DESCRIBING SPACETIMES BY THEIR CURVATURE

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1.Introduction

In general relativity the geometry of spacetime is usually described by a Lorentz metric on a four-dimensional manifold M. In this paper we discuss the extent to which one can describe the geometry purely in terms of curvature. In Riemannian geometry this question is as old as the subject itself since it is mentioned in Riemann's inaugural lecture ([1], p.280). The word 'curvature' can be interpreted in different ways and these are not equivalent in the absence of further information concerning a metric. It could for instance mean the Riemann tensor in the form R^a_{bcd}, the same tensor in the form R_{abcd} or the sectional curvature. A more exotic possibility is to interpret it as the lasso form defined by Gross in [2]. This last interpretation is in some ways very satisfactory but has the disadvantage that one has to work with an object defined not on M itself but on its loop space. It will not be discussed further here. We will concentrate on one possibility: the Riemann tensor in the form R^a_{bcd}. (Another possibility, that of sectional curvature, was discussed in [3] and [4].)

The definition of the curvature in terms of the metric can be thought of as an inhomogeneous second order partial differential equation for the metric. The basic questions one would like to answer concerning this differential equation are the usual ones of existence, uniqueness, continuous dependence of the solution on the curvature and differentiability of the solution. By far the most difficult of these four questions is that of existence; a survey of what is known about this question for curvature quantities containing in general less information than those considered here (e.g. the Ricci tensor) can be found in the book of Kazdan[5]. The uniqueness question is equivalent to the question of whether all the information about the metric of a spacetime is contained in its curvature. This is in fact true in the generic case. Now the curvature has a physical interpretation in terms of geodesic deviation. Thus one would like to conclude from the uniqueness result that measurements of geodesic deviation provide the same information as measurements of distances and time intervals. Since, however, physical measurements are all subject to error a minimal requirement for obtaining a statement of physical significance is that a small error in the curvature produces a small error in the metric determined. In other words it is necessary to know that the determination of the metric by the curvature is continuous.

The question of differentiability is connected with the difficult issue of what degree of differentiability one should demand of physical fields. That this is not merely a matter of personal taste can be seen in the analysis of the initial value problem for Einstein's equation or, in another context, in the way that quantum theory in many cases requires the use of distributional fields since more regular ones are of measure zero. Actually the analysis which we carry out here is rather insensitive to the type of differentiability used.

2. Topological considerations

In order to discuss continuity of the determination of the metric by the curvature it is first necessary to define a topology on the set of metrics on M and one on the set of curvatures. This is also necessary in order to define the notion of a property of metrics being generic. To say that a property holds generically means intuitively that it holds almost always. There are two possibilities of making such an idea precise: using measure theory or using topology. In the first case one can define a property to be generic if it holds except on a set of measure zero; in the second case a property is defined to be generic if the set of points for which it is true is open and dense. In fact the 'open and dense' condition is often replaced by the requirement that the set under consideration is a countable intersection of open dense sets. If this holds its complement is said to be of first category. There are two reasons for requiring in the present context that a set should be open and dense in order to be considered generic. Firstly this definition fits naturally with the interpretation in terms of physics (cf. Hawking[6]). Secondly the notion that a set of first category is in some sense 'small' must be used with care. Even in the simple case of the real line with the usual topology and Lebesgue measure it turns out that the whole space can be written as a union of a set of measure zero and a set of first category (Oxtoby[7]). Actually in the present case the option of a definition based on measure theory is not available since the spaces being considered are infinite-dimensional and it is extremely difficult to define measures with reasonable properties in infinite dimensions. Indeed defining appropriate measures on spaces of physical fields is an outstanding problem in constructive quantum field theory. Thus we adopt the topological definition of genericity.

Before considering the specific topologies of interest in the present context we recall some definitions from general topology. A base for a topology on a set X is a collection of subsets U_α such that whenever U_α and U_β belong to this collection and $x \in U_\alpha \cap U_\beta$ there is a U_γ belonging to the collection with $x \in U_\gamma \subset U_\alpha \cap U_\beta$. The topology generated by this base consists of all unions of sets belonging to the base. A sub-base for a topology can be any collection of subsets of X. The collection of finite intersections of elements of the sub-base is a base and the topology it generates is also said to be generated by the sub-base. Let $\{X_i\}$ be a sequence of topological spaces and denote by π_i the projection of the Cartesian product $\prod_i X_i$ onto X_i. The product topology on $\prod_i X_i$ is generated by the sub-base of sets of the form $\pi_i^{-1}(U)$ for some open subset U of X_i. The box topology on $\prod_i X_i$ is generated by the base consisting of the all sets of the form $\prod_i U_i$ where U_i is an open subset of X_i for each i. The box topology is in general strictly finer than the product topology although the two coincide for finite products of spaces. Now let $\{X_i\}$ and $\{Y_i\}$ be two sequences of topological spaces and suppose a map $f_i : X_i \to Y_i$ is given for each i. Then there is a natural product map $f = \prod_i f_i : \prod_i X_i \to \prod_i Y_i$. If the product sets are both given the product topology or both given the box topology then f is continuous if and only if all the f_i are continuous.

Given a manifold M choose a locally finite (and hence countable) open cover $\{U_i\}$ of M such that for each i the closure $\bar{U}_i$ is compact and contained in a chart domain. These sets can be thought of as regions over which individual observers can make measurements. (The word 'observer' is used here in a non-technical sense.) It is not the aim here to discuss what exactly an observer can measure; we will assume merely that a measurement of a physical field determines a bound on some norm defined for fields on U_i. If for instance we assume, following Lerner[8], that it is possible to estimate the field and its first k partial derivatives pointwise by a measurement then a measurement will give a bound on the uniform C^k norm of the field on U_i. For suitable regularity assumptions on the fields it is possible to regard the space of fields on the whole manifold as a subset of the product of the spaces of fields on the $\bar{U}_i$. Consider for example a C^k scalar field. Then there is an embedding $i : C^k(M) \hookrightarrow \prod_i C^k(\bar{U}_i)$ given by $i(f) = (f|\bar{U}_1, f|\bar{U}_2, \ldots)$. If the product space is given the product topology then the topology induced on C^k by the embedding is the compact open C^k topology. If the product is given the box topology then the resulting topology on $C^k(M)$ is called the Whitney C^k topology. It is easy to show that these topologies depend only on the manifold M and not on the cover $\{U_i\}$ chosen. Note that from the point of view of the physical interpretation an open set of fields in the compact open topology is determined by a finite set of measurements; to determine a Whitney open set on the other hand requires in general an infinite set of measurements. Since a single observer can only carry out a finite number of measurements in his lifetime this requires an infinite set of communicating observers. In practice this means that one is considering the theory not only in relation to all observations available directly or indirectly to a given observer but rather in relation to all observations which are in principle possible.

The Whitney C^0 topology has another description; it is identical with the so-called graph topology on $C^0(M)$. A base for this topology consists of the sets $\{f \in C^0(M) : \text{graph } f \subset U\}$ where U runs over all open subsets of $M \times \mathbb{R}$. So far we have only defined the C^k topologies for functions but in general the physical fields will be sections of some fibre bundle. The definition can be extended to that case by assuming that for each of the open sets U_i making up the cover of M the closure $\bar{U}_i$ is contained in a contractible subset of M. It then follows that any fibre bundle has a trivialisation over each $\bar{U}_i$ and after choosing charts on the typical fibre the case of sections of a bundle has been reduced to the case of functions. Alternatively a description analogous to the graph topology can be used but this requires the use of jet bundles. For the details see e.g. Lerner[8]. The definition using an open cover has the advantage that it generalises easily to norms other than the C^k norms. Consider for instance the Sobolev H^k norms. The construction above gives rise to two topologies on the space H^k_{loc} of functions (or sections of a bundle) locally in H^k. These could be referred to as the compact open and Whitney H^k topologies respectively. More generally the procedure defines two topological spaces of sections of any vector bundle corresponding to any Banach space valued section functor in the sense of Palais[9]. Topological spaces of sections of any fibre bundle are defined provided in addition Palais' axiom 5 is satisfied. For Sobolev spaces this is fulfilled provided $k > n/2$ where n is the dimension of the manifold M. It is also fulfilled for the Hölder spaces of functions of class $C^{k+\alpha}$. If one wishes to talk about the Whitney C^∞ topology then some care is necessary since there are two inequivalent ways one might think of defining it. On the one hand one could take as a base the collection of all Whitney C^k open sets for all k. This is the standard definition. Alternatively one could define a topology on $C^\infty(\bar{U}_i)$ having as base the C^k open sets for all k and apply the general construction given above to obtain a topology on $C^\infty(M)$. This last topology is strictly finer than the Whitney topology unless M is compact. Its restriction to the space $C_0^\infty(M)$ of smooth functions of compact support is the topology used in distribution theory, with respect to which the dual of that space is the space of distributions. No analogous ambiguity arises in the case of the compact open C^∞ topology.

It should be remarked that the Whitney topologies are by usual standards rather badly behaved, being in general neither first countable nor locally connected. This means that sequences are not sufficient for dicussing convergence and continuity in such spaces and that they cannot easily be considered as some sort of infinite-dimensional manifold (but see Michor[10]). The connected component of $C^k(M)$ with Whitney C^k topology containing the zero function is $C_0^k(M)$, the C^k functions of compact support. This can be proved by the kind of technique used to investigate connectedness in the box topology by Bourbaki[11], p.156. Note that C^k with Whitney topology is not a topological vector space although C_0^k is. (A topological vector space is always connected.)

The intuitive meaning of the statement that two functions are close in the compact open topology is that they are locally close together; this topology gives no control over the behaviour of the function 'at infinity'. The Whitney topology, on the other hand gives very strong control at infinity. Consider for instance the case where M is an open subset of $\mathbb{R}^n$. Then the zero function on M has an open neighbourhood V in $C^0(M)$ with Whitney topology such that every function in V extends continuously to a function on $\bar{M}$ which is zero on the boundary of M.

3.Uniqueness, stability and differentiability

This section summarises the results on uniqueness, stability and differentiability for prescribing Riemann tensors in the generic case obtained in [12,13] together with some generalisations. Results of a similar nature concerning prescribed curvature in gauge theories have been obtained by Mostow and Shnider[14,15]. Note first that the maps sending a metric to its Levi-Civita connection or Riemann tensor are continuous if we use one of the topologies constructed in the last section on all relevant spaces. This was proved in detail for the C^k topologies in [4]; for the Sobolev H^k topologies it follows from results of Palais[9] provided that k is large enough so that the Levi-Civita connection and Riemann tensor are well-defined.

Sufficient conditions for a Riemann tensor to determine the metric it arises from up to a constant factor in the case of various signatures and dimensions have been derived by several authors; for a review see Hall[16]. Intuitively it seems that these conditions are fulfilled in the generic case. This can be made precise as follows.

Theorem 1 Let M be a manifold of dimension $n \geq 3$. Let G be the set of C^2 metrics of a given signature which are determined up to a constant factor by their Riemann tensors. Then if Γ^k denotes the set of C^k metrics of that signature with Whitney C^k topology there is a subset V of G such that the set $V \cap \Gamma^k$ is open and dense for each $k \geq 3$.
Proof(sketch) The result was proved for Lorentz metrics in dimension 4 in [12] and generalised to any dimension greater than or equal to 4 and to positive definite metrics in [13]. In general there are two parts to the proof. The first is to show that for a generic set of metrics the Riemann tensor has maximal rank (when considered as a map $F^{cd} \mapsto R^a_{bcd}F^{cd}$)

on an open dense subset of M. The second step is to show that such a Riemann tensor determines the metric it comes from up to a constant factor. For the second step the arguments in [12,13] extend without difficulty to the other dimensions and signatures. (It is necessary to exercise a little care since the arguments fail in dimension 2). The extension of the first step is not so straightforward. The set of Riemann tensors at one point of M which have less than maximal rank is an algebraic subset of the vector space of all Riemann tensors at that point. Such a set can be stratified i.e. written as a union of smooth submanifolds. In order to carry through step 2 of the proof one needs to know that these manifolds can be chosen so that they 'fit together nicely' in order that a transversality theorem can be applied. In particular it suffices that they should form a 'submanifold complex'as defined in [12]. By a result of Whitney[17] the stratification can always be chosen to be (a)-regular and Trotman[18] has shown that an (a)-regular stratification is always a submanifold complex. Thus step 2 of the proof can be carried out in general. The rather long and complicated argument given in [12] to establish the existence of a suitable stratification is therefore not necessary for the proof of this theorem. That argument does, however, provide other useful information.

Remark Using the Sobolev embedding theorem one can show that the result of Theorem 1 implies the corresponding result with C^k replaced by H^k provided $k > n/2+3$. It is not easy to see how one could prove the result for Sobolev spaces directly. It is remarkable that the theory of partial differential equations which generally uses spaces similar to the Sobolev spaces and transversality or singularity theory which prefers C^k spaces seem to have made so little contact with each other.

In the proof of Theorem 1 the metric is determined directly on the open dense subset of M where the rank of the Riemann tensor is maximal and then determined on the rest of M by continuity. This rather indirect procedure suffices for proving uniqueness but when it comes to proving continuous determination and differentiability it leads to serious difficulties. (For a detailed discussion of analogous difficulties which arise in gauge theories and a way of overcoming them see the work of Mostow and Shnider[14,15].) Fortunately it turns out that these problems can be evaded in a relatively straightforward way provided the dimension of M is at least 4 and the signature Lorentz or definite. The idea is to prove something concerning the rank of the map $L : F^{cd} \mapsto R^a_{bcd}F^{cd}$ at every point of M and not just on an open dense set. Note that tensors of the form T^a_b have a Lie algebra structure given by the commutator. If R^a_{bcd} is the Riemann tensor of a Lorentz metric then the Lie algebra generated by the tensors in the range of L at each point of M is isomorphic to a Lie subalgebra of $so(1, n-1)$ and hence is at most of dimension $\frac{1}{2}n(n-1)$. Similarly for the Riemann tensor of a positive definite metric the range of L at each point generates an algebra isomorphic to a Lie subalgebra of $so(n)$. In [13] the following definition was introduced.

Definition A Riemann tensor on a manifold M of dimension n is called regular if:
(i) at each point of M its rank is at least $\frac{1}{2}(n-1)(n-2)+1$
(ii)at each point of M its range generates a Lie algebra whose dimension is maximal i.e. $\frac{1}{2}n(n-1)$.

This is a convenient class of Riemann tensors for the discussion of continuous determination and as the following result shows the condition that a metric have a regular Riemann tensor is generic.

Theorem 2 Let M be a manifold of dimension greater than or equal to 4 and let Γ^k be the space of C^k Lorentz or positive definite metrics with Whitney C^k topology. The set of metrics with regular Riemann tensors is open and dense in Γ^k for each $k \geq 2$.

This result was stated in [13] in the case $k = \infty$. The proof given there yields the above result for $k \geq 3$ directly. This implies density for $k = 2$. Openness for $k = 2$ follows from the fact that the condition that a Riemann tensor be regular defines a Whitney C^0 open set and the continuity of the map from metric to Riemann tensor with respect to the C^2 topology on metrics and the C^0 topology on Riemann tensors. As in the case of Theorem 1 this theorem implies corresponding results with C^k replaced by H^k or $C^{k+\alpha}$.

The condition of regularity is sufficient for obtaining results on differentiability.

Theorem 3 If a C^2 metric has a Riemann tensor which is regular and C^k then it is itself C^k.

Proof Firstly, it follows immediately from the determination process for the conformal class in [13] that the conformal class of the metric is C^k. Let $\hat{g}_{ab}$ denote the metric and $\hat{R}^a_{bcd}$ its Riemann tensor. Then there exists a C^k metric g_{ab} conformal to $\hat{g}_{ab}$. This means that $\hat{g}_{ab} = e^{2U} g_{ab}$ for some C^2 function U. It remains to show that U is in fact C^k. Define:

$$Y_{ab} = U_{;ab} - U_{,a}U_{,b} + \frac{1}{2} g_{ab} g^{cd} U,cU,d$$

where a semi-colon denotes the covariant derivative associated to the metric g_{ab}. Then by a standard formula for conformally related metrics:

$$\hat{R}_{ab} = R_{ab} - 2Y_{ab} - Y g_{ab}$$

The tensor R_{ab} is C^{k-2} while $\hat{R}_{ab}$ is C^k. It follows that Y_{ab} is C^{k-2}. Now:

$$U_{,ab} = Y_{ab} + \Gamma^c_{ab} U_{,c} + U_{,a}U_{,b} - \frac{1}{2} g_{ab} g^{cd} U_{,c} U_{,d}$$

A simple bootstrapping argument using this last formula shows that $U_{,ab}$ is C^{k-2} and hence that U is C^k.

The same proof shows that if $\hat{R}^a_{bcd}$ is of Sobolev class H^k or of Hölder class $C^{k+\alpha}$ then $\hat{g}_{ab}$ is of the same class. Because the Riemann tensor of a C^{k+2} metric is in general only C^k one might hope to prove more differentiability for the metric than is done in Theorem 3. Unfortunately this does not work for the following reason[5]. Let g_{ab} be a C^{k+2} metric on M and let $\phi : M \to M$ be a diffeomorphism which is C^{k-1} but not C^k. Then ϕ_*, the derivative of ϕ, will be C^{k-2} but not C^{k-1}. Thus while the tensor obtained by transforming the Riemann tensor of g_{ab} with ϕ is C^{k-2} the transformed metric will not in general be C^{k-1}. If one only demands that the metric is more differentiable than the Riemann tensor after possibly carrying out some diffeomorphism then stronger results can be obtained, at least in the positive definite case[19].

We next discuss the continuous determination of the metric by the Riemann tensor. As in the discussion of differentiability it is convenient to split the determination process into two steps. The first is to determine the conformal class of the metric. Then one can choose some test metric in the conformal class and attempt to determine the conformal factor relating this test metric to the metric being sought. This process was discussed in the C^∞ case in [13]. The approach used there in fact shows without further work that the determination of conformal classes by regular Riemann tensors is continuous when both conformal classes and Riemann tensors are given the Whitney C^k topology. It also shows that if test metrics in the conformal classes are chosen in a continuous way then the form dU corresponding to the conformal factor e^{2U} is determined continuously, albeit with a certain loss of differentiability of the topology in the following sense. In order to obtain continuity when the Riemann tensors are given the C^k topology the 1-forms must be given the C^{k-3} topology. This result relies only on the principle that smooth maps of bundles give rise to continuous maps of the corresponding spaces of sections. Thus a similar result holds with C^k replaced by H^k or $C^{k+\alpha}$ or other spaces and it is equally valid for the Whitney type of topologies and the compact open type. Since the Levi-Civita connection of the desired metric can be calculated directly from the test metric and dU it follows that a regular Riemann tensor determines continuously the Levi-Civita connection of any metric it arises from with the loss of three derivatives in the topology as above.

The situation is rather more complicated when it comes to determination of the metric itself since as already pointed out it is at best determined up to a constant factor. Thus the question of whether this determination is continuous must be formulated with some care. In [13] this was done as follows. Is it possible to choose a constant scaling for the metric giving rise to each regular Riemann tensor so that the resulting map from Riemann tensors to metrics is continuous? This is a trickier question than those considered so far and we will only discuss it for the C^∞ topologies. In [13] it was shown that constant factors can be chosen so as to make the map continuous in the compact open topology and an example was given to show that this may be impossible in the case of the Whitney topology. We will now discuss more generally those manifolds M for which the map can be made continuous in the Whitney C^∞ topology.

In order to prove the theorem which follows it is necessary to consider carefully the topology of M 'at infinity'. It is well known that any manifold can be written as the union of a sequence $\{K_n\}$ of compact subsets with $K_n \subset \text{Int } K_{n+1}$ for each n. We require a refinement of this result. A compact subset K of a manifold M is called *full* if the closure of each component of $M \backslash K$ is non-compact. Each compact set is contained in a full compact set and the sequence $\{K_n\}$ just mentioned can be chosen in such a way that each K_n is full ([11], chapter 1,§11,Ex. 14). If U_n is an open cover of M as used in the definition of the Whitney topology a function or form is said to vanish at infinity if the sequence $\{\epsilon_n\}$ of C_0 norms of its restrictions to the sets $\bar{U}_n$ tends to zero as $n \to \infty$. A function f is constant at infinity if $f - \alpha$ is zero at infinity for some α. The (necessarily unique) value of α for which this holds will be denoted by $f(\infty)$.

Lemma Let M be a manifold which can be written as $\bigcup_n K_n$ where $\{K_n\}$ is a sequence of compact subsets of M such that $K_n \subset \text{Int } K_{n+1}$ and $\text{Int } K_{n+2} \backslash K_n$ is connected for each n. Let $\Omega^1(M)$ denote the space of smooth 1-forms on M and let $d : C^\infty(M, \mathbb{R}) \to \Omega^1(M)$ be the exterior derivative. If C is the range of d then there exists a map $i : C \to C^\infty(M, \mathbb{R})$, continuous in the Whitney C^∞ topology, such that $d \circ i$ is equal to the identity.

Proof Let $\Omega_0^1(M)$ denote the space of smooth 1-forms vanishing at infinity and let C_0 be its intersection with C. It is elementary to show that Ω_0^1 is an open and closed linear subspace of $\Omega^1(M)$. Thus in order to prove the lemma it suffices to define a map i with the desired properties on C_0. It can then straightforwardly be defined on all other cosets of C_0 in C.

Choose a Riemannian metric on M. Instead of using the maximum of the components of a form on each U_n in the definition of the Whitney C^0 topology it is possible to use the maximum of its pointwise norm with respect to the given metric. This defines the same topology. We can assume without loss of generality that the cover U_n chosen is such that $U_n \subset K_n$. Let D_n be the diameter of the set $A_n = \mathrm{Int}K_{n+2} \backslash K_n$ with the induced Riemannian metric. Thus each pair of points in A_n can be joined by a smooth path of length less than $2D_n$. Choose a decreasing sequence of positive constants c_n such that $c_n < \frac{1}{2n^2} D_n$. Let x_n be a sequence of points of M such that $x_n \in A_n \cap A_{n-1}$ for each n. Then if df belongs to the Whitney open neighbourhood of zero determined by this sequence (using the cover U_n and the given Riemannian metric):

$$|f(x_{n+1}) - f(x_n)| \leq 2D_n c_n \leq \frac{1}{n^2}$$

It follows that $f(x_n)$ is a Cauchy sequence and hence converges to some number α. We will now show that f is constant at infinity with $f(\infty) = \alpha$. Without loss of generality we may assume that $\alpha = 0$. Given $\epsilon > 0$ it must be shown that there exists N such that $\|f|\bar{U}_n\| < \epsilon$ for $n \geq N$. Choose N so large that $\frac{1}{N^2} < \frac{\epsilon}{2}$ and $|f(x_n)| < \frac{\epsilon}{2}$ for $n \geq N$. Then if $x \in A_n$ and $n \geq N$ then $|f(x)| < |f(x_n)| + \frac{1}{N^2} < \epsilon$. It follows that if $\omega \in C_0$ then $\omega = df$ for some f vanishing at infinity. Define $i(\omega)$ for $\omega \in C_0$ to be the unique function vanishing at infinity with exterior derivative ω. It remains to show that this map i is continuous.

Let $\{\epsilon_n\}$ be a decreasing sequence of positive constants defining a Whitney open neighbourhood W of the zero function. We seek a sequence δ_n so that the image under i of the Whitney open neighbourhood of 0 defined by this sequence is contained in W. Let k_n be the smallest integer such that A_{k_n} has non-empty intersection with U_k for some $k \geq n$. Let δ_n be chosen so that, for each n, $\delta_{k_n} < \frac{1}{4} c_n \epsilon_n$. This is possible since k_n tends to infinity as $n \to \infty$. If $x \in U_n$ then $x \in A_k$ for some $k \geq k_n$. Thus:

$$\begin{aligned} |f(x)| \leq & |f(x_k) - f(x)| + |f(x_k)| \\ \leq & \frac{1}{2n^2} \epsilon_n + \frac{1}{4} \sum \frac{1}{4} \frac{1}{2n^2} \epsilon_n \\ < \epsilon_n \end{aligned}$$

and the proof is complete.

This lemma is what we need to prove the desired result on continuous determination in the Whitney topology.

Theorem 4 Let M be a simply connected manifold of dimension n. Then the map from regular curvature tensors to the metrics they arise from cannot be made continuous by a suitable choice of constant conformal factors if the de Rham cohomology group $H^{n-1}(M)$ is non-trivial. If, on the other hand, M can be written as a union of compact subsets K_n satifying the hypotheses of the preceding lemma then it can be made continuous.

Proof Suppose first that $H^{n-1}(M)$ is non-trivial. Choose a non-zero cohomology class belonging to it. By Poincaré duality there exists a corresponding class in $H^1_C(M)$, the first cohomology group of M with compact supports (see Bott and Tu[20]). This class can be represented by a 1-form ω with compact support. Since M is simply connected $H^1(M) = \{0\}$ and so there is a function f such that $df = \omega$. On the other hand the fact that ω has a non-trivial equivalence in $H^1_C(M)$ implies that f cannot itself have compact support. Moreover $f + \alpha$ is not of compact support for any $\alpha \in \mathbb{R}$. It follows that $M \backslash (\mathrm{supp}\omega)$ must have at least two components where f has different values. In fact a similar statement is true if $\mathrm{supp}\omega$ is replaced by a full compact set K containing $\mathrm{supp}\omega$. But each component of $M \backslash K$ has non-compact closure and so the argument used in [13] to show that determination is not always continuous can be used to show that in this case constant factors cannot be chosen so as to make the map from regular Riemann tensors to the metrics they arise from continuous.

Suppose now that M can be written as a union of compact subsets satisfying the hypotheses of the lemma. It has already been shown that a regular Riemann tensor determines the conformal class continuously and that when we choose test metrics in each conformal class continuously it determines the gradient of the conformal factor continuously. The lemma shows that if constant scalings are chosen correctly the conformal factor itself is determined continuously. Hence the metric is determined continuously in the Whitney C^∞ topology.

Remark It is likely that if $H^{n-1}(M)$ is trivial then there will exist a sequence of compact sets satisfying the hypotheses of the lemma but this is not entirely straightforward to prove. In any case such a sequence exists for $M = \mathbb{R}^n$ and so the second part of the theorem shows that there exist examples of manifolds where regular Riemann tensors continuously determine the metric in the Whitney topology.

4.Existence

There are some obvious algebraic conditions which a tensor K^a_{bcd} must satisfy in order to be a Riemann tensor. The symmetries $K^a_{bcd} = -K^a_{bdc}$ and $K^a_{[bcd]} = 0$ must be fulfilled and there must exist a Lorentz metric g_{ab} such that $g_{ab}K^b_{cde} - g_{cb}K^b_{ade} = 0$. From now on a tensor which satisfies these algebraic conditions will be referred to as a curvature candidate.

There are other restrictions satisfied by Riemann tensors which are not so easy to understand. Consider for instance the Bianchi identity $K^a_{b[cd;e]} = 0$. This involves not only the tensor K^a_{bcd} itself but also a metric (or at least a connection) because of the covariant derivatives which occur. It seems that the only reasonable way to express the restriction on K^a_{bcd} resulting from the Bianchi identity is: there exists some metric such that, with the covariant derivative associated to that metric, $K^a_{b[cd;e]} = 0$. It is not immediately clear that this represents any restriction. Note however that it implies the contracted

Bianchi identity $K^a_{b;a} = 0$ (where $K_{ab} = K^c_{acb}$) and DeTurck[21] has shown that there exist symmetric tensors K_{ab} which do not satisfy the contracted Bianchi identity for any metric. The other condition which comes to mind is the Ricci identity for the Riemann tensor itself. This can be written schematically as $DR = R * R$ where D is the skewed second covariant derivative and $*$ is a certain bilinear map. We will say that a curvature candidate K satisfies the Ricci identity with respect to a given metric if $DK = K * K$ with the operator D built from the covariant derivative associated to that metric. It seems to be a common belief that if a tensor K satisfies both the Ricci and Bianchi identities then it must be a curvature tensor. This statement does not make sense as it stands since these identities cannot be expressed in terms of K alone. However we can pose the following question: if a curvature candidate K satifies the Ricci and Bianchi identities with respect to some metric is it necessarily the Riemann tensor of some (possibly different) metric?

In fact it turns out that the answer to this question is no; counterexamples were constructed in [22]. A curvature candidate K was constructed with the following properties.
(i) K satisfies the Ricci and Bianchi identities with respect to some metric
(ii) K is not the Riemann tensor of any metric
(iii)given any point p there exists a metric g_p defined on a neighbourhood of p whose Riemann tensor R_p agrees with K to infinite order at p.
The construction of the examples will not be repeated here; the only additional feature of these examples which will be important in what follows is that they are, intuitively speaking, very special. Thus the question arises if these are really just pathological examples and if perhaps for a generic curvature candidate such a phenomenon cannot occur. The answer to this question is unfortunately as yet unknown. A possible way of attacking the algebraic problem will now be described.

Let K be a curvature candidate satisfying the Ricci identity $DK = K * K$ with respect to some metric with Riemann tensor R. The usual Ricci identity which all tensors satisfy is $DK = R * K$. Subtracting these two equations gives $(K - R) * K = 0$. If we could show that for generic K the equation $P * K = 0$ has no non-trivial solutions then this would imply $K = R$ so that K would be a Riemann tensor. Let L_K be the map sending P to $P * K$. The maps L_K can be thought of as sections of an appropriate vector bundle E. The condition that $P * K = 0$ has a non-trivial solution P can be expressed by the vanishing of certain determinants. Thus there is an algebraic subset A of E such that $P * K = 0$ has a non-trivial solution if and only if the image of the section K is contained in A. Suppose now that the following is true:
There exists a curvature candidate K such that at some
point p the equation $P * K = 0$ implies $P = 0$ at p. $(*)$
Condition $(*)$ is equivalent to the statement that the algebraic set A is not the whole of E. If this is true then by the applying results of Whitney[17] as in the proof of Theorem 1 it can be shown that A can be written as a union of smooth submanifolds of E of codimension at least 1. Moreover these submanifolds fit together in a nice enough way so that the transversality theorem can be applied. The conclusion from the application of the transversality theorem is that there is an open dense subset of curvature candidates in the Whitney C^∞ topology for which the only solution of $P * K = 0$ is the zero solution. Thus under the assumption that $(*)$ is true we have proved that in the generic case if a

curvature candidate K satisfies the Ricci identity with respect to some metric then K is the Riemann tensor of that metric. It remains to examine the validity of the condition $(*)$.

The condition $(*)$ is a purely algebraic one; it concerns relations between tensors at a single point. The map $P \mapsto P * K$ can be represented by a matrix(depending on K). The condition $(*)$ says that for some K this matrix has maximal rank. In order to verify the condition it is only necessary to find some such K explicitly. Unfortunately it is in the nature of the problem that even if almost all K give rise to matrices of maximal rank, those which one can write down explicitly and which are simple enough so that the matrix can be calculated by hand do not. (It should be borne in mind that the matrix is 96×576 in 4 dimensions.) A way of getting round this problem has been suggested by Chris Clarke. This uses a computer programme as follows. First the components of K are obtained from a random number generator. Whether or not the rank of the matrix is maximal is checked by a numerical procedure. By estimating errors one then proves that when this procedure gives the answer that the rank is maximal and a certain quantity calculated by the programme is small enough then the rank really is maximal. This idea has not yet been carried out in practice.

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