IX. Souslin Hypothesis Does Not Imply "Every Aronszajn Tree Is Special"

§0. Introduction

We prove that the Souslin Hypothesis does not imply "every Aronszajn tree is special"; solving an old problem of Baumgartner, Malitz and Reinhardt. For this end we introduce variants of the notion "special Aronszajn tree" and discuss them (this is §3, see references there). We also introduce a limit of forcings bigger than the inverse limit, and prove it preserves properness and related notions not less than inverse limit, and the proof is easier in some respects, and was done already in 78; see §1, §2. We can get away without using it for the present theorems, but we want to represent it somewhere. The Aronszajn trees are addressed in §4; we choose a costationary $S \subseteq \omega_1$ and make all $\aleph_1$-trees S-st-special, while on "$\omega_1 \setminus S$ the tree remains Souslin". If $S = \emptyset$ this means that every $\aleph_1$-tree is special when restricted to some unbounded set of levels, in fact while there is no antichains whose set of levels is stationary. See more in 4.9.

§1. Free Limits

1.1 Discussion and Definitions. For A a set of propositional variables, λ a regular cardinal, let: $L_\lambda(A)$ be the set of propositional sentences generated from A, by negation and conjunction and disjunctions on sets of power $< \lambda$.

Let $L_\mu(A) = \bigcup_{\lambda<\mu} L_\lambda(A)$ for μ a limit cardinal ($> \aleph_0$) or ∞. Let φ, ψ, θ denote sentences; Φ, Ψ set of sentences.

We define (in $L_\infty(A)$) $\vdash \psi$, or $\Phi \vdash \psi$ as usual (the rules of the finite case, and $\Phi \vdash \bigwedge \Phi$, and from $\Phi \vdash \varphi_i$ for $i \in I$ deduce $\Phi \vdash \bigwedge_{i \in I} \varphi_i$,) and let $\bigvee_i \varphi_i = \neg \bigwedge_i \neg \varphi_i$.

Always $\vdash$ means in $L_\infty(A)$ even if we deal with $L_\lambda(A)$.

The following is well known.

1.2 Theorem. The following are equivalent for Φ, φ:

(1) $\Phi \vdash \varphi$;

(2) there is no model of $\Phi \bigcup \{\neg\varphi\}$ with truth values in a complete Boolean algebra;

(3) if λ is such that $|\Phi|$, and the power of any set on which we make conjunction inside some sentence $\theta \in \Phi \bigcup \{\varphi\}$ are $\leq \lambda$ and $P = \text{Levy}(\aleph_0, \lambda)$ i.e. the collapsing of λ to ω by finite functions, *then*

$$\Vdash_P \text{ `` there is no model of } \Phi \bigcup \{\neg\varphi\}\text{''}.$$

1.2A Remark. This can be proven by a small fragment of ZFC, admissibility axioms, at least when we prove only (1) $\Leftrightarrow$ (3). Hence (by proving not (1) implies not (3)):

1.3 Conclusion. If A is a transitive admissible set, $\Phi, \varphi \in A$ then "$\Phi \vdash \varphi$" has the same truth value in V and in A.

1.4 Definition. For given A and $\theta \in L_\infty(A)$, let $FF_\lambda(\theta)$ be $\{\psi : \psi \in L_\lambda(A),\ \theta \nvdash \neg\psi\}$ partially ordered by $\psi_1 \leq \psi_2$ if $\theta \wedge \psi_2 \vdash \psi_1$. ($FF$ denotes free-forcing; we can identify φ, ψ if $\varphi \leq \psi \leq \varphi$.)

Reversing the definition of $\leq$ and adding a minimal element, we get a Boolean algebra in which every set of $< \lambda$ elements has a least upper bound provided that we identify ψ_1, ψ_2 when $\theta \vdash \psi_1 \equiv \psi_2$.

1.5 Definition. For any forcing notion P let $\theta[P]$ be the following sentence: $\bigwedge\{(c \to \neg d) \wedge (b \to a) : a, b \in P, a \leq b, c, d \in P, c, d \text{ incompatible}\} \wedge \bigwedge\{\bigvee_{a \in \mathcal{I}} a : \mathcal{I} \subseteq P \text{ a maximal set of pairwise incompatible elements}\}$.

1.6 Definition. Let $P_i (i < \delta)$ be $\lessdot$-increasing, δ an ordinal (λ an infinite regular cardinal). Then their λ-free limit ($\operatorname{Flim}^{\lambda}_{i<\delta} P_i$) is $FF_\lambda(\bigwedge_{i<\delta} \theta[P_i])$ (where the set of propositional variables is $\bigcup_{i<\delta} P_i$). If we omit λ we mean $\lambda = \aleph_1$.

1.7 Claim. $P \lessdot Q$ implies $\theta[Q] \vdash \theta[P]$, and $P \lessdot FF_\lambda(\theta[P])$.

Proof. The first statement is trivial, for the second see the proof of Claim 1.8.

$\square_{1.7}$

1.7A Remark. Our notation may be confusing, as for conditions $p, q \in P, p \wedge q$ is "p and q", i.e., both are in the generic set so in our order $p \wedge q$ is above p and above q as it give more information.

1.8 Claim. If as in Definition 1.6, P_δ is the λ-free limit of $P_i (i < \delta)$ *then* $P_i \lessdot P_\delta$ for $i < \delta$.

Proof. Let us check the conditions.

proof of clause (b) Let $\mathcal{I} \subseteq P_i$ be a maximal set of pairwise incompatible elements of P_i. Suppose $\varphi \in \operatorname{Flim}^{\lambda}_{i<\delta} P_i$ is incompatible with each $a \in \mathcal{I}$. As $\varphi \in \operatorname{Flim}^{\lambda}_{i<\delta} P_i$, by definition $\bigwedge_{i<\delta} \theta[P_i] \nvdash \neg\varphi$. So by 1.2, after some forcing there is a model of φ, $\bigwedge_{j<\delta} \theta[P_j]$. But $\bigvee_{a \in \mathcal{I}} a$ is a conjunct of the second sentence, so in the model some $q \in \mathcal{I}$ is true. So after some forcing, there is a model of $\varphi \wedge q, \bigwedge_{j<\delta} \theta[P_j]$, so by 1.2, $\bigwedge_{j<\delta} \theta[P_j] \nvdash \neg(\varphi \wedge q)$, so $\varphi \wedge q \in FF_\lambda(\bigwedge_{j<\delta} \theta[P_j])$; so φ, q are compatible in $FF_\lambda(\bigwedge_{j<\delta} \theta[P_j]) = \operatorname{Flim}^{\lambda}_{i<\delta} P_i$.

proof of clause (a) Let $a, b \in P_i$, if they are compatible in P_i, for some $c \in P_i$, $a \leq c, b \leq c$, and this clearly holds in P_δ by its definition.

If they are incompatible in P_i then $a \to \neg b$ appears as a conjunct in $\theta[P_i]$ and we can finish. Similarly for $a, b \in P_i, a \leq b$ in P_i implies $a \leq b$ in P_δ. $\square_{1.8}$

1.9 Remark. We can change the definition of $FF_\lambda(\theta)$ (hence of $F\mathrm{lim}$) by changing $\vdash$. The natural way is to let K be a class of complete Boolean algebras, and $\Phi \vdash_K \varphi$ iff any Boolean valued model of Φ is a Boolean valued model of φ provided that the complete Boolean algebra is from K. So $FF^K_\lambda(\theta) = \{\varphi \in L_\lambda : \theta \nvdash_K \neg\varphi\}$.

The most interesting K's seems

$K_1 = \{B : \text{forcing with } B \setminus \{0\} \text{ satisfies } X\}$

where $X =$ does not add reals, does not collapse $\aleph_1$, does not collapse stationary subsets of $\aleph_1$, the $UP_\ell(S)$ condition ($\mathrm{Min}S \geq \aleph_2, \ell = 0, 2$, see XV §3).

§2. Preservation by Free Limit

2.1 Theorem. If each P_i is a forcing notion; and P_i $(i < \delta)$ is $\lessdot$-increasing, each P_i is proper as well as $P_j/P_i (i < j < \delta)$ and for $\alpha < \delta, \mathrm{cf}(\alpha) = \aleph_0 \Rightarrow P_\alpha = \mathrm{Flim}^{\aleph_1}_{i<\alpha} P_i$ *then* their $\aleph_1$-free limit $P = \mathrm{Flim}^{\aleph_1}_{i<\delta} P_i$ is proper. Also P/P_i is proper for any $i < \delta$.

2.1A Remark. Similarly for μ-proper by [Sh:100] terminology if we take μ^+-free limit. We can restrict ourselves to non-limit i, j.

Proof. Let $N \prec (H(\chi), \in)$ be countable such that $\langle P_i : i < \delta \rangle \in N, p \in P \cap N$ and χ big enough (see III). Let $\{\mathcal{I}_n : n < \omega\}$ be a list of all pre-dense subsets of P which belong to N. Let $\delta(*) = \sup[\delta \cap N]$, and let $P_\delta = P$, so $N \cap P = N \cap P_\delta \subseteq N \cap P_{\delta(*)}$ and if $\delta(*) < \delta$ then $N \cap P_{\delta(*)} = \bigcup_{i<\delta(*)} N \cap P_i$. As necessarily $\mathrm{cf}(\delta(*)) = \aleph_0$ clearly $P_{\delta(*)} = \mathrm{Flim}^{\aleph_1}_{i<\delta(*)} P_i$. So it is enough to prove $p \wedge \bigwedge_n (\bigvee_{a \in N \cap \mathcal{I}_n} a) \in P_{\delta(*)}$ (in Boolean algebras terms: is not zero), for p being any member of $P \cap N$.

Now assume w.l.o.g. that everything is in some countable transitive model M (e.g. work in $V' = V^{\mathrm{Levy}(\aleph_0, \mu)}$, χ strong limit such that $\langle P_i : i < \delta \rangle \in H(\mu)$, $\mu < \chi$, let $M = H(\mu)^V$ and remember 1.3). We can find $\alpha_n < \alpha_{n+1}, \alpha_n \in N \cap \delta$, $\sup[\delta(*) \cap N] = \bigcup_n \alpha_n$.

Let $\langle \Phi_n : n < \omega \rangle$ be a list of all countable (in M) subsets of $P_{\delta(*)}$ which belongs to N.

We now define by induction on n, in V, G_n, p_n such that:

(1) $G_n \subseteq P_{\alpha_n}$, $G_n \subseteq G_{n+1}$,

(2) G_n is P_{α_n}-generic for M and $G_n \cap N$ is $(P_{\alpha_n} \cap N)$-generic for N,

(3) $p_n \leq p_{n+1}$, $p = p_0$, $p_n \in N \cap P$,

(4) p_n is compatible (in P) with every member of G_n,

(5) p_{2n+1} is $\geq q_n$ for some $q_n \in \mathcal{I}_n \cap N$,

(6) either $p_{2n+2} \vdash \wedge \Phi_n$ or $p_{2n+2} \vdash \neg r_n$ for some $r_n \in \Phi_n$.

The proof is trivial (provided you know about the composition of forcings and completeness theorem for the propositional calculus $L_{\omega_1,\omega}$).

In the end $G = \bigcup_n G_n$ gives us a model of $\bigwedge_{j<\delta(*)} \theta[P_j]$ (by: members of G are true, members of $\bigcup_{j<\delta} P_j \setminus G$ are false). This holds by clause (2).

For $r \in P \cap N, r$ is true in the model iff $p_n \geq r$ for some n (this is proved by induction on the complexity of r, (see conditions (4) and (6)). In the model p_n is true (for each $n < \omega$), hence $\bigvee_{a \in \mathcal{I}_n \cap N} a$ is true (for each $n < \omega$) hence $p \wedge \bigwedge_n (\bigvee_{a \in \mathcal{I}_n \cap N} a)$ is true there (p true as $p_0 = p$).

So in V there is a model of $\bigwedge_{j<\delta(*)} \theta[P_j]$, $p \wedge \bigwedge_n (\bigvee_{a \in \mathcal{I}_n \cap N} a)$ so $p \wedge \bigwedge_n (\bigvee_{a \in \mathcal{I}_n \cap N} a) \in P_{\delta(*)}$ as required. $\square_{2.1}$

2.1B Remark. Part of the proof is essentially a repetition of the completeness theorem for $L_{\omega_1,\omega}$ (propositional calculus). But note that in this proof there was no need (as in the ones for inverse limit) to use names. Also, almost all previous theorems on preservation hold for free iterations.

2.2 Definition. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i \leq i_0 \rangle$ be an ω_1-free iteration if :

(a) P_i is $\lessdot$-increasing,

(b) $P_{i+1} = P_i * \underset{\sim}{Q}_i = \{\langle p, q \rangle : p \in P_i, \Vdash_{P_i} \text{"}q \in \underset{\sim}{Q}_i\text{"}\}$, with the order $\langle p, q \rangle \leq \langle p^\dagger, q^\dagger \rangle \Leftrightarrow p \leq p^\dagger \wedge [p^\dagger \Vdash_P q \leq q^\dagger]$; and we identify $p \in P_i$ with $\langle p, \emptyset \rangle$,

(c) for limit δ, P_δ is the $\aleph_1$-free limit of $\langle P_i : i < \delta \rangle$.

2.3 Definition. For an $\aleph_1$-free iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < i_0 \rangle$ let $\mathrm{Flim}^{\aleph_1}\bar{Q}$ be $P_\alpha * \underset{\sim}{Q}_\alpha$ if $i_0 = \alpha + 1$ and $\mathrm{Flim}^{\aleph_1}_{i<i_0} P_i$ if i_0 is a limit ordinal, and we let $P_{i_0} = \mathrm{Flim}^{\aleph_1}\bar{Q}$.

2.4 Definition. We say that $\aleph_1$-free iteration preserves a property if whenever each Q_i (in V^{P_i}) has it, then so does P_i.

2.5 Theorem. Properness is preserved by $\aleph_1$-free iteration.

Proof. See 2.1; and prove by induction on β that for $\alpha < \beta$,

$(*)$ If $\langle P_i : i < i_{i_0} \rangle \in N \prec (H(\lambda), \in), ||N|| = \aleph_0, \alpha \leq \beta \leq i_0, \alpha \in N, \beta \in N, p \in P_\beta \cap N, q \in P_\beta$, and for every pre-dense $\mathcal{I} \subseteq P_\alpha$, $[\mathcal{I} \in N, \Rightarrow \mathcal{I} \cap N$ is pre-dense above q,$]$ and every $q^\dagger, q \leq q^\dagger \in P_\alpha$ is compatible with p *then* for some $r \in P_\beta$, for every pre-dense $\mathcal{I} \subseteq P_\beta$ we have $[\mathcal{I} \in N \Rightarrow \mathcal{I} \cap N$ pre-dense above $r]$, $p \leq r$ and for every $q^\dagger : q \leq q^\dagger \in P_\beta \Rightarrow q^\dagger$ compatible with r. $\square_{2.5}$

The following Definition and Theorem are not really necessary for the rest of the chapter, but will help in understanding §4.

2.6 Definition.

1) P is strongly proper when: *if* λ is large enough (i.e. $\lambda > (2^{|P|})^+$), $P \in N \prec (H(\lambda), \in)$, $||N|| = \aleph_0, p \in P \cap N$ and $\mathcal{I}_n \subseteq N$ pre-dense in $N \cap P$ (but we do not ask $\mathcal{I}_n \in N$), *then* for some $q, p \leq q \in P$, each $\mathcal{I}_n$ is pre-dense above q.
2) P is strongly α-proper *if* for large enough λ, $P \in N_i \prec (H(\lambda), \in)$, $||N_i|| = \aleph_0$, $\langle N_j : j \leq i \rangle \in N_{i+1}$ for $i < \alpha$, N_i increasing continuous, $p \in P \cap N_0$, $P, i \in N_i$, $\mathcal{I}^i_n \subseteq P \cap N_i$ is a pre-dense subset of $P \cap N_i$ (for $n < \omega$) and $\langle \mathcal{I}^j_n : j \leq i, n < \omega \rangle \in N_{i+1}$ (for $i < \alpha$) *then* there is a $q \in P, p \leq q$, $\mathcal{I}^n_i$ pre-dense above q (in P, for each $n < \omega, i \leq \alpha$). Note that we can replace $(H(\lambda), \in)$ by $(H(\lambda), \in, <^*_\lambda)$.

2.7 Theorem. Strong properness is preserved by $\aleph_1$-free iteration.

2.7A Remark. A similar theorem holds for strong α-properness and for CS iteration.

Proof. Let $\langle P_i, \underset{\sim}{Q}_j : i \le i_0, j < i_0\rangle$ be an $\aleph_1$-free iteration. We prove by induction on $\alpha \le i_0$ that for any $\beta < \alpha$:

$(*)_{\beta,\alpha}$ Let $\langle P_i, \underset{\sim}{Q}_j : i \le i_0, j < i_0\rangle \in N \prec (H(\lambda), \in)$ (where $\lambda > (2^{|P|})^+$), $||N|| = \aleph_0$, C a countable family of $\aleph_0$ pre-dense subsets of $P_\alpha \cap N$, closed under the operations listed below. Suppose $\beta < \alpha, p \in P_\alpha \cap N, \alpha \in N$, $\beta \in N, q \in P_\beta$, no $q^\dagger, q \le q^\dagger \in P_\beta$ is incompatible with p, and $[\mathcal{I} \subseteq P_\beta \& \mathcal{I} \in C \Rightarrow \mathcal{I}$ pre-dense above $q]$. *Then* there is $q_\alpha, p \le q_\alpha \in P_\alpha$, $q \le q_\alpha$, no $q \le q^\dagger \in P_\beta$ is incompatible with q_α and $[\mathcal{I} \in C \Rightarrow \mathcal{I}$ pre-dense above $q_\alpha]$.

The family of operations under which C is closed is (for $p \in N \cap P_{i_0}, \gamma \in N \cap i_0$ and $\mathcal{I} \in N$ a pre-dense subset of P_{i_0}):

(Op 1) $Op_1(\mathcal{I}, \gamma, p) = \{r : r \in P_\gamma \cap N$ and *either* for some $r^* \in P_{i_0}$ and $r_1 \in \mathcal{I}$ we have $r_1 \le r^*$ and $p \le r^*$ but no $r^\dagger, r \le r^\dagger \in P_\gamma \cap N$ is incompatible with r^* *or* r is incompatible with $p\}$ for $\gamma \in N$, $\mathcal{I} \in C$, $p \in P_\alpha$. (Note that for $p = \emptyset$ the last phrase is vacuous.)

For $\alpha = 0$. Totally trivial. For $\alpha = \gamma + 1$. So $\beta \le \gamma$, also as $\alpha \in N$ clearly $\gamma \in N$ and by the induction hypothesis for γ, $(*)_{\beta,\gamma}$ holds, so w.l.o.g. $\beta = \gamma$. So we want to use the hypothesis $P_\alpha = P_\gamma * \underset{\sim}{Q}_\gamma$, $\underset{\sim}{Q}_\gamma$ is strongly proper; then we use $\langle q, \underset{\sim}{r}\rangle$, $\underset{\sim}{r} \in Q_\gamma$ a name of an appropriate element of $\underset{\sim}{Q}_\gamma$. We have to prove that the appropriate subsets of $N[\underset{\sim}{G}_{P_\gamma}] \cap \underset{\sim}{Q}_\gamma[\underset{\sim}{G}_{P_\gamma}]$ are pre-dense subsets. But as C is closed under (Op 1) this is easy.

For α limit. Let $\alpha_n \in N$, $\bigcup_{n<\omega} \alpha_n$ is α or at least $[\bigcup_n \alpha_n, \alpha) \cap N = \emptyset$ (note $[\alpha^\dagger, \alpha)$ is interval of ordinals).

We work as in 2.1 using the induction hypothesis. $\square_{2.7}$

2.8 Claim. 1) If we iterate ω-proper, ${}^\omega\omega$-bounding forcings it does not matter whether we use $\aleph_1$-free iteration or countable support one (in the latter we get a dense subset of the first).

2) We can replace "ω-proper" by proper.

Proof. Left to the reader. $\square_{2.8}$

2.8A Remark. For the proof, see V §3 (and for (2) see XVIII §2). The parallel of 2.7 for countable support was noted by Harrington and the author.

By the way we note that unlike $\aleph_1$-c.c. forcing:

2.9 Example. There are proper forcing P, Q such that $P \lessdot Q$ but Q/P is not proper.

Proof. We let $P_0 =$ adding a subset $\underset{\sim}{r}$ of ω_1 with a condition being a countable characteristic function.

Let $Q_0 \in V^{P_0}$, $Q_0 = \{f : \mathrm{Dom}(f) = \alpha < \omega_1, \mathrm{Rang}(f) = \{0,1\}, f^{-1}(\{0\})$ is a closed set of ordinals (not just closed subset of α!) included in $r\}$. (r denotes the generic subset of ω_1 which P_0 produces.)

Now $P_0, P_0 * Q_0$ are proper but in V^{P_0}, $Q_0 \cong P_0 * Q_0/P_0$ is not proper as it destroys the stationarity of $\omega_1 \setminus r_0$.

§3. Aronszajn Trees: Various Ways to Specialize

We introduce new variants of the notion "special Aronszajn tree", define some old ones (special, r-special) and prove some known theorems and some easy ones. See Kurepa [Ku35], Baumgartner, Malitz and Rienhard [BMR], Baumgartner [B] and also Devlin and Shelah [DvSh:65]. Recall

3.1 Definition.

(1) An ω_1-tree $T = (|T|, <_T)$ is a partially ordered set, such that (when no confusion arises, we write $<$ instead of $<_T$ and T instead of $|T|$):

  (a) for every $x \in T, \{y \in T : y < x\}$ is well-ordered, and its order type which is denoted by $\mathrm{rk}(x) = \mathrm{rk}_T(x)$, is countable,

(b) $T_\alpha = \{x \in T : \mathrm{rk}(x) = \alpha\}$ is countable, $\neq \emptyset$,

(c) if $\mathrm{rk}(x) = \mathrm{rk}(y)$ is a limit ordinal *then* $x = y \Leftrightarrow \{z : z < x\} = \{z : z < y\}$,

(d) if $x \in T_\alpha, \alpha < \beta$, *then* for some $y \in T_\beta, x < y$, in fact there are at least two distinct such y's.

If we wave (c) and (d) we call it an almost ω_1-tree; similarly for the other definitions.

(2) A set $B \subseteq T$ is a *branch* if it is totally (i.e. linearly) ordered (hence well ordered) and maximal; it is an α-branch if it has order type α.

(3) An Aronszajn tree is an ω_1-tree with no ω_1-branch.

(4) An ω_1-tree is Souslin or ω_1-Souslin tree if there is no uncountable antichain (= set of pairwise incomparable elements).

3.1A Remark. Condition (1)(d) is not essential, except to make every Souslin tree an Aronszajn tree. So except this implication all the definitions and results in this section hold for almost ω_1-trees.

3.2 Definition.

(1) For a set $S \subseteq \omega_1$ which is unbounded, we call an ω_1-tree S-special if there is a monotonic increasing function f from $\bigcup_{\alpha \in S} T_\alpha$ to $\mathbb{Q}$ (the rationals), i.e., $x < y \Rightarrow f(x) < f(y)$.

(2) A special ω_1-tree is an ω_1-special ω_1-tree (this is the classical notion).

(3) r-special, $S - r$-special are defined similarly when the function maps T into $\mathbb{R}$ (the reals).

(4) We say f specializes (S-specialize, etc.) T. We can replace S by a function h, $\mathrm{Dom}(h) = \omega_1$, $\mathrm{Rang}(h) = S, h$ increasing.

3.3 Definition. For a stationary $S \subseteq \omega_1$ we call an ω_1-tree S-st-special if there is a function f, $\mathrm{Dom}(f) = \bigcup_{\alpha \in S \setminus \{0\}} T_\alpha$, and $x \in T_\alpha \Rightarrow f(x) \in \alpha \times \omega$ (cartesian product) such that $x < y \Rightarrow f(x) \neq f(y)$ when defined. If S is a set of limit ordinals we can assume $x \in T_\alpha \Rightarrow f(x) < \alpha$. So if $S \subseteq \omega_1$ is not stationary this says nothing.

3.4 Claim.

(1) If T is S-special or $S-r$-special ($S \subseteq \omega_1$ unbounded) or S-st-special ($S \subseteq \omega_1$ stationary) ω_1-tree *then* T is an Aronszajn tree but not Souslin. Any ω_1-Souslin tree is an Aronszajn tree.

(2) The following implications among properties of ω_1-trees hold (where $S_2 \subseteq S_1 \subseteq \omega_1$, S_2 unbounded in ω_1, $S_1 = \{\alpha(i) : i < \omega_1\}$, α_i increasing with i)

(a) S_1-special $\Rightarrow$ S_2-special,
S_1-special $\Rightarrow$ $S_1 - r$-special,
$S_1 - r$-special $\Rightarrow$ $S_2 - r$-special,
$S_1 - r$-special $\Rightarrow$ $S_1 \cap \{\alpha(i+1) : i < \omega_1\}$-special,

(b) for S_1 stationary: S_1-special $\Rightarrow$ $S_1 - st$-special,

(c) $S_1 - st$-special $\Rightarrow$ $S_2 - st$-special, (if S_1, S_1 are stationary subsets of ω_1).

(d) for $C \subseteq \omega_1$ closed unbounded: $S_1 \cap C - st$-special $\Leftrightarrow$ $S_1 - st$-special,

(e) if $(\forall i) h_1(i) \leq h_2(i)$ and T is h_1-special then T is h_2-special.

Proof. Trivial:

(1) for S-special, and $S-r$-special – well known, for $S-st$-special by the Fodor lemma.

(2) Trivial – check. E.g. the last phrase in (a), if f is $S_1 - r$-specialize T, define $f^* : \bigcup_{i<\omega_1} T_{\alpha(i+1)} \to \mathbb{Q}$ by: if $x \in T_{\alpha(i+1)}$ then $f^*(x)$ is a rational $< f^*(x)$ but $> f(y)$ where $y \in T_{\alpha(i)}, y < x$. $\square_{3.4}$

Remark. By 3.4(2)(d) dealing with $S - st$-special we can assume all members of S are limit, and so $\mathrm{Rang}(f) \subseteq \omega_1$ in the Definition.

3.5 Claim.

(1) T is S-special *iff* $S \subseteq \omega_1$ is unbounded and there is $f : \bigcup_{\alpha \in S} T_\alpha \to \omega$, $[x < y \,\&\, \mathrm{rk}(x) \in S \,\&\, \mathrm{rk}(y) \in S \Rightarrow f(x) \neq f(y)]$.

(2) T is $\omega_1 - st$-special *iff* T is special.

Remark. See Claim 3.11.

Proof. (1) Well known (the "only if" part is trivial; for the "if" part, with given $f : \bigcup_{\alpha \in S} T_\alpha \to \omega$ we define $f_n : \{x : x \in \bigcup_{\alpha \in S} T_\alpha$ and $f(x) \leq n\} \to \mathbb{Q}$ by induction on n such that $f_n \subseteq f_{n+1}$, f_n satisfies the requirement (see Definition 3.2) and $\mathrm{Rang}(f_n)$ is finite. Now $\bigcup_{n<\omega} f_n$ is as required).

(2) The "if" part is trivial.

So suppose f $\omega_1 - st$-specialize T. For every $x \in T$, let $K_x = \{t \in (rk(x) + 1) \times \omega$: for no $y \leq x$ is $f(y) = t\}$. We now define by induction on $\alpha < \omega_1, g_\alpha$ and $A_{x,t}$ (for $t \in K_x, x \in \bigcup_{\beta<\alpha} T_\beta$) such that:

(a) g_α is a function from $T_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} T_\beta$ to ω,

(b) $x < y, x \in T_{<\alpha}, y \in T_{<\alpha} \Rightarrow g_\alpha(x) \neq g_\alpha(y)$,

(c) $\beta < \alpha \Rightarrow g_\beta \subseteq g_\alpha$,

(d) $A_{x,t}$(for $t \in K_x, x \in T_{<\alpha}$) is an infinite subset of ω,

(e) for every $x \in T_{<\alpha}, t \neq s \in K_x \Rightarrow A_{x,t} \cap A_{x,s} = \emptyset$,

(f) $t \in K_x, x \in T_{<\alpha} \Rightarrow A_{x,t} \cap \{g_\alpha(y) : y \leq x\} = \emptyset$,

(g) if $x < y \,\&\, x \in T_{<\alpha} \,\&\, y \in T_{<\alpha}, t \in K_x \cap K_y$ then $A_{x,t} = A_{y,t}$.

For $\alpha = 0, 1, \alpha$ limit – no problem.

For $\alpha + 1 > 1$ – let $x \in T_\alpha \subseteq T_{<(\alpha+1)}$, and $s = f(x)$, so by K's definition for some $y = y_x < x$, $s \in K_y$. We choose $g_{\alpha+1}(x) \in A_{y,s}$ ($= A_{z,s}$ for every z satisfying $y \leq z < x$) and let $g_{\alpha+1} \restriction T_{<\alpha} = g_\alpha$.

For $t \in K_x \setminus \bigcup_{z<x} K_z$ (there are $\aleph_0$ such t's) we choose $A_{x,t} \subseteq A_{y,s} \setminus \{g_{\alpha+1}(x)\}$ infinite pairwise disjoint. If $t \in K_z \setminus \{f(z') : z' \leq x\}$ for some $z < x$ we let $A_{x,t} = A_{z,t}$.

Now by 3.5(1) clearly $g = \bigcup_{\alpha<\omega_1} g_\alpha$ shows T is special. $\square_{3.5}$

3.6 Claim. Let $S \subseteq \omega_1$ be unbounded.

(1) If every Aronszajn tree is S-special *then* every Aronszajn tree is special.

(2) If every Aronszajn tree is $S-r$-special *then* every Aronszajn tree is special.

Proof. (1) Let T be an Aronszajn tree, $S = \{\alpha(i) : i < \omega_1\}$, $\alpha(i)$ increasing. Define T^* (a partial order): The set of elements is $\{\langle x, \gamma\rangle : x \in T, \gamma \leq \alpha(\mathrm{rk}_T(x))$ and $y < x \Rightarrow \alpha(\mathrm{rk}_T(y)) < \gamma\}$; the order in T^* is: $\langle x, \gamma\rangle <_{T^*} \langle x^\dagger, \gamma^\dagger\rangle$ if $x < x^\dagger$ or $x = x^\dagger, \gamma < \gamma^\dagger$.

Now T^* is almost an Aronszajn tree; the only missing part is in Definition 3.1, part (d) ("in fact there are at least two distinct such y's") the problem is e.g. when i is a limit ordinal, $\alpha(i) > \bigcup_{j<i} \alpha(j)$, in level $\bigcup_{j<i} \alpha(j)$. We can add more elements and find an Aronszajn tree, T^{**} such that $T^* \subseteq T^{**}$, and if g^{**} S-specializes the tree T^{**}, we let $g : T \to \mathbb{Q}$ be defined by $g(x) \stackrel{\text{def}}{=} g^{**}(\langle x, \alpha(\mathrm{rk}_T(x))\rangle)$, it specializes T.

(2) By 3.6(1) and 3.4(2) (a), last clause. $\square_{3.6}$

3.7 Lemma.

(1) ($\diamondsuit_{\omega_1}$) There is an r-special Aronszajn tree which is not special.

(2) Moreover (in (1)) there is no antichain $\mathcal{I}$ such that $\mathrm{rk}(\mathcal{I}) = \{\mathrm{rk}(x) : x \in \mathcal{I}\}$ contains a closed unbounded subset of ω_1.

(3) ($\diamondsuit^*_{\omega_1}$) There is an r-special Aronszajn tree, such that for no antichain $\mathcal{I} \subseteq T$ is $\mathrm{rk}(\mathcal{I}) = \{\mathrm{rk}(x) : x \in \mathcal{I}\}$ stationary.

Remark. Part (1) was proved by Baumgartner [B1].

Proof. We define by induction on $\alpha < \omega_1$ the tree $(T_{<\alpha}, <_T \restriction T_{<\alpha})$ and $f : T_{<\alpha} \to \mathbb{R}$ satisfying $x < y \Rightarrow f(x) < f(y)$ such that if $\beta < \gamma < \alpha$, $x \in T_\beta$, ε a real positive number (> 0), then for some $y, x < y \in T_\gamma$, $f(y) < f(x) + \varepsilon$; and $x \in T_{\alpha+1} \Leftrightarrow f(x) \in \mathbb{Q}$ and if $\delta < \omega_1$ is a limit ordinal, $x \in T_\delta$ then $f(x) = \sup\{f(y) : y < x\} \in \mathbb{R}$.

For $\alpha = 0$, α-successor of successor or α limit, no problem.

For $\alpha + 1$, α limit, we are given antichains $\mathcal{I}^\alpha_n \subseteq T_{<\alpha}$ for $(n < \omega)$ (by $\diamondsuit_{\aleph_1}$ or $\diamondsuit^*_{\aleph_1}$) and we can define $T_{<\alpha+1}$ (and hence $f \restriction T_{<\alpha+1}$) such that

$(*)$ if $x \in T_\alpha, n < \omega$ and $\{y \in T_{<\alpha} : y < x\} \cap \mathcal{I}^\alpha_n = \emptyset$ *then* for some $y < x$, and $\varepsilon > 0$, $f(y) < f(x) < f(y) + \varepsilon$, and there is no z, $z \in \mathcal{I}^\alpha_n, y < z \in T_{<\alpha}$ and $f(y) < f(z) < f(y) + \varepsilon$.

Now $T = \bigcup_{\alpha<\omega_1} T_{<\alpha}$ and f are defined in the end. Suppose $\mathcal{I} \subseteq T$ is an antichain Now $C = \{\alpha < \omega_1 : \alpha$ limit, and if $x \in T_{<\alpha}, \varepsilon > 0$, and there is $y \in \mathcal{I}, x < y, f(x) < f(y) < f(x) + \varepsilon$ then there is such $y \in T_{<\alpha}\}$ is closed unbounded (note that it suffices to consider $\varepsilon \in \{1/n : n$ positive natural number $\})$.

Now if $\alpha \in C, \mathcal{I} \cap T_{<\alpha} = \mathcal{I}^\alpha_{n(0)} \in \{\mathcal{I}^\alpha_n : n < \omega\}$, $\alpha \in \mathrm{rk}(\mathcal{I})$ then by $(*)$ we get $\mathcal{I} \cap T_\alpha = \emptyset$ (if $y \in \mathcal{I}, y \in T_\alpha$, by $(*)$ we know $\{z : z < y\} \cap \mathcal{I}^\alpha_{n(0)} \neq \emptyset$; let z be in it, then $z < y$ both in $\mathcal{I}$, but $\mathcal{I}$ is an antichain).

Now by defining $\mathcal{I}^\alpha_n$ using $\diamondsuit_{\omega_1}$ or $\diamondsuit^*_{\omega_1}$ we get (1), (2) and (3). $\square_{3.7}$

3.8 Claim. $(\diamondsuit^*_{\aleph_1})$ Let h be a function from ω_1 to ω_1. There is a tree T which is h_1-special iff $\{i : h(i) < h_1(i)\}$ contains a closed unbounded subset of ω_1 (see Definition 3.2(4)).

Proof. Similar to the proof of 3.7. $\square_{3.8}$

3.9 Lemma. Let $S \subseteq \omega_1$ be stationary, and assume $\diamondsuit^*_{\omega_1 \setminus S}$ hold. There is an $S - st$-special tree which is $S_1 - st$-special iff $S_1 \setminus S$ is not stationary; moreover there is no antichain $\mathcal{I}$, such that $\mathrm{rk}(\mathcal{I}) \setminus S$ is stationary. (If $S = \omega_1$ we do not need any hypothesis, $\diamondsuit^*_\emptyset$ is meaningless anyhow and this is the classical theorem on the existence of special Aronszajn trees of Aronszajn himself.) Also we can make the tree such that it is not h-special for any h.

Proof. We define by induction on $\alpha < \omega_1$, $(T_{<\alpha}, <_T \restriction T_{<\alpha})$, and $(f \restriction T_{<\alpha}) : T_{<\alpha} \to \alpha \times \omega_1$; such that $x \in T_{<\alpha} \setminus T_0 \,\&\, \mathrm{rk}(x) \in S \Rightarrow f(x) \in \mathrm{rk}(x) \times \omega$; $x \in T_0 \Rightarrow f(x) \in \{0\} \times \omega = 1 \times \omega$; $\mathrm{rk}(x) < \omega_1 \setminus S \Rightarrow f(x) \in (\mathrm{rk}(x) + 1) \times \omega$ and $x < y \Rightarrow f(x) \neq f(y)$, such that

(a) $\beta \in S$, $x \in T_\beta \Rightarrow |\beta \times \omega \setminus \{f(y) : y < x\}| = \aleph_0$, (if β is a non-limit ordinal this holds trivially.)

(b) if $x \in T_\beta, \beta < \gamma < \alpha$, $\{\langle \xi, n\rangle\} \bigcup A \subsetneq ((\beta + 1) \times \omega \setminus \{f(z) : z \leq x\})$, $\langle \xi, n\rangle \notin A$ and A is finite, *then* there is $y \in T_\gamma$ such that $x < y$ and $\{f(z) : z \leq y\} \cap A = \emptyset$ but $\langle \xi, n\rangle \in \{f(z) : z \leq y\}$.

We can demand

(c) if α is limit, $\alpha \notin S$ and $x \in T_\alpha$ *then* $f(x) \notin \alpha \times \omega$

and note that we have

(d) if α is limit, $\alpha \in S$ and $x \in T_\alpha$ *then* $f(x) \in (\alpha \times \omega \setminus \{f(z) : z \leq x\})$.

We let $T_\alpha = [\alpha\omega, (\alpha+1)\omega)$ (so T has infinitely many minimal elements; we can add a root). Let $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$ for $\alpha < \omega_1 \setminus S$ be countable such that $\langle \mathcal{P}_\alpha : \alpha < \omega_1 \setminus S \rangle$ a witness for $\diamondsuit^*_{\omega_1 \setminus S}$. Let us carry the definition.

Case 1: $\alpha = 1$

$\leq_T \restriction T_{<1}$ is the equality.

Let $f \restriction T_0$ be any function from T_0 to $1 \times \omega$.

Case 2: α limit: trivial.

Case 3: $\alpha = \beta + 2$.

Let $\langle A_x : x \in T_\beta \rangle$ be a partition of $T_{\beta+1} = [(\beta+1)\omega, (\beta+2)\omega)$ to (pairwise disjoint) infinite sets. Define the order on $T_{<\alpha}$ by:

for $x, y \in T_{<\alpha} : x <_T y$ iff

$$x <_{T_{<(\beta+1)}} y \text{ or } (\exists z \in T_\beta)(x \leq_{T_{<(\beta+1)}} z \;\&\; y \in A_z).$$

Case 4: $\alpha = \delta + 1, \delta \in S$ (δ limit).

As in the construction of special Aronszajn trees using (b) (and taking care of it).

Case 5: $\alpha = \delta + 1, \delta \notin S$ (δ limit).

Let $\{B^\delta_n : n < \omega\}$ be a list of maximal antichains of $T_{<\delta}$ including all maximal antichains which belongs to $\mathcal{P}_\delta$. Let $\{(\gamma_n, k_n) : n < \omega\}$ enumerate $\delta \times \omega$.

Choose $\langle \beta_n : n < \omega \rangle$ such that $\beta_0 = \beta, \beta_n < \beta_{n+1} < \delta$ and $\bigcup_{n<\omega} \beta_n = \delta$. For each $\beta < \delta, x \in T_\beta$ and finite $A \subseteq ((\beta+1) \times \omega \setminus \{f(z) : z \leq x\})$, we choose by induction on $n, y_n = y^\delta_n[A, x]$, and $\varepsilon_n = \varepsilon_n[A, x]$ such that:

(i) $y_0 = x, \quad y_n \in T_{\varepsilon_n}, y_n \leq_{T_{<\delta}} y_{n+1}, \delta > \varepsilon_n \geq \beta_n$

(ii) A is disjoint to $\{f(z) : z \leq y_n\}$

(iii) let $\ell(n) < \omega$ be minimal such that:

$$\langle \gamma_{\ell(n)}, k_{\ell(n)} \rangle \notin \{f(z) : z \leq y_n\} \cup A,$$

then $(\gamma_{\ell(n)}, k_{\ell(n)}) \in \{f(z) : z < y_{n+1}\}$.

(iv) either $\{z : z < y_{n+1}\} \cap B_n^\delta \neq \emptyset$, or there is no $y \in T_{<\delta}$ satisfying $y_n < y$, $\{f(z) : z \leq y\} \cap A = \emptyset$, $\quad \{z : z < y\} \cap B_n^\delta \neq \emptyset$.

The induction step is by part (b) of the induction hypothesis.

Let $\langle (x_i^\delta, \beta_i^\delta, A_i^\delta) : i < \omega \rangle$ list all triplets (x, β, A) as above (i.e. $x_i^\delta \in T_{\beta_i^\delta}, \beta_i^\delta < \delta, A_i^\delta$ finite $\subseteq ((\beta_i^\delta + 1) \times \omega \setminus \{f(z) : z \leq x_i^\delta\})$.

Now we define

$$y_1 \leq_{T_{<\alpha}} y_2$$

iff for some $\beta < \delta$, $\quad y_1 \leq_{T_{<\beta}} y_2$, or for some $i < \omega$ we have $y_2 = \delta + i$, $\bigvee_{n<\omega} y_1 \leq y_n^\delta[A_i^\delta, x_i^\delta]$. Let $f(\delta + i) = (\delta, 0)$. It is easy to see that we finish this case, too. So we have carried the inductive construction.

Now let us see that $T = T_{<\omega_1} = \bigcup_{\alpha<\omega_1} T_{<\alpha}$ is as required (clearly $f : T \to \omega_1 \times \omega$). Being S-st special is by the requirements

$$\mathrm{rk}(x) \in S \Rightarrow f(x) \in \mathrm{rk}(x) \times \omega$$
$$x < y \Rightarrow f(x) \neq f(y).$$

Now suppose $\mathcal{I} \subseteq T$ is an antichain, $S_1 \stackrel{\text{def}}{=} \mathrm{rk}(\mathcal{I}) \setminus S$ stationary.
For each $\alpha \in S_1$, let $x_\alpha \in \mathcal{I}$, $\quad \mathrm{rk}(x_\alpha) = \alpha$. Note: $f(x_\alpha) = (\alpha, 0)$. Now, by the definition of $\diamondsuit^*_{\omega_1 \setminus S}$, for some club C of ω_1:

$$\delta \in C \cap S_1 \Rightarrow \{x_\alpha : \alpha < \delta \cap S_1\} \in \mathcal{P}_\delta.$$

Choose $\delta(*) \in C \cap S_1$ such that $M_{\delta(*)} \prec M_{\omega_1}$ where for $\delta \leq \omega_1$ we define

$$M_\delta = \Big(T_{<\delta}, <_{T_{<\delta}}, f{\restriction}T_{<\delta}, \{x_\alpha : \alpha \in S_1 \cap \delta\}\Big).$$

So $\{x_\alpha : \alpha < \delta(*) \cap S_1\} \in \mathcal{P}_{\delta(*)}$ hence for some n we have $\{x_\alpha : \alpha < \delta(*) \cap S_1\} = B^{\delta(*)}_n$ and for some $i < \omega$ we have $x_{\delta(*)} = \delta(*) + i$. Now $x_{\delta(*)}$ (which is well defined as $\delta(*) \in S_1$) is in $T_{\delta(*)}$ and $y^{\delta(*)}_m[A^{\delta(*)}_i, x^{\delta(*)}_i] <_T x_{\delta(*)}$.

As $\alpha \in S_1 \Rightarrow f(x_\alpha) = \langle \alpha, 0 \rangle$, by the choice of $y^{\delta(*)}_{n+1}[A^{\delta(*)}_i, x^{\delta(*)}_i]$ (clause (iv)) and by the choice of C

$$\{z : z < y^{\delta(*)}_{n+1}[A^{\delta(*)}_i, x^{\delta(*)}_i]\} \cap \{x_\alpha : \alpha < \delta(*)\} \neq \emptyset$$

contradicting $\{x_\alpha : \alpha \in S_1\}$ being an antichain. $\square_{3.9}$

3.10 Lemma. ($\Diamond_{\omega_1}$) There is a special Aronszajn tree T, such that for no antichain $\mathcal{I} \subseteq T$ is $\mathrm{rk}(\mathcal{I})$ closed unbounded. (For stationary: there is necessarily: this is mentioned in Devlin and Shelah [DvSh:65] p. 25).

Remark. E.g., MA + $2^{\aleph_0} > \aleph_1$ implies that this fails.

Proof. We define by induction on α, $(T_{<\alpha}, <_T \restriction T_{<\alpha})$ and $f : T_{<\alpha} \to \mathbb{Q}$ monotonic, so that $\beta < \gamma < \alpha$, $x \in T_\beta$, $\varepsilon > 0$ implies that for some $y \in T_\gamma$ we have $x < y \,\&\, f(x) < f(y) < f(x) + \varepsilon$. For limit $\delta < \omega_1$ we are given an antichain $\mathcal{I}^\alpha \subseteq T_{<\alpha}$ (by $\Diamond_{\aleph_1}$) and demand that for $x \in T_\alpha$, either

$$(\exists y \in \mathcal{I}^\alpha) y < x$$

or $(\exists y < x)$ [there is no z, $y < z \in \mathcal{I}^\alpha, f(z) \leq f(x) \in \mathbb{Q}$].
[How? For each $y \in T_{<\delta}$ and rational $\varepsilon > 0$, choose if possible, $z = z_{y,\varepsilon}$ such that $y \leq_{T_{<\delta}} z \in \mathcal{I}_\delta$, $f(z) < f(y) + \varepsilon$, if not let $z_{y,\varepsilon} = y$. Let $q_{y,\varepsilon} = f(z_{y,\varepsilon}) + (f(y) + \varepsilon - f(z_{y,\varepsilon}))/2$. So we can demand that for every $x \in T_\delta$ for some $y \in T_{<\delta}$ and rational $\varepsilon > 0$ we have $z_{y,\varepsilon} <_{T_{<(\delta+1)}} x$ and $f(x) = q_{y,\varepsilon}$.]
The checking is easy. See the last two paragraphs of the proof of 3.7. $\square_{3.10}$

3.11 Lemma. T is $\{\alpha + 1 : \alpha < \omega_1\}$-special iff T is r-special.

Remark. Proved by Baumgartner [B1].

Proof. The direction $\Leftarrow$ already appears.

For $\Rightarrow$ let f $\{\alpha+1 : \alpha < \omega_1\}$-specialize T.

Let $g : \mathbb{Q} \to \mathbb{Q}$ and $\varepsilon : \mathbb{Q} \to \{1/n : n > 0 \text{ natural}\}$ be such that the intervals $[g(q)-\varepsilon(q), g(q)+\varepsilon(q)]$ are pairwise disjoint and g is order preserving (possible: let $\mathbb{Q} = \{q_n : n < \omega\}$ and define $f(q_n)$, $\varepsilon(q_n)$ by induction on n).

Now define f^* as follows: $x \in T_{\alpha+1} \Rightarrow f^*(x) = g(f(x))$

$$x \in T_\alpha, \alpha \text{ limit} \Rightarrow f^*(x) = \operatorname{Sup}\{g(f(y)) : y < x, y \in T_{\beta+1}, \beta < \alpha\}\ .$$

Now f^* r-specializes T; the only point to check is:

$$x \in T_{\alpha+1}, \alpha \text{ limit} \Rightarrow g(f(x)) > \operatorname{Sup}\{g(f(y)) : y < x, y \in T_{\beta+1}, \beta+1 < \alpha\}$$

which follows by g's definition (the sup is $\leq g(f(x)) - \varepsilon(f(x))$ as for every $y < x$, $g(f(x))$ is smaller than it). $\square_{3.11}$

§4. Independence Results

It is well known that

4.1 Claim. If T is an $\aleph_1$-Souslin tree, $\lambda > \aleph_1$, $N \prec (H(\lambda), \in)$, $\|N\| = \aleph_0$ $T \in N, x \in T_\delta$, $\delta = \omega_1 \cap N$ *then* $B_T(x) = \{y \in T_{<\beta} : y < x\}$ is generic for (T, N), i.e., for every $\mathcal{I} \in N, \mathcal{I} \subseteq T$ which is pre-dense in T, we have:

$$\mathcal{I} \cap B_T(x) = \mathcal{I} \cap N \cap B_T(x) \neq \emptyset.$$

4.2 Definition. 1) For an Aronszajn tree T,
$Q(T) = \{(h, f) : h$ is a partial function from ω_1 to ω_1;

$$\alpha < \beta \ \&\ \{\alpha, \beta\} \in \operatorname{Dom}(h) \Rightarrow 0 < \alpha \leq h(\alpha) < \beta \leq h(\beta);$$

f is a finite function,

$\mathrm{Dom}(f) \subseteq \bigcup_{\alpha \in \mathrm{Dom}(h)} T_{h(\alpha)}$;
$x \in T_{h(\alpha)} \Rightarrow f(x) \in \alpha \times \omega$;

$$x < y \,\&\, [x, y \in \mathrm{Dom}(f)] \Rightarrow f(x) \neq f(y)\}$$

The order on $Q(T)$ is defined by $(h, f) \leq (h^\dagger, f^\dagger)$ if $h \subseteq h^\dagger, f \subseteq f^\dagger$;

We let $(h, f) \bigcup (h^\dagger, f^\dagger) \stackrel{\mathrm{def}}{=} (h \bigcup h^\dagger, f \bigcup f^\dagger)$, $(h, f) \bigcup h^\dagger \stackrel{\mathrm{def}}{=} (h \bigcup h^\dagger, f)$, $(h, f) \bigcup f^\dagger \stackrel{\mathrm{def}}{=} (h, f \bigcup f^\dagger)$.

2) We say that T is (S, h)-st-specialized by f if: S is a stationary subset of ω_1, and h is a function from S to ω_1 satisfying $(\forall \alpha \in S)(\alpha \leq h(\alpha))$ and h is a function with domain $\bigcup\limits_{\alpha \in S} T_{h(\alpha)}$ satisfying

$$\alpha \in S \;\&\; x \in T_\alpha \Rightarrow f(x) \in \alpha \times \omega \quad \text{and}$$

$$\alpha < \beta \;\&\; x \in T_{h(\alpha)} \;\&\; y \in T_{h(\beta)} \;\&\; x <_T y \Rightarrow F(x) \neq f(y).$$

So T is (S, h)-st-special if some f (S, h)-st-specialized it.
(Easily implies T is not Souslin.)

4.3 Definition. For an Aronszajn tree T and stationary set S, $Q(T, S) = \{(h, f) : (h, f) \in Q(T)$, and $\alpha \in (\mathrm{Dom}(h)) \cap (S \setminus \{0\})$ implies $h(\alpha) = \alpha\}$, order – as before.

Explanation. Our aim is to get a universe in which SH (Souslin Hypothesis) holds (i.e., there is no Souslin tree) but not every Aronszajn tree is special. The question was raised by Baumgartner Malitz and Reinhardt [BMR], and later independently by U. Abraham, and is natural as, until now, the consistency of SH was proved by making every Aronszajn tree special; see the proof of Solovay and Tennenbaum [ST], Martin and Solovay [MS], Baumgartner, Malitz and Reinhard [BMR] without CH, and Jensen proof in Devlin and Johnsbraten [DeJo] with CH (and Laver and Shelah [LvSh:104] for $\aleph_2$-Souslin tree). For this aim we have introduced in §3 various notions of specializations (each implying the tree is not Souslin). So the program is to make every tree special in some weaker than the usual sense. The notion r-special which had been introduced by Kurepa [Ku35] is not suitable, as if every Aronszajn tree is r-special then every Aronszajn tree is special (see 3.6(2)). Similarly "h-special" for any fixed increasing $h : \omega_1 \to \omega$ is not suitable by 3.6(1) (see Definition 3.2(4)).

So a natural candidate is "h-special for some h" (i.e., for every tree there is an h for which it is h-special). Forcing with $Q(T)$ does the job for T – we take generic h and f. (It would be more natural to let f go to $\mathbb{Q}$ and be monotonically increasing, but by 3.5(2) the forcing $Q(T)$ makes T h-special for some h, and this way we have more uniformity with Definition 4.3.) So we should iterate such forcings, but retain some T as not special.

A second way is to make each T $S - st$-special for some fixed stationary S; for this $Q(T, S)$ is tailored. (Note that the f we get from a generic subset of $Q(T, S)$ has domain $\bigcup_{\alpha \in S_1} T_\alpha$ where $S \setminus S_1$ non-stationary.) For $S = \emptyset$ we get the previous case, so we shall ignore $Q(T)$.

This leads to a secondary problem: Can every Aronszajn tree be $S_1 - st$-special, but some Aronszajn trees are not $S_2 - st$-special ($S_2 \setminus S_1$ stationary, of course)? We answer positively.

4.4 Claim. 1) For T an Aronszajn tree, $S \subseteq \omega_1$, $Q(T, S)$ is proper.
2) For T an Aronszajn $\aleph_1$-tree, and $S \subseteq \omega_1$ we have:

$$\Vdash_{Q(T,S)} \text{“}T \text{ is not Souslin tree, in fact for some function } \underset{\sim}{h} \text{ is } (\omega_1, \underset{\sim}{h})\text{-st-special”}.$$

3) In part (2), if S is stationary then $\Vdash_{Q(T,S)}$ "T is S-st-special".

Proof. We can assume w.l.o.g. $|T_0| = \aleph_0$. Let $\lambda > (2^{\aleph_1})$, $N \prec (H(\lambda), \in)$ be countable, $T, S \in N$, $p_0 = (h, f) \in Q(T, S) \cap N$, and let $\delta = N \cap \omega_1$.

Then $p_1 = (h \bigcup \{\langle \delta, \delta \rangle\}, f) \in Q(T, S)$ exemplifies what is required.

For checking, we really repeat the proof of Baumgartner Malitz and Reinhardt [BMR] that the standard forcing (now) for specializing an Aronszajn tree satisfies the $\aleph_1$-c.c. (or read the proof of demand (iii) in the proof of 4.6 – it is just harder).

2) Let $\underset{\sim}{h} = \bigcup\{h : (h, f) \in \underset{\sim}{G}_{Q(T,S)}\}$, and $\underset{\sim}{C} = \mathrm{Dom}(\underset{\sim}{h})$ and $\underset{\sim}{f} = \bigcup\{f : (h, f) \in$

$\underset{\sim}{G}_{Q(T,S)}\}$. We know (see III) that it is forced that $\underset{\sim}{C}$ is a club of ω_1, so T becomes $(\underset{\sim}{C}, \underset{\sim}{h})$-st-special.

3) Should be clear. $\square_{4.4}$

4.5 Definition. We call a forcing notion P, (T^*, S)-preserving (do you have a better name?), where T^* is an Aronszajn tree, $S \subseteq \omega_1$, if: for every $\lambda > (2^{|P|+\aleph_1})^+$, $\langle P, T^*, S\rangle \in N \prec (H(\lambda), \in), N$ countable, $\delta \stackrel{\text{def}}{=} N \cap \omega_1 \notin S$ and $p \in N \cap P$, there is p_1 which is preserving for (p, N, P, T^*, S); i.e.,

(i) $p \le p_1 \in P$,

(ii) p_1 is (N, P)-generic,

(iii) for every $x \in T^*_\delta$, *if*

$(*)$ $x \in A \to (\exists y < x)(y \in A)$ holds for every $A \subseteq T^*, A \in N$,

then

$(**)$ for every P-name $\underset{\sim}{A}, \underset{\sim}{A} \in N$ such that $\Vdash_P$ "$\underset{\sim}{A} \subseteq T^*$" the following holds:

$$p_1 \Vdash \text{“}x \in \underset{\sim}{A} \to (\exists y < x) y \in \underset{\sim}{A}\text{”}.$$

4.6 Lemma. If T^*, T are Aronszajn trees, $S \subseteq \omega_1$, *then* $Q(T, S)$ is (T^*, S)-preserving.

Remark. If T^* is Souslin tree then $(*)$ from Definition 4.5 is satisfied by every countable $N \prec (H(\lambda), \in)$ and $x \in T^*_\delta$ when $N \cap \omega_1 = \delta$ (this follows by 4.1).

Proof. Let $P \stackrel{\text{def}}{=} Q(T, S)$. Let $N \prec (H(\lambda), \in)$, $\delta \stackrel{\text{def}}{=} N \cap \omega_1 \notin S$, $||N|| = \aleph_0$, $\langle T^*, T, S\rangle \in N$ hence $P \in N$, $p = (h_0, f_0) \in P \cap N$ (as in Definition 4.5.), and (remembering $\delta = N \cap \omega_1$) let $\delta^* = \sup\{f(\delta) + 1 : f \in N, f(\delta)$ is an ordinal $< \omega_1\}$.

Define $p_1 = (h_0 \bigcup \{\langle \delta, \delta^*\rangle\}, f_0)$, demands (i) of Definition 4.5 is trivial, and demand (ii): its proof is easier than that of demand (iii), so let us check condition (iii). So suppose $x \in T^*_\delta$ and

$(*)$ if $A \subseteq T^*, A \in N, x \in A$ then $(\exists y)(y <_{T^*} x \,\&\, y \in A)$.

Let $\underset{\sim}{A}$ be a $Q(T,S)$-name of a subset of T^*, and $\underset{\sim}{A} \in N$. We shall prove that for every $p_2, p_1 \le p_2 \in Q(T,S)$, for some $p_3, p_2 \le p_3 \in Q(T,S)$, and $p_3 \Vdash$ "$x \notin \underset{\sim}{A}$" or $p_3 \Vdash$ "$y \in \underset{\sim}{A}$ for some $y <_{T^*} x$".

Let $p_2 = (h_2, f_2)$, if $p_2 \Vdash_P$ "$x \notin \underset{\sim}{A}$", then we can choose $p_3 = p_2$. Otherwise there is $p_2^\dagger \in P$, such that

$$p_2 \le p_2^\dagger \text{ and } p_2^\dagger \Vdash_P \text{“}x \in \underset{\sim}{A}\text{”}.$$

Let $p_2^\dagger = (h_2^\dagger, f_2^\dagger)$, $p_2^\dagger = p_2^a \bigcup p_2^b$, where $p_2^a = (h_2^a, f_2^a)$, $p_2^b = (h_2^b, f_2^b)$ where $h_2^a = h_2^\dagger \restriction \delta$, $h_2^b = h_2^\dagger \restriction [\delta, \omega_1)$ (closed open interval) and

$$f_2^a = f_2^\dagger \restriction T_{<\delta}, \quad f_2^b = f_2^\dagger \restriction (T \setminus T_{<\delta}).$$

Note that by the definition of $Q(T,S)$:

4.6A Fact.

(1) $p_2^a \in P \cap N$,

(2) $z \in \mathrm{Dom}(f_2^b) \Rightarrow \mathrm{rk}_T(z) \ge \delta^*$.

Now let $\alpha_0 = \mathrm{SupRang}(h_2^a)$ (which is $< \delta$) and we define a function F as follows:

$$\begin{aligned}
&\mathrm{Dom}(F) = \{y \in T^* : \mathrm{rk}(y) >_{T^*} \alpha_0\},\\
&F(y) = \mathrm{Sup}\{\alpha^* < \omega_1 : \text{there is } ({}^*h_2^b, {}^*f_2^b) \text{ (in } Q(T,S)) \text{ such that:}\\
&\qquad \text{(a) } \mathrm{Min}(\mathrm{Dom}({}^*h_2^b)) = \mathrm{rk}_{T^*}(y),\\
&\qquad \text{(b) } {}^*h_2^b(\mathrm{rk}_{T^*}(y)) = \alpha^*,\\
&\qquad \text{(c) } (h_2^a \textstyle\bigcup {}^*h_2^b, f_2^a \bigcup {}^*f_2^b) \Vdash_{Q(T,S)} \text{“}y \in \underset{\sim}{A}\text{”}\}
\end{aligned}$$

(so we demand also that $(h_2^a \bigcup {}^*h_2^b, f_2^a \bigcup {}^*f_2^b)$ is in $Q(T,S)$).

Now clearly $F \in N$ (as it is defined by a (first-order) formula in $(H(\lambda), \in)$ whose parameters are in N). Clearly $F(y) \le \omega_1$ (for $y \in T^* \setminus T^*_{\le \alpha_0}$). Let $A^* = \{y \in T^* : \mathrm{rk}_{T^*}(y) > \alpha_0, F(y) = \omega_1\}$. (Note that $A^* \subseteq T^*$ is a set, not a P-name of a set.).

Let F^* be a function from ω_1 to ω_1 defined by:

$$F^*(\alpha) = \mathrm{Sup}\{F(y)+1 : y \in T^*_{\le\alpha}, \mathrm{rk}_{T^*}(y) > \alpha_0, y \notin A^*, \text{ i.e., } F(y) < \omega_1\}.$$

As $|T^*_{\le\alpha}| \le \aleph_0$, we have $F^* : \omega_1 \to \omega_1$, and clearly $F^* \in N$ (same reason).

By the definition of δ^*, $F^*(\delta) < \delta^*$. But (h_2^b, f_2^b) exemplify $F(x) \ge \delta^*$, so necessarily $F(x) = \omega_1$. So by the definition of A^* above $x \in A^*$. Hence by the hypothesis $(*)$ there is $y <_{T^*} x$ such that $y \in A^*$. So (in $H(\lambda)$, hence in N) we can define a sequence $\bar{p} = \langle (h_2^{b,i}, f_2^{b,i}) : i < \omega_1 \rangle$ such that:

(a)$'$ $\mathrm{Min}(\mathrm{Dom}(h_2^{b,i})) = \mathrm{rk}_{T^*}(y) > \alpha_0$,

(b)$'$ $h_2^{b,i}(\mathrm{rk}(y)) \ge \alpha_0 + i$,

(c)$'$ $(h_2^a \bigcup h_2^{b,i}, f_2^a \bigcup f_2^{b,i}) \Vdash_{Q(T,S)}$ "$y \in \underset{\sim}{A}$".

For $i < \delta$ let $p_3^i = (h_2^a \bigcup h_2^b \bigcup h_2^{b,i}, f_2^a \bigcup f_2^b \bigcup f_2^{b,i})$. If $p_3^i \in Q(T,S)$ then by clause (c)$'$ and as $y <_T x$, this condition is as required.

Why can p_3^i be not in $Q(T,S)$? The first coordinate $(h_2^a \bigcup h_2 \bigcup h_2^{b,i})$ is O.K., as $h_2^a \subseteq h_2^{b,i} \in N$.

What about the second? Note that $f_2^a \bigcup f_2^b, f_2^a \bigcup f_2^{b,i}$ are O.K. as $p_2 \in Q(T,S)$ and by (c)$'$ above correspondingly. Hence the only danger is that there are $z_1 \in \mathrm{Dom}(f_2^b), z_2 \in \mathrm{Dom}(f_2^{b,i})$, $z_2 <_T z_1$ (as $f_2^{b,i} \in N$, $rk(z_1) \ge h_2^b(\delta) = \delta^*$ this is the only bad possibility).

But remember that in $H(\lambda)$ we have $z \in \mathrm{Dom}(f_2^{b,i}) \Rightarrow rk(z) \ge i$, so by a lemma on Aronszajn trees due to Baumgartner, Malitz and Reinhart (in their proof of $MA \vdash$ "every Aronszajn tree is special") which appears in the proof of III 5.4, there is a sequence $\langle i_n : i < \omega \rangle (i_n < \omega_1)$ such that

$$m \ne n \ \&\ z_1 \in \mathrm{Dom}(f_2)^{b,i_m} \ \&\ z_2 \in \mathrm{Dom}(f_2^{b,i_n}) \Rightarrow \begin{pmatrix} z_1 \not<_T z_2 \\ z_2 \not<_T z_1 \end{pmatrix}.$$

So again there is such a sequence in N, and all but at most $|\mathrm{Dom}(f_2^b)|$ are O.K., i.e., $p_3^i \in Q(T,S)$. So we finish. $\square_{4.6}$

4.7 Theorem. Let T^* be a Souslin tree. Suppose $P_\alpha(\alpha \le \alpha_0)$, $\underset{\sim}{Q}_\alpha(\alpha < \alpha_0)$ form an $\aleph_1$-free iteration (i.e., $P_{\alpha+1} = P_\alpha * \underset{\sim}{Q}_\alpha$, $P_\delta = \mathrm{Flim}^{\aleph_1}_{\alpha<\alpha_0} P_\alpha$) and for every α at least one of the following holds:

(α) $\underset{\sim}{Q}_\alpha$ is (in V^{P_α}) (T^*, S)-preserving,

(β) there is a P_α-name $\underset{\sim}{\mathcal{I}}_\alpha$ of an antichain of T^* (in V^{P_α}), $\underset{\sim}{S}_\alpha \stackrel{\text{def}}{=} \text{rk}(\underset{\sim}{\mathcal{I}}_\alpha) \subseteq \omega_1 \backslash S$ where $\text{rk}(\underset{\sim}{\mathcal{I}}_\alpha) = \{\text{rk}(x) : x \in \underset{\sim}{\mathcal{I}}_\alpha\}$, and in V^{P_α}:

$\underset{\sim}{Q}_\alpha = Q_{club}(\omega_1 \setminus S_\alpha) = \{g : \text{for some } i < \omega_1, \text{Dom}(g) = i+1, \text{Rang}(g) = \{0,1\},$ $\{j \leq i : g(i) = 1\}$ is closed and is $\subseteq \omega_1 \setminus S_\alpha\}$,
Then P_{α_0} is (T^*, S)-preserving.

4.7A Remark. 1)We can amalgamate conditions (α) and (β) but it has no use.
2) See on such theorems in XVIII§3.
3) Note that the forcing notion $Q(T,S)$ (T is an Aronszajn tree, $S \subseteq \omega_1$ co-stationary) adds an antichain $\mathcal{I}$ of T such that $\text{rk}(\mathcal{I}) \setminus S$ is stationary. This is because by the proof of 4.4 $\Vdash_{Q(T,S)}$ "$\{\delta < \omega_1 : \text{for some } (h,f) \in G_{Q(T,S)}$, we have $\delta \in \text{Dom}(h)$, $h(\delta) = \delta \in S\}$ is stationary" (together with Fodor's lemma).

Proof. We prove by induction on $\alpha \leq \alpha_0$ the following:

$\oplus_\alpha$ Suppose $\beta < \alpha \leq \alpha_0$, $N \prec (H(\lambda), \in)$, $\beta \in N, \alpha \in N$, $\langle P_i : i \leq \alpha \rangle \in N$, $\delta = N \cap \omega_1 \notin S$, $p \in P_\alpha \cap N$, $q_1 \in P_\beta$, and

(i) (a) $p{\restriction}\beta \leq q_1$ (natural meaning: no $q^\dagger$, $q_1 \leq q^\dagger \in P_\beta$ is incompatible with p; if we deal with complete BA, $p{\restriction}\beta$ is the projection).
(b) Moreover, if $p{\restriction}\beta \leq p^\dagger \in P_\beta \cap N$, then $q_1, p^\dagger$ are compatible;

(ii) q_1 is (N, P_β)-generic,
(iii) if $x \in T^*_\delta$ and $(\forall A \subseteq T^*)(A \in N \& x \in A \to (\exists y < x) y \in A)$ then for very P_β-name $\underset{\sim}{A} \in N$, we have $q_1 \Vdash_{P_\beta}$ "$x \in \underset{\sim}{A} \to (\exists y <_{T^*} x) y \in \underset{\sim}{A}$".
Then there is $p_1 \in P_\alpha$ such that
(i)$'$ (a) $p_1{\restriction}\beta = q_1$, $p \leq p_1$ (natural meaning).
(b) if $p{\restriction}\alpha \leq p^\dagger \in P_\alpha \cap N$ then $p_1, p^\dagger$ are compatible,
moreover (this implies (a)+(b)): *if* $p{\restriction}\alpha \leq p^\dagger \in P_\alpha \cap N$, $q_1 \leq q^\dagger \in P_\beta, p^\dagger{\restriction}\beta \leq q^\dagger$ *then* $p^\dagger, p_1, q^\dagger$ are compatible ($=$ have an upper bound),

(ii)′ p_1 is (N, P_α)-generic,

(iii)′ the parallel of (iii) with $\beta \mapsto \alpha, q_1 \mapsto p_1$.

Case 1. $\alpha = 0$. Trivial.

Case 2. $\alpha + 1$. By the similarity between the assumptions on q_1 and the conclusion on p_1, we can assume w.l.o.g. $\beta = \alpha$. Let $G \subseteq P_\alpha$ be generic over V, $q_1 \in G$. Then $N[G] \prec (H(\lambda), \in)$ (see III 2.11).

Now in $V[G]$ (hence in $H(\lambda)[G]$) we can find in $\underset{\sim}{Q}_\alpha[G_{P_\alpha}]$ a condition $p_1^\dagger \geq p(\alpha)$, which is $(N[G], Q_\alpha[G])$-generic, as in Definition 4.5. Why?

Note that we can ignore (i)′(b), as we can take a disjunction over countably many possibilities one for each $r \in N[G]$, $P_{\alpha+1}/G$, $r \geq p$. More accurately, maybe in Q_α it does not exist, but we can make a trivial change in Q_α to ensure it, without affecting the iteration (in fact, the forcing notion we actually use has such a condition anyway).

Now our proof splits according to which of the conditions (α) or (β) from the theorem, $\underset{\sim}{Q}_\alpha[G]$ satisfies.

(α) Straightforward, by Definition 4.5.

(β) By the choice of T^* (a Souslin tree), by Claim 4.1 we have: $[x \in A \in N$ and $x \in T^*_\delta \Rightarrow (\exists y < x) y \in A]$. So by the assumption on q_1 for every $A \in V[G], A \subseteq T^*$, $A \in N[G]$, of course there is a P_α-name $\underset{\sim}{A} \in N$, $\underset{\sim}{A}[G] = A$; now we know $q_1 \Vdash_{P_\alpha}$ "$x \in \underset{\sim}{A} \rightarrow (\exists y < x) y \in \underset{\sim}{A}$", hence in $V[G], x \in A \cap T^*_\delta \Rightarrow (\exists y < x) y \in A$.

In particular, we can take $A = \underset{\sim}{\mathcal{I}}_\alpha[G] \in N[G]$ (remember $\underset{\sim}{\mathcal{I}}_\alpha[G] \in V^{P_\alpha}$ and as $\langle P_i : i \leq \alpha \rangle \in N$ hence w.l.o.g. $\langle \underset{\sim}{I}_j : j < \alpha$ and if $\underset{\sim}{Q}_j$ satisfies clause (β), $\underset{\sim}{I}_j$ is as there, otherwise $\underset{\sim}{I}_j = \emptyset \rangle$). So clearly if $x \in T^*_\delta \cap \underset{\sim}{\mathcal{I}}_\alpha[G]$ then $x \in A$ implies $\underset{\sim}{\mathcal{I}}_\alpha[G]$ is not an antichain, contradiction. So $T^*_\delta \cap \underset{\sim}{\mathcal{I}}_\alpha[G] = \emptyset$, so $\delta \notin \underset{\sim}{S}_\alpha[G] = \mathrm{rk}(\underset{\sim}{\mathcal{I}}_\alpha)[G]$, and then the desired conclusion is quite easy (remember Q_α's definition).

So we have p_1 as required. Now p_1 is in V^{P_α}, so in V we have a P_α-name $\underset{\sim}{p}_1$ for it, and let $p_1 = (q_1, \underset{\sim}{p}_1^\dagger) \in P_\alpha * \underset{\sim}{Q}_\alpha$ which by the usual thing for composition of forcing, is as required.

Case 3. α limit.

Choose α_n for $n < \omega$ such that $\beta = \alpha_1 < \dots \alpha_n < \alpha_{n+1} < \dots; \alpha_n \in N$ and $\alpha(*) \stackrel{\text{def}}{=} \sup(\alpha \cap N) = \bigcup_{0<n<\omega} \alpha_n$.

We define by induction on $n < \omega, n \geq 1$, $q_n \in P_{\alpha_n}$, $q_{n+1} \restriction \alpha_n = q_n$, q_1 is given and q_{n+1} is obtained from q_n by the induction hypothesis, with $p, q_n, q_{n+1}, \alpha_n, \alpha_{n+1}$ here standing for $p, q_1, p_1, \alpha, \beta$ there.

Let $\langle (\underset{\sim}{A}_n, x_n) : n < \omega \rangle$ be a list of all pairs $(\underset{\sim}{A}, x)$, where $\underset{\sim}{A}$ is a P_α-name of a subset of T^*, $x \in T^*_\delta$ and $\underset{\sim}{A}$ is in N; let $\langle \mathcal{J}_n : n < \omega \rangle$ be a list of all pre-dense subsets of P_α which belong to N. Let

$p_1 \stackrel{\text{def}}{=} p \wedge \bigwedge_n q_n \wedge \bigwedge_n (\bigvee_{r \in \mathcal{J}_n \cap N} r) \wedge \bigwedge_{n<\omega} [\bigvee \{p \in P_\alpha \cap N : p \Vdash_{P_\alpha}$ "$y \in \underset{\sim}{A}_n$" for some $y <_{T^*} x_n\} \vee \bigvee \{q_n \wedge \bigwedge_{r \in \mathcal{J}} r : \mathcal{J} \subseteq N \cap P_\alpha, \mathcal{J}$ is definable in $(N, \{y : y < x\})$ and $q_n \wedge \bigwedge_{r \in \mathcal{J}} r \Vdash_{P_\alpha}$ "$x_n \notin \underset{\sim}{A}_n$"$\}]$.

There are two facts on p_1 we have to prove:

(A) $p_1 \in P_\alpha = \operatorname{Flim}^{\aleph_1}_{i<\alpha(*)} P_i$, i.e., $\bigwedge_{i<\alpha(*)} \theta[P_i] \nvdash p_1$ (as clearly p_1 has the right form),

(B) (i)$'$, (ii)$'$, (iii)$'$ (of $\oplus_\alpha$ above) hold.

For proving both facts we do the following. We assume everything is in some countable transitive model M (or $M \Rightarrow V, V \Rightarrow V^*$, in V^* we have $|H(\lambda)^V|$ is countable which is easy by forcing).

Let $p^\dagger, q^\dagger$ be as in (i)$'$ (the "moreover" version).

We let $G_{\alpha_1} \subseteq P_{\alpha_1} = P_\beta$ be generic (i.e., M-generic) such that $p^\dagger \restriction \beta, q^\dagger \in G_{\alpha_1}$.

We shall find $G_{\alpha(*)} \subseteq P_{\alpha(*)}$ such that for each n the set $G_{\alpha(*)} \cap P_{\alpha_n}$ is generic (for (M, P_{α_n})), and the truth values it gives to all $p \in \bigcup_{n<\omega} P_{\alpha_n}$ make $p_1 \wedge p^\dagger$ true (so we have, in V, a model exemplifying $\bigwedge_{i<\alpha(*)} \theta[P_i] \nvdash \neg(p_1 \wedge p^\dagger)$ (= fact (A)), and $G_{\alpha_1} \subseteq G_{\alpha(*)}$.

As for fact (B), clause (B) (ii)$'$ holds trivially by the definition of p_1 (i.e., $\bigwedge_n (\bigvee_{r \in \mathcal{J}_n} r)$). Similarly the last conjunct takes care of (B) (iii)$'$.

The "moreover" phrase of (B) (i)$'$ holds by the free choice of $p^\dagger, q^\dagger$ (and the way $G_{\alpha_1}, G_{\alpha(*)}$ are chosen), hence $p_1 \restriction \beta = q_1$; the other inequality follows by p_1's definition. So it is enough to find $G_{\alpha(*)}$.

We define by induction G_{α_n}, p_n such that (as in the proof of 2.2):

(1) $G_{\alpha_n} \subseteq P_{\alpha_n}, G_{\alpha_n} \subseteq G_{\alpha_{n+1}}$,

(2) G_{α_n} is P_{α_n}-generic over M,

(3) $p_n \leq p_{n+1}, p_0 = p^\dagger, p_n \in P_\alpha \cap N$,

(4) p_n is compatible (in P_α) with every member of $G_{\alpha_n}, q_n \in G_{\alpha_n}$,

(5) p_{3n+1} is $\geq q_n^\dagger$ for some $q_n^\dagger \in \mathcal{J}_n \cap N$,

(6) $p_{3n+2} \vdash \wedge\Phi_n$ or $p_{3n+2} \vdash \neg r_n$ for some $r_n \in \Phi_n$, where $\langle \Phi_n : n < \omega \rangle$ is a list of all countable $\Phi \subseteq P_\alpha, \Phi \in N$,

(7) in $M[G_{\alpha_n}]$ for every $A \in N[G_{\alpha_n}]$, $A \subseteq T^*$ we have $[x \in T^*_\delta \ \&x \in A \rightarrow (\exists y < x) y \in A]$ holds ($q_n \in G_{\alpha_n}$ do the job),

(8) $p_{3n+3} \Vdash_{P_\alpha}$ "$(\exists y < x_n) y \in \underset{\sim}{A}_n$" or $p_{3n+3} \wedge \bigwedge_{r \in \mathcal{J}} r \Vdash_{P_\alpha}$ "$x \notin \underset{\sim}{A}_n$", for some $\mathcal{J}$, such that $\mathcal{J} \subseteq G_{\alpha_{n-1}}$, and $\mathcal{J}$ is definable in $(N, \{y : y <_{T^*} x_n\})$ (remember $\{(\underset{\sim}{A}_n, x_n) : n < \omega\}$ list the pairs $(\underset{\sim}{A}, x)$, $\underset{\sim}{A} \in N$ a P_α-name of a subset of T^*, and $x \in T^*_\delta$.)

As in the proof of 2.2, this suffices [for $\mathcal{J}$ as in (8), use the conjunct corresponding to $\mathcal{J} \cup \{p_{3n+3}\}$ in p_1]. The only nontrivial part in the definition is taking care of (8). So let $n = 3k+2, p_n, G_{\alpha_n}$, be defined, and we shall define $p_{n+1}, G_{\alpha_{n+1}}$. We define:

$\underset{\sim}{A}_k^\dagger = \{y \in T^*$: there is $r \in P_\alpha, r \geq p_n$, which is compatible with every member of $\underset{\sim}{G}_{\alpha_n}$(= the name of the generic subset of P_{α_n}) such that $r \Vdash_{P_\alpha}$ "$y \in \underset{\sim}{A}_k$"$\}$.

Clearly $\underset{\sim}{A}_k^\dagger$ is a P_{α_n}-name (as we use $\underset{\sim}{G}_{\alpha_n}$ in the definition but not $\underset{\sim}{G}_{\alpha_n+1}$) and if $p_n \restriction \alpha_n \leq r \in P_{\alpha_n}$ then:

$(*)$ $r \Vdash_{P_{\alpha_n}}$ "$y \notin \underset{\sim}{A}_k^\dagger$" implies $r \wedge p_n \Vdash_{P_{\alpha_n}}$ "$y \notin \underset{\sim}{A}_k$".

However the inverse implication does not follow. Now if we can choose p_{n+1}, such that $p_n \leq p_{n+1} \in P_\alpha \cap N, p_{n+1}$ compatible with every member of G_{α_n} (equivalently of $G_{\alpha_n} \cap N$) such that $p_{n+1} \Vdash_{P_\alpha}$ "$y \in \underset{\sim}{A}_k$" for some $y <_{T^*} x_k$, then we can proceed to define $G_{\alpha_{n+1}}$ with no problem.

So we assume that there is no such p_{n+1} and let $p_{n+1} = p_n$. Let

$$\mathcal{J} = \{\neg r : r \in P_{\alpha_n} \cap N \ , r \Vdash_{P_{\alpha_n}} \text{“} y \in \underset{\sim}{A}_k^\dagger \text{” for some } y <_{T^*} x_k\}.$$

Clearly $\mathcal{J}$ is definable in $(N, \{y : y <_{T^*} x_k\})$, $\mathcal{J} \subseteq P_{\alpha_n} \cap N$, and $\mathcal{J} \subseteq G_{\alpha_n}$ (as if $(\neg r) \in \mathcal{J}$, $(\neg r) \notin G_{\alpha_n}$ then $r \in G_{\alpha_n}$ so we would not have arrive here), and $p_n \leq p_{n+1} \in P_\alpha \cap N$ and $p_{n+1} \in P_\alpha / G_{\alpha_n}$ so it is enough to prove

(**) $p_n \wedge q_n \wedge \bigwedge_{r \in \mathcal{J}} r \Vdash_{P_\alpha}$ “$x_k \notin \underset{\sim}{A}_k$”.

Now $\underset{\sim}{A}_k^\dagger$ is a P_{α_n}-name of a subset of T^* (and it belongs to N), so by the choice of q_n:

$$q_n \Vdash_{P_{\alpha_n}} \text{“} x_k \in \underset{\sim}{A}_k^\dagger \rightarrow (\exists y <_{T^*} x_k) y \in \underset{\sim}{A}_k^\dagger \text{”}.$$

However for each $y <_{T^*} x_k$,

$$\mathcal{J}_y = \{r \in P_{\alpha_n} : r \Vdash_{P_{\alpha_n}} \text{“} r \in \underset{\sim}{A}_k^\dagger \text{” or } r \Vdash_{P_{\alpha_n}} \text{“} y \notin \underset{\sim}{A}_k^\dagger \text{”}\}$$

is a dense subset of P_{α_n} which belongs to N hence $\mathcal{J}_y \cap N$ is pre-dense above q_n (in P_{α_n}) (as $y \in N$). So q_n forces that if $y \in \underset{\sim}{A}_k^\dagger (y <_{T^*} x_k)$ then some $r \in \mathcal{J}_y \cap N$ is in the generic subset of P_{α_n}, and $r \Vdash_{P_{\alpha_n}}$ “$y \in \underset{\sim}{A}_k^\dagger$”. Hence $q_n \wedge p_n \in P_\alpha$ forces that: if $x_k \in \underset{\sim}{A}_k$, then necessarily $x_k \in \underset{\sim}{A}_k^\dagger$ (see (*)) hence some $y <_{T^*} x_k$ is in $\underset{\sim}{A}_k^\dagger$. Hence some $r \in \mathcal{J}_y \cap N$ for which $r \Vdash_{P_{\alpha_n}}$ “$y \in \underset{\sim}{A}_k^\dagger$” is in the generic set , and clearly $\neg r \in \mathcal{J}$. So clearly (as $p_n \in \mathcal{J}$) $q_n \wedge p_n \wedge \bigwedge_{r \in \mathcal{J}} r$ forces that: $x_k \in \underset{\sim}{A}_k$ leads to a contradiction (as r and $\neg r$ are incompatible) so it forces $x_k \notin \underset{\sim}{A}_k$ i.e. (**) holds as promised, so we have succeeded to define $P_{n+1} = P_{3k+3}$ as required. There is no problem to define $G_{\alpha_{n+1}}$, so we finish proving (8) hence the theorem. $\square_{4.7}$

4.8 Conclusion. Assume $S \subseteq \omega_1$ is co-stationary. For some forcing notion P, not collapsing $\aleph_1$, in V^P we have: every Aronszajn tree is $S - st$-special, but some Aronszajn tree T^* is not $S^* - st$-special for any $S^* \subseteq \omega_1 \setminus S$ stationary, moreover for every antichain $\mathcal{I}$ of T^*, $\mathrm{rk}(\mathcal{I}) \setminus S$ is not stationary. Also there is no Souslin tree. Assuming $2^{\aleph_0} = 2^{\aleph_1}$, $2^{\aleph_1} = \aleph_2$ we have: P is proper $\aleph_2$-c.c. of cardinality $\aleph_2$.

Remark. S is co-stationary – otherwise it is not interesting, but there is no other restriction e.g. S may be empty. See more 4.9(2).

Proof. Trivial by the previous Theorems 4.6, 4.7, but note that for ensuring in a transparent way that T^* remains an Aronszajn tree we would start the iterated forcing by $Q(T^*, S)$. As for the $\aleph_2$-chain condition, see VIII §2. Remember also that our forcings are proper and proper forcings preserves stationarity of subset of ω_1 (see III). In more details, by some preliminary forcing without loss of generality $V \models$ "$\diamondsuit_{\aleph_1} + 2^{\aleph_1} = \aleph_2$" and let T^* be a Souslin tree. We can define an $\aleph_1$-free iteration $\langle P_i, \underset{\sim}{Q}_j : i \leq \omega_2,\ j < \omega_2 \rangle$ as in 4.7, such that:

(a) $Q_0 = Q(T^*, S)$

(b) each Q_α satisfies one of the following:

  (α) Q_α is proper and (T^*, S)-preserving of cardinality $\aleph_1$.

  (β) for some P_α-name of an antichain $\underset{\sim}{\mathcal{I}}_\alpha$ of T^*, $\mathrm{rk}(\underset{\sim}{\mathcal{I}}_\alpha) \cap S = \emptyset$ and $\underset{\sim}{Q}_\alpha = Q_{\mathrm{club}}(\omega_1 \setminus \mathrm{rk}(\underset{\sim}{\mathcal{I}}_\alpha)) = \{g :$ for some $i < \omega_1$ g is a function from $i+1$ to $\{0,1\}$, $g^{-1}(\{1\})$ closed and included in $\omega_1 \setminus \mathrm{rk}(\underset{\sim}{\mathcal{I}}_\alpha)\}$.

(c) for every $\gamma < \omega_2$ and P_γ-name $\underset{\sim}{\mathcal{I}}$ of an antichain of T^* such that $\Vdash_{P_\gamma}$ "$\mathrm{rk}(\underset{\sim}{\mathcal{I}}_\alpha) \cap S = \emptyset$", *for some* $\beta < \omega_2$, $\underset{\sim}{\mathcal{I}}_\beta = \underset{\sim}{\mathcal{I}}$ (and $\gamma \in (\beta, \omega_2)$ and $\Vdash_{P_\beta}$ "$\underset{\sim}{Q}_\beta = Q_{\mathrm{club}}(\omega_1 \setminus \mathrm{rk}(\underset{\sim}{\mathcal{I}})$").

(d) for every $\gamma < \omega_2$ and P_γ-name $\underset{\sim}{T}$ of an ω_1-tree for some $\beta < \omega_2$ we have $\Vdash_{P_{\beta_2}}$ "$\underset{\sim}{Q}_\beta = Q(\underset{\sim}{T}_\beta, S)$, $\underset{\sim}{T}_\beta$ is an Aronszajn tree, and if $\underset{\sim}{T}$ is an Aronszajn tree (in V^{P_β}) then $\underset{\sim}{T}_\beta = \underset{\sim}{T}$".

Now:

(i) P_α (and P_α / P_β for $\beta < \alpha$) is S-proper [Why? as in both cases in (b), $\underset{\sim}{Q}_\alpha$ is S-proper]

(ii) P_α is (T^*, S)-preserving [Why? By 4.7].

(iii) P_α does not collapse $\aleph_1$ [Why? By (ii) as $S \subseteq \omega_1$ is co-stationary.]

(iv) in V^{P_α} (if $\alpha > 0$) T^* is an Aronszajn tree [Why? Q_0 ensures it: if $h^* = \cup\{h : (\exists f)[(h,f) \in \underset{\sim}{G}_{Q_0}]\}$, $f^* = \cup\{f : (\exists h)[(h,f) \in \underset{\sim}{G}_{Q_0}]\}$, $\mathrm{Dom}(h^*) = \{\alpha(j) : j < \omega_1\}$ is increasing and continuous (by density) we have: f^* is a function with domain $\bigcup_{i<\omega_1} T^*_{h^*(\alpha(i))}$ and $x \in T^*_{h^*(\alpha(i))} \Rightarrow h(x) \in \alpha(i) \times \omega$,

$[x \in \mathrm{Dom}(f^*) \,\&\, y \in \mathrm{Dom}(f^*) \,\&\, x <_{T^*} y \Rightarrow f^*(x) \neq f^*(y)]$ and $i \in S \Rightarrow \alpha(i) = i$ on S. So by Fodor's lemma, T^* has no uncountable antichains).]

(v) We can define $\langle P_i, \underset{\sim}{Q}_j : i \leq \omega_2, j < \omega_2 \rangle$ to satisfy condition (a), (b), (c), (d). [The least trivial point is to ensure an instance of condition (d), given by the bookkeeping, which is fine as for Aronszajn tree T, $Q(T,S)$ is proper (by 4.4.) and (T^*, S)-preserving by (4.6). We succeed in having the bookkeeping as $2^{\aleph_1} = \aleph_2$.]

(vi) P_{ω_2} satisfies the $\aleph_2$-c.c. [Why? As in III, using "(T^*, S)-preservance" here similarly to the way we use "S-properness" there. Remember we have assume $V \models \diamondsuit_{\aleph_1}$ hence $2^{\aleph_0} = \aleph_1$.]

(vii) P_{ω_2} collapses no cardinal and changes no cofinality.

(viii) in $V^{P_{\omega_2}}$, T^* is S-st-special [Why? use Q_0].

(ix) in $V^{P_{\omega_2}}$, for every antichain $\mathcal{I}$ of T^*, $\mathrm{rk}(\mathcal{I}) \setminus S$ is non-stationary (remember: $\mathrm{rk}(\mathcal{I}) = \{\mathrm{rk}(x) : x \in \mathcal{I}\}$) [Why? by condition (c)]).

(x) in $V^{P_{\omega_2}}$, every ω_1-Aronszajn is S-st-special [Why? By clause (d) and the definition of $Q(T,S)$]

(xi) in $V^{P_{\omega_1}}$ there is no ω_1-Souslin tree provided that [Why? By 3.4(1) and clause (x) above when S is stationary or 4.4(2).]

(xii) P_α preserve stationarity of subsets of ω_1, moreover is proper. [Why? It is $(\omega_1 \setminus S)$-proper by its being (T^*, S)-preserving clause (ii), and it is S-proper by clause (i)]

Putting together (i)—(xii) we have clearly finished. $\square_{4.8}$

4.9 Concluding Remarks.

(1) We can ask: can we do it with G.C.H. and can we get independence of other variants of "every Aronszajn tree is non-Souslin, special, etc." but we have not tried. For G.C.H. it is natural to use a variant of the forcing used in V §6 for the consistency of G.C.H. + SH with ZFC.

(2) By the definition of the forcing $Q(T,S)$; and by 3.5(2) (applied to an almost subtree), in 4.8 we get that every Aronszajn tree is $S^\dagger$-special for some $S^\dagger$ (the range of the generic h). So for S empty, we get: every Aronszajn tree is $S^\dagger$-

special for some $S^\dagger$ (equivalently h-special for some $h : \omega_1 \to \omega_1$) but some tree is not $S^* - st$-special for any stationary $S^* \subseteq \omega_1$.

(3) Note that case (β) in 4.7, is needed for the part of conclusion of 4.8 saying: for no antichain $\mathcal{I} \subseteq T^*$ is $\mathrm{rk}(\mathcal{I}) \setminus S$ stationary (we are adding a closed unbounded subset of ω_1 disjoint to any such $\mathrm{rk}(\mathcal{I}) \setminus S$). Waving this we can omit (β) in 4.7.

(4) Abraham noted that "T is h-special for some h" is equivalent to "T is $S-r$-special for some closed unbounded $S \subseteq \omega_1$". Note that we can define $S - P$-special for every partial order P, and if $\alpha_i \in P (i < \omega_1)$ implies $(\exists i < j < \omega_1)$ $\alpha_i \leq \alpha_j$ then "T $S - P$-special" implies "T is not Souslin". Note also that "$S - r$-special for some closed unbounded S" implies $\omega_1 - \mathbb{R} \times \mathbb{Q}$-special [$\mathbb{R}$-reals, $\mathbb{Q}$-rationals, the order-lexicographic]. So we have proved, e.g., "every Aronszajn tree is $\omega_1 - \mathbb{R} \times \mathbb{Q}$ special" does not imply "every Aronszajn tree is special".

(5) We can also try to get a model of ZFC where, e.g,

- (A) (for some stationary co-stationary $S \subseteq \omega_1$) every Aronszajn tree is $S - st$-special, but some Aronszajn tree T^* is not h-special for any h; or
- (B) there is no Souslin tree but some Aronszajn tree is not h-special for any h.

For (A) it is natural to define $Q^\dagger(T,S) = \{(h,f) : (h,f) \in Q(T,S)$, $\mathrm{Dom}(f) \subseteq \bigcup_{h(\alpha)=\alpha} T_\alpha\}$. But T is the union of $\aleph_0$ disjoint copies of T^*, so $Q(T,S)$ cause "T^* is h-special for some h".

(6) We can generalize Definition 4.5. Let $\bar{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ be a sequence of countable models, the universe of M_α is γ_α, $\gamma_\alpha (\alpha < \omega_1)$ increasing continuous, and let $\varphi(x, y, U)$ be a quantifier free formula, where x, y are individual variables and U is a monadic predicate. We call a forcing notion $(\bar{M}, \varphi)$-preserving if for λ large enough and N a countable elementary submodel of $(H(\lambda), \in)$, $P \in N, p \in N$, there is $q \geq p$ which is (N, P)-generic and, letting $\delta = N \cap \omega_1$:

if $x \in M_{\delta+1}$ and $(\forall A \in N)(\exists y \in M_\delta)\varphi(x, y, A)$

then

$q \Vdash_P$ "$(\forall A \in N[G])(\exists y \in M_\delta)\varphi(x, y, A)$.

(7) Note that *if* $\alpha(i) < \omega_1$ is (strictly) increasing continuous in i, T is an ω_1-tree, h is a function, $\mathrm{Dom}(h) = \bigcup_{i<\omega} T_{\alpha(i+1)}$, $[x \in T_{\alpha(i+1)} \Rightarrow h(x) < \alpha(i) \times \omega]$ and $h(x) = h(y) \Rightarrow \neg(x <_T y)$, *then* there is $h^* : \bigcup_{i<\omega_1} T_{\alpha(i+1)} \to \mathbb{Q}$ such that $(*)$ $x, y \in \mathrm{Dom}(h^*) \,\&\, x <_T y \Rightarrow h^*(x) \neq h^*(y)$ and even $(*)^+$ $x, y \in \mathrm{Dom}(h^*) \,\&\, x <_T y \Rightarrow h^*(x) < h^*(y)$.
The proof with $(*)$ is similar to the proof of 3.5(2). So to derive $(*)^+$ first prove with $(*)$ and then use 3.5(1).