ON THE ENERGY OF THE GRAVITATIONAL FIELD AT SPATIAL INFINITY

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Abstract: Different frameworks for defining energy at spatial infinity are reviewed. Some new results concerning the definitional uniqueness of energy-momentum are presented.

1. INTRODUCTION

There have long been at least five different frameworks which allow to define energy and momentum at spatial infinity, but it is only recently that one has been able to answer the question, can one define global gravitational energy in a unambiguous way. The aim of this paper is to review the known and present some new results about the definitional uniqueness of energy-momentum in each of these frameworks. The approaches discussed here can be naturally divided into two groups, the first in which a Cauchy surface is the fundamental object one deals with, the second in which a four dimensional space-time is the starting point of the analysis. The problems arising in three dimensions, discussed in chapter 2, are much better understood because in that case one can ignore our lack of knowledge of the long time behaviour of solutions of Einstein equations. In the Cauchy setting one can uniquely define a number m and an $O(3)$ vector p^i which are usually called energy and momentum. Unfortunately our understanding of the four dimensional picture, discussed in chapter 3, is far from being complete. In the natural four dimensional coordinate framework we present two sets of conditions under which definitional uniqueness of a four-vector p^μ can be established. It should be stressed that one

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would like to be able to prove uniqueness under weaker conditions. We also present a condition under which the ambiguities in the Ashtekar-Hansen structure of spatial infinity can be analyzed, but the general case for this definition still remains an open problem.

In this paper we shall not attempt to derive (whatever this means) the various expressions which are used to define energy-momentum, the reader is referred to [As3] [BOM] [Fa] for a satisfactory analysis of this issue in the ADM spirit, to [Ki1] [Ki2] [Ch1] [Ch2] for more fancy four-dimensional symplectic methods. In this paper we shall be concerned with spatial infinity - the reader interested in energy at null infinity is referred to the review papers by Trautman [Tr], Goldberg [Go], and Ashtekar [As4], which also contain many references to early work on problems relevant to spatial infinity.

2. THE 3+1 APPROACH

As noted in the introduction, this is the approach in which the most complete results have been obtained. We shall distinguish between the ADM coordinate approach and Geroch's attempt to geometrize the ADM ideas.

2.1 The ADM Framework

In this framework ([ADM], cf. also [As3] [AG] [RT] [BOM]) one assumes the existence of preferred asymptotically flat coordinate systems in which the metric has some decay properties towards the flat metric as r goes to infinity. More precisely, one considers coordinate systems defined on $\mathbb{R}^3\backslash K$, where K is conditionally compact (i.e. open and its closure is compact), in which[1]

(2.1.1)
$$g_{ij} - \delta_{ij} = o(r^{-1/2}) \ , \ \ \partial_k g_{ij} = o(r^{-3/2}) \ ,$$
$$\partial_k g_{ij} \in L^2(\mathbb{R}^3\backslash K) \ , \ \ {}^3R(g) \in L^1(\mathbb{R}^3\backslash K) \ ,$$

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[1] $f = O(r^{\alpha})$ if $|f| \leq C(1+r)^{\alpha}$ for some $0 \leq C \in \mathbb{R}$; $f = o(r^{\alpha})$ if $\lim r^{-a} f = 0$, at infinity or at the origin, whichever case occurs being usually obvious from the context, $r(x) = \{\Sigma(x^i)^2\}^{1/2}$. The signature is $-+++$.

and the metric is uniformly elliptic in $\mathbb{R}^3\backslash K$:

$$g_{ij}(x)X^iX^j \geq c\sum (X^i)^2\ , \quad \text{for all}\ \ x \in \mathbb{R}^3\backslash K\ ,\ \ X^i \in \mathbb{R}^3\ ,$$

for some $c \in \mathbb{R}^+\backslash\{0\}$. One then defines

$$p^0 = \lim \frac{1}{16\pi}\oint_{S(R)}(g_{ik,i}-g_{ii,k})dS_k \tag{2.1.2}$$

and the limit is taken as R goes to infinity. We have:

PROPOSITION 2.1.1: ([*Ba*] [*Ch2*] [*ŌM*] [*Sol*]) *Under* (2.1.1) *the integral* (2.1.2) *converges to a finite value.*

PROPOSITION 2.1.2: ([*Ba*] [*Ch2*]) *The integral* (2.1.2) *is an invariant in the class of coordinate systems satisfying* (2.1.1).

The decay of the metric $g_{ij} - \delta_{ij} = o(r^{-1/2})$ is the best possible [DvS][2]. Proposition 2.1.2 is a consequence of the following "asymptotic symmetries lemma", which shows the strong rigidity of the structure underlying (2.1.1):

PROPOSITION 2.1.3: ([*Ba*] [*Ch2*]) *All* C^1 *coordinate transformations*[3] *preserving* (2.1.1) *are of the form*

$$y^i = \omega^i_j\, x^i + \zeta^i,\ \ \omega^i_j \in O(3),\ \ \zeta^i = o(r^{1/2}),\ \ \zeta^i{}_{,j} = o(r^{-1/2}),\ \ \zeta^i{}_{,jk} = o(r^{-3/2}).$$

If one moreover requires

$$g_{ij} = \delta_{ij} + h_{ij}(\theta,\varphi)\, r^{-1} + o(r^{-1}),\ \ \partial_k g_{ij} = h_{kij}(\theta,\varphi) r^{-2} + o(r^{-2}),$$

then ζ^i must be of the form[4]

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2 It has been pointed out to the author by J. Bicak that this has probably been first observed by Møller.

3 It can be shown that if $y(x)$ is C^1 and the matrices g representing the metric are C^k in both x and y coordinate systems, than $y(x)$ must be C^{k+1} .

4 This can be established using proposition 2.1.3 and the methods of [Ch2].

$$\zeta^i = C^i \ell n(r) + \zeta^i(\theta,\varphi) + o(r^{-1})\ , \quad C^i \text{ constant.}$$

Let us note that due to the constraint equation

$$^3R(g) = K^{ij}K_{ij} - K^2 + 16\pi\mu\ , \quad \mu = T_{\mu\nu}n^\mu n^\nu\ ,$$

the condition $^3R(g) \in L^1$ will be satisfied if $K_{ij} \in L^2(\mathbb{R}^3\backslash K)$ (finite "kinetic energy" of the gravitational field) and if $\mu \in L^1(\mathbb{R}^3\backslash K)$, where μ is the matter density (finite amount of matter). These conditions are sufficient to ensure that the integrals

$$p^i = \lim_{R\to\infty} \frac{1}{8\pi} \oint_{S(R)} K^{ij} dS_j \tag{2.1.3}$$

converge and transform as an $O(3)$ vector under the transformations of proposition 2.1.3. if the matter current $j^i = T^i{}_\mu n^\mu$ belongs to $L^1(\mathbb{R}^3\backslash K)$. Proposition 2.1.2 and the above remarks imply that in the class of metrics for which there exist coordinate systems satisfying (2.1.1) one can in an invariant way define the two numbers p^0, $|\vec{p}| = \{\Sigma(p^i)^2\}^{1/2}$, and the number

$$m = \{(p_0)^2 - |\vec{p}|^2\}^{1/2}$$

(cf. [Ba] [BM] [Ch3] [OM] where it is shown that the square root makes sense under the weak conditions considered here if the dominant energy condition holds), which we shall call the mass of a three-dimensional asymptotically flat end. Let us note that that one can also meaningfully speak of infinite mass under certain conditions, the interested reader is referred to [Ch3].

2.2 The Geroch Approach.

At the heart of Geroch's description of spatial infinity [Ge1] [Ge2] is the observation, that if one applies the inversion $x^i \to y^i = x^i/r(x)^2$ to an asymptotically flat metric in the sense of the previous section one will obtain

$$g = g_{ij}dx^i dx^j = \{\delta_{ij} + O(r(x)^{-\alpha})\}\, dx^i dx^j = r(y)^{-4}\, \{\delta_{ij} + O(r(y)^{\alpha})\}\, dy^i dy^j\ , \tag{2.2.1}$$

so that the metric $\bar{g}_{ij} = r(y)^4 g_{ij}$ is continuous at the origin. Geroch considers the set $\mathcal{M} = M \cup \{i_0\}$, where i_0 is a point (say the origin of the coordinates y^i) and describes the asymptotics of the metric g in terms of local properties - in a neighbourhood of the origin - of the metric $\bar{g}$. In order to do this one needs to introduce some structure on $\mathcal{M}$. There is a standard one point compactification topology with which $\mathcal{M}$ can be equipped, the simplest way of defining a differentiable structure on $\mathcal{M}$ is to declare the coordinates y^i we just have introduced by inversion to be smooth, but there is no uniqueness in this structure: one can perform a coordinate transformation of the type considered in Proposition 2.1.3, $x^i \to \bar{x}^i = x^i + O(r^{1-\alpha})$, and then define $\bar{y}^i = \bar{x}^i/r^2(\bar{x})$. One has $\bar{y}^i = y^i + O(r(y)^{1+\alpha})$ which is a differentiable transformation, but not in general C^2. One can include such coordinate transformations in the basic structure by enlarging the atlas on $\mathcal{M}$ in a way which we describe below, but are these all ambiguities that arise? The following example, due to B. Schmidt, shows that there can be non-uniqueness of C^1 structure when considering metrics which are merely continuous. The standard flat metric on $\mathbb{R}^2$

$$ds^2 = d\rho^2 + \rho^2 d\Psi^2$$

expressed in terms of coordinates

$$r = \rho \,,\ \varphi = \Psi - \ell n^{\alpha}\rho \,,\ 0 < \alpha < 1 \,, \tag{2.2.2}$$

takes the form

$$ds^2 = (1+\alpha^2 \ell n^{2(\alpha-1)} r)\, dr^2 + 2\alpha\, r\, \ell n^{\alpha-1} r\, dr d\varphi + r^2 d\varphi^2. \tag{2.2.3}$$

In "rectangular" coordinates $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$, the metric (2.2.3) is of the form

$$ds^2 = \{\delta_{ij} + O(\ell n^{\alpha-1} r)\}\, dx^i dx^j = g_{ij} dx^i dx^j \,,$$

and the coefficients g_{ij} are continuous at the origin since $\alpha - 1 < 0$. It is easy to check that the functions g_{ij} are smooth for $r > 0$ and that the transformation (2.2.2) is continuous, smooth for $r > 0$, however it is not C^1 on $\mathbb{R}^2$: the derivatives $\partial y^i/\partial x^j$, where $y^1 = r\cos\Psi$, $y^2 = r\sin\Psi$, are bounded but have no limit at $\rho = 0$. It follows from this example that the requirement of continuity of a metric may be compatible with many differentiable structures (we just have exhibited a one parameter family of them). As will be discussed below, the addition of the requirement of Hölder continuity of the metric together with an associated condition on the rate of blow-up of its derivatives fixes the $C^{1,\alpha}$ structure of $\mathcal{M}$ uniquely. Before presenting our results in detail some terminology will be needed. Let f be a C^1 function defined in a neighbourhood O of the origin in $\mathbb{R}^3$. We shall say that f is of class $A_{k,\alpha}$, $k \geq 1$, $\alpha \in (0,1]$, if there exists a constant C such that

$$|\partial_i f(x) - \partial_i f(0)| \leq Cr^{\alpha}, \ldots, |\partial_{i_1 \ldots i_\ell} f(x)| \leq Cr^{\alpha+1-\ell}, \quad 2 \leq \ell \leq k, \tag{2.2.4}$$

f will be said of class $\bar{A}_k$ if $f \in A_{k,1}$ and if the limits

$$\lim_{r\to 0} r^{-1}(\partial_i f(r\vec{n}) - \partial_i f(0)), \ldots, \lim_{r\to 0} r^{k-2}\partial_{i_1} \ldots \partial_{i_k} f(r\vec{n}) \tag{2.2.5}$$

exist, where $\vec{n}$ is any unit vector. A pair $(\mathcal{M}, i_0)$ will be called an $A_{k,\alpha}(\bar{A}_k)$ manifold if $\mathcal{M}$ is equipped with an atlas in which the transition functions are of $A_{k,\alpha}(\bar{A}_k)$ class, i_0 being the point where the derivatives are allowed to blow up. We shall say that a function f is of class $B_{k,\alpha}$ if f satisfies

$$|f(x) - f(0)| \leq Cr^{\alpha}, \quad |\partial_i f| \leq Cr^{\alpha-1}, \ldots, \quad |\partial_{i_1} \ldots \partial_{i_k} f| \leq Cr^{\alpha-k}.$$

On an $A_{k,\alpha}$ manifold one can define $B_{\ell,\alpha}$ tensors, $\ell \leq k-1$, i.e. tensors the components of which are $B_{\ell,\alpha}$ in local charts. Analogously $\bar{B}_\ell$ functions and tensors are defined by the requirement that the following limits as r tends to zero

$$\lim r^{-1}\{f(r\vec{n}) - f(0)\}, \ldots, \lim r^{\ell-1}\partial_{i_\ell} f(r\vec{n})$$

exist and are finite. Following Geroch [Ge1] [Ge2] we define:

DEFINITION: A three dimensional Riemannian manifold (M,g) will be called asymptotically flat if there exists an $A_{k,\alpha}(\bar{A}_k)$ manifold $\mathscr{M} = M \cup \{i_0\}$, $k \geq 2$, with a $B_{k-1,\alpha}(\bar{B}_{k-1})$ metric $\bar{g}$ and a conformal diffeomorphism $\Psi : M \to \mathscr{M}\backslash\{i_0\}$, such that the conformal factor Ω defined by $g = \Omega^{-2}\Psi^*\bar{g}$ can be extended to a function $\Omega : \mathscr{M} \to \mathbb{R}^+ \cup \{0\}$ satisfying

$$\Omega > 0 \text{ on } \mathscr{M}\backslash\{i_0\},\ \Omega(i_0) = 0,$$
$$\nabla\Omega \in A_{k-1,\alpha}(\mathscr{M},i_0)(\bar{A}_{k-1}(\mathscr{M},i_0)),$$
$$\nabla_a\nabla_b\Omega(i_0) = 2g_{ab}(i_0).$$

In [Ch4] the following has been proved:

PROPOSITION 2.2.1: A $B_{k-1,\alpha}$ *conformal structure on an* $A_{k,\alpha}(\mathscr{M},i_0)$ *manifold,* $\alpha \in (0,1)$, $k \geq 3$, *defines the* $A_{k,\alpha}$ *differentiable structure uniquely.*

PROPOSITION 2.2.2: A $\bar{B}_{k-1}(\mathscr{M},i_0)$ *conformal structure on a three dimensional manifold* $\mathscr{M}$ *defines uniquely a three parameter family of* $\bar{A}_k(\mathscr{M},i_0)$ *differentiable structures. Two inequivalent structures differ by a transformation of the form*

$$y^i = x^i + (C^i r^2 - 2C^k x^k x^i)\, \ell n(r) \tag{2.2.6}$$

where C^i *is a constant vector.*

Proposition 2.2.2 implies in particular the existence of a three parameter family of inequivalent conformal completions of an asymptotically flat (in the sense of section 2.1) Riemannian manifold. Let us derive an expression for the ADM mass in the conformal framework. It is convenient to start with the Ashtekar-Hansen formula for the ADM mass ([AH], cf. also section 3.2, equation (3.2.8)):

$$p^0 = \lim_{r\to\infty} \frac{1}{8\pi} \oint_{S(r)} r^3\, {}^4R_{0i0j} n^i n^j d^2S,\ n^i = x^i/r\,,\ d^2S = \sin\theta\, d\theta\, d\varphi\,, \tag{2.2.7}$$

which holds if (2.1.1) holds and if one moreover assumes

$$\partial_\mu\partial_\nu g_{k\ell} = o(r^{-5/2}),\ \ T_{\mu\nu} = o(r^{-3}). \tag{2.2.8}$$

From the Codazzi-Mainardi equations and the decay conditions on the metric and on the energy-momentum tensor one has

$$(2.2.9)\quad p^0 = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r} r^3\,{}^3R_{ij}n^in^jd^2S = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r} r^3({}^3R_{ij}-{}^3Rg_{ij}/4)n^in^jd^2S\,.$$

The formula for the transformation of the Ricci tensor under conformal changes of the metric yields

$$R_{ij} - \tfrac{1}{4}Rg_{ij} = \bar{R}_{ij} - \tfrac{1}{4}\bar{R}\bar{g}_{ij} + \Omega^{-1}\bar{\nabla}_i\bar{\nabla}_j\Omega - \tfrac{1}{2}\Omega^{-2}|\bar{\nabla}\Omega|^2\bar{g}_{ij}$$

(we have dropped the superscript 3 on the Ricci tensor since no confusion is likely to occur), so that after performing an inversion $y^i = x^ir(x)^{-2}$, setting $\Omega = r(y)^2$, one obtains

$$(2.2.10)\qquad p^0 = \frac{1}{16\pi}\lim_{d\to 0}\oint_{S_d}\Omega^{-1}(\bar{R}_{ij} - \tfrac{1}{4}\bar{R}\bar{g}_{ij} + \Omega^{-1}\bar{\nabla}_i\bar{\nabla}_j\Omega - \tfrac{1}{2}\Omega^{-2}|\bar{\nabla}\Omega|^2\bar{g}_{ij})\,\bar{\nabla}^i\Omega\,dS^j\,,$$

where S_d denotes either a coordinate ball $r(y) = d$ or a geodesic ball (with respect to the metric $\bar{g}$) of radius d. The convergence of the integral (2.2.10) and its independence upon the choice of Ω in the appropriate class can be either derived in the conformal framework making use of proposition 2.2.1, or follows directly from the results of the previous section. A formula for p_i can be derived by inversion in a straightforward manner. Let us finally note the following propositions:

PROPOSITION 2.2.3: Let $(\mathcal{M}_1,i_1,g_1)$, $(\mathcal{M}_2,i_2,g_2)$ *be two* $A_{k,\alpha}$ *conformal completions,* $\alpha \in (0,1)$, $k \geq 2$, *of an asymptotically flat three dimensional Riemannian manifold* (M,g). *There exists an* $A_{k,\alpha}$ *conformal diffeomorphism between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

PROPOSITION 2.2.4: A $\bar{B}_{k-1}(\mathcal{M},i_0)$, $k \geq 2$, *asymptotically flat metric on a manifold* $\mathcal{M}$ *defines uniquely a three parameter family of inequivalent* $\bar{A}_k$ *differentiable structures on* $\mathcal{M}$.

Remark: Proposition 2.2.3 and 2.2.4 improve by one order of differentiability the results of propositions 2.2.1 and 2.2.2 when a preferred (physical) metric is singled out. It should be noted that the proof of propositions 2.2.3-4 is considerable simpler than the proof of 2.2.1-2.

Proof: We shall prove proposition 2.2.3, proposition 2.2.4 can be established in a similar way. In local $A_{2,\alpha}$ charts $\{x^i\}$, $\{y^i\}$ in neighbourhoods of i_1 and i_2 we have

$$g^1 = g^1_{ij}dx^idx^j = \{g^1_{ij}(0)+O(r(x)^\alpha)\}dx^idx^j,\ \Omega_1 = g^1_{ij}(0)x^ix^j+O(r(x)^{2+\alpha}),$$

$$g^2 = g^2_{ij}dy^idy^j = \{g^2_{ij}(0)+O(r(y)^\alpha)\}dy^idy^j,\ \Omega_2 = g^2_{ij}(0)y^iy^j+O(r(y)^{2+\alpha}),$$

and performing linear coordinate transformations if necessary we have $g^1_{ij}(0) = g^2_{ij}(0) = \delta_{ij}$. The inversions $x^i \to \bar{x}^i = x^i/r(x)^2$, $y^i \to \bar{y}^i = y^i/r(y)^2$ give two asymptotically flat coordinate systems for the physical metric g, and the result follows by proposition 2.1.3.

This last proof clearly illustrates that the results about the coordinate structure, as in the previous section, are equivalent to results about Geroch's completions - in particular propositions 2.3.1 and 2.3.2 may be used to prove proposition 2.1.3 by reversing the argument of the last proof.

3. THE FOUR DIMENSIONAL APPROACH

We shall start this chapter with the natural coordinate approach which goes back to Einstein [Ei] and Klein [Kl]. A fundamental contribution to this formulation is due to von Freud [vF].

3.1 The Einstein-von Freud definition of energy-momentum.

In this approach one assumes that in a neighbourhood of the slice $t = 0$ there exists a four-dimensional coordinate system $\{x^\mu\}$, $x^\mu \in \Omega_x$, in which the metric satisfies

(3.1.1) $$|g_{\mu\nu}-\eta_{\mu\nu}| \le C(1+r)^{-\alpha}\,,\ |\partial_\sigma g_{\mu\nu}| \le C(1+r)^{-\alpha-1}\,,\ \alpha > 0\,,$$

(3.1.2) $$T_{\mu\nu} = O(r^{-3-\epsilon})\,,\ \epsilon > 0\,,$$

(cf. [Ch3][OM]). If $\alpha > 1/2$ one defines

(3.1.3) $$p_\mu Y^\mu = \lim_{R\to\infty} \frac{3}{16\pi} \oint_{\substack{r(x)=R\\ t=0}} X^\mu \delta^{\alpha\beta\gamma}_{\lambda\nu\mu}\, \eta^{\lambda\rho}\, \eta_{\gamma\sigma} g^{\sigma\nu}{}_{,\rho} dS_{\alpha\beta}\,,$$

where X^μ is any vector field of the form $X^\mu = Y^\mu + O(r^{-\alpha})$, $Y^\mu{}_{,\nu} = 0$. Under (3.1.1) and (3.1.2) when $\alpha>1/2$ (3.1.3) reduces to (2.1.2)-(2.1.3) so that the integrals converge as noted in section 2.1. Because of the unpleasant transformation properties of the integrand of (3.1.3) the number $m^2 = -\eta^{\alpha\beta} p_\alpha p_\beta$ could a priori assume quite arbitrary values (cf. e.g. [DvS]), it turns out that under some hypotheses which we present below one can establish rigidity of the structure associated with (3.1.1), allowing to deduce that m is uniquely defined by the four-dimensional asymptotic structure of the space-time. Let us note the following result [Ch5]:

PROPOSITION 3.1.1: *Let* $\{x^\mu\} = \Omega_x$, $\{y^\mu\} = \Omega_y$ *be two coordinate systems, suppose that (3.1.1) holds throughout* Ω_x *and* Ω_y *with some constants* C_x, C_y, $\alpha \in (0,1]$, *suppose that the slices* $x^0 = \text{const.}$ *and* $y^0 = \text{const.}$ *are spacelike submanifolds, let* Ω_x *and* Ω_y *contain boost-type domains:*

$$\Omega_x \supset \Omega^x_{\theta_x,R_x,T_x} := \{x^\mu : r(x) \ge R_x\,,\ |x^0| \le \theta_x r(x) + T_x\}$$

$$\Omega_y \supset \Omega^y_{\theta_y,R_y,T_y} := \{y^\mu : r(y) \ge R_y\,,\ |y^0| \le \theta_y r(y) + T_y\}$$

(T_x and T_y are allowed to be equal to plus infinity)[5]. Let F denote the coordinate transformation $y^\mu = \Phi^\mu(x^\alpha)$, wherever defined, let $N^x_R = \{x^\mu : x^0 = 0\ ,\ r(x) \ge R\}$.

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[5] The subscripts x and y in $T_x, R_y,$ etc. do not denote a pointwise dependence of the constants T_x and R_y but are meant to indicate that the constant in question is associated with the coordinate system $\{x^\mu\}$ or $\{y^\mu\}$.

If

$$\Phi(N^x_R) \subset \Omega^y_{\theta_y, R_y, T_y}$$

then N^x_R is the graph of a Lorentz transformation and a slowly growing term:

$$y^\mu = \Lambda^\mu_\nu x^\nu + \zeta^\mu(x), \quad |\zeta^\nu{}_{,\rho}| \le Cr^{-\alpha}, \quad |\zeta^\nu{}_{,\alpha\beta}| \le Cr^{-1-\alpha}, \tag{3.1.4}$$

$$|\zeta^\mu| \le Cr^{1-\alpha} \text{ if } 0 < \alpha < 1, \quad |\zeta^\mu| \le C\ell n(r) \text{ if } \alpha = 1.$$

Remark: If one assumes that the equalities

$$g_{\mu\nu} - \eta_{\mu\nu} = k_{\mu\nu}(\theta,\varphi)r^{-1} + o(r^{-1}), \quad g_{\mu\nu,\sigma} = k_{\mu\nu\sigma}(\theta,\varphi)r^{-2} + o(r^{-2}), \tag{3.1.5}$$

hold throughout Ω_x and Ω_y, it is simple to show that

$$\begin{aligned} y^\mu &= \Lambda^\mu_\nu x^\nu + \zeta^\mu(\theta,\varphi) + C^\mu \ell n(r) + o(1), \quad C^\mu{}_{,\nu} = 0, \\ y^\mu{}_{,\nu} - \Lambda^\mu_\nu &= \chi^\mu_\nu(\theta,\varphi)r^{-1} + o(r^{-1}), \\ y^\mu{}_{,\alpha\beta} &= \chi^\mu_{\alpha\beta}(\theta,\varphi)r^{-2} + o(r^{-2}). \end{aligned} \tag{3.1.6}$$

Outline of the Proof: The idea of the proof is to use the transformation laws of the connection interpreted as a set of differential equations for the functions $y(x)$

$$\frac{\partial^2 y^\mu}{\partial y^\alpha \partial y^\beta} = \Gamma^\sigma_{\alpha\beta}(x)\frac{\partial y^\mu}{\partial x^\sigma} - \Gamma'^\mu_{\nu\rho}(y(x))\frac{\partial y^\nu}{\partial x^\alpha}\frac{\partial y^\rho}{\partial x^\beta} \tag{3.1.7}$$

to obtain a priori estimates on the behaviour of $\partial y^\mu/\partial x^\alpha$ and $\partial^2 y^\mu/\partial x^\alpha \partial x^\beta$. The first step of the proof is to show that $r(y(x))$ goes to infinity as $r(x)$ does. Next one shows that

$$c^{-1}r(x) \le r(y(x)) \quad \text{for } x \in N^x_R. \tag{3.1.8}$$

The proof of (3.1.8) involves consideration of Riemannian distance with respect to a rather unnatural supplementary metric one introduces on the hypersurface $x^o = 0$ – this is a technical trick and the author believes there exists a more natural argument which leads to this inequality. One notes that (3.1.7) and (3.1.8) imply an alternative:

either the derivatives $\partial y^\mu/\partial x^\alpha$ are uniformly bounded or the sum of their squares uniformly tends to infinity as $r(x)$ does. If the derivatives are bounded the final result follows in a straightforward manner from (3.1.7)-(3.1.8), and to prove that they cannot blow to infinity one shows that there exists at least one curve extending to infinity along which the derivatives remain bounded. One checks that if there exists a spacelike curve in space-time which is well behaved in both coordinate systems, i.e. if there exist constant vectors η^μ_x , η^μ_y such that

$$(3.1.9) \quad x^\mu = \eta^\mu_x s + o(s) \ , \ y^\mu = \eta^\mu_y s + o(s) \ , \ dx^\mu/ds = \eta^\mu_x + o(1) \ , \ dy^\mu/ds = \eta^\mu_y + o(1),$$

then all derivatives $\partial y^\mu/\partial x^\alpha$ tend to finite limits along this curve as s goes to infinity in virtue of (3.1.7) and (3.1.8). The obvious idea for a candidate curve is to consider geodesics (but maybe a simpler choice exists?) and one is therefore led to ask how do spacelike geodesics behave asymptotically. The estimates (3.1.9) are established in the appendix B of [Ch5] for spacelike geodesics which remain simultaneously in some x and some y coordinates boost-type domains - this last step requires the introduction of the hypothesis that the x and y coordinates cover at least some boost-type domain. To complete the proof one shows that under the hypotheses above there exists at least one geodesic which lies both in an x and a y boost-type domain.

Proposition 3.1.1 allows us to prove a reasonably satisfactory result about the definitional uniqueness of m for vacuum space-times:

PROPOSITION 3.1.2: *Let* (M,g) *be a vacuum space-time, let* $\{y^\mu\}$ *be a coordinate system on* M *covering some boost-type domain* $\Omega^y_{\theta_y,R_y,T_y}$, $T_y \in (-\infty,\infty]$, *let* (3.1.1) *hold in* $\Omega^y_{\theta_y,R_y,T_y}$ *with some constants* $C \in \mathbb{R}^+$, $\alpha \in (1/2,1]$. *Let* N_x *be a spacelike submanifold of* M , $N_x \subset \Omega^y_{\theta_y,R_y,T_y}$, *and suppose that there exists a coordinate system* $\{x^i\}$ *on* N_x , $\{x^i\} \supset N^x_{R_x} = \{x^i : r(x) \geq R_x\}$, *such that the induced metric* g_{ij} *and the extrinsic curvature tensor* K^{ij} *satisfy*

(3.1.10) $$|g_{ij} - \delta_{ij}| \le Cr^{-\alpha}, \ldots, |\partial_{i_1} \ldots \partial_{i_5} g_{ij}| \le Cr^{-\alpha-5}\,,$$
$$|K^{ij}| \le Cr^{-\alpha-1}, \ldots, |\partial_{i_1} \ldots \partial_{i_4} K^{ij}| \le Cr^{-\alpha-4}\,.$$

Then the invariant mass of N_x *is equal to the invariant mass of the slice* $y^0=0$.

Proof: A simple extension of the boost theorem [COM] shows that one can construct a four dimensional coordinate system $\{x^\mu\}$ covering some boost type domain in which (3.1.1) holds. Proposition 3.1.1 implies that the coordinate transformation $y^\mu(x^\alpha)$ is a boost plus a "supertranslation", and it is well-known (cf. e.g. [Ch3]) that such transformations leave m invariant.

The hypotheses of proposition 3.1.1 consist of:

h1) a condition on the largeness of the coordinate system $\{x^\mu\}$,

h2) a condition on the largeness of the "reference" coordinate system $\{y^\mu\}$,

h3) a condition on the asymptotic behaviour of the metric in the respective domains of definition of coordinates x and y , Ω_x and Ω_y,

h4) a condition on the way the slice $x^0 = 0$ is embedded in Ω_y.

As is shown in the proof of proposition 3.1.2, the boost-type domain condition is not too restrictive when dealing with vacuum asymptotically flat metrics. We also know from the boost theorem [COM] that (3.1.1) will hold in boost-type domains with slope $\theta < 1$. In the non-vacuum case no boost theorem is available, so that no result along the lines of proposition 3.1.2 has been established. Hypotheses h1) and h4) appear to be superfluous in the sense that one would expect them to follow from asymptotic flatness. Let us show that in the $1 + 1$ dimensional case the asymptotic symmetries theorem is valid under much weaker assumptions:

PROPOSITION 3.1.3: *Let* $\dim M = 1 + 1$, *let* $\{x^\mu\} = \Omega_x \subset \mathbb{R}^2$, $\{y^\mu\} = \Omega_y \subset \mathbb{R}^2$ *be two coordinate systems on* M, *let* Φ *denote the coordinate transformation* $y^\mu = \Phi^\mu(x^\alpha)$, *wherever defined. Let the slices* $x^0 = 0$, $y^0 =$ const. *be spacelike, suppose* *that there exists a constant* C *such that* (3.1.1) *holds for* $x \in N^x_R = \{x^0 = 0\ ,$ $r(x) \geq R\} \subset \Omega_x$ *and for* $y^\mu \in \Phi(N^x_R)$. *There exists a Lorentz matrix* Λ^μ_ν *and a constant* C′ *such that* (3.1.4) *holds for* $x \in N^x_R$.

Proof: Let $t := x^0$, $t := y^0$, $x := x^1$, $y := y^1$. The curve $x \to y^\mu(t{=}0,x)$ has infinite proper length therefore multiplying y^μ by minus one if necessary we can assume that the limit as x goes to infinity of $y(t{=}0,x)$ is plus infinity. Let $E = \ell n D$, $D := \{\Sigma(\partial y^\mu/\partial x^\alpha)^2\}^{1/2}$. From (3.1.7) we have

$$dE = D^{-2}\sum \frac{\partial y^\mu}{\partial x^\beta}\,\frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta}\,dx^\alpha = f_\alpha dx^\alpha - f'_\alpha dy^\alpha\ , \tag{3.1.11}$$

where

$$f_\alpha = D^{-2}\sum \frac{\partial y^\mu}{\partial x^\beta}\Gamma^\sigma_{\ \alpha\beta}\frac{\partial y^\mu}{\partial x^\sigma} \leq C(1+|x^1|)^{-\alpha-1}\ ,$$

$$f'_\alpha = D^{-2}\sum \frac{\partial y^\mu}{\partial x^\beta}\Gamma'^{\,\mu}_{\ \alpha\rho}\frac{\partial y^\rho}{\partial x^\beta} \leq C(1+|y^1|)^{-\alpha-1}\ .$$

Let Γ_x be the straight line $\{x^0 = 0\ ,\ x^1 = s\ ,\ s \in [s_0,x^1]\}$. We have

$$E(0,x^1) - E(0,s_0) = \int_{\Gamma_x} f_\alpha dx^\alpha - \int_{\Gamma_x} f'_\alpha dy^\alpha\ .$$

The first integral is clearly finite whatever x^1. To evaluate the second let us note that by lemma 1 of [Ch5] Γ_x can be parametrized by $y = y^1$, so that

$$\left|\int_{\Gamma_x} f'_\alpha dy^\alpha\right| \leq \int_{\Gamma_x} |f'_0(dt/dy) + f'_1|\ dy^1\ .$$

Since Γ_x is spacelike we have $|dt/dy| \leq C$ in virtue of our hypotheses so that the last integral is also finite whatever x^1 , therefore E is bounded and (3.1.4) follows from lemma 3 of [Ch5] .

Returning to four dimensions, let us write the condition $N^x_R \subset \Omega_x$ in the form

$$y^0|_{N^x_R} \leq \theta_y r(y)|_{N^x_R} + T_y, \tag{3.1.12}$$

There exists a condition similar in spirit to (3.1.12) whose geometric meaning is a little more obscure and which allows to weaken h1) and relax h2)[6] .

Proposition 3.1.4: *Let* $\{x^\mu\} = \Omega_x$ *and* $\{y^\mu\} = \Omega_y$ *be two coordinate systems, let* Φ *denote the coordinate transformation* $y^\mu = \Phi^\mu(x^\alpha)$, *wherever defined. Suppose that* $N^x_R = \{x^0 = 0\ ,\ r(x) \geq R\} \subset \Omega_x$ *and that for* $x^\alpha \in N^x_R$, $y^\mu \in \Phi(N^x_R)$ (3.1.1) *holds with some constant*[7] C , *and assume that* $r(y(x))$ *goes to infinity as* $r(x)$ *does*[8]. *If*

$$y^0|_{N^x_R} = o(r(x)^{1+\alpha}) \ , \qquad (3.1.13)$$

there exists a Lorentz matrix Λ^μ_ν *such that* (3.1.4) *holds on* N^x_R .

Proof: Suppose that the derivatives $\partial y^\mu/\partial x^\alpha$ are not uniformly bounded on N^x_R , therefore by proposition 2 of [Ch5] we have, for $r \geq r_0$, $|\partial t/\partial t| \geq C_1 r(x)^\alpha$, $\Sigma|\partial t/\partial x^i| \geq C_2 r(x)^\alpha$, where $t := x^0$, $t(x^i) := y^0$ $(t = 0\ ,\ x^i)$. Consider any integral curve $\Gamma_{x_0} \subset N^x_R$ of grad(t) parametrized by the distance parameter s starting at x_0 , let $n^i = dx^i/ds = \mathrm{grad}^i(t)/|\mathrm{grad}(t)|$. Along Γ_{x_0} we have $dt/ds = n^i \mathrm{grad}_i(t) = |\mathrm{grad}(t) \geq C' r_o^{1+\alpha}$ which implies $t(s) \geq c's + t(0)$, in particular Γ_{x_0} cannot stay within a ball of finite coordinate radius R and thus r(s) tends to infinity as s does. Let x be any point lying on Γ_{x_0}. From $g_{ij}X^iX^j \geq c_x\,\Sigma(X^i)^2$ we have

$$\begin{aligned} t(x) &= t(x_0) + \int_{\Gamma_{x_0}} dt/ds\ ds \\ &\geq t(x_0) + \int_{\Gamma_{x_0}} c' r^\alpha ds = t(x_0) + \int_{\Gamma_{x_0}} c' r^\alpha \Big(g_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\Big)^{1/2} du \\ &\geq t(x_0) + \int_{\Gamma_{x_0}} c' c_x^{1/2} r^\alpha dr/du\ du \\ &\geq 2c(r^{1+\alpha} - r_0^{1+\alpha}) + t(x_0) \ , \end{aligned}$$

6 The proof of this proposition is due to R. Geroch.

7 It is *not* assumed that (3.1.1) holds throughout Ω_x and Ω_y .

8 cf. also lemma 2 of [Ch5] .

u being any parameter along Γ_{x_0} , and for sufficiently large $r(x)$ one has $t(x) \geq c'' r^{1+\alpha}(x)$ which contradicts (3.1.13) therefore all the derivatives $\partial y^\mu / \partial x^\alpha$ remain bounded and the result follows from lemma 3 of [Ch5] .

Let us note that for sufficiently large $r(y)$ the function y^0 is equivalent to the Lorentzian distance $d(y)$ from the point $\{y^\mu\}$ to the hypersurface $y^0 = 0$:

$$c^{-1} d(y) \leq |y^0| \leq c\, d(y) .$$

On the other hand for $r(x) > r_1$, for some r_1 , $r(x)$ is equivalent to the Riemannian distance $\sigma(x)$ on N_R^x from some fixed point $x_0 \in N_R^x$ to $x \in N_R^x$:

$$c'^{-1} \sigma(x) \leq r(x) \leq c' \sigma(x) .$$

These remarks shed some light on the geometric content of the condition (3.1.13) .

3.2 The Ashtekar-Hansen Approach

The relationship between the Ashtekar-Hansen approach and the coordinate approach of Section 3.1 is similar to the one between the Geroch and the ADM approach: one replaces some coordinate conditions in "exterior regions" by coordinate conditions in the vicinity of a point (differential geometry). While, however, the Geroch approach is entirely equivalent to the ADM one, nothing is known about generic existence of space-times satisfying the Ashtekar-Hansen (AH) conditions. The AH requirements turn out to be satisfied by the Kerr family of metrics, so that the whole framework is not vacuous, moreover the conditions given by these authors contain a useful knowledge about the global structure of space-time very difficult to capture in terms of coordinate systems of the previous section, so that there is little doubt that the AH conditions deserve some attention. The most important still unsolved problem of the Ashtekar-Hansen approach is the potential lack of uniqueness of their structure. In this section we present a condition under which the AH completions of a space-time turn out to be quasi-unique.

The AH conditions are essentially the requirement that the metric be better behaved, in local coordinates in a neighbourhood of "spatial infinity" conformally transformed to a point, than what one obtains by a straightforward Lorentzian inversion, applied to metrics as considered in the previous section: in terms of coordinates $y^\mu = x^\mu/(x^\alpha x_\alpha)$, $x^\alpha x_\alpha \equiv \eta_{\mu\nu}x^\mu x^\nu$, a metric of the form (3.1.1) defined on a boost-type domain $\Omega_{\theta,R,T}$ with slope θ smaller than one takes the form

$$(3.2.1) \qquad ds^2 = (y^\alpha y_\alpha)^{-2}(\eta_{\mu\nu}+h_{\mu\nu})dy^\mu dy^\nu, \quad |h_{\mu\nu}| \le C(\theta)r(y)^\alpha ,$$

with $h_{\mu\nu}$ defined on a wedge $W_{\theta,R^{-1}}$:

$$W_{\theta,\epsilon} = \{y^\mu : |y^0| < \theta r(y) , r < \epsilon\} ,$$

and $C(\theta)$ will generically blow up to infinity as θ goes to 1 . If $g_{\mu\nu}$ is the Kerr metric and if the coordinates x^μ are carefully chosen (cf. [AH]) one obtains a metric defined in $W_{1,\epsilon}$ with

$$(3.2.2) \qquad |h_{\mu\nu}| \le Cr$$

with some constant C , but it seems that one cannot avoid a blow up of the first derivatives of the metric as one approaches the light cone of $i_0 = \{y^\mu = 0\}$:

$$(3.2.3) \qquad |\partial_\sigma h_{\mu\nu}(t,r,\theta,\varphi)| \le C_1(|t|/r) , \; |\partial_\sigma\partial_\mu h_{\alpha\beta}(t,r,\theta,\varphi)| \le C_2 (|t|/r)^{-1}$$

with C_1 and C_2 being, say, monotonically increasing functions from $[0,1)$ to $\mathbb{R}^+$ which tend to infinity in a reasonably mild way as one tends to 1: if we define, for $0 < \eta < 1$,

$$\mathscr{C}_2(\eta) = \int_0^\eta C_2(s)ds ,$$

we have, in the Schwarzschild case (cf. [Ch6])

$$(3.2.4) \qquad \mathscr{C}_2(\eta) \le cC_1(\eta) , \; \int_0^1 C_1(\eta)d\eta < \infty ,$$

with some constant c. Proceeding in the spirit of Ashtekar and Hansen we shall say that a space-time (M,g) is asymptotically flat if there exists a triple $(\mathcal{M},\mathcal{g},i_0)$, called a completion of (M,g), such that

1) $\mathcal{M} = M \cup \{i_0\}$
2) there exists no timelike curve in $\mathcal{M}$ from any point in M to i_0,
3) there exists a coordinate system in M covering at least a wedge $W_{1,\epsilon}$, $\epsilon>0$, such that $i_0 = \{y^\mu=0\}$ and in which (3.2.2)- (3.2.4) hold[9],
4) In M the metric g is conformal to $\mathcal{g}$, $g_{ij} = \Omega^{-2}\mathcal{g}_{ij}$, and in the coordinate system as in point 3) Ω satisfies

$$\lim_{i_0} \Omega = 0, \ \lim_{i_0} \partial_\alpha \Omega = 0 ,$$

and
$$\forall\, y^\mu \in W_{1,\epsilon}, \ |\partial_\alpha\partial_\beta\Omega - 2\eta_{\alpha\beta}| \le Cr$$

with some constant C. Ashtekar and Hansen also assume that for all $|\eta| < 1$ the limits

$$\lim_{r\to 0} h_{\alpha\beta}(t = r\eta,r,\theta,\varphi)/r, \ \lim_{r\to 0} \partial_\mu h_{\alpha\beta}(t=r\eta,r,\theta,\varphi), \ \lim_{r\to 0} \partial_\mu\partial_\nu h_{\alpha\beta}(t=r\eta,r,\theta,\varphi)r , \tag{3.2.5}$$

exist and are finite. One can in an obvious way formalize along the lines of section 2.2 the structures described above, we shall leave this as an easy exercise to the reader. It is natural to consider two completions as equivalent if the appropriate coordinate systems in the vicinity of i_0 are related to each oterh via a transformation of the form

$$y^\mu = \Lambda^\mu_\nu x^\nu + r^2\zeta^\mu(t/r,\theta,\varphi) + o(r^2)$$

with Λ^μ_ν - a Lorentz matrix, ζ^μ - bounded and appropriately differentiable. Two completions will be called almost equivalent if they differ by a coordinate transformation of the form

[9] The ultimate motivation for (3.2.4) is, that it is satisfied by the completions of the Kerr metrics, and that this condition allows to prove theorem 3.2.2.

$$y^{\mu} = \Lambda^{\mu}_{\nu}x^{\nu} + (C^{\mu}x^{\alpha}x_{\alpha} - 2x^{\mu}x^{\alpha}C_{\alpha})\ell n(r) + r^2\zeta^{\mu}(t/r,\theta,\varphi) + o(r^2) ,$$

C^{μ} - a constant vector. A completion $(\mathscr{M},\mathscr{g},i_0)$ will be called geodesically regular if for every spacelike geodesic Γ of the physical metric g extending up to i_0 there exists numbers $\theta(\Gamma) < 1$ and $s_0 \in R$ such that for all $y^{\mu}(s) \in \Gamma$ and $s \geq s_0$ we have[10]

$$\text{(3.2.6)} \qquad r(y(s)) \leq \theta(\Gamma)|y^0(s)| .$$

In [Ch6] the following has been proved:

PROPOSITION 3.2.1: *Every completion of a Kerr metric is geodesically regular.*

The above is one of the motivations for the introduction of the notion of geodesic regularity. We also have [Ch6][11]:

THEOREM 3.2.2: *Let* (M,g) *admit a geodesically regular completion* $(\mathscr{M},\mathscr{g},i_0)$. *Then all completions of* (M,g) *are geodesically regular and almost equivalent.*

From Theorem 3.2.2. and the results of [As2] it follows:

THEOREM 3.2.3: *Let* $(\mathscr{M},\mathscr{g},i_0)$ *be a geodesically regular completion of a space-time* (M,g). *The total energy-momentum of* (M,g) *is a uniquely defined four vector at* i_0 .

Let us for completeness recall a simple derivation (cf. [Ch3]) of the Ashtekar-Hansen formula for the four-momentum. Simple algebra leads to the identity

$$\text{(3.2.7)}\ 2d(\sqrt{-\det g}\ \epsilon_{\mu\nu\alpha\beta}X^{\nu}X^{\mu}g^{\alpha\gamma}\Gamma^{\beta}_{\gamma\rho}dx^{\rho}) = -\sqrt{-\det g}\ \epsilon_{\mu\nu\alpha\beta}X^{\nu}X^{\mu}R^{\alpha\beta}{}_{\rho\sigma}dx^{\rho}\wedge dx^{\sigma}$$
$$+ 3\delta^{\alpha\beta\gamma}_{\gamma\mu\nu}X^{\nu}\eta^{\gamma\rho}\eta_{\gamma\sigma}g^{\sigma\mu}{}_{,\rho}\ \epsilon_{\alpha\beta\chi\psi}dx^{\chi}\wedge dx^{\psi} + O(r^{-1-2\alpha})$$

where X^{μ} is any constant coefficients vector, d denotes exterior differentiation, and we have assumed that (3.1.1) and

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10 This condition could be violated by spacelike geodesics which are asymptotically tangent to the light cone of i_0 .

11 cf. [Ch6] for more general results if (3.2.4) is not assumed to hold.

$$|g_{\mu\nu,\alpha\beta}| \le C(1+r)^{-2-\alpha}$$

hold. If one also assumes that (3.1.2) holds then $R^{\alpha\beta}{}_{\mu\nu}$ may be replaced by the Weyl tensor $C^{\alpha\beta}{}_{\mu\nu}$ with a supplementary $O(r^{-2-\epsilon})$ term added to $O(r^{-1-2\alpha})$. When integrated over a sphere $r(x) = \text{const}$ the left hand side of (3.2.7) gives zero by Stokes theorem and the limit of the second term at the right hand side of (3.2.7) gives $32\pi p_\mu X^\mu$ so that if $2\alpha > 1$ we have

$$p_\mu X^\mu = \lim_{R\to\infty} \frac{1}{32\pi} \oint_{S(R)} \sqrt{-\det g}\, \epsilon_{\mu\nu\alpha\beta} X^\mu x^\nu C^{\alpha\beta}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma .$$

The double dual identity for the Weyl tensor

$$\epsilon_{\mu\nu\alpha\beta} C^{\alpha\beta}{}_{\rho\sigma} = C^{\alpha\beta}{}_{\mu\nu}\, \epsilon_{\rho\sigma\alpha\beta}$$

gives the Ashtekar-Hansen formula:

(3.2.8) $$p_\mu X^\mu = \lim_{R\to\infty} \frac{1}{16\pi} \oint_{S(R)} \sqrt{-\det g}\, R_{\mu\nu\alpha\beta} X^\mu x^\nu dS^{\alpha\beta} , \quad dS^{\alpha\beta} = \tfrac{1}{2} \epsilon^{\alpha\beta}{}_{\mu\nu} dx^\mu \wedge dx^\nu .$$

3.3 The Background Metric formulation.

The idea of using a background metric to define energy-momentum goes back to Rosen (cf. [Ro] and references therein) and seems to have been rediscovered more or less independently by several authors ([AD] [Ch1] [Fr] [GPP] [Ka] [NAS] [Sor]). The idea is to replace (3.1.3) by

(3.3.1) $$p(X) = \lim_{R\to\infty} \frac{3}{16\pi} \oint_{\substack{r(x)=R \\ t=0}} X^\mu \delta^{\alpha\beta\gamma}_{\lambda\nu\mu} f^{\lambda\rho} f_{\gamma\sigma} g^{\sigma\nu}{}_{|\rho} \sqrt{-\det f}\, dS_{\alpha\beta} ,$$

where $f_{\lambda\rho}$ is a background metric, a bar denotes covariant differentiation with respect to the background and X^μ is a Killing vector of the background metric. It seems that in the asymptotically flat case the most natural thing to ask is that the background metric $f_{\mu\nu}$ be equal to the Minkowski metric $\eta_{\mu\nu}$ in the coordinate system in which (3.1.1) holds, so that one is in fact back in the situation of section 3.1, gaining however the flexibility of using non-asymptotically Minkowskian coordinate systems in actual computations. (3.3.1) can also be used to define energy in non-asymptotically flat space-times ([AD] [AM] [Ch1] [HT] [DrS]), one should however bear in mind that the expression (3.3.1) is potentially background-dependent. For this reason a definition of energy via (3.3.1) for a given set of boundary conditions should be implemented by a proof of background independence, in the appropriate class of backgrounds singled out by the boundary conditions. In this respect the analysis of [AD] or [HT] is far from being complete, the numbers one obtains in the asymptotically anti-de Sitter space-times can be potentially as meaningless as p_μ calculated for asymptotically Minkowskian space-times in coordinate systems in which $g_{\mu\nu}$ tends to $h_{\mu\nu}$ as $r^{-1/2}$ (cf. [DvS] [Ch3] [OM]).

Returning to Minkowski space-time, it appears natural to admit in (3.3.1) any background for which, in a coordinate system in which the physical metric satisfies (3.1.1), the background metric also satisfies (3.1.1) (cf. also [Pe] for a similar approach at null infinity). Let us note the following:

PROPOSITION 3.3.1: *Let* $\Omega_x \subset \mathbb{R}^{n+1}$, $n \geq 1$, *be connected and simply connected, let* $f_{\mu\nu}$ *be a flat metric defined on* Ω_x *satisfying* (3.1.1), *suppose that* $N^x_R = \{x^0 = 0 , r(x) \geq R\} \subset \Omega_x$. *There exist functions* $z^a : \Omega_x \to \mathbb{R}^{n+1}$ *satisfying* (3.1.4) *along* N^x_R[12] *such that*

$$f_{\mu\nu}dx^\mu dx^\nu = -(dz^0)^2 + \Sigma(dz^i)^2 .$$

Proof: Let $x_0 \in \Omega_x$, let $e^a(x_0)$ be any orthonormal frame at x_0, define $e^a(x)$ as the parallel transport of $e^a(x_0)$ along any curve from x_0 to x. By simple connectedness of Ω_x and by flatness of $f_{\mu\nu}$ the tetrads e^a are well defined and covariantly constant. This implies in particular

$$[e_a,e_b] = \nabla_{e_a} e_b - \nabla_{e_b} e_a = 0 , \tag{3.3.2}$$

[12] If $\Omega_{\theta,R,T} \subset \Omega_x$ then (3.1.4) will hold in $\Omega_{\varphi,R,T}$.

where e_a are vector fields dual to e^a. Define the functions z^a as solutions of the equations

(3.3.3) $$dz^a = e^a \, , \, z^a(x_0) = z_0 \, .$$

The integrability conditions of (3.3.3) are satisfied by (3.3.2). (3.3.3) shows that

$$\eta_{ab} dz^a dz^b = \eta_{ab} e^a e^b = f_{\mu\nu} dx^\mu dx^\nu \, ,$$

in particular $\det(\partial z^a / \partial x^\mu)^2 = -\det f_{\mu\nu} \neq 0$ so that the transformation $z^a(x^\mu)$ is a diffeomorphism. Let us show that the tetrads e^a are of the form

(3.3.4) $e^a_\mu = \Lambda^a_\mu + h^a_\mu \, , \, \Lambda^a_\mu$ - a Lorentz matrix, $|h^a_\mu| \leq Cr^{-\alpha}$, $|\partial_\sigma h^a_\mu| \leq Cr^{-\alpha-1}$.

Since the frame is covariantly constant we have

(3.3.5) $$e^a_{\mu,\nu} = \Gamma^\sigma_{\mu\nu} e^a_\sigma \, .$$

Let $e = \Sigma e^a_\mu e^a_\mu$. (3.3.5) gives

$$d(\ell n(e)) = e^{-1} de = e^{-1} \Sigma e^a_\mu e^a_{\mu,\nu} dx^\nu = e^{-1} \Sigma e^a_\mu \Gamma^\sigma_{\mu\nu} e^a_\sigma dx^\nu = O(r^{-1-\alpha}) dx^\nu \, ,$$

so that by Appendix A of [Ch5], e has a finite angle-independent limit at infinity, in particular all the coefficients e^a_μ are bounded. (3.3.5) and Appendix A of [Ch5] imply (3.3.4), a simple analysis of (3.3.3) and (3.3.4) yields (3.1.4).

Proposition 3.3.1 implies that $p(X)$ is background independent for $\alpha > 1/2$ in the class of backgrounds described above: in the coordinate system given by proposition 3.3.1 we have

$$p(X) = \lim_{R\to\infty} \frac{3}{16\pi} \oint_{\substack{r(x)=R \\ t=0}} X^\mu \delta^{\alpha\beta\gamma}_{\lambda\nu\mu} \eta^{\lambda\rho} \eta_{\lambda\sigma} (g^{\sigma\nu}{}_{,\rho} - f^{\sigma\nu}{}_{,\rho}) dS_{\alpha\beta}$$

($g^{\sigma\nu}{}_{|\rho} = (g^{\sigma\nu} - f^{\sigma\nu})_{|\rho} = g^{\sigma\nu}{}_{,\rho} - f^{\sigma\nu}{}_{,\rho}$ + "$\Gamma(g-f)$" terms which are $O(r^{-1-2\alpha})$ and which will give no contribution in the limit if $\alpha > 1/2$), the term involving derivatives of $f^{\sigma\nu}$ is the negative of the four momentum of the flat metric $f_{\mu\nu}$ which proposition 3.3.1. It should be emphasized that the family of admissible backgrounds, as introduced above, is potentially dependent upon the choice of the coordinates one started with, to free oneself from those one can make appeal to the results of section 3.1.

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