Chapter VI

Other Quantifiers: An Overview

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Generalized quantifiers were introduced by Mostowski [1957] as a means of generating new logics. In the meantime, their study has greatly developed, so that today there are more quantifiers in the literature than there are abstract model theorists under the sun. In any logic $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^i)_{i\in I}$ one does not need to introduce specific formation rules for renaming and substitution; for, upon adding to the finite set of logical symbols of $\mathscr{L}_{\omega\omega}$ one new symbol for each Q^i, all sentences in $\mathscr{L}$ are obtainable by an induction procedure on strings of symbols, pretty much as in $\mathscr{L}_{\omega\omega}$. One can gödelize sentences and start studying the axiomatizability and decidability of theories in $\mathscr{L}$. One might even go as far as to write down the proof of a theorem in $\mathscr{L}$ and then have it published in some mathematical journal. For infinitary logics this all seems to be a bit more problematic.

There are several ways to introduce quantifiers. For instance, nonlinear prefixes of existentially and universally quantified variables may be regarded as quantifiers as is discussed in Section 1. Quantifiers are also used for transforming concepts such as isomorphism, well-order, cardinality, continuity, metric completeness, and the "almost all" notion into primitive logical notions such as $=$ (see Sections 2 and 3).

There is no reason why quantifiers introduced via the above definability criteria should also preserve the nice algebraic properties of $\mathscr{L}_{\omega\omega}$. Indeed, in many cases they do not. However, in a final section of this chapter we will briefly describe a novel approach to quantifiers, an approach that is based on the fact that every separable Robinson equivalence relation $\sim$ on structures is canonically representable as $\mathscr{L}$-equivalence, $\equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}_{\omega\omega}\{Q \mid \equiv_{\mathscr{L}(Q)}$ is coarser than $\sim\}$. In addition to this, $\mathscr{L}$ turns out to have compactness and interpolation: The open, interior quantifiers and their n-dimensional variants can be introduced in this way, starting from a suitable approximation of homeomorphism.

We do not aim at an encyclopedic coverage here. Rather, we only aim to present an anthology of the most significant facts and techniques in the variegated realm of quantifiers. In line with this, highly developed quantifiers or special topics are discussed in detail in Chapters IV, V, VII, and XV.

Throughout this chapter $\mathscr{L}^{\mathrm{mII}}$ will be taken to mean second-order logic with universal and existential quantifiers over *unary* relations. Moreover, we will also write $\mathscr{L}(Q^i)_{i\in I}$ instead of $\mathscr{L}_{\omega\omega}(Q^i)_{i\in I}$.

1. *Quantifiers from Partially Ordered Prefixes*

In this section we will present the logic $\mathscr{L}^{\circledast}$ with quantifiers which arise from nonlinear prefixes (see Section 1.1). The logic $\mathscr{L}(Q^{\mathrm{H}})$ with the smallest such quantifier often gives a full account of the whole $\mathscr{L}^{\circledast}$ (see Section 1.2). Further topics on $\mathscr{L}(Q^{\mathrm{H}})$ are discussed in Section 1.3.

1.1. Partially Ordered Quantifiers

Let φ be a first-order formula in prenex normal form. Each existentially quantified variable x in the prefix of φ only depends on the universally quantified variables which precede x. We can naturally consider formulas with nonlinearly ordered prefixes such as, for example,

$$(1)\qquad \left.\begin{matrix}\forall x, x' \; \exists y \\ \forall t \quad \exists z, z'\end{matrix}\right\} \psi(x, x', t, y, z, z'),$$

which is equivalent to $\exists f, g, g' \, \forall x, x', t \, \psi(x, x', t, f(x, x'), g(t), g'(t))$. Another example is Henkin's prefix (see also Chapter II):

$$(2)\qquad \left.\begin{matrix}\forall x \; \exists y \\ \forall x' \; \exists y'\end{matrix}\right\} \psi(x, x', y, y'), \quad \text{viz.,} \quad \exists f, f' \, \forall x, x' \, \psi(x, x', f(x), f'(x')).$$

The smallest logic which is closed under this prefix is $\mathscr{L}(Q^{\mathrm{H}})$, where $Q^{\mathrm{H}} = \{\langle A, R\rangle \mid R \subseteq A^4 \text{ and } R \supseteq f \times g \text{ for some } f, g\colon A \to A\}$ = Henkin's quantifier. Similarly, the prefix in (1) results in a quantifier Q which is given by

$$Q = \{\langle A, R\rangle \mid R \subseteq A^6 \text{ and } R \supseteq f \times g \text{ for some } f\colon A^2 \to A \text{ and } g\colon A \to A^2\}.$$

We will agree to say that the (variable binding) pattern of Q^{H} is $\{\langle 1, 1\rangle, \langle 1, 1\rangle\}$, and that the pattern of Q in the discussion above is $\{\langle 2, 1\rangle, \langle 1, 2\rangle\}$. More generally, we set

1.1.1 Definition. Let $\pi = \{\langle n_1, m_1\rangle, \ldots, \langle n_r, m_r\rangle\}$ be a sequence of pairs of natural numbers ≥ 1. Then the *partially ordered quantifier* Q_π *with pattern* π is given by

$$Q_\pi = \{\langle A, R\rangle \mid R \subseteq A^s \text{ and } R \supseteq f_1 \times \cdots \times f_r \text{ for some } f_1\colon A^{n_1} \to A^{m_1}, \ldots, f_r\colon A^{n_r} \to A^{m_r}\},$$

where $s = n_1 + m_1 + \cdots + n_r + m_r$. We will also say that Q_π has *r rows*.

Partially ordered quantifiers do express some genuine mathematical notion, namely, uniformization. As a matter of fact, the quantifier $\forall x \, \exists y Rxy$ expresses the fact that the binary relation R can be uniformized, just as the quantifier

$$\begin{Bmatrix} \forall x\ \exists y \\ \forall x'\ \exists y' \end{Bmatrix} S(x, x', y, y')$$

expresses the fact that the 4-ary relation S contains the product of two binary uniformizable relations. Similar considerations hold for every partially ordered quantifier.

The syntactical rules for forming formulas in $\mathscr{L}(Q)$, with $Q = Q_\pi$, are naturally obtained by generalizing the rules for $\mathscr{L}(Q^{\mathrm{H}})$. Thus, Q binds s distinct variables, and if we display Q as

$$(3)\qquad Q = \begin{Bmatrix} \forall x_1^1 \cdots x_{n_1}^1\ \exists y_1^1 \cdots y_{m_1}^1 \\ \cdots \quad \cdots \quad \cdots \\ \forall x_1^r \ldots x_{n_r}^r\ \exists y_1^r \ldots y_{m_r}^r \end{Bmatrix},$$

then we immediately obtain the semantics of Q. In this development, the existentially quantified variables in a row are thought of as only depending on the universally quantified variables in the same row. Let us denote by $\mathscr{L}^{\circledast}$ the smallest logic in which all partially ordered prefixes of the form (3) are allowed. If this is done, we then have:

1.1.2 Theorem. *For an arbitrary class K, if K is* PC *in $\mathscr{L}_{\omega\omega}$ then K is* EC *in $\mathscr{L}^{\circledast}$. Indeed, $K =$ $\mathrm{Mod}_{\mathscr{L}^{\circledast}}\,\psi$, for some ψ of the form $Q\chi$ where Q is a partially ordered quantifier as in* (3) *above, and $\chi \in \mathscr{L}_{\omega\omega}$ is quantifier-free.*

Proof. Upon replacing relations by their characteristic functions, $K = \mathrm{Mod}$ $\exists g_1 \ldots g_j \theta$, where θ is a first-order formula in prenex normal form. Using Skolem functions, θ becomes equivalent to $\exists f_1 \ldots f_n \forall x_1 \ldots x_m \alpha$, where α is quantifier-free. The terms in α can be safely assumed to have the form $f(y_1 \ldots y_k)$, where $y_1, \ldots, y_k$ are variable symbols, so that no function symbol occurs in the argument of f. Indeed, one might use the equivalence between, for example, $\forall y, z\beta(f(g(y, z), h(y)))$ and $\forall y, z, t, u[t = g(y, z) \wedge u = h(y) \rightarrow \beta(f(t, u))]$. By similarly adding new universally quantified variables, we can also assume, without loss of generality, that in the argument of any two different functions there are no common variables and also that the n variables occurring in the argument of each n-ary function are all distinct. We finally make sure that a function symbol does not occur in two different terms. Thus, we replace, for example, $\exists f \forall x, y, z\varphi(f(x, y), f(y, z))$ by writing $\exists f, g\ \forall x, y, t, z\{[x = t \wedge y = z \rightarrow f(x, y) = g(t, z)] \wedge [t = y \rightarrow \varphi(f(x, y), g(t, z))]\}$. Now, K is reduced to the desired form. ◻

1.2. The Relationship Between $\mathscr{L}^{\mathrm{mII}}$, $\mathscr{L}^{\circledast}$ and $\mathscr{L}(Q^{\mathrm{H}})$

Walkoe [1970] observed that if Q is any partially ordered quantifier such that $\mathscr{L}(Q) \not\equiv \mathscr{L}_{\omega\omega}$, then $\mathscr{L}(Q^{\mathrm{H}}) \leq \mathscr{L}(Q)$. Thus, Q^{H} is the weakest partially ordered quantifier. The two theorems of this subsection tell us to which extent Q^{H} alone can replace the denumerable set of all partially ordered quantifiers. We shall also investigate the relationship between partially ordered quantifiers and second-order logic. In this latter respect, Väänänen [1977c] proved that there is no generalized quantifier Q such that $\mathscr{L}(Q) \equiv$ full second-order logic.

1.2.1 Theorem. *$\mathscr{L}^{\circledast}$ is equivalent to $\mathscr{L}(Q^{\mathrm{H}})$ in first-order Peano arithmetic. That is, for every φ in $\mathscr{L}^{\circledast}$, there is a ψ in $\mathscr{L}(Q^{\mathrm{H}})$ having the same models as φ among the models of Peano arithmetic.*

Proof. By making repeated use of pairing functions (say, by using formula $\chi(x, y, z)$ in the language of Peano arithmetic, which defines a bijection from M^2 onto M in each model $\mathfrak{M}$ of Peano arithmetic), we can safely assume that every quantifier Q in φ has only one universally quantified variable and only one existentially quantified variable in each row. Moreover, it is no loss of generality to assume that Q has only two rows. As a matter of fact, we have the equivalence between

$$\begin{Bmatrix} \forall x_1 \ \exists z_1 \\ \cdots \ \ \cdots \\ \forall x_n \ \exists z_n \end{Bmatrix} \theta \quad \text{and} \quad \begin{Bmatrix} \forall x_1 \cdots x_n \ \exists z_1 \cdots z_n \\ \forall x'_1 \cdots x'_n \ \exists z'_1 \cdots z'_n \end{Bmatrix} \left[\theta \wedge \bigwedge_{i=1}^{n} (x_i = x'_i \rightarrow z_i = z'_i) \right].$$

We can now use pairing functions again to contract the latter prefix into Q^{H}. This concludes the proof of the theorem. $\square$

Remark. Theorem 1.2.1 can be generalized (without altering the proof) to any arbitrary first-order theory where a definable pairing function is available.

Recall the definitions of $\leq_{\mathrm{RPC}}$ and of the Δ-closure $\Delta\mathscr{L}$ of a logic $\mathscr{L}$ from Chapter II. Intuitive notions stemming from first-order logic might suggest that $\Delta\mathscr{L}^{\mathrm{mII}} = \mathscr{L}^{\mathrm{mII}}$. However, this is not the case. Indeed, recall that in the definition of $\leq_{\mathrm{RPC}}$, extra universes are allowed which, in settings where Löwenheim–Skolem fails, cannot be coded as extra relations on some given universe.

1.2.2 Theorem. $\Delta\mathscr{L}(Q^{\mathrm{H}}) = \Delta\mathscr{L}^{\circledast} = \Delta\mathscr{L}^{\mathrm{mII}}$.

The proof proceeds through the following two claims:

Claim 1. $\mathscr{L}^{\circledast} \leq_{\mathrm{RPC}} \mathscr{L}^{\mathrm{mII}}$.

Proof. It suffices to show that for every $\varphi \in \mathscr{L}^{\circledast}(\tau)$, Mod φ is in $\mathrm{RPC}_{\mathscr{L}^{\mathrm{mII}}}$. For the moment, assume that τ has just one sort s, and that only Q^{H} occurs in φ. Now, $\neg Q^{\mathrm{H}}$ asserts the nonexistence of functions, while $\mathscr{L}^{\mathrm{mII}}$ can only express the nonexistence of sets. To overcome this difficulty, we add a binary function symbol J to τ, and let the first-order sentence α assert that J maps the set of all pairs in s one-one onto a new sort s'. For X any set-variable of $\mathscr{L}^{\mathrm{mII}}$, let $\beta(X)$ assert that X represents via J (that is, $J^{-1}[X]$ is) a function: namely, $\beta(X)$ is $\forall x \, \exists! y \, \exists z (z \in X \wedge z = J(x, y))$. If X represents a function $\hat{X}$, then the fact that $\hat{X}$ maps x into y, for short $X(x) = y$, is simply expressed by the $\mathscr{L}^{\mathrm{mII}}$-formula

$J(x, y) \in X$. Now, let $\varphi' \in \mathscr{L}^{\mathrm{mII}}$ be obtained from φ via the following inductive procedure: $\psi' = \psi$ if ψ is atomic, $(\neg\psi)' = \neg(\psi')$, $(\psi \wedge \chi)' = \psi' \wedge \chi'$, $(\exists x\psi)' = \exists x(\psi')$. For the crucial Q^{H}-clause, where ψ is given by

$$\left\{\begin{matrix}\forall x & \exists y \\ \forall x' & \exists y'\end{matrix}\right\} \theta(x, x', y, y'), \quad \text{viz.,} \quad \exists g, g' \, \forall x, x' \, \theta(x, x', g(x), g'(x')),$$

we let ψ' be given by $\exists X, X'[\beta(X) \wedge \beta(X') \wedge \forall x, x', y, y'(y = X(x) \wedge y' = X'(x') \rightarrow \theta')]$. Clearly, the τ-reducts of the models of $\alpha \wedge \varphi'$ are exactly the models of φ so that Mod $\varphi \in \mathrm{RPC}_{\mathscr{L}^{\mathrm{mII}}}$ as required. If φ has many sorts, or if φ has a p.o. quantifier $Q \neq Q^{\mathrm{H}}$, then we proceed similarly, using maps $J_Q \colon A_s^{n+1} \rightarrow A_{s''}$ to code into subsets of a new sort s'' each n-ary function asserted to exist by Q.

Claim 2. $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(Q^{\mathrm{H}})$.

Proof. If $A \neq \varnothing$ and $\{\varnothing, A\} \subseteq S \subseteq P(A)$, where P denotes power set, then $S \neq P(A)$ iff $\exists f \colon A \rightarrow \{0, 1\}$ such that $\forall r \in S$, $r \neq f^{-1}(1)$; that is to say, iff $\exists f \colon A \rightarrow \{0, 1\}$ and $\exists g \colon S \rightarrow A$ such that $\forall r \in S$, $\forall x \in A$ $[x = g(r) \rightarrow (g(r) \in r \leftrightarrow f(x) = 0)]$. Using relativized Q^{H} we can equivalently say the following:

$$(1) \qquad \left\{\begin{matrix}\forall x \in A & \exists y \in \{0, 1\} \\ \forall r \in S & \exists t \in A\end{matrix}\right\} \quad t = x \rightarrow (t \in r \leftrightarrow y = 0).$$

Now, to prove our claim, it is enough to show that for every $\varphi \in \mathscr{L}^{\mathrm{mII}}(\tau)$, Mod $\varphi \in \mathrm{RPC}_{\mathscr{L}(Q^{\mathrm{H}})}$. To this purpose, add to τ new unary relations A and S, as well as one binary relation E and the constants 0 and 1. Let the roles of S, A, 0, 1, E be described by sentence α which is given by the conjunction of the following formulas: $\forall x((Sx \vee Ax) \wedge \neg(Sx \wedge Ax))$, $S0 \wedge S1$, $\forall x(Ax \rightarrow Ex1)$, $\neg\exists x(Ax \wedge Ex0)$, $\forall^S r$, $\forall^S r'$ $[r = r' \leftrightarrow \forall^A x(Exr \leftrightarrow Exr')]$, *where* $\forall^Z x\theta$ as usual means $\forall x(Zx \rightarrow \theta)$. Let β be a reformulation of (1) without relativizations, that is,

$$\left\{\begin{matrix}\forall x & \exists y \\ \forall r & \exists t\end{matrix}\right\} \{Ax \wedge Sr \rightarrow [At \wedge (y = 0 \vee y = 1) \wedge (t = x \rightarrow (Etr \leftrightarrow y = 0))]\}.$$

Let φ' be obtained from φ by relativizing to A (that is, to $\{x \mid Ax\}$) each quantified individual variable in φ, and by relativizing to S each quantified set variable in φ (we can add more A's and S's if more sorts occur in φ), and finally by replacing $y \in X$ throughout by Eyx_X, where x_X is an individual variable. By the above discussion, the τ-reducts of models of $\alpha \wedge \neg\beta \wedge \varphi'$, upon restriction to $\{x \mid Ax\}$, are exactly the models of φ. As a matter of fact, $\alpha \wedge \neg\beta$ ensures that in our transcription of second-order variables as variables ranging over S we are missing no subset of A. Thus, we have proved that $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(Q^{\mathrm{H}})$. Those who do care to relativize classes may add one more sort s'' as well as a function symbol f and assert that f is an isomorphic embedding of the structure on sort s'' onto the restriction to $\{x \mid Ax\}$ of τ-reducts of models of $\alpha \wedge \neg\beta \wedge \varphi'$. $\square$

1.2.3 Corollary. *$\mathscr{L}(Q^{\mathrm{H}})$, $\mathscr{L}^{\circledast}$ and $\mathscr{L}^{\mathrm{mII}}$ have the same Löwenheim and the same Hanf numbers. Moreover, they have recursively isomorphic sets of valid sentences.*

Proof. The proof of this result is routine as it follows from standard facts of abstract model theory and from an easy inspection of the above proof (see also Proposition XVII.4.4.2(i)). □

1.2.4 Remark. The gödelized set V^{mII} of valid sentences in $\mathscr{L}^{\mathrm{mII}}$ is not definable in n-th order arithmetic. Indeed, it is not a Σ_n^m subset of the natural numbers, for any $n, m \in \omega$ (see Montague [1965] and also Tharp [1973]). For the Hanf number of $\mathscr{L}^{\mathrm{mII}}$ see Barwise [1972b] and Väänänen [1979b]. The reader should also consult Theorem 2.1.5(i) of the present chapter for more on this notion.

1.3. Further Topics on $\mathscr{L}(Q^{\mathrm{H}})$

In the light of Theorem 1.2.2 the implicit expressive power of $\mathscr{L}(Q^{\mathrm{H}})$ is very strong (see also Theorem 2.1.1 and Proposition 2.1.3). Concerning the explicit expressive power of $\mathscr{L}(Q^{\mathrm{H}})$, we first observe that $\mathscr{L}(Q^{\mathrm{H}}) \geq \mathscr{L}(Q_0)$. Indeed,

$$\begin{aligned} Q_0 x\varphi(x) \quad \text{iff} \quad & \exists t\{\varphi(t) \wedge \exists f, g\, \forall u, v[(u = v \leftrightarrow f(u) = g(v)) \\ & \wedge (\varphi(u) \rightarrow \varphi(f(u)) \wedge f(u) \neq t)]\}. \end{aligned}$$

1.3.1 Proposition. *$\mathscr{L}(Q^{\mathrm{H}})$ is neither (ω, ω)-compact nor axiomatizable, nor does it have the weak Beth property.*

Proof. There is a sentence of $\mathscr{L}(Q^{\mathrm{H}})$ characterizing up to isomorphism the standard model of arithmetic, since Q_0 is EC in $\mathscr{L}(Q^{\mathrm{H}})$. Thus, $\mathscr{L}(Q^{\mathrm{H}})$ cannot be countably compact and, using Gödel's incompleteness theorem, $\mathscr{L}(Q^{\mathrm{H}})$ is not axiomatizable. Failure of the weak Beth property is now a particular case of a result in abstract model theory which holds for every finitely generated logic in which the class $\{\mathfrak{A} \mid \mathfrak{A} \cong \langle\omega, <\rangle\}$ is EC (see, for example, Makowsky–Shelah [1979b, Theorem 6.1], or Theorems XVII.4.1.1 and 4.2.9). □

1.3.2 Theorem. $\mathscr{L}(Q^{\mathrm{H}}) \geq \mathscr{L}(Q_\alpha)$ *iff* $\alpha = 0$.

Proof. We must prove only the ($\Rightarrow$)-direction. To this purpose, it suffices to show, by induction on the complexity of formulas, that for each formula φ in the pure identity language of $\mathscr{L}(Q^{\mathrm{H}})$—that is to say, only the equality ($=$) occurs in φ—there is a formula $\hat{\varphi}$ in the pure identity language of $\mathscr{L}_{\omega\omega}$ equivalent to φ upon restriction to infinite sets (that is, $\kappa \models_{\mathscr{L}(Q^{\mathrm{H}})} \varphi \leftrightarrow \hat{\varphi}$, for each $\kappa \geq \omega$). The only nontrivial step in the proof arises in the case where φ has the form $Q^{\mathrm{H}}\psi$. In this case, one then uses upward and downward Löwenheim–Skolem methods for $\mathscr{L}_{\omega\omega}$ to establish that φ does not distinguish between infinite sets. By contrast, for $\alpha > 0$, Q_α does distinguish between infinite sets. □

1.4. Bibliographical Notes

Henkin's quantifier was introduced in Henkin [1961], while Ehrenfeucht proved that $\mathscr{L}(Q^{\mathrm{H}})$ is neither countably compact, nor axiomatizable (see Henkin [1961]). Theorems 1.1.2 and 1.2.1 are proved in Enderton [1970] and in Walkoe [1970]. The proof of Theorem 1.3.2 given above is due to Lopez-Escobar [1969], who also proved the failure of interpolation. Paulos [1976] proved that both Δ-closure and Beth property fail for $\mathscr{L}(Q^{\mathrm{H}})$. Failure of the weak Beth property is proved in Gostanian–Hrbacek [1976] who used general ideas from Craig [1965]. The reader should also consult Kreisel [1967], Mostowski [1968] and Lindström [1969] for more in this connection. Back-and-forth games for $\mathscr{L}(Q^{\mathrm{H}})$-equivalence are used by Krynicki [1977b] in connection with Theorem 1.3.2. Here the reader should also see Krawczyk–Krynicki [1976] and Weese [1980]. Theorem 1.2.2 is proved in Krynicki [1978] and Krynicki–Lachlan [1979]. In the latter paper, the reader can also find decidability (undecidability) results on $\mathscr{L}(Q^{\mathrm{H}})$. Partially ordered quantifiers are used in Barwise [1976] to find nice first-order axiomatizations for certain classes of structures such as, for example, the class of structures having a nontrivial automorphism f such that $f^2 =$ identity. In Walkoe [1970, 1976] and in Keisler–Walkoe [1973] partially ordered quantifiers are used to prove the following result about *ordinary* model theory: Let Q' and Q'' be first-order prefixes, with $Q' \neq Q''$ and Q' and Q'' having the same length. Then, for some quantifier-free formula φ in $\mathscr{L}_{\omega\omega}$, there is no quantifier-free formula ψ in $\mathscr{L}_{\omega\omega}$ such that $Q'\varphi$ is equivalent to $Q''\psi$. See Harel [1979], Cowles [1981], and Barwise [1979] for further information about Q^{H}.

2. Quantifiers for Comparing Structures

The quantifiers presented in this section express the fact that two structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic: In Section 2.1 both $\mathfrak{A}$ and $\mathfrak{B}$ are sets, and in Section 2.2 we add one binary relation; while in Section 2.3 we keep $\mathfrak{A}$ fixed.

2.1. Equicardinality Quantifiers

Recall that Härtig's quantifier I is defined by $I = \{\langle A, R, S\rangle \,|\, |R| = |S|\}$, so that $Ixy\varphi(x), \psi(y)$ says that $|\{x \,|\, \varphi(x)\}| = |\{y \,|\, \psi(y)\}|$. Rescher's quantifier Q^{R} is given by $Q^{\mathrm{R}} = \{\langle A, R, S\rangle \,|\, |R| < |S|\}$. Chang's quantifier Q^{C} binds only one variable and $Q^{\mathrm{C}}x\varphi(x)$ says that $Ixy\varphi(x), (y = y)$. Clearly, $\mathscr{L}(Q^{\mathrm{C}}) \leq \mathscr{L}(I)$. Also observe that $\mathscr{L}(Q_0) \leq \mathscr{L}(I)$. As a matter of fact, we have

(1) $\qquad Q_0 x\, \varphi(x) \quad \text{iff} \quad \exists z[\varphi(z) \wedge Ixy\, \varphi(x), \varphi(y) \wedge y \neq z].$

These points clear, we can now consider

2.1.1 Theorem. $\mathscr{L}(Q_0) < \mathscr{L}(I) < \mathscr{L}(Q^R) < \mathscr{L}(Q^H)$.

Proof. For the proof that $\mathscr{L}(Q_0) \leq \mathscr{L}(I)$ holds we only need give due regard to (1) above. It is trivially true that $\mathscr{L}(I) \leq \mathscr{L}(Q^R)$. Also, $\mathscr{L}(Q^R) \leq \mathscr{L}(Q^H)$ holds, for we have

$$(Ixy\, \varphi(x), \psi(y)) \vee (Q^R xy\, \varphi(x), \psi(y)) \quad \text{iff}$$
$$\exists f, f'\, \forall x, x' \{(x = x' \leftrightarrow f(x) = f'(x')) \wedge [\varphi(x) \rightarrow \psi(f(x))]\}.$$

$\mathscr{L}(Q_0)$ is not equivalent to $\mathscr{L}(I)$, since the former—as a sublogic of $\mathscr{L}_{\infty\omega}$—has the Karp property, while the second logic does not. (*Proof*: The two-cardinal structures $\langle \omega_1, U \rangle$ and $\langle \omega_1, V \rangle$ with $|U| = \omega$ and $|V| = |\omega_1 \sim V| = \omega_1$ are partially isomorphic, but not $\mathscr{L}(Q^C)$-equivalent, and hence not $\mathscr{L}(I)$-equivalent). The fact that $\mathscr{L}(I)$ is not equivalent to $\mathscr{L}(Q^R)$ has been proved by Hauschild [1981] (In this connection, the reader should also see Weese [1981b]). The fact that $\mathscr{L}(Q^R)$ is not equivalent to $\mathscr{L}(Q^H)$ has been proven by Cowles [1981]. $\square$

The following sentence of $\mathscr{L}(I)$ characterizes $\langle \omega, < \rangle$ up to isomorphism:

$\forall x \neg Iuv(u < x), (v \leq x) \wedge$ "$<$ is a discrete linear order with first element".

Using the above sentence, we immediately obtain

2.1.2 Proposition. *$\mathscr{L}(I)$ and $\mathscr{L}(Q^R)$ are neither (ω, ω)-compact nor axiomatizable nor do they satisfy the weak Beth property.*

Proof. The proof is the same as that given for Proposition 1.3.1. $\square$

As for the implicit expressive power of $\mathscr{L}(I)$ we have

2.1.3 Proposition. *The following are* RPC *in $\mathscr{L}(I)$ and, hence in $\mathscr{L}(Q^R)$ also:*

(i) *the class of well-ordered structures;*
(ii) *the class of well-ordered structures which are isomorphic to some cardinal;*
(iii) *the class of well-founded structures;*
(iv) *the class $\{\langle A, E \rangle \mid \langle A, E \rangle \cong \langle L(\alpha), \in \rangle$, for some ordinal $\alpha\}$;*
(v) *the class $\{\langle A, E \rangle \mid \langle A, E \rangle \cong \langle L(\kappa), \in \rangle$, for some cardinal $\kappa\}$.*

Proof. (i), we note that $<$ well-orders its universe of sort s iff there is an additional sort s' and a binary relation Rxx', where $x \in s$ and $x' \in s'$, such that the function $f(x) = |\{x' \mid Rxx'\}|$ is strictly increasing; that is to say, we have formally that

$$x < y \rightarrow [(Rxx' \rightarrow Ryx') \wedge \neg Iu'v' Rxu', Ryv'].$$

To prove (ii), we add the clause that $\forall z \neg Ixy(x = x), (y < z)$ to the above sentence. The proof of (iii) is the same as for (i). To prove (iv), we use Mostowski's collapsing lemma and standard results on constructible sets to exhibit a finite subtheory of ZF + $V = L$ whose well-founded models are exactly those that are isomorphic to $\langle L(\alpha), \in\rangle$, for some $\alpha \in \mathrm{On}$. Now, recall that well-foundedness is RPC in $\mathscr{L}(I)$, by (iii). To prove (v), we use (iv) and (ii). $\square$

2.1.4 Theorem. ($V = L$). *$\Delta\mathscr{L}(I) \equiv \Delta\mathscr{L}(Q^{\mathrm{R}}) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$. $\mathscr{L}(I)$, $\mathscr{L}(Q^{\mathrm{R}})$ and $\mathscr{L}^{\mathrm{mII}}$ have the same Löwenheim number and the same Hanf number. Moreover, they have recursively isomorphic sets of valid sentences.*

Proof. That $\mathscr{L}(I) \leq_{\mathrm{RPC}} \mathscr{L}(Q^{\mathrm{R}})$ is trivially true. The fact that $\mathscr{L}(Q^{\mathrm{R}}) \leq_{\mathrm{RPC}} \mathscr{L}^{\mathrm{mII}}$ is proven by use of pairing functions, as in Claim 1 of Theorem 1.2.2. We must now show that $V = L$ implies that $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(I)$. Let α be a sentence of $\mathscr{L}(I)$ of type τ whose E-reducts are exactly the structures that are isomorphic to $\langle L(\kappa), \in\rangle$, for some cardinal κ, as in Proposition 2.1.3(v) (E is meant as membership). Expand τ by adding a function symbol f, and let sentence β assert that "f is increasing and maps the ordinals one–one onto the infinite cardinals." By 2.1.3(i)(ii), ordinals and cardinals are true (up to isomorphism) ordinals and cardinals, so that f is isomorphic to the aleph function and κ is a fixed point, $\omega_\kappa = \kappa$. Hence, using GCH—a consequence of $V = L$—we have that $\kappa = \beth_\kappa$. Now add two constants c and p and let sentence θ assert that "c is a cardinal and p is the power set of c". In every model of $\alpha \wedge \beta \wedge \theta$, c is indeed isomorphic to a cardinal, and p is isomorphic to the set of constructible subsets of c (use, for example, Theorem 7.4.3(vii) in Chang, Keisler [1977], to the effect that, since $\kappa = \beth_\kappa$, then $L(\kappa) = R(\kappa) \cap L$; recall also that $c < \kappa$). Now, given $\varphi \in \mathscr{L}^{\mathrm{mII}}$ of type τ_φ, we construct $\varphi' \in \mathscr{L}(I)$ as is done in Claim 2 of Theorem 1.2.2 by relativizing each quantified individual variable to $\{x \mid x < c\}$, i.e. to $\{x \mid Exc\}$, and relativizing each set variable to $\{r \mid Erp\}$, and using E instead of $\in$. By $V = L$, p is the power set of c, so that the τ_φ-reducts of models of $\varphi' \wedge \alpha \wedge \beta \wedge \theta$, upon restriction to $\{x \mid Exc\}$ are exactly the models of φ. Whence we have that $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(I)$. The proof of the theorem is completed by using standard tools. $\square$

Remark. Thus, we see that under the assumption that $V = L$, the gödelized set V_I of valid sentences of $\mathscr{L}(I)$ is not a Σ^m_n subset of ω, $\forall n, m \in \omega$ (see Remark 1.2.4). As was remarked by Väänänen [1980b, p. 198], $\Delta\mathscr{L}(I) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$ continues to hold if $V = L$ is weakened to $V = L[0^\#]$, or even to $V = L^\mu$.

2.1.5 Theorem. (i) *If λ is the smallest inaccessible (hyperinaccessible, Mahlo, hyper-Mahlo) cardinal, then the Hanf number of $\mathscr{L}(I)$ is $> \lambda$.*

(ii) *The gödelized set V_I of valid sentences of $\mathscr{L}(I)$ is neither a Σ^1_2, nor a Π^1_2 subset of ω.*

(iii) *The fact that the Löwenheim number of $\mathscr{L}(I)$ is $< 2^\omega$ and V_I is a Δ^1_3 subset of ω is consistent, if* ZF *is consistent.*

(iv) *The fact that $\mathscr{L}(I)$ and $\Delta\mathscr{L}(I)$ have different Hanf numbers is consistent, if* ZF *is consistent.*

Proof. (i) Let φ be a sentence of $\mathscr{L}(I)$ of type $\tau = \{E, \ldots\}$ such that the E-reducts of the models of φ are the well-founded models of ZFC + "there are no inaccessible cardinals". The existence of φ then follows from Proposition 2.1.3 together with standard results from axiomatic set theory. Now, $\langle R(\lambda), E\rangle \models \varphi$, where λ is the first inaccessible cardinal and E means membership. Note that $|R(\lambda)| = \lambda$. We claim that for no $\mu > \lambda$, φ has a model of cardinality μ. Otherwise (*absurdum* hypothesis) let $\mathfrak{B} = \langle B, E, \ldots\rangle \models \varphi$ with $|B| = \mu$. By Mostowski's collapsing lemma, we have that $\mathfrak{B} \upharpoonright E$ is (isomorphic to) a transitive model of ZFC. Also, $\lambda \in B$ holds; for otherwise, by the assumed inaccessibility of λ, we would have $|B| < \mu$. For a suitable transitive well-founded (end) extension $\mathfrak{D}$ of $\mathfrak{B}$ we have that $\mathfrak{D} \models$ "λ is inaccessible". Now, "x is not inaccessible" is a Σ_1 predicate. Hence, we cannot have $\mathfrak{B} \models$ "λ is not inaccessible", by a familiar persistence argument. Thus, $\mathfrak{B} \models$ "λ is inaccessible and there are no inaccessibles"—a contradiction. In case λ is hyperinaccessible, *etc.*, the proof is the same, since we only need the fact that each of these properties is inherited by transitive submodels.

(ii) Assume that V_I is either Σ_2^1 or Π_2^1 (*absurdum* hypothesis). By Shoenfield's absoluteness lemma, V_I is an element v of, say $L(\omega_1)$. Let ψ be a sentence in $\mathscr{L}(I)$ of type $\tau = \{E, \ldots\}$ such that the E-reducts of the models of ψ are the sets $\langle L(\kappa), E\rangle$ as in Proposition 2.1.3(v). Let χ assert further that an uncountable ordinal is in the universe so that $\kappa > \omega_1$. Now, $x \in V_I$ holds true iff $\psi \wedge \chi \rightarrow x \in v$ holds true. Proceeding as in Tarski's diagonal argument, we now let $y \in W$ mean that y is the Gödel number of a formula $\beta(x)$ having one free variable such that $\beta(y)$ is false. By the above discussion, W is an element w of $L(\omega_1)$ and $x \in W$ holds true iff $\psi \wedge \chi \rightarrow x \in w$ holds true. Let z be the Gödel number of the formula $\theta(x)$ which asserts that "$\psi \wedge \chi \rightarrow x \in w$." Then $z \in W$ iff $z \notin W$. This is, of course, a contradiction.

(iii) This is proven in Väänänen [1980b, Corollary 3.2.3]. The reader should see Example XVII.2.4.3 and Proposition XVII.2.4.7 of this volume.

(iv) is proven in Väänänen [1983]. See also Theorem XVII.4.5.4 of the present volume. ☐

Let us end this subsection with a brief examination of Q^{C}. On finite structures, Q^{C} may be replaced by $\forall$. On structures of cardinality ω_α, Q^{C} behaves like Q_α: indeed many of the techniques used for the Q_α—notably for Q_1—apply equally well to Q^{C}, as is shown in detail in the textbook by Bell and Slomson [1969, Chapter 13]. These techniques are also extensively discussed in Chapters IV and V of this volume. We will thus limit ourselves to stating, without proof, the following results about Q^{C}.

2.1.6 Theorem. (i) *Let T be a countable set of sentences in $\mathscr{L}(Q^{\mathrm{C}})$ having a denumerable model. Then T has a model of every infinite cardinality.*

(ii) *Assume that all singular cardinals are strong limit. Then $\mathscr{L}(Q^{\mathrm{C}})$ is both axiomatizable and (ω, ω)-compact relative to infinite structures.*

(iii) *Assume* GCH, *then $\mathscr{L}(Q^{\mathrm{C}})$ is compact relative to infinite structures.* ☐

The logic $\mathscr{L}(Q^{\mathrm{C}})$ is not closed under relativization (and hence, it is not Δ-closed). Indeed, if $\mathscr{L}(Q^{\mathrm{C}})$ allowed relativization, then the relativization of $Q^{\mathrm{C}}x\,\varphi(x)$ to

$\{y \mid \psi(y)\}$ would be equivalent to $Ixy\, \varphi(x) \wedge \psi(x), \psi(y)$, and we could then characterize the standard model of arithmetic in $\mathscr{L}(Q^{\mathrm{C}})$ as is done in $\mathscr{L}(I)$ by using Section 2.1(1). Thus, we would contradict Theorem 2.1.6(i).

Evidently, $\mathscr{L}(Q^{\mathrm{C}})$ is not (ω, ω)-compact, for $Q_0 z(z = z)$ can be expressed as $\exists x Q^{\mathrm{C}} z(z \neq x)$. In the above theorem, compactness relative to infinite structures means that for every set T of sentences in $\mathscr{L}(Q^{\mathrm{C}})$, if each finite $T' \subseteq T$ has an infinite model (that is, a model whose universe is infinite), then T itself has an infinite model.

2.2. Similarity Quantifier and Its Variants

The quantifier I says that two *sets* are isomorphic; the *similarity* quantifier S says that two structures with a binary relation are isomorphic; that is,

$$\mathfrak{A} \models Sxyuv\, \varphi(x, y), \psi(u, v) \quad \text{iff} \quad \langle A, \varphi^{\mathfrak{A}} \rangle \cong \langle A, \psi^{\mathfrak{A}} \rangle,$$

where $\varphi^{\mathfrak{A}} = \{\langle a, b \rangle \in A^2 \mid \mathfrak{A} \models \varphi(a, b)\}$. Let α be given by

$$\forall m, n, p[m < n < p \rightarrow \neg Sxyx'y'(m < x < y < p), (n < x' < y' < p)].$$

Then a discrete linear ordering with first element is a model of α iff it is isomorphic to $\langle \omega, < \rangle$. By arguing as in Proposition 1.3.1, we see that $\mathscr{L}(S)$ is neither (ω, ω)-compact nor axiomatizable, nor does it have the weak Beth property.

Concerning the implicit expressive power of $\mathscr{L}(S)$, in Väänänen [1980a] it is proven that $\Delta\mathscr{L}(S) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$. The easy direction of this theorem uses pairing functions as in Claim 1 of Theorem 1.2.2. For the other direction, we first show that well-foundedness is RPC-definable in $\mathscr{L}(S)$. As a matter of fact, the quantifier I is clearly RPC in $\mathscr{L}(S)$. But I is also the complement of an RPC-class in $\mathscr{L}(S)$, since $\langle A, U, V \rangle \models \neg IxyUxVy$ iff the disjoint sum B of U and V satisfies $\langle B, U^2 \rangle \ncong \langle B, V^2 \rangle$. Therefore, I is EC in $\Delta\mathscr{L}(S)$ and, by using Proposition 2.1.3, well-foundedness is RPC in $\mathscr{L}(S)$, as was required. To conclude the proof that $\Delta\mathscr{L}(S) = \Delta\mathscr{L}^{\mathrm{mII}}$, we now try to express genuine power set in $\Delta\mathscr{L}(S)$, and, finally, argue as in Claim 2 of Theorem 1.2.2.

Thus, if we try to express isomorphism as a primitive logical notion, we may well attain the implicit expressive power of $\mathscr{L}^{\mathrm{mII}}$ by means of a single quantifier. Note here the analogy with the case of $\mathscr{L}(Q^{\mathrm{H}})$ in the framework of partially ordered quantification.

2.2.1 Variants of S. We can consider isomorphism between certain binary relations such as orderings or equivalence relations. Thus, we might define, say, S_{DLO} and S_{EQ} as follows:

$$\mathfrak{A} \models S_{\mathrm{DLO}} xyvw\, \varphi(x, y), \psi(v, w) \quad \text{iff} \quad \langle A, \varphi^{\mathfrak{A}} \rangle \cong \langle A, \psi^{\mathfrak{A}} \rangle,$$

and $\psi^{\mathfrak{A}}$ is a dense linear ordering over its field;

$$\mathfrak{A} \models S_{\mathrm{EQ}}xyvw\ \varphi(x, y), \psi(v, w) \quad \text{iff} \quad \langle A, \varphi^{\mathfrak{A}} \rangle \cong \langle A, \psi^{\mathfrak{A}} \rangle,$$

and $\psi^{\mathfrak{A}}$ is an equivalence relation over its field.

The list of such variants of the quantifier S is potentially infinite. However, we shall limit our attention to S_{DLO} and S_{EQ}. It is not difficult to see that $\Delta\mathscr{L}(S_{\mathrm{EQ}}) \geq \Delta\mathscr{L}(I)$ ($|A| = |B|$ iff the equivalence relation given by equality on A is isomorphic to equality on B), and that $\Delta\mathscr{L}(S_{\mathrm{EQ}}) \leq \Delta\mathscr{L}(I)$ (two equivalence relations $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic iff for every λ, $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of equivalence classes of power λ). Therefore, assuming $V = L$ we can apply Theorem 2.1.4 to the effect that $\Delta\mathscr{L}(S_{\mathrm{EQ}}) = \Delta\mathscr{L}^{\mathrm{mII}}$. Turning to S_{DLO}, we immediately see that $\mathscr{L}(Q_1) \leq \Delta\mathscr{L}(S_{\mathrm{DLO}})$ ($|A| \geq \omega_1$ iff there are two nonisomorphic dense linear orders without endpoints on A). It is also proven in Väänänen [1980a] that $\mathscr{L}(Q_0) \leq \Delta\mathscr{L}(S_{\mathrm{DLO}})$ and that $\mathscr{L}(S_{\mathrm{DLO}}) \leq \Delta\mathscr{L}(S_{\mathrm{EQ}})$ is an independent statement of ZF.

2.3. *The Quantifiers $Q^{I\mathfrak{A}}$ and $Q^{P\,I\mathfrak{A}}$*

For $\mathfrak{A}$ an arbitrary structure of finite relational type τ, let $Q^{I\mathfrak{A}}$ have as its defining class $I\mathfrak{A} = \{\mathfrak{B} \mid \mathfrak{B} \cong \mathfrak{A}\}$. Clearly, $\mathscr{L}(Q^{I\mathfrak{A}}) \equiv \mathscr{L}_{\omega\omega}$ iff $\mathfrak{A}$ is finite iff $\mathscr{L}(Q^{I\mathfrak{A}})$ is compact. Next, we will consider the denumerable case.

2.3.1 Theorem. *Let $\mathscr{L} = \mathscr{L}(Q^{I\mathfrak{A}})$, with $|A| = \omega$. Then $\mathscr{L}$ does not have the Craig property. Furthermore, $\mathscr{L}$ is (ω, ω)-compact iff there is a first-order sentence α with no finite models whose denumerable models are exactly the models in $I\mathfrak{A}$.*

Proof. The proof is by cases. We will begin with

Case 1. $\exists\alpha \in \mathscr{L}_{\omega\omega}$ whose denumerable models are exactly those in $I\mathfrak{A}$.
Then, let $\psi \in \mathscr{L}$ be defined by $\mathfrak{B} \models \psi$ iff $\mathfrak{B}$ has two sorts s and s' and f maps $\mathfrak{B} \upharpoonright s$ one-one into $\mathfrak{B} \upharpoonright s'$ and $\mathfrak{B} \upharpoonright s' \in I\mathfrak{A}$. Then we see that the class of countable sets is $\mathrm{RPC}_{\mathscr{L}}$. Now, let $\varphi \in \mathscr{L}$ be defined by $\mathfrak{D} \models \varphi$ iff $\mathfrak{D} \notin I\mathfrak{A}$ and $\mathfrak{D} \models \alpha$ and g maps D one-one into $D' \subsetneqq D$. We then see that the class of uncountable sets is $\mathrm{RPC}_{\mathscr{L}}$. Therefore, Q_1 is EC in $\Delta\mathscr{L}$ so that $\Delta\mathscr{L} \geq \Delta\mathscr{L}(Q_1)$. The proof for Case 1 can now be completed as follows:

Subcase 1.1. α may be assumed to have no finite models.
Then $\mathfrak{B} \in I\mathfrak{A}$ iff $\mathfrak{B} \models_{\mathscr{L}(Q_1)} \alpha \wedge \neg Q_1 x(x = x)$. Hence, $I\mathfrak{A} \in \mathrm{EC}_{\mathscr{L}(Q_1)}$. Whence $\mathscr{L} \leq \mathscr{L}(Q_1)$. By the above discussion, we have $\Delta\mathscr{L} \equiv \Delta\mathscr{L}(Q_1)$. This shows that $\mathscr{L}$ is (ω, ω)-compact (as is $\mathscr{L}(Q_1)$ and Δ-closure preserves compactness) and that $\mathscr{L}$ does not have the interpolation property ($\Delta\mathscr{L}(Q_1)$ does not, see Hutchinson [1976]).

Subcase 1.2. every α as in Case 1 has some finite model.
Then α need have arbitrarily large finite models; let $\theta \in \mathscr{L}$ be given by $\mathfrak{B} \models \theta$ iff $\mathfrak{B}$ has sorts s, s', s'' and f maps $\mathfrak{B} \upharpoonright s$ one-one into $\mathfrak{B} \upharpoonright s'$, g maps $\mathfrak{B} \upharpoonright s'$ one-one into

$\mathfrak{B} \upharpoonright s''$, $\mathfrak{B} \upharpoonright s'' \in I\mathfrak{A}$, $\mathfrak{B} \upharpoonright s' \notin I\mathfrak{A}$, $\mathfrak{B} \upharpoonright s' \models \alpha$; thus $\mathfrak{B} \upharpoonright s$ can be of every finite (but of no infinite) cardinality and $\neg Q_0$ is $\mathrm{RPC}_{\mathscr{L}}$; trivially Q_0 is $\mathrm{RPC}_{\mathscr{L}}$, so that $\Delta\mathscr{L} \geq \Delta\mathscr{L}(Q_0)$ and $\mathscr{L}$ cannot be (ω, ω)-compact (as $\Delta\mathscr{L}$ is not, and Δ-closure preserves compactness). Actually we can find a recursively enumerable (r.e.) set of $\mathscr{L}$-sentences which is a counterexample to compactness, i.e. $\mathscr{L}$ is not r.e. compact. Then $\mathscr{L}$ does not have the Beth property (hence interpolation fails for $\mathscr{L}$), by a well-known general fact in abstract model theory, to the effect that the Beth property implies r.e. compactness in every finitely generated logic (see, for example, Väänänen [1977b], or Makowsky-Shelah [1979b, Theorem 6.1], or Theorem XVII.4.2.9 of the present volume).

Case 2. $\neg\exists\alpha \in \mathscr{L}_{\omega\omega}$ whose denumerable models are exactly those in $I\mathfrak{A}$.

Subcase 2.1. $\exists\mathfrak{B}$ denumerable such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B} \not\cong \mathfrak{A}$.
Let $\{I_n\}_{n<\omega}$: $\mathfrak{A} \cong_\omega \mathfrak{B}$, as given by the Fraïssé–Ehrenfeucht characterization of $\equiv$ (see Chapter II.4.2). Rename the sorts and symbols of $\mathfrak{B}$. Let $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{B}, I_0, \omega, <, L, J, f \rangle$, where $L^{\mathfrak{M}}np$ iff $p \in I_n$ (for $p \in I_0$, $n \in \omega$), $J^{\mathfrak{M}}pab$ iff $p(a) = b$ (for $a \in A$, $b \in B$), $f^{\mathfrak{M}}$ maps A one–one onto B. Take a finite subtheory T of $\mathrm{Th}_{\mathscr{L}}\,\mathfrak{M}$ such that for every $\mathfrak{M}' \models T$, $\mathfrak{M}' = \langle \mathfrak{A}', \mathfrak{B}', I'_0, D', <', L', J', f' \rangle$, $\langle D', <' \rangle$ is still a discrete linear order with first element, L', J', still codes in $\mathfrak{M}'$ a D'-sequence of sets of partial isomorphisms with the back-and-forth property so that $\mathfrak{A}' \equiv \mathfrak{B}'$, f' maps A' one–one onto B', $\mathfrak{A}' \in I\mathfrak{A}$ and $\mathfrak{B}' \notin I\mathfrak{A}$. For details about T, see, for example, Flum [1975b, proof of Lindström's theorem]. If $\mathscr{L}$ is (ω, ω)-compact (*absurdum* hypothesis) then it would be consistent to assume that $\langle D', <' \rangle$ has an infinitely descending chain. Hence, $\mathfrak{A}' \cong_p \mathfrak{B}'$. Whence, $\mathfrak{A}' \cong \mathfrak{B}'$ by Karp's back-and-forth argument, since f' ensures that $\mathfrak{B}'$ is denumerable also. But, then, the basic isomorphism axiom for $\mathscr{L}$ implies that $\mathfrak{B}' \in I\mathfrak{A}$—a contradiction. We have thus actually proved that $\mathscr{L}$ is not r.e. compact. Hence, by the well-known general results quoted above (see Theorem XVII.4.2.9), $\mathscr{L}$ does not have the Beth (resp., Craig) property.

Subcase 2.2. $\neg\exists\mathfrak{B}$ denumerable such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B} \not\cong \mathfrak{A}$.
For $n = 1, 2, \ldots$, there are $\mathfrak{B}_n \not\cong \mathfrak{A}$, $|B_n| = \omega$, and $\{I_0, \ldots, I_n\}$ such that $\{I_0, \ldots, I_n\}: \mathfrak{A} \cong_n \mathfrak{B}_n$ (otherwise $\exists\alpha \in \mathscr{L}_{\omega\omega}$ whose denumerable models are exactly those in $I\mathfrak{A}$, by the Fraïssé–Ehrenfeucht characterization of $\equiv$, thus, we contradict our assumptions). So let $\mathfrak{M}_n = \langle \mathfrak{A}, \mathfrak{B}_n, I_0, \omega, <, L, J, f, s \rangle$ as in the above proof of Subcase 2.1, where s is the successor function. Let T_n be a finite theory such that for every $\mathfrak{M}'_n \models T_n$, L' and J' code a finite sequence of sets of partial isomorphisms $\{I'_0, \ldots, I'_{s\ldots s(0)}\}: \mathfrak{A}' \cong_n \mathfrak{B}'_n$, with $|B'_n| = |A|$, $\mathfrak{A}' \in I\mathfrak{A}$, $\mathfrak{B}'_n \notin I\mathfrak{A}$. Now, $T = \bigcup T_n$ is inconsistent, by the Fraïssé–Ehrenfeucht characterization of $\equiv$ as well as by our assumptions, and T yields a counterexample to the r.e. compactness of $\mathscr{L}$. Thus, the Beth property also must fail for $\mathscr{L}$, for we can argue as at the end of Case 1. The examination of Subcase 2.2 concludes the proof of our theorem. $\square$

For $\mathfrak{A}$ a structure of finite relational type τ, let $Q^{PI\mathfrak{A}}$ have as its defining class $PI\mathfrak{A} = \{\mathfrak{B} \mid \mathfrak{B} \cong_p \mathfrak{A}\} = \{\mathfrak{B} \mid \mathfrak{B} \equiv_{\mathscr{L}_{\infty\omega}} \mathfrak{A}\}$. Clearly, we have that $\mathscr{L}(Q^{PI\mathfrak{A}}) \equiv \mathscr{L}_{\omega\omega}$ iff $PI\mathfrak{A} \in \mathrm{EC}_{\mathscr{L}_{\omega\omega}}$.

2.3.2 Theorem. *Assume that $PI\mathfrak{A} \notin EC_{\mathscr{L}_{\omega\omega}}$, where $\mathfrak{A}$ need not be denumerable. Then, $\Delta\mathscr{L}(Q^{PI\mathfrak{A}}) \geq \Delta\mathscr{L}(Q_0)$. In particular, $\mathscr{L}(Q^{PI\mathfrak{A}})$ is not (ω, ω)-compact and does not have the Beth property. Moreover $\mathscr{L}(Q^{PI\mathfrak{A}})$ is not axiomatizable.*

Proof. The proof is by cases. We begin with

Case 1. $\exists\mathfrak{B}$ such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B} \not\cong_p \mathfrak{A}$.
Let $\{I_n\}_{n<\omega}: \mathfrak{A} \cong_\omega \mathfrak{B}$ and $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{B}, I_0, \omega, <, L, J \rangle$ with L and J coding $\{I_n\}_{n<\omega}$ as in the proof of Subcase 2.1 of Theorem 2.3.1. By a similar argument, we exhibit a finite subtheory T of $\mathrm{Th}_{\mathscr{L}}\,\mathfrak{M}$ from which a counterexample to r.e. compactness can be obtained. Hence, the Beth property fails also for $\mathscr{L} = \mathscr{L}(Q^{PI\mathfrak{A}})$. A closer examination of T shows that $\langle \omega, < \rangle$ is $\mathrm{RPC}_{\mathscr{L}}$; and, hence, $\mathscr{L}$ is not axiomatizable, by Gödel's incompleteness theorem.

Case 2. $\mathfrak{B} \equiv \mathfrak{A}$ implies $\mathfrak{B} \cong_p \mathfrak{A}$.
Then, for $n = 1, 2, \ldots$, there is a $\mathfrak{B}_n$ such that $\mathfrak{B}_n \cong_n \mathfrak{A}$, and $\mathfrak{B}_n \notin PI\mathfrak{A}$ (otherwise, $PI\mathfrak{A}$ would be EC in $\mathscr{L}_{\omega\omega}$). Now argue as in Subcase 2.2 of Theorem 2.3.1, to obtain a counterexample T to r.e. compactness and hence to the Beth property in $\mathscr{L}(Q^{PI\mathfrak{A}})$. Indeed, T is a recursive set of sentences so that, by a trick method which goes back to Craig and Vaught [1958], one can code T into a single sentence whose $<$-reducts are all isomorphic to $\langle \omega, < \rangle$. Thus, $\langle \omega, < \rangle$ is RPC in $\mathscr{L}(Q^{PI\mathfrak{A}})$, and the proof is concluded by arguing as in Case 1. □

Remarks. Barwise [1974a] proved that $\Delta\mathscr{L}(Q_0) = \mathscr{L}_{\omega^+}$, where $\omega^+ = \omega_1^{\mathrm{CK}}$ is the smallest admissible set to which ω belongs (see also XVII.3.2.2). More generally, for $U \subseteq \omega$, let $\langle \omega, U \rangle^+$ denote the smallest admissible set having ω and U as its elements: then we have

2.3.3 Theorem. $\Delta\mathscr{L}(Q^{PI\langle \omega, <, U \rangle}) \equiv \Delta\mathscr{L}(Q^{I\langle \omega, <, U \rangle}) \equiv \mathscr{L}_{\langle \omega, U \rangle^+}$.

Proof. The reader is referred to Makowsky–Shelah–Stavi [1976, Theorem 4.1]. See also Theorem XVII.3.2.3 of this volume. □

2.4. Bibliographical Notes

The quantifiers Q^{R} and I were introduced respectively by Rescher [1962] and Härtig [1965]. Failure of (ω, ω)-compactness and axiomatizability for $\mathscr{L}(I)$ was proven by Yasuhara [1969] and Issel [1969]. The latter author also proved that ω_ω is the Hanf and the Löwenheim number of the fragment of $\mathscr{L}(I)$ with equality and otherwise only unary relation symbols. Proposition 2.1.3(i) goes back to Lindström [1966a, p. 192]. For Theorem 2.1.4, see, for example, Väänänen [1978] and Pinus [1979b]. Lower bounds for the Hanf number of $\mathscr{L}(I)$ were also discussed by Fuhrken [1972] and Pinus [1978]. Further information on $\mathscr{L}(I)$ can be obtained from Väänänen's papers quoted in Section 2.1 as well as from Väänänen [1978, 1979b]. Named after C. C. Chang, the quantifier Q^{C} is studied in detail in Bell–Slomson [1969]; the fragment containing $=$ but otherwise only unary relations, was studied by Slomson [1968], who proved that ω is both its

Löwenheim and its Hanf number. He also proved the decidability of this fragment—a proof of the decidability of the corresponding fragments of $\mathscr{L}(Q^{\mathrm{H}})$ and $\mathscr{L}(I)$ can be found in Krynicki–Lachlan [1979]. An axiomatization of the fragment of $\mathscr{L}(Q^{\mathrm{C}})$—without equality—was given by Yasuhara [1966a]. The quantifiers S, S_{DLO}, S_{EQ} and their relativized versions are presented in Väänänen [1980a]. The quantifiers $Q^{I\mathfrak{A}}$ and $Q^{PI\mathfrak{A}}$ are studied in Makowsky–Shelah–Stavi [1976]. The reader should also see Makowsky [1973] for more in this connection.

3. Cardinality, Equivalence, Order Quantifiers and All That

In this section, we will consider quantifiers which assert that a structure has a certain property. In Section 3.1 we will study properties of sets and equivalence relations. In Section 3.2, we shall focus attention on linear orderings. Other cases are examined in Section 3.3.

3.1. Cardinality and Equivalence Quantifiers

Let Q have a class of sets as its defining class. By the isomorphism property, Q must express some property of cardinals. As a typical example, consider the quantifier Q_α which asserts that "there are at least ω_α-many elements", where α is an ordinal ≥ 0. The Q_α's are extensively studied in Chapters IV and V. The following result extends to quantifiers of the form $Qx_1 \ldots x_n \varphi_1(x_1), \ldots, \varphi_n(x_n)$.

3.1.1 Theorem. *Assume that each quantifier Q^i occurring in* (i) *through* (iii) *below is a class of sets. Furthermore:*

(i) *Let $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$. If $\mathscr{L}$ is (ω, ω)-compact and Δ-closed, then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*
(ii) *Let $\mathscr{L} = \mathscr{L}(Q^1, \ldots, Q^n)$. If $\mathscr{L}$ obeys interpolation, then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*
(iii) *For $\alpha \geq 1$ a fixed ordinal, let $\mathscr{L} = \mathscr{L}(Q_\alpha, Q^i)_{i\in I}$. Then $\mathscr{L}$ is not Δ-closed.*

For the proof of this result we need the following

3.1.2 Lemma. *For $\kappa, \lambda \geq \omega$, let $\mathfrak{A}^\kappa_\lambda = \langle A, E\rangle$, where E is an equivalence relation on A having λ equivalence classes, each of cardinality κ. Let $\mathscr{L}^0 = \mathscr{L}(Q^j)_{j\in J}$, where each Q^j is a class of sets. Then $\mathfrak{A}^\kappa_\kappa \equiv_{\mathscr{L}^0} \mathfrak{A}^\kappa_\omega$.*

Proof of Lemma 3.1.2. Let $\mathscr{L}_C = \mathscr{L}_{\infty\omega}(Q_\alpha)_{\alpha\in On}$. Then, $\mathfrak{A}^\kappa_\kappa \equiv_{\mathscr{L}_C} \mathfrak{A}^\kappa_\omega$, as was observed by Caicedo [1979, p. 93] with the help of a back-and-forth argument (this refines Keisler's proof that $\mathfrak{A}^{\omega_1}_\omega \equiv_{\mathscr{L}(Q_1)} \mathfrak{A}^{\omega_1}_{\omega_1}$; see II.4.2.8). We also have that $\mathscr{L}_C$-equivalence is finer than $\mathscr{L}^0$-equivalence, as was proven in the same paper by Caicedo ([1979, Lemma 4.2]). Also see Väänänen [1977c].

Proof of Theorem 3.1.1. (i) Assume that $\mathscr{L} \not\equiv \mathscr{L}_{\omega\omega}$. Then, by definition of $\mathscr{L}$, there is a sentence φ in the pure identity language of $\mathscr{L}$ which is not equivalent to any $\mathscr{L}_{\omega\omega}$-sentence. We now consider

Case 1. For some $\lambda > \omega$, λ and ω are separated by φ (say, $\omega \models_{\mathscr{L}} \varphi$ and $\lambda \not\models_{\mathscr{L}} \varphi$). Using a *choice function* from λ into $\mathfrak{A}^{\lambda}_{\lambda}$ (that is, a bijection from λ onto $\mathfrak{A}^{\lambda}_{\lambda}/E$) and a choice function from ω into $\mathfrak{A}^{\lambda}_{\omega}$, we see that $\mathfrak{A}^{\lambda}_{\lambda}$ and $\mathfrak{A}^{\lambda}_{\omega}$ belong to complementary RPC classes in $\mathscr{L}$. So, if we use Δ-closure, $\mathfrak{A}^{\lambda}_{\lambda}$ and $\mathfrak{A}^{\lambda}_{\omega}$ can be separated by some sentence in $\mathscr{L}$—thus contradicting Lemma 3.1.2.

Case 2. For every $\lambda > \omega$, $\omega \models_{\mathscr{L}} \varphi$ iff $\lambda \models_{\mathscr{L}} \varphi$ (say, $\omega \models_{\mathscr{L}} \varphi$).

Subcase 2.1. $\exists n < \omega$ such that φ has no model of cardinality $> n$.
Then, without loss of generality, φ has no finite models, so that $\mathscr{L} \geq \mathscr{L}(Q_0)$, and $\mathscr{L}$ is not even r.e. compact.

Subcase 2.2. Both φ and $\neg\varphi$ have arbitrarily large finite models.
Then, the theory whose sentences are $\neg\varphi$, $\exists^{\geq 1}x(x = x)$, $\exists^{\geq 2}x(x = x), \ldots$ is a counterexample to r.e. compactness.

Subcase 2.3. φ has arbitrarily large finite models, but $\neg\varphi$ does not.
Then φ is first-order, contradicting our assumption.

(ii) By inspection of the proof of (i), we see that r.e. compactness and Δ-closure are actually sufficient to imply that $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$. But, if $\mathscr{L}$ obeys interpolation, then $\mathscr{L}$ has both Beth and Δ-closure. Hence, $\mathscr{L}$ is r.e. compact, since $\mathscr{L}$ is finitely generated by assumption.

(iii) Using choice functions, we see that $\mathfrak{A}^{\omega_\alpha}_{\omega_\alpha}$ and $\mathfrak{A}^{\omega_\alpha}_{\omega}$ belong to complementary RPC classes of $\mathscr{L}(Q_\alpha)$, and hence of $\mathscr{L}$ also. If $\mathscr{L}$ were Δ-closed, then some sentence in $\mathscr{L}$ would separate these two structures, thus contradicting Lemma 3.1.2. This completes the proof of the theorem. $\square$

Let $X = \{\langle A, E\rangle \mid E \text{ is an equivalence relation on } A\}$. Then Q is an *equivalence quantifier* iff its defining class is a subclass of X.

3.1.3 Theorem. *Let $\mathscr{L}$ be a compact logic with the interpolation and the Feferman–Vaught property* (FVP). *Let Q be an equivalence quantifier which is* EC *in $\mathscr{L}$. Then Q is* EC *in $\mathscr{L}_{\omega\omega}$.*

Proof. We pose a denial, and let K be a class of equivalence relations which is EC in $\mathscr{L}$ but not in $\mathscr{L}_{\omega\omega}$. Then K must separate two elementarily equivalent structures $\mathfrak{A} = \langle A, E\rangle$ and $\mathfrak{A}' = \langle A', E'\rangle$ (say, $\mathfrak{A} \in K$ and $\mathfrak{A}' \notin K$) by a familiar open cover argument using the compactness of $\mathscr{L}$ (for a similar argument see, for example, Theorem III.1.1.5). We now proceed by cases:

Case 1. Each equivalence class of $\mathfrak{A}$ and $\mathfrak{A}'$ has infinitely many elements.
Thus, let N and M be infinite sets such that $|N| = \omega$, $|M| > |A \cup A'|$, $N \equiv_{\mathscr{L}} M$. Such sets N and M clearly exist by the assumed compactness of $\mathscr{L}$. By FVP, we

have that $[\mathfrak{A}, \mathfrak{A}', N] \equiv_{\mathscr{L}} [\mathfrak{A}, \mathfrak{A}', M]$ (as three-sorted structures). By adding two functions f and f', we can expand $[\mathfrak{A}, \mathfrak{A}', M]$ to a model of the sentence φ which asserts that "f and f' are injections of A and A' respectively into the third sort s_3". On the other hand, $[\mathfrak{A}, \mathfrak{A}', N]$ can be expanded to a model of the conjunction ψ of the sentence asserting that "sort s_3 is injected by h and h' into each equivalence class of $\mathfrak{A}$ and $\mathfrak{A}'$, respectively" (where h is, for example, a binary function $h(x, z)$, x in the first sort, $z \in s_3$, and $h(x, \cdot)$ maps s_3 one–one into the equivalence class of x in $\mathfrak{A}$) and of the sentence which asserts that "either g_0 is a bijection of $\mathfrak{A}/E$ onto $\mathfrak{A}'/E'$, or g and g' are injections of s_3 into $\mathfrak{A}/E$ and $\mathfrak{A}'/E'$, respectively". Since $\mathscr{L}$ has compactness and interpolation, then $\mathscr{L}$ satisfies Robinson's consistency, to the effect that $\varphi \wedge \psi$ has a model $[\mathfrak{B}, \mathfrak{B}', P, \ldots]$ which is also a model of $\mathrm{Th}_{\mathscr{L}}[\mathfrak{A}, \mathfrak{A}', N]$ (that is, a model of $\mathrm{Th}_{\mathscr{L}}[\mathfrak{A}, \mathfrak{A}', M]$). In this model, we have that $\mathfrak{B} \cong \mathfrak{B}'$ by the Cantor–Bernstein theorem, and $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}$, $\mathfrak{B}' \equiv_{\mathscr{L}} \mathfrak{A}'$, thus contradicting the isomorphism axiom for $\mathscr{L}$, since K separates $\mathfrak{A}$ and $\mathfrak{A}'$.

Case 2. Each equivalence class of $\mathfrak{A}$ and $\mathfrak{A}'$ has finitely many elements.
Then let $n = 1, 2, \ldots$. Let κ_n, κ_n' be such that in $\mathfrak{A}$ there are κ_n equivalence classes with n elements and in $\mathfrak{A}'$ there are κ_n' such classes. If κ_n is finite, then $\kappa_n = \kappa_n'$ (since $\mathfrak{A} \equiv \mathfrak{A}'$). If κ_n is infinite, then ω can be injected into $\mathfrak{A}/E$ and into $\mathfrak{A}'/E'$. Let $[\mathfrak{A}, \mathfrak{A}', N] \equiv_{\mathscr{L}} [\mathfrak{A}, \mathfrak{A}', M]$ be as above. Then $[\mathfrak{A}, \mathfrak{A}', N]$ can be expanded to a model of the sentence asserting that "f is an injection showing that there are more ($\geq$) than $|N|$ equivalence classes with n elements in $\mathfrak{A}$, and f' does the same for $\mathfrak{A}'$, or else g_0 is a bijection showing that such classes are as many in $\mathfrak{A}$ as in $\mathfrak{A}'$". On the other hand, $[\mathfrak{A}, \mathfrak{A}', M]$ can be expanded to a model of the sentence which asserts that "h and h' show that there are less ($\leq$) than $|M|$ equivalence classes in $\mathfrak{A}$ and $\mathfrak{A}'$ with n elements, or else g is a bijection showing that such classes are as many in $\mathfrak{A}$ as in $\mathfrak{A}'$". Using Robinson's theorem as was done in Case 1, we exhibit a model $[\mathfrak{B}, \mathfrak{B}', P, \ldots]$ of all these sentences together, and of $\mathrm{Th}_{\mathscr{L}}[\mathfrak{A}, \mathfrak{A}', N]$ as well so that $\mathfrak{B} \cong \mathfrak{B}'$ (since $\kappa_n = \kappa_n'$ for all $n \in \omega$), $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ and $\mathfrak{A}' \equiv_{\mathscr{L}} \mathfrak{B}'$, again contradicting $\mathfrak{A} \in K$ and $\mathfrak{A}' \notin K$.

Case 3. Neither Case 1, nor Case 2 occurs.
Then let $\mathfrak{A}_1$ be the substructure of $\mathfrak{A}$ only containing the equivalence classes having infinitely many elements, and let $\mathfrak{A}_2$ be the substructure of $\mathfrak{A}$ containing the equivalence classes with finitely many elements. Let $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ be similarly defined with regard to $\mathfrak{A}'$. Then $\mathfrak{A}_1 \equiv \mathfrak{A}_1'$, and $\mathfrak{A}_2 \equiv \mathfrak{A}_2'$ (by using standard results of first-order model theory, as $\mathfrak{A} \equiv \mathfrak{A}'$); so, by the arguments given for Cases 1 and 2, we see that $\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_1'$ and $\mathfrak{A}_2 \equiv_{\mathscr{L}} \mathfrak{A}_2'$. By FVP, we have that $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv_{\mathscr{L}} [\mathfrak{A}_1', \mathfrak{A}_2']$. Now consider structure $\mathfrak{M} = [\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}, f, g]$, where f and g are the canonical embeddings of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively into $\mathfrak{A}$. Let $\mathfrak{M}'$ be similarly defined, using new symbols for f', g' and $\mathfrak{A}'$. If $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{A}'$ (*absurdum* hypothesis), then $[\mathfrak{A}_1, \mathfrak{A}_2]$ and $[\mathfrak{A}_1', \mathfrak{A}_2']$ have expansions $\mathfrak{M}$ and $\mathfrak{M}'$ with $\mathrm{Th}_{\mathscr{L}} \mathfrak{M} \cup \mathrm{Th}_{\mathscr{L}} \mathfrak{M}'$ inconsistent. Hence, by Robinson's consistency, they are $\mathscr{L}$-inequivalent, thus contradicting the fact that $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv_{\mathscr{L}} [\mathfrak{A}_1', \mathfrak{A}_2']$. Therefore, $\mathfrak{A}$ and $\mathfrak{A}'$ must be $\mathscr{L}$-equivalent. This, in turn, contradicts our initial *absurdum* hypothesis according to which K separates $\mathfrak{A}$ and $\mathfrak{A}'$. $\square$

As actual examples of equivalence quantifiers, consider the Q^{E}_α ($\alpha \geq 0$), which are defined by $Q^{\mathrm{E}}_\alpha = \{\langle A, E\rangle \mid E$ is an equivalence relation on A with at least ω_α distinct equivalence classes$\}$. See Feferman [1975]. We thus have

3.1.4 Theorem. (i) *$\mathscr{L}(Q_\alpha) \not\geq \mathscr{L}(Q^{\mathrm{E}}_\alpha)$, whenever $\alpha > 0$;*

(ii) *$\Delta\mathscr{L}(Q_\alpha) \equiv \Delta\mathscr{L}(Q^{\mathrm{E}}_\alpha)$;*

(iii) *$\mathscr{L}(Q_\alpha)$ and $\mathscr{L}(Q^{\mathrm{E}}_\alpha)$ have recursively isomorphic sets of valid sentences, the same Hanf and same Löwenheim numbers (the latter being equal to ω_α), and equal compactness spectrum;*

(iv) *interpolation fails for $\mathscr{L}(Q^{\mathrm{E}}_\alpha)$, if ω_α is regular.*

Proof. (i) Immediate from Lemma 3.1.2.

(ii) Using choice functions, one sees that $\Delta\mathscr{L}(Q^{\mathrm{E}}_\alpha) \leq \Delta\mathscr{L}(Q_\alpha)$; on the other hand, using the equivalence relation given by identity ($=$), one also sees that $\Delta\mathscr{L}(Q^{\mathrm{E}}_\alpha) \geq \Delta\mathscr{L}(Q_\alpha)$.

(iii) Immediate from (ii), together with standard results of abstract model theory; there is no problem for the Hanf number in this case, see Corollary XVII.4.3.4 and Section 4.5 in the same Chapter.

(iv) for ω_α regular, $\Delta\mathscr{L}(Q_\alpha)$ does not obey interpolation (see Hutchinson [1976b]). ☐

Thus each $\mathscr{L}(Q^{\mathrm{E}}_\alpha)$ is closely related to $\mathscr{L}(Q_\alpha)$; the latter logics are studied extensively in Chapters IV and V.

3.2. Order Quantifiers

Let $X = \{\langle A, R\rangle \mid R$ is a partial ordering relation$\}$. Then Q is an *order quantifier* iff its defining class is a subclass of X. One notable example of an order quantifier, $Q^{\mathrm{cf}\,\omega} = \{\langle A, <\rangle \mid <$ is a linear ordering of cofinality $\omega\}$, has been discussed in detail in Chapter II, where it is proved that $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ is compact, axiomatizable and has Löwenheim number equal to ω_1. From a general result due to Ebbinghaus [1975b] one can infer that $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ does not have the interpolation property (Counterexample II.7.1.3(c)). It is an open problem whether there exist extensions of $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ generated by a set of quantifiers and satisfying Robinson's consistency theorem. Such extensions (if any) would have many syntactic and algebraic properties in common with first-order logic.

The order quantifier Q^D gives rise to a logic having many properties in common with $\mathscr{L}(Q_1)$, where $Q^D = \{\langle A, <\rangle \mid <$ is a dense linear ordering with a countable dense subset$\}$. As a matter of fact we have:

3.2.1 Theorem. (i) *$\mathscr{L}(Q^D)$ is (ω, ω)-compact, axiomatizable and its Löwenheim number is ω_1;*

(ii) *$\mathscr{L}(Q^D)$ does not have the interpolation property.*

Proof. (i) $Q^D xy\, \varphi(x, y)$ iff $\{\langle x, y\rangle \mid \varphi(x, y)\}$ is a dense linear ordering and aa s "s is dense in the ordering $\varphi(x, y)$". Now refer to Section IV.4.

(ii) From Ebbinghaus [1975b]. ☐

By contrast with Q^D, an order quantifier having many properties in common with the quantifier I is $R = \{\langle A, <\rangle \mid <$ has the order type of a regular cardinal$\}$. Note that $\Delta\mathscr{L}(Q^R) \leq \Delta\mathscr{L}(R) \leq \Delta\mathscr{L}^{\text{mII}}$ (for the first inclusion, note that $|A| < |B|$ iff there is a regular cardinal, namely $|A|^+$, and injections of $|A|^+$ into B and of A into an initial segment of $|A|^+$; for the second inclusion, proceed as in Claim 1 of Theorem 1.2.2). Also, by saying that a discrete linear ordering with first element has the order type of a regular cardinal, we can characterize $\langle \omega, <\rangle$ in $\mathscr{L}(R)$. Hence, the latter is not (ω, ω)-compact, not axiomatizable, and does not have the weak Beth property (see Proposition 2.1.2). In addition, Proposition 2.1.3 and Theorem 2.1.5 above can be applied to $\mathscr{L}(R)$ as well.

3.2.2 Theorem. (i) *If $V = L$, then $\Delta\mathscr{L}(I) \equiv \Delta\mathscr{L}(R) \equiv \Delta\mathscr{L}^{\text{mII}}$;*

(ii) *the fact that $\Delta\mathscr{L}(R) \not\equiv \Delta\mathscr{L}(I)$ is consistent, if* "ZF + *there are uncountably many measurable cardinals" is consistent;*

(iii) *the fact that $\Delta\mathscr{L}(R) \not\equiv \Delta\mathscr{L}^{\text{mII}}$ is consistent, if* ZF *is consistent.*

Proof. The argument for (i) is by the above discussion and by Theorem 2.1.4.

(ii) See Väänänen [1978, 3.1];

(iii) See Väänänen [1980b, Corollary 3.2.5 and the remark following it]. See also Chapter XVII, *passim.* □

Our final example of an order quantifier is the *well-order quantifier W*, which is defined by $\mathfrak{A} \models Wxy\,\varphi(x, y)$ iff $\{\langle x, y\rangle \in A^2 \mid \mathfrak{A} \models \varphi(x, y)\}$ well-orders its field. Clearly, we have that $\langle \omega, <\rangle$ can be characterized by a sentence of $\mathscr{L}(W)$, whence $\mathscr{L}(W)$ is not (ω, ω)-compact, not axiomatizable, and does not have the weak Beth property. Theorem 2.1.5(i) can be applied to $\mathscr{L}(W)$ with the same proof.

3.2.3 Theorem. *Let $\mathscr{L} = \mathscr{L}(W)$. Then we have:*

(i) *The gödelized set of valid sentences of $\mathscr{L}$ is the complete Π^1_2 subset of ω;*

(ii) *the Löwenheim number of $\mathscr{L}$ is ω;*

(iii) *$\Delta\mathscr{L} < \Delta\mathscr{L}_{\omega_1\omega_1}$, and $\Delta\mathscr{L} < \Delta\mathscr{L}(I)$;*

(iv) *assuming that $V = L$, the Hanf number of $\mathscr{L}$ equals the Löwenheim number of $\mathscr{L}^{\text{mII}}$;*

(v) *the smallest logic $\mathscr{L}' \geq \mathscr{L}$ having the Beth property is not Δ-closed;*

(vi) *the smallest logic $\mathscr{L}'' \geq \mathscr{L}$ having the weak Beth property is not a sublogic of $\mathscr{L}_{\infty\infty}$.*

Proof. (i) follows from Kotlarski [1978, p. 126]. In this connection, the reader should also see Corollary XVII.4.3.7 of the present volume.

(ii) We extend the usual proof of the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega\omega}$ by witnessing also that $\neg Wxy\varphi(x, y)$ with the help of an infinitely descending chain of constants.

(iii) is immediate from (ii).

(iv) See Väänänen [1979b, p. 316].

(v) See Makowsky–Shelah [1979b, p. 222]. Note that, as a consequence, the Beth property does not imply Craig interpolation.

Finally, for (vi) see Theorem XVII.4.1.3. □

Let us conclude this subsection with a note on some general facts about binary quantifiers. Assert that Q is *binary* iff Q has the form $Qx_1y_1 \ldots x_ny_n\, \varphi_1(x_1, y_1), \ldots, \varphi_n(x_n, y_n)$. Krynicki–Lachlan–Väänänen [1984] have proven negative results concerning binary quantifiers along the lines of the negative results about monadic and equivalence quantifiers that were given in Section 3.1 above. For example, binary quantifiers cannot count the dimension of a vector space in much the same way as monadic quantifiers cannot count the number of equivalence classes of an equivalence relation. Furthermore, there exists a ternary quantifier which is not definable by using binary quantifiers only.

3.3. Other Quantifiers

In this subsection we briefly deal with other quantifiers occurring in the literature. The reader is referred to Chapter IV for the "almost all" quantifier aa, as well as for the Magidor–Malitz quantifiers. Other classes of quantifiers are considered in Chapter III. Quantifiers arising in connection with infinitary languages are dealt with in Part C. For second-order quantifiers see Chapters XII and XIII. Quantifiers for enriched structures are studied in Chapter XV (but see also Section 4 below).

To introduce our next class of quantifiers we need the following:

3.3.1 Definition. A class K of structures is *inductive* iff it is closed under unions of chains (with respect to the substructure relation $\subseteq$). For λ a cardinal, K is λ-*inductive* iff K is closed under λ-unions, where $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta (\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq, \ldots)$ is a λ-*union* iff for every $B \subseteq A (= \bigcup A_\beta)$ with $|B| < \lambda$, there is $\beta < \alpha$ such that $B \subseteq A_\beta$.

This notion clear, then we have

3.3.2 Theorem. *If K is an arbitrary class of structures of type τ, with K closed under isomorphism, and if λ is an arbitrary cardinal, the following are equivalent*:

(i) *Both K and its complement $\bar{K}$ are λ-inductive*;
(ii) $\forall \mathfrak{A} \in \mathrm{Str}(\tau)\, \exists \mathfrak{A}_0 \subseteq \mathfrak{A}$, *with* $|A_0| < \lambda$ *such that* $\forall \mathfrak{B}$, $\mathfrak{A}_0 \subseteq \mathfrak{B} \subseteq \mathfrak{A}$ *implies* $(\mathfrak{A}_0 \in K$ *iff* $\mathfrak{B} \in K)$.

Proof. See Makowsky [1975b, Theorem 2.16]. □

Following Makowsky [1975b], we call any class (or quantifier) Q, λ-*securable* iff Q satisfies either of conditions (i) or (ii) in Theorem 3.3.2 above. ω-securable classes are called *continuous* by Tharp [1974]. From the definition it follows that $\exists$ and $\forall$ are 2-securable, Q_α is $\omega_{\alpha+1}$-securable, W is ω_1-securable, Q^{C} is never λ-securable. Moreover, Q^{D} is ω_2-securable if there is no Suslin tree (see Makowsky [1975b]). We also have

3.3.3 Theorem. *Let K be an arbitrary class of type τ, with K closed under isomorphism*:

(i) *K is n-securable, for some $n \in \omega$, iff both K and $\bar{K}$ can be defined by* $\forall\exists$-*sentences of* $\mathscr{L}_{\omega\omega}$;

(ii) *K is ω-securable iff both K and $\bar{K}$ are inductive;*
(iii) *if K is ω-securable and has type $\tau' = \{U_1, \ldots, U_m\}$, where each U_i is a unary relation, then K is* EC *in $\mathscr{L}_{\omega\omega}$;*
(iv) *for λ a regular cardinal, let $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$, where each Q^i is λ-securable; then, if $\lambda \leq \omega$, the Löwenheim number of $\mathscr{L}$ is ω; if $\lambda > \omega$, then each consistent sentence of $\mathscr{L}$ has a model of cardinality $< \lambda$; in particular, the Löwenheim number of $\mathscr{L}(W)$ is ω;*
(v) *$\mathscr{L}_{\omega_1\omega}$ is the smallest Δ-closed logic containing all the ω-securable quantifiers.*

Proof. (i) See Makowsky [1975b, Corollary 3.11]. Observe here that Tharp [1973] proved that if K is n-securable, then K is EC in $\mathscr{L}_{\omega\omega}$.

(ii) See Makowsky [1975b, Theorem 2.14]; but also see Miller [1979].

(iii) See Tharp [1974, Theorem 5].

(iv) See Tharp [1974, Theorem 7], for the case $\lambda = \omega$; see Makowsky [1975b, Theorem 2.1], however, for the general case. Recall that W is an ω_1-securable quantifier.

(v) See Makowsky [1975b, Corollary 5.6]. ☐

We now deal with quantifiers which are used to express the fact that "there exist large sets of indiscernibles". Given a structure $\mathfrak{A} \in \mathrm{Str}(\tau)$, let $q^I_{\mathfrak{A}} = \{B \subseteq A \mid B$ contains an infinite set of order indiscernibles in $\mathfrak{A}\}$, and let $q^F_{\mathfrak{A}} = \{B \subseteq A \mid B$ contains arbitrarily large finite sequences of indiscernibles in $\mathfrak{A}\}$. The resulting logics, $\mathscr{L}(Q^I)$ and $\mathscr{L}(Q^F)$ are syntactically the same as, for example, $\mathscr{L}(Q_1)$. Moreover, their semantics is obtained by letting, for instance,

$$\mathfrak{A} \models Q^I x\, \varphi(x) \quad \text{iff} \quad \{x \in A \mid \mathfrak{A} \models \varphi(x)\} \in q^I_{\mathfrak{A}}.$$

Notice the dependence of Q^I on the whole of $\mathfrak{A}$, rather than on its universe only. Steinhorn [1980] has a number of categoricity and quantifier elimination results on Q^I and Q^F (see also [1981]). He also proves that $\mathscr{L}(Q^F)$ does not have the interpolation property.

Thomason [1966] introduced a logic $\mathscr{L}_q$ with free variables for quantifiers. The idea here was to examine those properties which are common to all generalized quantifiers. If Q is any such variable, then $Qx_1 \ldots x_n\, \varphi_1(x_1), \ldots, \varphi_n(x_n)$ is a formula of $\mathscr{L}_q$. If $\psi(\ldots Q)$ is a sentence of $\mathscr{L}_q$, then a model of $\psi(\ldots Q)$ consists of an ordinary structure $\mathfrak{M}$ together with a quantifier (in the sense of Mostowski) $\hat{Q}$ which serves as an interpretation of Q. Sentence $\psi(\ldots Q)$ is valid in $\mathscr{L}_q$ iff $(\mathfrak{M}, \hat{Q})$ satisfies $\psi(\ldots Q)$ for all structures $\mathfrak{M}$ and all interpretations $\hat{Q}$. Yasuhara [1969] wrote down a sentence characterizing the natural numbers in $\mathscr{L}_q$. Therefore, $\mathscr{L}_q$ is neither (ω, ω)-compact, nor axiomatizable. The sets of valid sentences of $\mathscr{L}(I)$ and $\mathscr{L}_q$ are recursively isomorphic.

Now let $\mathscr{L}_Q$ be just as $\mathscr{L}_q$, but with quantifier variables to be interpreted over binary quantifiers. Then, in Väänänen [1980a], it is proved that $\mathscr{L}_Q$ and $\mathscr{L}(Q^{\mathrm{H}})$ have recursively isomorphic sets of valid sentences. Roughly speaking, $\mathscr{L}_q$ is to $\mathscr{L}(I)$ as $\mathscr{L}_Q$ is to $\mathscr{L}(Q^{\mathrm{H}})$.

3.4. Bibliographical Notes

Theorem 3.1.1 is due to Caicedo [1979]. Using the Feferman–Vaught property, Makowsky [1978c] proves a stronger form of Theorems 3.1.3 and 3.1.1 for arbitrary monadic and equivalence quantifiers. The equivalence quantifiers Q_α^{E} were first introduced by Feferman [1975], after Keisler's counterexample to Craig's interpolation in $\mathscr{L}(Q_1)$. The quantifier $Q^{\mathrm{cf}\,\omega}$ is studied in Shelah [1975d]. For other compact quantifiers, see Rubin–Shelah [1980], where it is proved that compactness does not imply axiomatizability (if $V = L$). For the quantifier R, see Väänänen [1978, 1979b, 1980b]. For further information about Q^{D} see Makowsky–Shelah–Stavi [1976]. For free quantifier variables and their associated logics, see Thomason–Randolph Johnson Jr. [1969], Yasuhara [1966b], Bell–Slomson [1969], Väänänen [1979d, 1980a], and Anapolitanos–Väänänen [1981].

4. Quantifiers from Robinson Equivalence Relations

Although compactness and interpolation are often regarded as desirable properties of logics, in general quantifiers do not take care of such properties. For example, none of the logics described in the preceding sections has the Robinson property. A logic $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$ has compactness and interpolation iff $\mathscr{L}$ has the Robinson property (see Chapter XIX): the latter only depends on $\equiv_{\mathscr{L}}$. Thus, we may naturally ask which equivalence relations $\sim$ with the Robinson property (for short, Robinson equivalence relations) do generate a nice logic $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$. Recall that $\sim$ is *bounded* iff for every type τ there is κ_τ such that the number of equivalence classes of $\sim$ of type τ is κ_τ. $\sim$ is *preserved under reduct* iff $\mathfrak{A} \sim \mathfrak{B}$ implies $\mathfrak{A} \restriction \tau \sim \mathfrak{B} \restriction \tau$, for each, $\tau \subseteq \tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$. Preservation under renaming is defined analogously. A quantifier Q belongs to *hull*$(\sim)$ iff $\equiv_{\mathscr{L}(Q)}$ is *coarser* than $\sim$ (that is, $\mathfrak{A} \sim \mathfrak{B}$ implies $\mathfrak{A} \equiv_{\mathscr{L}(Q)} \mathfrak{B}$). We say that $\sim$ is *separable by quantifiers* iff whenever $\mathfrak{A}$, $\mathfrak{B}$ are structures of type τ and not-$\mathfrak{A} \sim \mathfrak{B}$ there is $\tau' \subseteq \tau$ and $Q \in \mathrm{hull}(\sim)$ of type τ' such that $\mathfrak{A} \restriction \tau' \in Q$ and $\mathfrak{B} \restriction \tau' \notin Q$ (intuitively, Q separates $\mathfrak{A}$ and $\mathfrak{B}$). We finally let $\mathscr{L}(\sim) = \mathscr{L}\{Q \mid Q \in \mathrm{hull}(\sim)\}$. These ideas clear, we now recall the following results from Chapter XIX:

Theorem. *Let $\sim$ be an arbitrary bounded Robinson equivalence relation on the class of all structures and assume that $\sim$ is preserved under reduct and renaming, is coarser than $\cong$ and finer than $\equiv$. Then, adopting the above notation we have*:

(i) *$\mathscr{L}(\sim)$ is the strongest logic $\mathscr{L}$ such that $\equiv_{\mathscr{L}}$ is coarser than $\sim$*;
(ii) *if, in addition, $\sim$ is separable by quantifiers, then $\mathscr{L}(\sim)$ is the unique (up to equivalence) logic $\mathscr{L}$ such that $\equiv_{\mathscr{L}} = \sim$. Furthermore, $\mathscr{L}(\sim)$ is compact and has the Craig interpolation property.* $\square$

Corollary. *The following hold up to equivalence:*

(i) $\mathscr{L}_{\omega\omega}$ *is the unique logic* $\mathscr{L}$ *such that* $\equiv_{\mathscr{L}} = \equiv$;
(ii) *topological logic* $\mathscr{L}_t$ *is the unique logic* $\mathscr{L}$ *such that* $\equiv_{\mathscr{L}} = \equiv^t$ *holds, where* $\equiv^t$ *is topological* ω*-partial homeomorphism; the open and the interior quantifiers and their n-dimensional versions are in* hull($\equiv^t$);
(iii) *the same as the first part of* (ii) *for n-dimensional monotone logic* ($n = 1, 2, 3, \ldots$). □

Note that in two-dimensional monotone logic we have a model-theoretical framework for such notions as uniform continuity and metric completeness (see Robinson [1973, p. 511]). For topological and monotone logic see Chapter XV, and Flum–Ziegler [1980]. The equivalence "Robinson Consistency = Compactness + Craig Interpolation" was first proved in Mundici [1982b] (and was announced in Mundici [1979a, b]) and, independently, in Makowsky–Shelah [1983]. The above theorem, as well as (i) of the corollary were first proven in Mundici [1982a]. Parts (ii) and (iii) of the corollary can be found in Mundici [1982c, II and 198?b].

By the above theorem, any separable Robinson equivalence relation $\sim$ canonically generates a nice set $\{Q^i\}_{i \in I}$ of quantifiers. In order to eliminate redundancy, we may restrict attention to subsets of hull($\sim$) of minimal cardinality but which are still capable of generating $\mathscr{L}(\sim)$. Once $\mathscr{L}(\sim)$ is written out as $\mathscr{L}\{Q \mid Q \in B\}$, for B any such minimal set, the quantifiers in B are enough to give a full account of all the syntactic as well as algebraic properties of $\mathscr{L}(\sim)$.

In the absence of a Kreisel-like program for quantifiers, the above theorem and corollary may also give some hints in the search of (sets of) quantifiers such as B. One might, for instance, investigate whether letting Q range over the elementary classes of $\Delta\mathscr{L}(Q^{\mathrm{cf}\,\omega})$, one can encounter an element of hull($\sim$), for $\sim$ a bounded separable Robinson equivalence relation $\neq\ \equiv$. As a first step in this direction, one would check whether the compact logic $\mathscr{L}(Q)$ obeys interpolation. The progression from the open and the interior quantifiers, to their multi-dimensional versions, and from the latter to topological logic $\mathscr{L}_t$ shows that this program is feasible. Incidentally, the rôle played by restricted second-order quantifiers for $\mathscr{L}_t$ shows that the usual first-order quantifiers do not have the sole right of producing good syntaxes (see also Chapters XII and XV in this respect).

In Mundici [1982e], the author tried to obtain Robinson equivalence relations and their associated quantifiers as a byproduct of more fundamental objects, such as (suitably generalized) back-and-forth approximations of isomorphism. Indeed, this can be done for $\equiv$ and $\equiv^t$. In addition, back-and-forth techniques already pervade (abstract) model theory.