists a continuous function ϕ on S^1 such that ϕ maps the unit circle S^1 onto Γ. In particular, if the origin is in the inside of Γ we then can choose a continuous function ϕ such that $\operatorname{ind}(T_\phi)=\operatorname{ind}(\widetilde{T})$. Theorem 2.1 and the BDF-theory [4] derive the following result.

Corollary 2.5. *Let f be a continuous function from $\mathbf{C}_*$ to $\mathbf{C}_*$. Then*

$$\operatorname{ind}(f(\widetilde{T})) = \gamma(f)\operatorname{ind}(\widetilde{T}) = \gamma(f)\operatorname{ind}(T).$$

In particular, if $\sigma_e(T)$ *is contained in the unit circle, then*

$$\operatorname{ind}(f(\widetilde{T})) = \gamma(f)\operatorname{ind}(\widetilde{T}) = \gamma(f)\operatorname{ind}(T),$$

where f *is in* $\mathcal{F}$.

3. The characteristic numbers of essentially normal operators

As is well known the index formula of Gohberg and Krein [9] gives indices of Toeplitz operators on the unit circle as minus the winding numbers of their symbols. A natural generalization of the notion of winding number may be the concept of characteristic number for an essentially normal operator. The characteristic number of an essentially normal operator has been defined in the introduction. However, the next equivalent definition may be more convenient for use. Assume first that $\mathcal{A}$ is a C^*-algebra of operators on a separable Hilbert space H which contains all compact operators and the identity operator. Since the image of all Fredholm operators in $\mathcal{A}$ is a multiplication group in the Calkin algebra, it follows that the indices of all Fredholm operators in $\mathcal{A}$ form a subgroup Λ of the integer group $\mathbf{Z}$, that is, there exists a unique non-negative integer m such that $\Lambda = m\mathbf{Z}$. The characteristic number $\gamma(\mathcal{A})$ of $\mathcal{A}$, by definition, is the above m. Thus, the characteristic number γ is an invariant of C^*-algebras in isomorphism sense. Let T be essentially normal. We define the characteristic number of T, $\gamma(T)$, by $\gamma(C^*(T))$. From the BDF-theory [4, Corollary 11.2], it is easily seen that $\gamma(T)=0$ if and only if T is a compact perturbation of a normal operator. If $\gamma(T)\neq 0$, then $\gamma(T)$ is characterized by the next proposition.

Proposition 3.1. *If* $\gamma(T)\neq 0$, *then* $\gamma(T)$ *is the greatest common divisor of* $\{|\operatorname{ind}(T-\lambda_n)|\}$, *where* $\{\lambda_n\}$ *is a set with exactly one point* λ_n *in each bounded component of* $\mathbf{C}\backslash\sigma_e(T)$. *In particular, if* $\sigma_e(T)$ *lies on a Jordan curve, then*

$$\gamma(T) = |\operatorname{ind}(T-\lambda)|,$$

where λ *is located in the inside of the Jordan curve.*

Proof. Let $\pi^1(\sigma_e(T))$ be the group of homotopy classes of continuous maps from $\sigma_e(T)$ to $\mathbf{C}_*$. Then $\pi^1(\sigma_e(T))$ is the free abelian group generated by the set $\{[z-\lambda_n]\}$ (see [4]). Thus for each invertible and continuous function f on $\sigma_e(T)$, f is homotopic to some $(z-\lambda_{l_1})^{k_1}\dots(z-\lambda_{l_m})^{k_m}$, and hence

$$\operatorname{ind}(f(\widetilde{T})) = \sum_{j=1}^{m} k_j \operatorname{ind}(T-\lambda_{l_j}).$$

The desired conclusion follows and the proof is complete.

To understand the characteristic number a little better we will turn to the following discussion.

It is well known that Toeplitz operators on the Hardy space $H^2(D)$ are completely characterized by $T_z^*TT_z=T$. This fact forces all Toeplitz operators on the Hardy space to form a rather small w^*-closed subspace of the space of all bounded operators of infinite codimension. However, Englis [7] proved that the set of all Toeplitz operators on the Bergman space $L_a^2(\Omega)$ is dense in the space of all bounded operators in the strong operator topology, and its norm closure contains all compact operators. In view of the result of Englis, one is convinced that the C^*-algebra $\mathcal{T}^\infty(\Omega)$ generated by all Toeplitz operators on the Bergman space is "quite huge". However, for a wide class of plane domains $\Omega\subset\mathbf{C}$, and bounded symmetric domains $\Omega\subset\mathbf{C}^n$, Englis [7], [8] proved that each Toeplitz operator on $L_a^2(\Omega)$ essentially commutes with some fixed shift (unilateral or bilateral shift), and hence $\mathcal{T}^\infty(\Omega)$ is a proper subset of all bounded operators. In view of Englis's results, it would be useful to know something more about the essential commutants of the unilateral and the bilateral shift. Englis said in [7] that it is not even clear whether these two essential commutants (of unilateral and bilateral shifts) are not in fact isomorphic as C^*-algebras. To answer Englis's question, let us state and prove the main result in this section. Let T be essentially normal. One then has a natural extension of $\mathcal{K}$ by $C(\sigma_e(T))$,

$$(*)\qquad 0\longrightarrow\mathcal{K}\longrightarrow C^*(T)\xrightarrow{\pi}C(\sigma_e(T))\longrightarrow 0.$$

If $\sigma_e(T)$ is countable, then $C^*(T)$ does not contain any unilateral or bilateral shifts by the above extension $(*)$. If $\sigma_e(T)$ is uncountable, then the characteristic number of T characterizes whether $C^*(T)$ contains any unilateral or bilateral shifts in the following way.

Theorem 3.2.

(1) *There exists a bilateral shift W in $C^*(T)$.*

(2) *If $\gamma(T)=0$, then $C^*(T)$ does not contain any unilateral shift of finite multiplicity.*

(3) *If $\gamma(T)\neq 0$, there is a unilateral shift of multiplicity m in $C^*(T)$ if and only if $\gamma(T)|m$.*

Proof. (1) Since $\sigma_e(T)$ is uncountable, there is a continuous function f from $\sigma_e(T)$ onto $[0,2\pi]$. Consider functions

$$f_k(z)=\exp\Big(\frac{i}{k}f(z)\Big),\quad k=1,\ 2,\dots .$$

It is easily seen that $f_k(z)$ is invertible in $C(\sigma_e(T))$. Take $A_k \in C^*(T)$ such that $\pi(A_k)=f_k$ for all natural numbers k. The above extension $(*)$ thus implies that $\sigma_e(A_1)=S^1$. Notice that

$$\operatorname{ind}(A_1)=\operatorname{ind}(A_k^k)=k\operatorname{ind}(A_k), \quad k=1,\ 2, \ldots,$$

and hence $\operatorname{ind}(A_1)=0$. Consequently, the BDF-theory [4] derives that A_1 is a compact perturbation of some bilateral shift. One thus concludes that there exists a bilateral shift W in $C^*(T)$.

(2) If $\gamma(T)=0$, then for any Fredholm operator A in $C^*(T)$, we have

$$\operatorname{ind}(A)=0.$$

Consequently, the conclusion of (2) follows.

(3) According to the definition of the characteristic number, the necessity is obvious. For the sufficiency, we only need to construct a unilateral shift S of multiplicity $\gamma(T)$ in $C^*(T)$. Let $A \in C^*(T)$ be such that $\operatorname{ind}(A)=\gamma(T)$. Write f for $\pi(A)$. Let $B \in C^*(T)$ be such that $\pi(B)=1/|f|$. This induces that $\operatorname{ind}(B)=0$ by [4, Corollary 11.3]. Set $L=AB$, then

$$\operatorname{ind}(L)=\operatorname{ind}(A)=\gamma(T)$$

and $\pi(L)$ is of unit modulus on $\sigma_e(T)$. Therefore, the image of L in the Calkin algebra is a unitary element. From [4, Theorem 3.1], L is a compact perturbation of the adjoint of a shift of multiplicity $\gamma(T)$, and hence the conclusion of (3) follows. The proof is complete.

Let $\mathcal{A}$ be a C^*-algebra of operators on a separable Hilbert space H. The essential commutant $\mathcal{A}'_e$ of $\mathcal{A}$ is defined by $\{B \mid AB-BA \in \mathcal{K}$ for all $A \in \mathcal{A}\}$. For an operator T on H, the essential commutant T'_e of T is defined by $\{B \mid BT-TB \in \mathcal{K}\}$. In particular, when T is essentially normal the Fuglede–Putnam theorem implies that $T'_e=[C^*(T)]'_e$.

Before continuing we need a striking result of Voiculescu [1, Corollary 2].

Lemma 3.3. *Every unital separable C^*-algebra in the Calkin algebra equals its own double commutant. Equivalently, if $\mathcal{A}$ is a unital separable C^*-algebra of operators on H which contains all compact operators, then we have*

$$(\mathcal{A}'_e)'_e=\mathcal{A}.$$

In what follows we always assume that the C^*-algebra $\mathcal{A}$ contains the identity operator and all compact operators on H, and $\mathcal{A}$ is separable.

Lemma 3.4. *The C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as C^*-algebras if and only if $(\mathcal{A}_1)'_e$ and $(\mathcal{A}_2)'_e$ are isomorphic as C^*-algebras.*

Proof. Let $\phi: \mathcal{A}_1 \to \mathcal{A}_2$ be an isomorphism of C^*-algebras. Applying [6, Corollary 5.41] we see that there exists a unitary operator U on H such that $\phi(A) = U^*AU$ for any $A \in \mathcal{A}_1$. A direct verification shows that $\phi_U: (\mathcal{A}_1)'_e \to (\mathcal{A}_2)'_e$ gives the desired isomorphism from $(\mathcal{A}_1)'_e$ onto $(\mathcal{A}_2)'_e$, where $\phi_U: (\mathcal{A}_1)'_e \to (\mathcal{A}_2)'_e$ is defined by $\phi_U(A) = U^*AU$ for any $A \in (\mathcal{A}_1)'_e$. Indeed, since $\mathcal{A}_2 = U^*\mathcal{A}_1U$, the equality

$$(\mathcal{A}_2)'_e = (U^*\mathcal{A}_1U)'_e = U^*(\mathcal{A}_1)'_eU$$

follows. The opposite direction is achieved by the same argument and Lemma 3.3.

The next corollary answers the question of Englis [7].

Corollary 3.5. *The essential commutants of unilateral and bilateral shifts are not isomorphic as C^*-algebras.*

Proof. Let S be a unilateral shift, and W a bilateral shift. Since

$$S'_e = [C^*(S)]'_e \quad \text{and} \quad W'_e = [C^*(W)]'_e,$$

Lemma 3.4 implies that $[C^*(S)]'_e$ and $[C^*(W)]'_e$ are isomorphic if and only if $C^*(S)$ and $C^*(W)$ are isomorphic. The latter case is impossible because $\gamma(S)=1$ and $\gamma(W)=0$. This completes the proof.

For an essentially normal operator T, the C^*-algebra T'_e is "quite huge". One thus would like to see if there exists any shift S (unilateral or bilateral shift) such that S essentially commutes with every operator R in T'_e, i.e., $SR-RS$ is compact for each R in T'_e. If $\sigma_e(T)$ is countable, then, by applying Lemma 3.3, there exists no shift which essentially commutes with every operator in T'_e. If $\sigma_e(T)$ is uncountable, Lemma 3.3 and Theorem 3.2 imply the following result.

Theorem 3.6.

(1) *There exists a bilateral shift W such that W essentially commutes with each operator in T'_e.*

(2) *There exists a unilateral shift S of multiplicity m which essentially commutes with each operator in T'_e if and only if $\gamma(T) \neq 0$ and $\gamma(T)|m$.*

Let Ω be a pseudoregular domain in $\mathbf{C}^n$, and let $C^*(\Omega)$ be the C^*-algebra generated by Toeplitz operators T_ϕ, $\phi \in C(\overline{\Omega})$, on the Bergman space $L^2_a(\Omega)$, and $\mathcal{K}$ the ideal of compact operators on $L^2_a(\Omega)$. From Salinas [13, Theorem 2.3], one has the canonical exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\Omega) \xrightarrow{\pi} C(\partial\Omega) \longrightarrow 0.$$

Consequently, T_{z_i} is essentially normal, and its essential spectrum is uncountable, where z_i is the ith coordinate function. Notice that the pseudoregular domains include the strongly pseudoconvex domains, pseudoconvex domains with real analytic boundary, and domains of finite type (see [13]).

Now applying Theorem 3.6, we have the following corollary.

Corollary 3.7. *Let Ω be a pseudoregular domain in $\mathbf{C}^n$. Since there is an essentially normal Toeplitz operator T_ϕ with uncountable essential spectrum on the Bergman space $L_a^2(\Omega)$ such that each Toeplitz operator essentially commutes with T_ϕ, it follows that there is a bilateral shift W such that $W^*T_fW-T_f$ is compact for each Toeplitz operator T_f. This implies that the C^*-algebra generated by all Toeplitz operators on the Bergman space $L_a^2(\Omega)$ is a proper subset of all bounded operators.*

In the last part of this section, we give some examples to illustrate the application of the characteristic number. These examples are closely related to the work of Barria and Halmos [2].

Example 1. On the Hardy space $H^2(D)$, E_n will denote the essential commutant of T_{z^n} for the natural number n. From Proposition 3.1 and Lemma 3.4, the following is immediate. Essential commutants E_n and E_m are isomorphic as C^*-algebras if and only if $n=m$. In particular, we have

$$E_n \cap E_m = E_{(n,m)},$$

where (n,m) denotes the greatest common divisor.

Example 2. Let B_n be the unit ball of $\mathbf{C}^n$, and S^{2n-1} be its boundary. We consider the Toeplitz algebra $\mathcal{T}(S^{2n-1})$ generated by all Toeplitz operators on the Hardy space $H^2(B_n)$ with continuous symbols on S^{2n-1}. From Coburn [5] or McDonald [11], we know that in the case $n>1$, $\mathrm{ind}(T_\phi)=0$ for T_ϕ with continuous symbol ϕ. This means that $\mathcal{T}(S^1)$ and $\mathcal{T}(S^{2n-1})$, $n>1$, are not isomorphic as C^*-algebras. The reason is that $\gamma(\mathcal{T}(S^1))=1$, $\gamma(\mathcal{T}(S^{2n-1}))=0$. In particular, $\mathcal{T}(S^{2n-1})$, $n>1$, does not contain any unilateral shift. Therefore, there is no unilateral shift which essentially commutes with each operator in $[\mathcal{T}(S^{2n-1})]_e'$, $n>1$.

4. The essential commutant of $\mathcal{E}+\mathcal{E}_*$

Let $L^2(B_n)$ be the space of square-integrable functions on B_n (we will consider only the usual Lebesgue measure on B_n). A unitary operator $U\colon L^2(B_n)\to L^2(B_n)$ is defined by

$$(Uf)(z)=f(\bar{z}), \quad z=(z_1,z_2,\dots,z_n).$$

For $\phi \in L^2(B_n)$, a small Hankel operator Γ_ϕ with symbol ϕ is defined by

$$\Gamma_\phi h = P(\phi U h), \quad h \in H^\infty(B_n),$$

where P is the orthogonal projection from $L^2(B_n)$ onto the Bergman space $L^2_a(B_n)$. Since $H^\infty(B_n)$ is dense in $L^2_a(B_n)$, this means that if Γ_ϕ is a bounded operator (when we put the L^2_a norm on H^∞), then Γ_ϕ extends to a bounded operator on $L^2_a(B_n)$. Therefore, in this section we consider bounded small Hankel operators while their symbols may be unbounded. By a straightforward verification, the following algebraic equations completely characterize small Hankel operators.

Proposition 4.1. *Let Γ be a bounded operator on $L^2_a(B_n)$. Then Γ is a small Hankel operator if and only if*

$$T^*_{z_j}\Gamma = \Gamma T_{z_j}, \quad j = 1,\ 2, \dots,\ n.$$

In [14], Zhu introduced the concept of reduced Hankel operator. A reduced Hankel operator $\widetilde{H}_\phi$ with symbol ϕ is defined by

$$\widetilde{H}_\phi : L^2_a(B_n) \longrightarrow \overline{L^2_a(B_n)},\ \widetilde{H}_\phi h = \overline{P}(\phi h), \quad h \in H^\infty(B_n),$$

where $\overline{P}$ denotes the orthogonal projection from $L^2(B_n)$ onto $\overline{L^2_a(B_n)}$ (the complex conjugate of $L^2_a(B_n)$).

A direct verification shows that the following is true.

Proposition 4.2. *The following relation is true*

$$\Gamma_\phi = U\widetilde{H}_{\hat{\phi}},$$

where $\hat{\phi}(z) = \phi(\bar{z})$ for $z \in B_n$.

Let $BC(B_n)$ be the space of bounded continuous functions on B_n, $C(\overline{B}_n)$ be the space of continuous functions of the closure $\overline{B}_n$ of B_n, and $C_0(B_n)$ be the space of continuous functions on B_n which vanish on the boundary S^{2n-1} of B_n. By Proposition 4.2 and Corollary in [14], the following assertions are immediate.

Corollary 4.3. *Given $\phi \in L^2(B_n)$, then*

(1) *Γ_ϕ is bounded if and only if $\overline{\hat{\phi}} \in L^\infty(B_n) + L^2_a(B_n)^\perp$ which holds if and only if $\overline{\hat{\phi}} \in BC(B_n) + L^2_a(B_n)^\perp$;*

(2) *Γ_ϕ is compact if and only if $\overline{\hat{\phi}} \in C(\overline{B}_n) + L^2_a(B_n)^\perp$ which holds if and only if $\overline{\hat{\phi}} \in C_0(B_n) + L^2_a(B_n)^\perp$.*

Set

$$\mathcal{E} = \{T \mid T_{z_j}T - TT_{z_j} \text{ is compact for } j = 1,\ 2, \dots,\ n\}.$$

Thus it is easily checked that $\mathcal{E}$ is the essential commutant of the C^*-algebra $\mathcal{T}(B_n)$ generated by all Toeplitz operators with continuous symbols. This implies that $\mathcal{E}$ contains the C^*-algebra $\mathcal{T}^\infty(B_n)$ generated by all Toeplitz operators. Set

$$\mathcal{E}_* = \{T \mid T_{z_j}^* T - T T_{z_j} \text{ is compact for } j=1,\ 2, \dots,\ n\}.$$

Thus $\mathcal{E}_*$ is a self-adjoint norm closed $\mathcal{E}$-bimodule, and $\mathcal{E}_*$ contains all small Hankel operators. By the Fuglede–Putnam theorem, we see that for $T \in \mathcal{E}_*$, $T_{\hat{f}} T - T T_f$ is compact for any $f \in C(\overline{B}_n)$, and hence derives that $\mathcal{E}_*^2 \subset \mathcal{E}$. Theorem 2 of [12] thus implies that $\mathcal{E} + \mathcal{E}_*$ is a C^*-algebra. We define the space $SC(B_n)$ of symmetric continuous functions by

$$SC(B_n) = \{f \in C(\overline{B}_n) \mid f(z) = f(\bar{z}) \text{ for all } z \in S^{2n-1}\}.$$

Let $\mathcal{T}(SC(B_n))$ denote the C^*-algebra generated by all Toeplitz operators with symbols in $SC(B_n)$ and all compact operators. Then we have the following result.

Theorem 4.4. *The essential commutant of $\mathcal{E} + \mathcal{E}_*$ is equal to $\mathcal{T}(SC(B_n))$, that is, an operator A essentially commutes with each operator in $\mathcal{E} + \mathcal{E}_*$ if and only if there exists some $f \in SC(B_n)$ such that $A = T_f +$ compact.*

Proof. Since

$$[\mathcal{E} + \mathcal{E}_*]_e' = \mathcal{E}_e' \cap [\mathcal{E}_*]_e' \quad \text{and} \quad \mathcal{E}_e' = \mathcal{T}(B_n),$$

we need only show that for $f \in C(\overline{B}_n)$, $T_f \Gamma_\phi - \Gamma_\phi T_f$ is compact for any $\phi \in L^\infty(B_n)$ if and only if $f \in SC(B_n)$. The sufficiency is obvious. To complete the proof of necessity, we first claim that if $g \in C(\overline{B}_n)$, and $\Gamma_\phi T_g$ is compact for any $\phi \in L^\infty(B_n)$, then $g|_{S^{2n-1}} = 0$.

Proof of the claim. According to the relation

$$U \Gamma_\phi T_g = \widetilde{H}_{\hat{\phi}} T_g,$$

we see that $\Gamma_\phi T_g$ is compact if and only if $\widetilde{H}_{\hat{\phi}} T_g$ is compact. For any $\phi \in L^\infty(B_n)$, a simple verification shows that

$$\widetilde{H}_{\hat{\phi}} T_g = \widetilde{H}_{\hat{\phi} g} - \widetilde{H}_{\hat{\phi}} H_g,$$

where H_g is defined by $H_g h = (I-P) g h$ for $h \in L_a^2(B_n)$. By [5], the Hankel operator H_g is compact. Consequently, for any ϕ, $\Gamma_\phi T_g$ is compact if and only if $\widetilde{H}_{\phi g}$ is compact which holds if and only if $\phi g \in C(\overline{B}_n) + L_a^2(B_n)^\perp$, [14]. From the above discussion, we see that

$$|g|^2 \phi \in C(\overline{B}_n) + L_a^2(B_n)^\perp$$

for any $\phi \in L^\infty(B_n)$. If there is a point $z_0 \in S^{2n-1}$ such that $g(z_0) \neq 0$, then the rotation invariance of $C(\overline{B}_n)+L^2_a(B_n)^\perp$ implies, by considering finitely many rotations g_j of g such that $\sum_{j=1}^k |g_j|^2 \neq 0$ on S^{2n-1}, that

$$\left(\sum_{j=1}^k |g_j|^2\right)\phi \in C(\overline{B}_n)+L^2_a(B_n)^\perp$$

for any $\phi \in L^\infty(B_n)$. Therefore, there exists some neighborhood V of S^{2n-1} such that $\sum_{j=1}^k |g_j|^2 > c$ on V for some positive constant c. It follows that

$$\phi\chi_V \in C(\overline{B}_n)+L^2_a(B_n)^\perp$$

for any $\phi \in L^\infty(B_n)$, where χ_V denotes the characteristic function of V. Since for any $\phi \in L^\infty(B_n)$, $\widetilde{H}_{(1-\chi_V)\phi}$ is compact, and hence $(1-\chi_V)\phi \in C(\overline{B}_n)+L^2_a(B_n)^\perp$ by [14]. Consequently,

$$\phi \in C(\overline{B}_n)+L^2_a(B_n)^\perp$$

for any $\phi \in L^\infty(B_n)$. This is obviously impossible, and hence leads to the desired conclusion, that is, $g|_{S^{2n-1}}=0$. The claim is proved.

Assume $f \in C(\overline{B}_n)$ such that $T_f\Gamma_\phi - \Gamma_\phi T_f$ is compact for any $\phi \in L^\infty(B_n)$. Since

$$T_f\Gamma_\phi - \Gamma_\phi T_f = \Gamma_\phi T_{\hat{f}} - \Gamma_\phi T_f + \text{compact} = \Gamma_\phi T_{\hat{f}-f} + \text{compact}.$$

By the above claim, the equality $\hat{f}=f$ on S^{2n-1} follows. This completes the proof of Theorem 4.4.

Corollary 4.5.

(1) *There exists a bilateral shift W such that W essentially commutes with each operator in $\mathcal{E}+\mathcal{E}_*$, that is, $WA-AW$ is compact for each $A \in \mathcal{E}+\mathcal{E}_*$.*

(2) *There is no unilateral shift of finite multiplicity which essentially commutes with each operator in $\mathcal{E}+\mathcal{E}_*$.*

Proof. (1) Apply Theorem 3.2.

(2) In the case $n=1$, since each operator A in $\mathcal{T}(SC(D))$ has the form $A=T_f+$ compact for some $f \in SC(D)$, Corollary 2.4 implies that the index of each Fredholm operator in $\mathcal{T}(SC(D))$ is zero. In the case $n>1$, since the index of each Fredholm Toeplitz operator is zero by [5] or [11], the conclusion follows.

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References

1. ARVESON, W., Notes on the extensions of C^*-algebras, *Duke Math. J.* **44** (1977), 329–354.
2. BARRIA, J. and HALMOS, P. R., Asymptotic Toeplitz operators, *Trans. Amer. Math. Soc.* **273** (1982), 621–630.
3. BEKOLLE, D., BERGER, C. A., COBURN, L. A. and ZHU, K. H., BMO in the Bergman metric on bounded symmetric domains, *J. Funct. Anal.* **93** (1990), 921–953.
4. BROWN, L. G., DOUGLAS, R. G. and FILLMORE, P. A., Unitary equivalence modulo the compact operators and extensions of C^*-algebras, in *Proceedings of a Conference on Operator Theory* (Fillmore, P. A., ed.), Lecture Notes in Math. **345**, pp. 58–128, Springer-Verlag, Berlin–Heidelberg, 1973.
5. COBURN, L. A., Singular integral operators and Toeplitz operators on odd spheres, *Indiana Univ. Math. J.* **23** (1973/74), 433–439.
6. DOUGLAS, R. G., *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.
7. ENGLIS, M., Density of algebras generated by Toeplitz operators on Bergman spaces, *Ark. Mat.* **30** (1992), 227–240.
8. ENGLIS, M., Toeplitz operators on Cartan domains essentially commute with a bilateral shift, *Proc. Amer. Math. Soc.* **117** (1993), 365–368.
9. GOHBERG, I. C. and KREIN, M. G., Systems of integral equations on a half-line with kernels depending on the difference of arguments, *Uspekhi Mat. Nauk* **13**:2 (1958), 3–72 (Russian). English transl.: *Amer. Math. Soc. Transl.* **14** (1960), 217–287.
10. LIU, J., ZHANG, Y. and GUO, K., A new C^*-algebra and its essential commutant, *Chinese Sci. Bull.* **44** (1999), 204–207.
11. MCDONALD, G., Fredholm properties of a class of Toeplitz operators on the ball, *Indiana Univ. Math. J.* **26** (1977), 567–576.
12. POWER, S. C., C^*-modules and an odd-even decomposition for C^*-algebras, *Bull. London Math. Soc.* **8** (1976), 268–272.
13. SALINAS, N., The ∂-formalism and the C^*-algebra of the Bergman n-tuple, *J. Operator Theory* **22** (1989), 325–343.
14. ZHU, K. H., Duality and Hankel operators on the Bergman spaces of bounded symmetric domains, *J. Funct. Anal.* **81** (1988), 260–278.

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