OPERATOR THEORETIC PROBLEMS FOR FOURIER MULTIPLIERS IN $L^p(\mathbf{R})$

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Abstract. This survey article (also containing a few new results) is devoted to a consideration of various properties of some well known classes of operators, such as decomposable operators, generalized scalar operators, well-bounded operators and AC-operators, for example, when attention is restricted to the particular (but important) setting of Fourier p-multiplier operators on the real line group. The aim is to highlight some relevant phenomena and to pose several natural questions which occur in this setting.

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1. Introduction. One of the great successes of operator theory in Hilbert spaces was the spectral theorem for normal operators, which can be viewed as a natural extension of the theory of diagonalizable matrices in finite dimensional spaces. In the late 1940's and early 1950's, N. Dunford extended the notion of a normal operator (actually any operator similar to a normal operator) to the Banach space setting. He considered any operator S having an integral representation of the form

$$S = \int_{\mathbf{C}} \lambda dE(\lambda), \tag{1.1}$$

where $E(\cdot)$ is a projection-valued measure on the Borel sets of $\mathbf{C}$ having its support equal to the spectrum $\sigma(S)$ of S and which is countably additive for the strong operator topology. The operator S is called *scalar-type spectral* and the measure $E(\cdot)$, necessarily unique, is called the *resolution of the identity* of S. Actually, Dunford's theory also allows for operators of the form $S+N$, where N is any operator commuting with S and which is *quasinilpotent* (i.e. $\sigma(N) = \{0\}$). Such operators $S+N$ are called *spectral*; see [15] for the general theory. Unfortunately, as attractive as the theory is it turned out to be more restrictive than had been anticipated, [18]. Indeed, the analogue in L^p-spaces of many normal operators in L^2-spaces fail to be spectral or, require additional (and somewhat stringent) assumptions (eg. [30]). Even such natural candidates as constant coefficient partial differential operators in $L^p(\mathbf{R}^n)$, for $p \neq 2$, turn out to be (unbounded) spectral operators only if the polynomial symbol defining the operator is constant; see [4].

The above remarks make it clear that for applications to concrete and natural examples the theory of spectral operators has serious limitations. This was soon realized by the mathematical community and myriad new theories arose. We wish to discuss a few of these.

It is clear from (1.1) that any scalar-type spectral operator S acting in a Banach space X can be viewed as possessing a special kind of *operator-valued distribution.* Indeed, let $C^\infty(\mathbf{C}) \simeq C^\infty(\mathbf{R}^2)$ denote the Fréchet algebra of all infinitely differentiable (and $\mathbf{C}$-valued) functions defined on $\mathbf{R}^2$, equipped with the sequence of seminorms

$$q_{n,k} : g \mapsto \sum_{|\alpha| \leq k} (\alpha)^{-1} \sup_{\|x\| \leq n} \left| \frac{\partial^\alpha g}{\partial x^\alpha}(x) \right|, \qquad n \geq 1, \quad k \geq 0,$$

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where $x = (x_1, x_2) \in \mathbf{R}^2$ and we use standard multi-index notation for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in \mathbf{N} \cup \{0\}$. Then there exists a *continuous homomorphism* $\Phi : C^\infty(\mathbf{R}^2) \to \mathcal{L}(X)$ satisfying $\Phi(\mathbb{1}) = I$ (the identity operator on X) and $\Phi(id_{\mathbf{C}}) = S$. Here $\mathbb{1}$ denotes the function constantly equal to 1 on $\mathbf{R}^2$, $id_{\mathbf{C}}$ is the identity function on $\mathbf{R}^2 \simeq \mathbf{C}$, and $\mathcal{L}(X)$ is the space of all bounded linear operators of X into itself (equipped with the operator norm topology). Of course, the particular Φ required is given by $g \mapsto \int_{\mathbf{C}} g(\lambda)dE(\lambda)$, for each $g \in C^\infty(\mathbf{C})$. Moreover, the distributional order of Φ is zero since $\|\Phi(g)\| \leq K \sup\{|g(\lambda)| : \lambda \in \sigma(S)\} \leq K q_{n,0}(g)$, where $K = 4\sup\{\|E(\delta)\| : \delta \subseteq \mathbf{C},\ \delta \text{ Borel } \}$ and n is the least positive integer for which $\sigma(S) \subseteq \{z \in \mathbf{C} : |z| \leq n\}$. Observations of this kind led I. Colojoară and C. Foiaş to initiate the theory of *generalized scalar operators*, namely, those operators $S \in \mathcal{L}(X)$ possessing a homomorphism $\Phi : C^\infty(\mathbf{R}^2) \to \mathcal{L}(X)$ with the above properties (but *not* required to have distributional order zero). Such a mapping Φ (which is a special case of the more general notion of a *functional calculus*) is called a *spectral distribution* for S. If S possesses a spectral distribution which takes all of its values in the *bicommutant* algebra $\{S\}^{cc} \subseteq \mathcal{L}(X)$ of S, then S is called a *regular* generalized scalar operator and such a spectral distribution is also called *regular*. For the theory of generalized scalar operators we refer to [13]; see also [17] and [35].

In a different direction, it is possible to view the restrictive nature of scalar-type spectral operators given by (1.1) as being due to the fact that the resolution of the identity of S is required to generate *unconditionally convergent* integrals (for the strong operator topology), thereby generating a functional calculus for S based on the algebra of all bounded Borel functions on $\sigma(S)$. Adopting a Riemann-Stieltjes type approach to integration D.R. Smart, [33], developed the notion of a *well bounded operator* $S \in \mathcal{L}(X)$ as one satisfying $\sigma(S) \subseteq \mathbf{R}$ and which admits a continuous functional calculus $\Phi : AC(J) \to \mathcal{L}(X)$, where $AC(J)$ is the algebra of all absolutely continuous functions defined on an interval $J \supseteq \sigma(S)$ and equipped with the usual variation norm for functions of bounded variation. If the underlying Banach space X is reflexive, then it is possible to retain an integral representation for S of the form

$$S = \int_J^{\oplus} \lambda dF(\lambda), \tag{1.2}$$

where the right-hand-side of (1.2) now exists as the strong operator limit of certain Riemann-Stieltjes sums with respect to an increasing, uniformly bounded *spectral family* $F : \mathbf{R} \to \mathcal{L}(X)$ which is concentrated on J; see [9] for these definitions. The main feature of such a theory is that it can accommodate functional calculi based on integration theories which do not require a σ-additive measure. Well bounded operators (on arbitrary spaces X) admitting a representation of the type (1.2) are said to be of *type* (B). For the general theory of well bounded operators we refer to [14]. It is worth noting that every well bounded operator is necessarily generalized scalar. Indeed, such an operator S admits a spectral distribution $\Phi : C^\infty(\mathbf{R}^2) \to \mathcal{L}(X)$, of order at most one, given by $\Phi(g) := (g|_J)(S)$, for $g \in C^\infty(\mathbf{R}^2)$, where $f \mapsto f(S)$ denotes the functional calculus for S based on $AC(J)$. Actually,

$$\|(g|_J)(S)\| = O\left(\sup_J |g| + \sup_J \left|\frac{\partial g}{\partial x}\right|\right), \qquad g \in C^\infty(\mathbf{R}^2).$$

In order to extend the theory to admit some operators with complex spectra, J.R. Ringrose, [27], introduced the notion of well boundedness on curves. Further generalizations are also known, such as *polar operators*, [9], (i.e. $S \in \mathcal{L}(X)$ is polar if there exist commuting well bounded operators R, A of type (B) such that $S = Re^{iA}$), *trigonometrically well bounded operators,* [11], and

AC-operators, [10], (i.e. those operators S possessing a continuous functional calculus based on the algebra $AC(J{\times}K)$ for suitable intervals $J, K \subseteq \mathbf{R}$). The estimate

$$\|(g|_{J\times K})(S)\| = O\left(\sup_{J\times K}|g| + \sup_{J\times K}\left|\frac{\partial g}{\partial x}\right| + \sup_{J\times K}\left|\frac{\partial g}{\partial y}\right| + \sup_{J\times K}\left|\frac{\partial^2 g}{\partial x \partial y}\right|\right),$$

for $g \in C^\infty(\mathbf{R}^2)$, shows that each AC-operator admits a spectral distribution of order at most two, [[10]; p.308]. In particular, AC-operators are generalized scalar.

Finally we mention an extensive class of operators introduced by C. Foiaş, [19], namely the *decomposable operators.* An operator $S \in \mathcal{L}(X)$ is decomposable (in the sense of Foiaş) if, for every finite open cover $\{U_j\}_{j=1}^r$ of $\mathbf{C}$ there are closed invariant subspaces $\{X_j\}_{j=1}^r$ for S such that $X = X_1 + \ldots + X_r$ and $\sigma(S|_{X_j}) \subseteq U_j$ for all $1 \leq j \leq r$. Despite the generality of the definition the class of decomposable operators has many desirable features; all of its members have the single-valued extension property, the spectrum always coincides with the approximate point spectrum, [[13]; p.31], and so on. Moreover, it contains many well known classes of operators such as the generalized scalar operators, spectral operators, compact operators, etc..

Our aim is to examine various aspects of operators from the above classes when we restrict our attention to the special setting of Fourier multiplier operators in L^p-spaces. There are several reasons for considering this more specialized setting. Firstly, multiplier operators play a fundamental role in harmonic analysis and so some more refined aspects of such operators in a specific setting are not without interest. Secondly, the setting of multiplier operators is rich enough in that it is a non-trivial class of operators and yet is concrete enough to provide examples illustrating much of the general theory of the various classes of operators alluded to above, especially from the viewpoint of functional calculi. Furthermore, there have been some significant advances made in recent years concerning certain global aspects of constant coefficient partial differential operators in Euclidean L^p-spaces, [4], [5], [6], [7], where the methods are based on local spectral theory and the construction of suitable functional calculi within the space of *multiplier operators.* Such a differential operator of the type mentioned can be viewed as an unbounded Fourier p-multiplier operator. The context of this note concerns only bounded operators. However, by passing to a consideration of the resolvent operators of the differential operator the same phenomena occurring for the differential operator are also exhibited by certain *bounded* Fourier p-multiplier operators. This does not lead to any significant loss of generality and still exhibits most of the essential phenomena that we wish to illustrate. For simplicity of presentation, attention will be restricted to the line group $\mathbf{R}$.

Several questions are formulated throughout the text. We hope these will be of some interest to people from a variety of areas such as operator theory, Banach algebras and/or harmonic analysis.

2. Fourier multiplier operators. The Fourier transform

$$\hat{f}(x) = (2\pi)^{-1/2}\int_{\mathbf{R}} e^{-ixt}f(t)dt, \qquad \text{a.e. } x \in \mathbf{R},$$

is defined for all $f \in L^1(\mathbf{R})$. The Hausdorff-Young theorem states that if $1 \leq p \leq 2$ and $f \in L^1(\mathbf{R}) \cap L^p(\mathbf{R})$, then $\|\hat{f}\|_q \leq \|f\|_p$ where q (called the conjugate index of

p) satisfies $p^{-1}+q^{-1}=1$. Accordingly, the mapping $f \mapsto \hat{f}$ extends by continuity to a bounded linear operator $\mathcal{F}: L^p(\mathbf{R}) \to L^q(\mathbf{R})$. Elements $\mathcal{F}(f)$, for arbitrary $f \in L^p(\mathbf{R})$, are also denoted by $\hat{f}$. An element of $\mathcal{L}(L^p(\mathbf{R}))$, $1 \le p < \infty$, is called a (Fourier) *p-multiplier operator* if it commutes with each translation operator τ_x, for $x \in \mathbf{R}$. Of course, τ_x is defined by $\tau_x f: y \mapsto f(x-y)$ for a.e. $y \in \mathbf{R}$, with $f \in L^p(\mathbf{R})$. It is known that $T \in \mathcal{L}(L^p(\mathbf{R}))$ is a p-multiplier operator iff there exists $\psi \in L^\infty(\mathbf{R})$, necessarily unique, such that

$$(Tf)\hat{} = \psi \hat{f}, \qquad f \in L^2(\mathbf{R}) \cap L^p(\mathbf{R}). \tag{2.1}$$

The function ψ is called a *p-multiplier* and the associated p-multiplier operator T is denoted by $S_\psi^{(p)}$. The space of all p-multiplier functions is denote by $\mathcal{M}^{(p)}$. The inequality

$$\|\psi\|_\infty \le \|S_\psi^{(p)}\|_{\mathcal{L}(L^p(\mathbf{R}))}, \qquad \psi \in \mathcal{M}^{(p)}, \tag{2.2}$$

is well known. So is the fact that $\mathcal{M}^{(2)} = L^\infty(\mathbf{R})$ and $\mathcal{M}^{(1)} = \{\hat{\mu} : \mu \in BM_r\} := BM_r\hat{}$. Here BM_r is the algebra (with respect to convolution) of all regular, complex Borel measures on $\mathbf{R}$ and $\hat{\mu}$ denotes the Fourier-Stieltjes transform of $\mu \in BM_r$. If we equip $\mathcal{M}^{(p)}$ with the norm

$$|||\psi|||_p := \|S_\psi^{(p)}\|_{\mathcal{L}(L^p(\mathbf{R}))}, \qquad \psi \in \mathcal{M}^{(p)}, \tag{2.3}$$

then $\mathcal{M}^{(p)}$ becomes a commutative, unital Banach algebra with respect to pointwise multiplication. Moreover, for each $\psi \in \mathcal{M}^{(p)}$, the functions $\mathrm{Re}(\psi)$, $\mathrm{Im}(\psi)$ and the complex conjugate $\overline{\psi}$ also belong to $\mathcal{M}^{(p)}$. Since $\mathcal{M}^{(p)}$ is isometrically isomorphic to $\mathcal{M}^{(q)}$, where q is the conjugate index of $p \in (1,\infty)$ we will often restrict our attention to $1 \le p \le 2$. The space $\{S_\psi^{(p)} : \psi \in \mathcal{M}^{(p)}\}$ of all *p-multiplier operators* will be denoted by $Op(\mathcal{M}^{(p)})$, $1 \le p < \infty$. The subalgebra of BM_r consisting of the *discrete measures* is denoted by $BM_r^{(d)}$.

Let $\mathcal{B}_p := \{E \subseteq \mathbf{R} : \chi_E \in \mathcal{M}^{(p)}\}$, where χ_E denotes the characteristic function of E. Since elements of $\mathcal{M}^{(1)}$ are continuous it is clear that $\mathcal{B}_1 = \{\emptyset, \mathbf{R}\}$ and since $\mathcal{M}^{(2)} = L^\infty(\mathbf{R})$ it is clear that $\mathcal{B}_2$ is the family of all measurable subsets of $\mathbf{R}$. What about $1 < p < 2$? Since $\mathcal{M}^{(p)}$ is an algebra of functions under pointwise multiplication it follows that $\mathcal{B}_p$ is an *algebra of subsets* of $\mathbf{R}$. It is known that all intervals in $\mathbf{R}$ belong to every $\mathcal{B}_p$, $1 < p < \infty$. Of course, there are sets in $\mathcal{B}_p$ which are more complex than finite disjoint unions of intervals. Let $\{\lambda_k\}_{k=0}^\infty$ be a Hadamard sequence of positive numbers, i.e. $\inf\{\lambda_{k+1}/\lambda_k : k \in \mathbf{N}_0\} \ge r > 1$, where $\mathbf{N}_0 := \{0\} \cup \mathbf{N}$. Define $\Delta_j = [\lambda_{j-1}, \lambda_j)$ if $j > 0$, $\Delta_0 = (-\lambda_0, \lambda_0)$ and $\Delta_j = (-\lambda_{|j|}, -\lambda_{|j|-1}]$ if $j < 0$. Then $\{\Delta_j : j \in \mathbf{Z}\}$ has the Littlewood-Paley property. So, if $\{\alpha_k\}_{k \in \mathbf{Z}}$ is any 0-1 valued sequence, then the function $\psi : \mathbf{R} \to \mathbf{C}$ given by $\psi := \Sigma_{k \in \mathbf{Z}} \alpha_k \chi_{\Delta_k}$ belongs to $\mathcal{M}^{(p)}$ and so both $\psi^{-1}(\{1\})$ and $\psi^{-1}(\{0\})$ belong to $\mathcal{B}_p$. There are still other types of sets in $\mathcal{B}_p$. For instance, if K is a connected subset of $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$ and $s \in \mathbf{R}$, then the function $t \mapsto \chi_K(e^{ist})$, for $t \in \mathbf{R}$, belongs to $\mathcal{M}^{(p)}$, $1 < p < \infty$, [20]. Accordingly, the periodic set $\{t \in \mathbf{R} : e^{ist} \in K\}$ belongs to $\mathcal{B}_p$ for all $1 < p < \infty$. So, the algebra of sets $\mathcal{B}_p$ is rather rich.

One of the interesting features about multiplier sets stems from Banach space geometry since elements of $\mathcal{B}_p$ yield a characterization of translation invariant, *complemented* subspaces of $L^p(\mathbf{R})$. Clearly if $\chi_E \in \mathcal{M}^{(p)}$, then $S_{\chi_E}^{(p)}$ is a projection onto some translation invariant, complemented subspace of $L^p(\mathbf{R})$. Conversely, a result of H. Rosenthal, [32], implies that every complemented, translation invariant subspace of $L^p(\mathbf{R})$, $1 < p < \infty$, is the range of some projection of the form $S^{(p)}_{\chi_E}$ with $E \in \mathcal{B}_p$. Recent work of I. Kluvánek, [23], [24], shows that $\mathcal{B}_p$ is also of interest for certain aspects of operator theory related to integration theory; see also [[20],[21],[25],[27],[28],[29]]. The following result is therefore of some interest.

Proposition 2.1.
(i) *Let $1 < p < \infty$, $p \neq 2$. If $E \in \mathcal{B}_p$, then there exists an open set $U \subseteq \mathbf{R}$ such that $\chi_E = \chi_U$ a.e.. In particular, E is (up to a Lebesgue null set) equal to a countable union of pairwise disjoint intervals.*
(ii) *For each $p \neq 2$ there exists an open set $V \subseteq [0,1]$ such that $V \notin \mathcal{B}_p$.*
(iii) *Let $1 < p < \infty$ and let $\{E(n)\}_{n=1}^{\infty} \subseteq \mathcal{B}_p$ be a sequence of sets such that*

$$E := \{t \in \mathbf{R} : \lim_{n\to\infty} \chi_{E(n)}(t) \text{ exists}\}$$

has the property that $\mathbf{R}\backslash E$ is a Lebesgue null set and $\sup_n |||\chi_{E(n)}|||_p < \infty$. If $E^+ := \{t \in \mathbf{R} : \lim_{n\to\infty} \chi_{E(n)} = 1\}$, then both $E^+, E \in \mathcal{B}_p$ and $\{S^{(p)}_{\chi_{E(n)}}\}_{n=1}^{\infty}$ converges to $S^{(p)}_{\chi_{E^+}}$ in the strong operator topology of $\mathcal{L}(L^p(\mathbf{R}))$.

Part (i) is due to V. Lebedev and A. Olevskiĭ, [26]. Part (ii) follows from a careful examination of the well known construction of a *set of uniqueness* due to A. Figà-Talamanca and G.I. Gaudry; see the proof of Theorem 3.3.8 in [27], for example. For (iii) we refer to [[28]; Theorem 6]. As a consequence we have the following fact.

Proposition 2.2. *Let $1 < p < \infty$, $p \neq 2$. Then the algebra of sets $\mathcal{B}_p$ fails to be a σ-algebra. Moreover, $\sup\{|||\chi_E|||_p : E \in \mathcal{B}_p\} = \infty$.*

Proof. Let $V \subseteq [0,1]$ be an open set such that $V \notin \mathcal{B}_p$; see Proposition 2.1(ii). Then $V = \cup_{n=1}^{\infty} I_n$ for pairwise disjoint intervals $I_n \subseteq [0,1]$, $n \in \mathbf{N}$. Since $I_n \in \mathcal{B}_p$, for each $n \in \mathbf{N}$, this already shows that $\mathcal{B}_p$ is not a σ-algebra. Define $E(n) := \cup_{j=1}^{n} I_j$, for $n \in \mathbf{N}$ and note, in the notation of Proposition 2.1(iii), that $E^+ := \{t \in \mathbf{R}; \lim_{n\to\infty} \chi_{E(n)}(t) = 1\}$ coincides with V up to a Lebesgue null set. If $\sup_n\{|||\chi_{E(n)}|||_p : n \in \mathbf{N}\} < \infty$, then it follows from Proposition 2.1(iii) that E^+ (and hence also V) belongs to $\mathcal{B}_p$ which is contrary to the choice of V. Accordingly, $\sup\{|||\chi_E|||_p : E \in \mathcal{B}_p\}$ is infinite. □

Note that Propositions 2.1 and 2.2 imply that $\cup_{1<p<2}\mathcal{B}_p$ is an increasing union of algebras of measurable sets (each one containing all intervals) which is not a σ-algebra.

From the above discussion it is clear that the algebra $\mathrm{sim}(\mathcal{B}_p)$ of all *simple functions* based on $\mathcal{B}_p$ (which is contained in $\mathcal{M}^{(p)}$ of course) contains many non-trivial elements. Indeed, it is known that $\{S^{(p)}_{\varphi} : \varphi \in \mathrm{sim}(\mathcal{B}_p)\}$, $1 < p < \infty$, is dense in $Op(\mathcal{M}^{(p)})$ with respect to the *strong operator topology*, [[25]; Examples 9 and 19]. The situation for the closure of $\mathrm{sim}(\mathcal{B}_p)$ in the *operator norm topology* (i.e. in $(\mathcal{M}^{(p)}, |||\cdot|||_p)$), denoted by $\overline{\mathrm{sim}}(\mathcal{B}_p)$, is quite different and will be discussed later.

At this point we merely record the fact if $\varphi \in \overline{\text{sim}}(\mathcal{B}_p)$, then also $\overline{\varphi}$, $\text{Re}(\varphi)$ and $\text{Im}(\varphi)$ belong to $\overline{\text{sim}}(\mathcal{B}_p)$, [[28]; Lemma 2].

We conclude this section by noting that there are no non-zero multiplier operators which are compact; in fact, a bit more is true. For a Banach space X let $K(X)$ denote the operator norm closed, two sided ideal of all compact operators. For each $T \in \mathcal{L}(X)$ define

$$\kappa(T) := \inf\{\|T - C\| : C \in K(X)\}.$$

Then T is called a *Riesz operator* if $\lim_{n\to\infty}[\kappa(T^n)]^{1/n} = 0$. Since $T^n \in K(X)$ for all $n \in \mathbf{N}$, whenever $T \in K(X)$, it is clear that every compact operator is a Riesz operator. The spectrum of a Riesz operator $T \in \mathcal{L}(X)$ is a countable set with zero as only possible limit point. Moreover, if $\lambda \in \sigma(T)\backslash\{0\}$, then the spectral projection P_λ corresponding to the spectral set $\{\lambda\}$ of T, which is given by

$$P_\lambda = \frac{1}{2\pi i}\int_\gamma (T - \mu I)^{-1} d\mu \tag{2.4}$$

for any contour γ in the resolvent set $\rho(T) := \mathbf{C}\backslash\sigma(T)$ separating λ from $\sigma(T)\backslash\{\lambda\}$, is necessarily a *finite rank projection.* For these basic facts, definitions and further properties of Riesz operators we refer to [14], for example.

Proposition 2.3. *Let $1 \leq p < \infty$ and $T \in Op(\mathcal{M}^{(p)})$ be a Riesz operator. Then $T = 0$.*

Proof. Suppose there exists a point $\lambda \in \sigma(T)\backslash\{0\}$. Since the integrand in (2.4) is a continuous function on the compact set γ and $d\mu$ is a finite measure, it follows that the integral (2.4) can be formed as an operator norm limit of Riemann sum approximations. But, T commutes with all translations and hence, so does $(T - \mu I)^{-1}$ for each $\mu \in \gamma$. Accordingly, all such Riemann sums are multiplier operators and hence, so is the limit operator P_λ.

Let $1 < p < \infty$. Since $P_\lambda = S^{(p)}_{\chi_E}$ for some $E \in \mathcal{B}_p$ and $P_\lambda \neq 0$, it follows from Proposition 2.1(i) that there must exist a non-trivial interval $J \subseteq E$. Then the range of the projection $S^{(p)}_{\chi_J}$, which is easily seen to be infinite dimensional, is contained in the range of $S^{(p)}_{\chi_E} = P_\lambda$. This contradicts the fact that P_λ is a finite rank projection. Accordingly, $\sigma(T) = \{0\}$. It is clear from (2.2) that the Banach algebra $Op(\mathcal{M}^{(p)})$ is semisimple (i.e. contains no non-zero quasinilpotent elements) and so $T = 0$.

For $p = 1$ we have seen that $\mathcal{B}_p = \{\emptyset, \mathbf{R}\}$ and so P_λ, if it is a non-zero idempotent multiplier operator, must equal the identity operator. This again contradicts P_λ being finite rank and so $\sigma(T) = \{0\}$. For the same reason as above $T = 0$. □

3. Decomposable multiplier operators. For a systematic study of decomposable multiplier operators on general LCA groups we refer to [[3], [16]], for example, and the references therein. As mentioned before our attention will be restricted to $\mathbf{R}$.

It is known that $\mathcal{M}^{(1)} \subseteq \mathcal{M}^{(p)}$, for all $1 \leq p < \infty$, with the inclusion being strict if $p \neq 1$. Indeed, since $\mathcal{M}^{(1)} = BM_r\hat{}$ is contained in the bounded continuous functions we see that $\chi_{[0,1]} \in \mathcal{M}^{(p)}\backslash\mathcal{M}^{(1)}$ for all $1 < p < \infty$. A general fact about multiplier operators is the inclusion

$$\text{ess range}(\varphi) \subseteq \sigma(S_\varphi^{(p)}), \qquad \varphi \in \mathcal{M}^{(p)}, \tag{3.1}$$

valid for $1 \leq p < \infty$, [36], where ess range(φ) denotes the essential range of the function φ. Let $\mathcal{D}^{(p)}$ denote the family of all functions $\varphi \in \mathcal{M}^{(p)}$ such that the multiplier operator $S_\varphi^{(p)} \in \mathcal{L}(L^p(\mathbf{R}))$ is decomposable. A multiplier function $\varphi \in \mathcal{M}^{(p)}$ is said to have the *spectral mapping property* if

$$\text{ess range}(\varphi) = \sigma(S_\varphi^{(p)}). \tag{3.2}$$

The first part of the following result shows that decomposable multiplier operators have a very desirable feature.

Proposition 3.1.
(i) *Suppose* $1 \leq p < \infty$. *Then every multiplier* $\varphi \in \mathcal{D}^{(p)}$ *satisfies the spectral mapping property.*
(ii) *If* $p \neq 2$, *then* $\mathcal{M}^{(1)} \backslash \mathcal{D}^{(p)} \neq \emptyset$ *and hence,* $\mathcal{D}^{(p)} \neq \mathcal{M}^{(p)}$.

For part (i) we refer to (the proof of) Lemma 3.2 in [3], or [[4]; Corollary 3.4], for example. If $p \neq 2$, then there exists $\mu_p \in BM_r$ such that ess range$(\hat{\mu}_p) \neq \sigma(S_{\hat{\mu}_p}^{(p)})$, [[37]; p.239]. This together with (i) establishes part (ii). It is worth noting, for $1 < p < \infty$ (and in $\mathbf{R}^n$ and $\mathbf{T}^n$), that M. Zafran even exhibited p-multipliers in $C^\infty(\mathbf{R})$ which vanish at infinity and fail the spectral mapping property, [[37]; Theorem 3.2].

It was noted in the Introduction that every decomposable operator has the single valued extension property (see [[13]; p.1] for the definition). Multiplier operators provide a natural class of examples showing that the converse is false. Indeed, *every element of* $Op(\mathcal{M}^{(p)})$, $1 \leq p < \infty$, *has the single valued extension property*; see the proof of [[3]; Theorem 3.4], for example. However, since $\mathcal{M}^{(p)} \backslash \mathcal{D}^{(p)} \neq \emptyset$ for all $1 \leq p < \infty$ (c.f. Proposition 3.1(ii)) it follows that every multiplier operator $S_\varphi^{(p)}$, for $\varphi \in \mathcal{M}^{(p)} \backslash \mathcal{D}^{(p)}$ and $1 \leq p < \infty$, has the single valued extension property but is not decomposable. Of course, examples of operators without the single valued extension property are also known, [[13]; p.10], [22].

In general, the sum and product of commuting decomposable operators need not be decomposable. However, for multiplier operators some positive conclusions can be made.

Proposition 3.2. ([3]; §3]) (i) $\mathcal{D}^{(1)}$ *is a (proper) closed subalgebra of* $\mathcal{M}^{(1)}$ *and contains both* $\hat{L}^1 := \{\hat{g} : g \in L^1(\mathbf{R})\}$ *and* $(BM_r^{(d)})\hat{} := \{\hat{\mu} : \mu \in BM_r^{(d)}\}$.
(ii) *If* $\varphi \in \mathcal{D}^{(1)}$ $(\subseteq \mathcal{M}^{(p)})$ *and* $\psi \in \mathcal{D}^{(p)}$, *then both* $\varphi\psi$ *and* $\varphi + \psi$ *belong to* $\mathcal{D}^{(p)}$ *for all* $1 \leq p < \infty$. *In particular (put* $\psi = \mathbb{1}$*), we have* $\mathcal{D}^{(1)} \subseteq \mathcal{D}^{(p)}$, $1 \leq p < \infty$.
(iii) $\mathcal{D}^{(p)}$ *is a closed set in* $\mathcal{M}^{(p)}$ *and hence, the closure of* $\mathcal{D}^{(1)}$ *in* $(\mathcal{M}^{(p)}, |||\cdot|||_p)$ *is contained in* $\mathcal{D}^{(p)}$, $1 \leq p < \infty$.
(iv) *Let* $p \in (1, \infty)$. *If* r *satisfies* $|\frac{1}{p} - \frac{1}{2}| < |\frac{1}{r} - \frac{1}{2}|$, *then* $\mathcal{M}^{(r)} \cap C_0(\mathbf{R}) \subseteq \mathcal{D}^{(p)}$, *where* $C_0(\mathbf{R})$ *is the space of continuous functions on* $\mathbf{R}$ *which vanish at infinity.*

For the inclusion $\hat{L}^1 \subseteq \mathcal{D}^{(1)}$ we also refer to [[13]; p.205].

Some of the conclusions of Proposition 3.2 as proved in [3] rely on a general limit result of C. Apostol, [[8]; Corollary 2.8], which is valuable to formulate in its own right in our setting.

Proposition 3.3. *Let $1 \leq p < \infty$, $\varphi \in \mathcal{M}^{(p)}$ and $\{\varphi_n\}_{n=1}^{\infty} \subseteq \mathcal{D}^{(p)}$ be a sequence such that*

$$\lim_{n\to\infty} (\sup\{|\lambda| : \lambda \in \sigma(S^{(p)}_{\varphi_n} - S^{(p)}_{\varphi}) = \sigma(S^{(p)}_{\varphi_n-\varphi})\}) = 0. \tag{3.3}$$

Then $\varphi \in \mathcal{D}^{(p)}$. In particular, if $\lim_{n\to\infty} |||\varphi_n - \varphi|||_p = 0$, then (3.3) is satisfied since the spectral radius of any bounded linear operator is at most equal to the norm of the operator. Or, if $\varphi_n - \varphi$ has the spectral mapping property, for each $n \in \mathbf{N}$, then $\lim_{n\to\infty} \|\varphi_n - \varphi\|_\infty = 0$ suffices for (3.3) to hold (c.f. (3.2)).

In view of Proposition 3.1(i) and the fact that the containment $\mathcal{D}^{(p)} \subseteq \mathcal{M}^{(p)}$ is strict, we refer to elements of $\mathcal{D}^{(p)}$ as "good p-multipliers". Some natural questions arise.

Qu.1. (a) *For $p \notin \{1, 2\}$ is $\mathcal{D}^{(p)}$ a vector subspace or a subalgebra of $\mathcal{M}^{(p)}$?*

(b) *The same question as in* (a) *arises for the set of p-multipliers with the spectral mapping property. Also, is the set of p-multipliers with the spectral mapping property closed in $\mathcal{M}^{(p)}$?*

Qu.2. *How extensive is the collection $\mathcal{D}^{(p)}$ and are there ways of identifying some natural subcollections of $\mathcal{D}^{(p)}$ within $\mathcal{M}^{(p)}$?*

Qu.3. *Is there a "reasonable" characterization of $\mathcal{D}^{(p)}$?*

The answers to Qu.1 and Qu.3 seem difficult and we are unable to give any insight in general, other than a suggestive remark concerning Qu.3. A plausible criterion in this regard might be that the elements of $\mathcal{D}^{(p)}$ are precisely those elements of $\mathcal{M}^{(p)}$ which satisfy the spectral mapping property. For the line group $\mathbf{R}$ this conjecture may be correct. However, for other LCA groups it is known that there exist groups G and elements $\hat{\mu} \in \mathcal{M}^{(1)}(G)$ with $\mu \in BM_r(G)$ such that the spectral mapping property $\sigma(S^{(1)}_{\hat{\mu}}) = \overline{\hat{\mu}(\Gamma)}$ holds (with Γ the dual group of G), but $S^{(1)}_{\hat{\mu}} \notin \mathcal{D}^{(1)}(G)$; see [[3]; p.32] and [[36]; Example 3.4]. This same example (c.f. [[3]; p.32]) shows that the answer to Qu.1(b) is negative for general LCA groups. However, the answer for the line group $\mathbf{R}$ is still unknown.

For the remainder of this section we wish to consider Qu.2 and allude to a positive answer for both parts of this question. Let $m^{(p)}$ denote the closure of $\hat{L}^1$ in $\mathcal{M}^{(p)}$. We have already seen that $m^{(p)} \subseteq \mathcal{D}^{(p)}$. Also, Proposition 3.2(i),(ii) shows that all trigonometric polynomials belong to $\mathcal{D}^{(p)}$, $1 \leq p < \infty$. Hence, any linear combination of translation operators is a decomposable multiplier operator, for all $1 \leq p < \infty$. We will see that $\mathcal{D}^{(p)}$ is significantly larger than both $m^{(p)}$ and the closure in $\mathcal{M}^{(p)}$ of the space of trigonometric polynomials. Our starting point is the observation that if a multiplier operator has a rich enough functional calculus, then it will necessarily be decomposable (the converse failing to be true in general). This approach is used effectively in [13]; see also [4], [5], [6], [7], for example.

Let BV denote the Banach algebra of all functions on $\mathbf{R}$ with bounded variation, equipped with the usual BV-norm $\|\cdot\|_{BV}$, [[13]; p.208]. Then for each $1 < p < \infty$

there is a constant $K_p > 0$, [[13]; p.208] such that

$$||S_\varphi^{(p)}||_{\mathcal{L}(L^p(\mathbf{R}))} \leq K_p||\varphi||_{BV}, \qquad \varphi \in BV.$$

In particular, $BV \subseteq \mathcal{M}^{(p)}$ for all $1 < p < \infty$. Since the inequality

$$||g \circ \varphi||_{BV} \leq ||\varphi||_{BV} \cdot \sup\{|g(x,y)| + \left|\frac{\partial g}{\partial x}(x,y)\right| + \left|\frac{\partial g}{\partial y}(x,y)\right| : |(x,y)| \leq ||\varphi||_{BV}\}$$

is valid for all $g \in C^\infty(\mathbf{R}^2)$, [[13]; p.210], it is clear that $g \mapsto S_{go\varphi}^{(p)}$, for $g \in C^\infty(\mathbf{R}^2)$, is a spectral distribution for $S_\varphi^{(p)}$. Accordingly, $S_\varphi^{(p)}$ is generalized scalar and so, in particular, decomposable, [[13]; p.65]. This shows that $BV \subseteq \mathcal{D}^{(p)}$, $1 < p < \infty$. It is clear from (2.2) and the fact that $\hat{L}^1 \subseteq c_0(\mathbf{Z})$ (via the Riemann-Lebesgue lemma), that $m^{(p)} \subseteq c_0(\mathbf{Z})$. Accordingly, the inclusion $BV \subseteq \mathcal{D}^{(p)}$ implies that $\mathcal{D}^{(p)} \backslash m^{(p)}$ is non-empty for all $1 < p < \infty$.

The class of decomposable multiplier operators given by BV can be further extended, [[3]; §3]. Recall the dyadic decomposition of $\mathbf{R}$ is given by the intervals $\{\Delta_j\}_{j\in\mathbf{Z}}$, where $\Delta_j = [2^{j-1}, 2^j)$ if $j > 0$, $\Delta_0 = (-1,1)$ and $\Delta_j = (-2^{-j}, -2^{-j-1}]$ if $j < 0$. Denote by $\mathfrak{M}$ the set of all bounded functions $\varphi : \mathbf{R} \to \mathbf{C}$ satisfying $\sup_{j\in\mathbf{Z}} \mathrm{Var}(\varphi_j) < \infty$, where $\mathrm{Var}(\varphi_j)$ denotes the variation of φ on Δ_j. Endowed with the norm

$$||\varphi||_{\mathfrak{M}} := ||\varphi||_\infty + \sup_{j\in\mathbf{Z}} \mathrm{Var}(\varphi_j), \qquad \varphi \in \mathfrak{M},$$

the space $\mathfrak{M}$ is a Banach algebra. Moreover, the Marcinkiewicz multiplier theorem implies that $\mathfrak{M} \subseteq \mathcal{M}^{(p)}$, $1 < p < \infty$, with a continuous inclusion. That is, there is a constant $\alpha_p > 0$ such that

$$||S_\varphi^{(p)}||_{\mathcal{L}(L^p(\mathbf{R}))} \leq \alpha_p||\varphi||_{\mathfrak{M}}, \qquad \varphi \in \mathfrak{M}.$$

Clearly $BV \subseteq \mathfrak{M}$ and the containment is proper. Indeed, $\varphi = \Sigma_{j=1}^\infty \chi_{\Delta_{2j}}$ belongs to $\mathfrak{M}\backslash BV$. Note that any bounded function $\varphi \in C^1(\mathbf{R})$ satisfying $\sup_{j\in\mathbf{Z}} \int_{\Delta_j} |\varphi'(x)|dx < \infty$ belongs to $\mathfrak{M}$; this is one of the traditional formulations of the Marcinkiewicz multiplier theorem. Since the inequality

$$||g \circ \varphi||_{\mathfrak{M}} \leq ||\varphi||_{\mathfrak{M}} \cdot \left(\sup_{(x,y)\in K} |g(x,y)| + \sup_{(x,y)\in K} \left|\frac{\partial g}{\partial x}(x,y)\right| + \sup_{(x,y)\in K} \left|\frac{\partial g}{\partial y}(x,y)\right|\right)$$

is valid for all $g \in C^\infty(\mathbf{R}^2)$, where $K = \text{ess range}(\varphi)$, the map $g \mapsto S_{go\varphi}^{(p)}$, for $g \in C^\infty(\mathbf{R}^2)$, is a spectral distribution for $S_\varphi^{(p)}$. So, again $\{S_\varphi^{(p)} : \varphi \in \mathfrak{M}\} \subseteq \mathcal{D}^{(p)}$, $1 < p < \infty$.

Our considerations of the operators $S_\varphi^{(p)}$, where $\varphi \in BV$ or more generally $\varphi \in \mathfrak{M}$, exhibit a common approach which we now wish to formulate as a general procedure for producing generalized scalar multiplier operators; see [[7]; Lemma 2.1] for an analogous result.

Proposition 3.4. *Let $1 \leq p < \infty$ and $\varphi : \mathbf{R} \to \mathbf{C}$ be a function such that $g \circ \varphi \in \mathcal{M}^{(p)}$ for all $g \in C^\infty(\mathbf{R}^2)$. Then $\Phi : g \mapsto S_{go\varphi}^{(p)}$ from $C^\infty(\mathbf{R}^2)$ into $\mathcal{L}(L^p(\mathbf{R}))$*

is a spectral distribution for $S_{\varphi}^{(p)}$. In particular, $S_{g\circ\varphi}^{(p)}$ is a generalized scalar (hence decomposable) operator for every $g \in C^{\infty}(\mathbf{R}^2)$.

Moreover, Φ is a regular spectral distribution for $S_{\varphi}^{(p)}$ if and only if $S_{\overline{\varphi}}^{(p)} \in \{S_{\varphi}^{(p)}\}^{cc}$. In particular, this is the case whenever φ is $\mathbf{R}$-valued.

Proof. Letting $g = id_{\mathbf{C}}$ shows that $\varphi \in \mathcal{M}^{(p)}$. Hence, $K :=$ ess range(φ) is a compact subset of $\mathbf{C}$. Since

$$\|g|_K\|_{\infty} = \sup_{x \in \mathbf{R}} |g(\varphi(x))| \leq q_{n,0}(g), \qquad g \in C^{\infty}(\mathbf{R}^2),$$

where $n \in \mathbf{N}$ satisfies $K \subseteq \{z \in \mathbf{C} : |z| \leq n\}$, it follows that the restriction map $g \mapsto g|_K$ is a continuous linear map from the Fréchet space $C^{\infty}(\mathbf{R}^2)$ into the Banach space of all bounded, $\mathbf{C}$-valued functions defined on K, endowed with the sup-norm. Since the mapping $S_h^{(p)} \mapsto h$ is continuous from $Op(\mathcal{M}^{(p)})$ into $L^{\infty}(\mathbf{R})$–see (2.2)–the continuity of Φ follows from the closed graph theorem. It is routine to check that Φ is a homomorphism and satisfies $\Phi(\mathbb{1}) = I$ and $\Phi(id_{\mathbf{C}}) = S_{\varphi}^{(p)}$. So, $S_{\varphi}^{(p)}$ is generalized scalar. Actually, it then follows that $S_{g\circ\varphi}^{(p)}$ is generalized scalar for every $g \in C^{\infty}(\mathbf{C})$; see [[13]; p.105].

Since every translation operator belongs to the commutant algebra $\{S_{\varphi}^{(p)}\}^{c} \subseteq \mathcal{L}(L^p(\mathbf{R}))$ it is clear that the bicommutant $\{S_{\varphi}^{(p)}\}^{cc} \subseteq Op(\mathcal{M}^{(p)})$. The question is to decide when Φ is a *regular* spectral distribution, that is, when $\{\Phi(g) : g \in C^{\infty}(\mathbf{R}^2)\} \subseteq \{S_{\varphi}^{(p)}\}^{cc}$. For any polynomial q in the two variables z and $\bar{z}$ it is clear from the homomorphism property of Φ that $\Phi(q) = S_{q(\varphi,\overline{\varphi})}^{(p)} = q(S_{\varphi}^{(p)}, S_{\overline{\varphi}}^{(p)})$. Since polynomials are dense in $C^{\infty}(\mathbf{R}^2)$, the map Φ is continuous, and $\{S_{\varphi}^{(p)}\}^{cc}$ is closed in $\mathcal{L}(L^p(\mathbf{R}))$, it follows that Φ takes all of its values in $\{S_{\varphi}^{(p)}\}^{cc}$ if and only if $S_{\overline{\varphi}}^{(p)} \in \{S_{\varphi}^{(p)}\}^{cc}$. $\square$

Multiplier operators can have spectral distributions which do *not* take all of their values in $Op(\mathcal{M}^{(p)})$. For instance, write $L^p(\mathbf{R}) = \mathbf{C}^2 \oplus Y$ and let $A = B \oplus 0_Y$, where $0_Y \in \mathcal{L}(Y)$ is the zero operator and $B \in \mathcal{L}(\mathbf{C}^2)$ is a non-zero operator such that $B^2 = 0$. Then $A \in \mathcal{L}(L^p(\mathbf{R}))$ is also non-zero and satisfies $A^2 = 0$. We have noted previously that there are no non-zero quasinilpotent elements in $Op(\mathcal{M}^{(p)})$ and so $A \notin Op(\mathcal{M}^{(p)})$. It is routine to check that

$$\Phi : g \mapsto g(1)I + \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right)(1)A, \qquad g \in C^{\infty}(\mathbf{R}^2),$$

is a spectral distribution for the multiplier operator I. Note that the function $\overline{id_{\mathbf{C}}}(x,y) := x - iy$ gets mapped by Φ to the *non-multiplier* operator $I + \frac{1}{2}(1-i)A$. Since $\{I\}^{cc} = \{\alpha I : \alpha \in \mathbf{C}\}$ we see that $\Phi(\overline{id_{\mathbf{C}}}) \notin \{I\}^{cc}$, that is, Φ is *not* a regular spectral distribution for $\Phi(id_{\mathbf{C}}) = I$.

On the other hand, if a multiplier operator $S_{\varphi}^{(p)}$ has a spectral distribution Φ such that also $\Phi(\overline{id_{\mathbf{C}}}) \in Op(\mathcal{M}^{(p)})$, then Φ necessarily takes all of its values in $Op(\mathcal{M}^{(p)})$. Indeed, for any polynomial q, the homomorphism properties of Φ and the fact that $\mathcal{M}^{(p)}$ is an algebra, imply that $\Phi(q) = q(S_{\varphi}^{(p)}, S_h^{(p)}) \in \mathcal{M}^{(p)}$, where $h \in \mathcal{M}^{(p)}$ satisfies $S_h^{(p)} = \Phi(\overline{id_{\mathbf{C}}})$. Again the density of polynomials in $C^{\infty}(\mathbf{R}^2)$, the continuity of Φ and the closedness of $Op(\mathcal{M}^{(p)})$ in $\mathcal{L}(L^p(\mathbf{R}))$ imply that $\Phi(g) \in Op(\mathcal{M}^{(p)})$, for all $g \in C^{\infty}(\mathbf{R}^2)$.

We should point out a relevant fact that this stage: if $S_\varphi^{(p)}$ is any generalized scalar multiplier operator whose spectrum $\sigma(S_\varphi^{(p)})$ has the property that the function $\overline{id_{\mathbf{C}}}$ on $\sigma(S_\varphi^{(p)})$ is the restriction of some function which is holomorphic in a neighbourhood of $\sigma(S_\varphi^{(p)})$, then $S_\varphi^{(p)}$ is necessarily a *regular* generalized scalar operator, [[13]; p.100]. This is the case for instance if ess range$(\varphi) = \sigma(S_\varphi^{(p)})$ is a subset of a line in $\mathbf{C}$ or of an arc on a circle in $\mathbf{C}$.

The above discussion poses some natural questions.

Qu.4. *Does there exist a multiplier $\varphi \in \mathcal{M}^{(p)}$ with $S_\varphi^{(p)}$ a generalized scalar operator such that $S_{\overline{\varphi}}^{(p)} \notin \{S_\varphi^{(p)}\}^{cc}$, i.e. $\{S_\varphi^{(p)}\}^c \neq \{S_{\mathrm{Re}(\varphi)}^{(p)}\}^c \cap \{S_{\mathrm{Im}(\varphi)}^{(p)}\}^c$?*

Qu.5. *Does there exist a multiplier $\varphi \in \mathcal{M}^{(p)}$ such that $S_\varphi^{(p)}$ has a spectral distribution taking all of its values in $Op(\mathcal{M}^{(p)})$ and $S_{\overline{\varphi}}^{(p)} \notin \{S_\varphi^{(p)}\}^{cc}$?*

Qu.6. *Does there exist a multiplier $\varphi \in \mathcal{M}^{(p)}$ such that $S_\varphi^{(p)}$ possesses a spectral distribution Φ (or even a regular spectral distribution) which takes all of its values in $Op(\mathcal{M}^{(p)})$, but $\Phi(\overline{id_{\mathbf{C}}}) \neq S_{\overline{\varphi}}^{(p)}$?*

If the answer to Qu.4 is negative one can ask the following question.

Qu.7. *Does there exist $\varphi \in \mathcal{M}^{(p)}$ such that $\{S_\varphi^{(p)}\}^c \supseteq \{S_{\mathrm{Re}(\varphi)}^{(p)}\}^c \cap \{S_{\mathrm{Im}(\varphi)}^{(p)}\}^c$ is a proper inclusion? Equivalently, $S_{\overline{\varphi}}^{(p)} \notin \{S_\varphi^{(p)}\}^{cc}$.*

In relation to these questions we observe that $\{S_\varphi^{(p)}\}^c$ always contains all translation operators and hence, $\{S_\varphi^{(p)}\}^{cc} \subseteq Op(\mathcal{M}^{(p)})$, $1 \le p < \infty$. It is easy to see that $\{S_\varphi^{(p)}\}^c$ need not be commutative in general (of course, $Op(\mathcal{M}^{(p)}) \subseteq \{S_\varphi^{(p)}\}^c$) and that $\{S_\varphi^{(p)}\}^{cc} \neq Op(\mathcal{M}^{(p)})$ in general. Indeed, let $R : L^p(\mathbf{R}) \to L^p(\mathbf{R})$ be the reflection operator defined by $Rf : x \mapsto f(-x)$, in which case $R^2 = I$. Then $R \in \mathcal{L}(L^p(\mathbf{R}))$ but $R \notin Op(\mathcal{M}^{(p)})$. Let $\varphi \in \mathcal{M}^{(p)}$ be any symmetric multiplier (i.e. $\varphi(-x) = \varphi(x)$ for a.e. $x \in \mathbf{R}$). Then $R \in \{S_\varphi^{(p)}\}^c$ and $\{S_\varphi^{(p)}\}^{cc} \neq Op(\mathcal{M}^{(p)})$. Since R does not commute with all translations it is clear that $\{S_\varphi^{(p)}\}^c$ is not commutative.

Let us return to Proposition 3.4 and consider some applications to producing further examples of elements in $\mathcal{D}^{(p)}$.

Let $\mathcal{N}$ denote the Banach algebra of all bounded functions $\varphi \in C^1(\mathbf{R}\backslash\{0\})$ which satisfy

$$||\varphi||_{\mathcal{N}} := ||\varphi||_\infty + \sup_{x\neq 0} |x\varphi'(x)| < \infty.$$

The Mihlin multiplier theorem ensures that $\mathcal{N} \subseteq \mathcal{M}^{(p)}$, $1 < p < \infty$, and that there exists a constant $\beta_p > 0$ such that

$$||S_\varphi^{(p)}||_{\mathcal{L}(L^p(\mathbf{R}))} \le \beta_p ||\varphi||_{\mathcal{N}}, \qquad \varphi \in \mathcal{N}.$$

Given $\varphi \in \mathcal{N}$ we observe that $K :=$ ess range$(\varphi) = \overline{\varphi(\mathbf{R}\backslash\{0\})}$ is a compact subset of $\mathbf{C}$. Accordingly,

$$\begin{aligned} ||g \circ \varphi||_{\mathcal{N}} &\le \sup_{z\in K} |g(z)| + \sup_{x\neq 0} |x| \cdot \left(\left| \frac{\partial g}{\partial x}(\varphi(x)) \cdot \varphi'(x) \right| + \left| \frac{\partial g}{\partial y}(\varphi(x)) \cdot \varphi'(x) \right| \right) \\ &\le \sup_{z\in K} |g(z)| + \sup_{z\in K} \left(\left| \frac{\partial g}{\partial x}(z) \right| + \left| \frac{\partial g}{\partial y}(z) \right| \right) \cdot \sup_{x\neq 0} |x\varphi'(x)| \end{aligned}$$

is finite for each $g \in C^\infty(\mathbf{R}^2)$, that is, $g \circ \varphi \in \mathcal{N} \subseteq \mathcal{M}^{(p)}$. So, Proposition 3.4 applies and we deduce, for $\varphi \in \mathcal{N}$ and $1 < p < \infty$, that each operator $S_\varphi^{(p)}$ is generalized scalar. In particular, the Schwartz space $\mathcal{S}(\mathbf{R})$ of all rapidly decreasing functions belongs to $\mathcal{N}$ and so all multiplier operators $S_\varphi^{(p)}$ for $\varphi \in \mathcal{S}(\mathbf{R})$ are generalized scalar. Note that the function $\varphi(x) = e^{i \log |x|}$, for $x \neq 0$, belongs to $\mathcal{N}$ but not to any of the spaces $\hat{L}^1$, BV, $\mathfrak{M}$ or $\mathcal{S}(\mathbf{R})$.

An important subalgebra of $\mathcal{N}$, which plays an important role in the theory of semigroups of linear operators, is the following one. For each $\omega \in (0, \frac{\pi}{2})$, let $S_\omega := \{z \in \mathbf{C}\backslash\{0\} : |\arg(z)| < \omega\}$ and then define $H^\infty(-S_\omega \cup S_\omega)$ to be the space of all bounded holomorphic functions defined on the open double cone $-S_\omega \cup S_\omega$. Then $H^\infty(-S_\omega \cup S_\omega) \subseteq \mathcal{N} \subseteq \mathcal{M}^{(p)}$, $1 < p < \infty$, in the sense that the restriction $\varphi_{\mathbf{R}}$ of any function $\varphi \in H^\infty(-S_\omega \cup S_\omega)$ to $\mathbf{R}\backslash\{0\}$ belong to $\mathcal{N}$. Indeed, applying the Cauchy integral formula to φ it can be verified that $|\varphi'_{\mathbf{R}}(x)| \leq \|\varphi\|_\infty / |x| \sin(\omega)$, for $x \neq 0$, and hence

$$|||\varphi_{\mathbf{R}}|||_p \leq \alpha_p \|\varphi\|_\infty / \sin(\omega), \qquad \varphi \in H^\infty(-S_\omega \cup S_\omega),$$

for some constant $\alpha_p > 0$ depending only on p. So, we see that all multipliers in $\mathcal{H} := \cup_{0<\omega<\frac{\pi}{2}} \{\varphi_{\mathbf{R}} : \varphi \in H^\infty(-S_\omega \cup S_\omega)\} \subseteq \mathcal{N}$ are generalized scalar, for all $1 < p < \infty$. In particular, $\mathcal{H} \subseteq \mathcal{D}^{(p)}$, $1 < p < \infty$. We point out that there exist C^∞-multipliers in $\mathcal{M}^{(p)}$, $1 < p < \infty$, which are not elements of $\mathcal{H}$. For instance, if $s \in \mathbf{R}\backslash\{0\}$, then the function $x \mapsto e^{isx}$, for $x \in \mathbf{R}$, which is the multiplier corresponding to the translation operator τ_s, cannot belong to $\mathcal{H}$ since this would imply that $z \mapsto e^{isz}$ is bounded in $-S_\omega \cup S_\omega$ for some $\omega \in (0, \frac{\pi}{2})$.

Further examples can be generated via a multiplier theorem of M. Schechter which states if $k \in \mathbf{N}$ and the function $\varphi \in C^k(\mathbf{R})$ satisfies estimates of the form

$$|\varphi^{(r)}(x)| \leq \frac{\beta}{|x|^{b+ar}}, \qquad |x| > 1, \qquad r \in \{0, 1, \ldots, k\},$$

for some constants $\beta > 0$, $b > 0$ and $a \leq 1$, then $\varphi \in \mathcal{M}^{(p)}$ for all $p \in (1, \infty)$ satisfying

$$(1 - a)|\tfrac{1}{2} - \tfrac{1}{p}| < b. \tag{3.4}$$

Combining this result with Proposition 3.4 it can be shown, [[7]; Proposition 2.5], that $S_\varphi^{(p)}$ is generalized scalar (for all p satisfying (3.4)) with a spectral distribution given by $g \mapsto S_{g \circ \varphi}^{(p)}$, for $g \in C^\infty(\mathbf{R}^2)$.

So far many of the generalized scalar multiplier operators we have produced arise from multipliers which possess some sort of smoothness properties. To produce further classes of non-smooth examples is straightforward. Given $\varphi \in \operatorname{sim}(\mathcal{B}_p)$ we can write $\varphi = \Sigma_{j=1}^n \alpha_j \chi_{E(j)}$, where $\{\alpha_j\}_{j=1}^n \subseteq \mathbf{C}$ are distinct and $\{E(j)\}_{j=1}^n$ is a finite family of non-null multiplier sets from $\mathcal{B}_p$ which is pairwise disjoint and satisfies $\cup_{j=1}^n E(j) = \mathbf{R}$. Then ess range$(\varphi) = \{\alpha_j\}_{j=1}^n$ and for each $g \in C^\infty(\mathbf{R}^2)$ we have $g \circ \varphi = \Sigma_{j=1}^n g(\alpha_j) \chi_{E(j)}$, from which it is immediate that $g \circ \varphi \in \mathcal{M}^{(p)}$. Accordingly, Proposition 3.4 applies to show that $S_\varphi^{(p)}$ is a generalized scalar operator, for all $1 < p < \infty$. Of course, this example is already known to us since $S_\varphi^{(p)}$ is a scalar-type spectral operator.

We end this section with four further questions. It is known that the sum and product of commuting *regular* generalized scalar operators T_1, T_2 (in any Banach space) are at least generalized scalar again, [[13]; p.100], but that they may fail to be regular, [1]. Without the regularity requirement of T_1 and T_2 their sum and product may even fail to be generalized scalar, [1]. However, such examples are constructed in non-reflexive Banach spaces. So, in the reflexive spaces $L^p(\mathbf{R})$, $1 < p < \infty$, and for multiplier operators (which have an additional rich structure of their own) we can ask the following questions.

Qu.8. *Let $1 < p < \infty$. Do there exist two regular generalized scalar operators from $Op(\mathcal{M}^{(p)})$ whose sum and/or product (necessarily generalized scalar) fail to be regular?*

It follows from the comments just prior to Qu.4 that if an example of two such regular generalized scalar multiplier operators $S^{(p)}_{\varphi_1}$, $S^{(p)}_{\varphi_2}$ exists (for Qu.8), then *not both* φ_1 and φ_2 can be $\mathbf{R}$-valued (or, have their values on a line in $\mathbf{C}$). To see this, let $\sigma_T(S^{(p)}_{\varphi_1}, S^{(p)}_{\varphi_2})$ denote the joint Taylor spectrum for the commuting operators $S^{(p)}_{\varphi_1}$ and $S^{(p)}_{\varphi_2}$. By the spectral mapping theorem and analytic functional calculus for pairs of commuting operators, [34], applied to $q_j(z_1, z_2) = z_j$ we see that $\sigma(S^{(p)}_{\varphi_j}) = \sigma(q_j(S^{(p)}_{\varphi_1}, S^{(p)}_{\varphi_2})) = q_j(\sigma_T(S^{(p)}_{\varphi_1}, S^{(p)}_{\varphi_2}))$, for $j = 1, 2$. It follows that $\sigma_T(S^{(p)}_{\varphi_1}, S^{(p)}_{\varphi_2}) \subseteq \sigma(S^{(p)}_{\varphi_1}) \times \sigma(S^{(p)}_{\varphi_2})$. By this observation and the spectral mapping theorem applied to the functions $(z_1, z_2) \mapsto z_1 + z_2$ and $(z_1, z_2) \mapsto z_1 z_2$ we have

$$\sigma(S^{(p)}_{\varphi_1} + S^{(p)}_{\varphi_2}) = \{z_1 + z_2 : (z_1, z_2) \in \sigma_T(S^{(p)}_{\varphi_1}, S^{(p)}_{\varphi_2})\} \subseteq \{z_1 + z_2 : z_j \in \sigma(S^{(p)}_{\varphi_j})\}$$

and

$$\sigma(S^{(p)}_{\varphi_1} S^{(p)}_{\varphi_2}) = \{z_1 z_2 : (z_1, z_2) \in \sigma_T(S^{(p)}_{\varphi_1}, S^{(p)}_{\varphi_2})\} \subseteq \{z_1 z_2 : z_j \in \sigma(S^{(p)}_{\varphi_j})\},$$

from which it is clear that $\sigma(S^{(p)}_{\varphi_1} + S^{(p)}_{\varphi_2}) \subseteq \mathbf{R}$ and $\sigma(S^{(p)}_{\varphi_1} S^{(p)}_{\varphi_2}) \subseteq \mathbf{R}$ whenever $\sigma(S^{(p)}_{\varphi_j}) \subseteq \mathbf{R}$, for $j = 1, 2$.

Qu.9. *Let $1 < p < \infty$. Do there exist two generalized scalar operators from $Op(\mathcal{M}^{(p)})$ whose sum and/or product fail to be generalized scalar?*

All of our examples of decomposable multiplier operators to date have been generalized scalar, which poses the following question.

Qu.10. *Let $1 < p < \infty$. Does there exist a decomposable p-multiplier operator which fails to be generalized scalar (or regular generalized scalar)?*

We point out that for $p = 1$ it is known that there exist decomposable 1-multiplier operators which fail to be *regular* generalized scalar, [[13]; p.205].

Qu.11. *Does there exist a decomposable 1-multiplier operator which fails to be generalized scalar?*

4. Well bounded multiplier operators. Let us first return to the semisimple Banach algebra $\overline{\text{sim}}(\mathcal{B}_p)$, $1 < p < \infty$. Since $Op(\overline{\text{sim}}(\mathcal{B}_p))$ is a commutative semisimple Banach algebra generated by a Boolean algebra of projections its maximal ideal space Δ is totally disconnected (as $\{S^{(p)}_{\chi_E} : E \in \mathcal{B}_p\}$ is Boolean algebra isomorphic to the closed-open subsets of Δ). It follows from [[2]; Corollary 4.7] that $\overline{\mathrm{sim}}(\mathcal{B}_p) \subseteq \mathcal{D}^{(p)}$, for $1 < p < \infty$. We collect some known results to show that $\overline{\mathrm{sim}}(\mathcal{B}_p)$ is quite an extensive family of decomposable multipliers.

A function $f \in BV$ has a decomposition of the form $f_1 + f_2 + f_3$ with $f_1 \in AC$, f_2 singular and continuous (i.e. its derivative is zero a.e.) and f_3 a jump function. If f vanishes at some point of $\mathbf{R}$ (or at $-\infty$), then there is a unique decomposition of this type with all three components f_1, f_2 and f_3 vanishing at that point. If f_2 is identically zero, then f is said to have zero continuous singular part.

For each $s \in \mathbf{R}$, the group character $e_s : x \mapsto e^{isx}$, for $x \in \mathbf{R}$, is the multiplier corresponding to the translation operator τ_s. The closure in $L^\infty(\mathbf{R})$ of $\mathrm{span}\{e_s : s \in \mathbf{R}\}$ is the space of *almost periodic functions* and is denoted by AP.

The following result shows that $\overline{\mathrm{sim}}(\mathcal{B}_p)$ is quite an extensive class of functions.

Proposition 4.1.
(i) $\mathcal{H} \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$, *for every* $1 < p < \infty$.
(ii) $\hat{L}^1 \subseteq m^{(p)} \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$, *for every* $1 < p < \infty$.
(iii) *If* $\varphi \in BV$ *has zero continuous singular part and* $\varphi(-\infty) = 0$, *then* $\varphi \in \overline{\mathrm{sim}}(\mathcal{B}_p)$ *for all* $1 < p < \infty$.
(iv) *Let* $v > 0$. *If* φ *is a* v*-periodic function on* $\mathbf{R}$ *which is absolutely continuous in an interval of length* v, *then* $\varphi \in \overline{\mathrm{sim}}(\mathcal{B}_p)$, $1 < p < \infty$.
(v) $AP \cap BM_r \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$, *for every* $1 < p < \infty$.
(vi) *Fix* $p \in (1, 2]$. *Then* $\cup_{1 \le q < p} \mathcal{M}^{(q)} \cap C_0(\mathbf{R}) \subseteq m^{(p)}$ *and so is contained in* $\overline{\mathrm{sim}}(\mathcal{B}_p)$.

Parts (iii) and (iv) can be found in [23] and the remainder in [29]. Some further comments are in order. Since $m^{(p)} \subseteq c_0(\mathbf{Z})$ it is clear that $m^{(p)}$ is a *proper* closed subalgebra of $\overline{\mathrm{sim}}(\mathcal{B}_p)$. If $1 < q < p \le 2$, then it is known that $\mathcal{M}^{(q)} \subseteq \mathcal{M}^{(p)}$ with $|||\varphi|||_p \le |||\varphi|||_q$ for all $\varphi \in \mathcal{M}^{(q)}$. Not only is the inclusion $\mathcal{M}^{(q)} \subseteq \mathcal{M}^{(p)}$ proper when $q < p$ (this is well known), but so is the inclusion $\overline{\mathrm{sim}}(\mathcal{B}_q) \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$, [[29]; Proposition 6]. It was noted in Section 3 that the closure $\overline{\mathrm{span}}\{e_s : s \in \mathbf{R}\}^{(p)}$ (in the space $\mathcal{M}^{(p)}$) of all trigonometric polynomials is contained in $\mathcal{D}^{(p)}$. Actually more is true; Proposition 4.1(iv) implies that $\overline{\mathrm{span}}\{e_s : s \in \mathbf{R}\}^{(p)} \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$. Since the elements of $\overline{\mathrm{span}}\{e_s : s \in \mathbf{R}\}^{(p)} \subseteq AP$ (see (2.2)) are all bounded continuous functions it is clear that the containment $\overline{\mathrm{span}}\{e_s : s \in \mathbf{R}\}^{(p)} \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$ is proper for all $p \in (1, 2) \cup (2, \infty)$. Note that Proposition 4.1(iii) implies that every bounded rational function belongs to $\overline{\mathrm{sim}}(\mathcal{B}_p)$, $1 < p < \infty$.

In view of Proposition 4.1(iii) and (iv) and the fact that the Cantor function belongs to $\overline{\mathrm{sim}}(\mathcal{B}_p)$, [[29]; p.399], can we expect a positive answer to the following question?

Qu.12. *Is* $BV \subseteq \overline{\mathrm{sim}}(\mathcal{B}_p)$, *for each* $1 < p < \infty$?
Qu.13. *Is the inclusion* $\overline{\mathrm{sim}}(\mathcal{B}_p) \subseteq \mathcal{D}^{(p)}$ *strict, for each* $p \in (1, 2) \cup (2, \infty)$?

The maximal ideal space of $\mathcal{M}^{(p)}$, $p \in (1, 2) \cup (2, \infty)$, seems most difficult to identify. Perhaps the following question is more realistic.

Qu.14. *Is there a "reasonably concrete description" of the maximal ideal space of* $\overline{\mathrm{sim}}(\mathcal{B}_p)$, $1 < p < \infty$, $p \ne 2$?
Qu.15. *If* $1 < q \le p \le 2$, *then the inclusion* $\mathcal{M}^{(q)} \subseteq \mathcal{M}^{(p)}$ *implies that* $\mathcal{B}_q \subseteq \mathcal{B}_p$. *Is the containment* $\mathcal{B}_q \subseteq \mathcal{B}_p$ *strict when* $q < p$?

We point out that there are sufficient idempotent multiplier operators to *determine* $Op(\mathcal{M}^{(p)})$ within the space $\mathcal{L}(L^p(\mathbf{R}))$ in the sense that $T \in \mathcal{L}(L^p(\mathbf{R}))$ belongs to $Op(\mathcal{M}^{(p)})$ iff $T \in \{S^{(p)}_{\chi_E} : E \in \mathcal{B}_p\}^c$, for $1 < p < \infty$. Since $\{\tau_s : s \in \mathbf{R}\} \subseteq \{S^{(p)}_{\chi_E} : E \in \mathcal{B}_p\}^c$, one direction is clear. Conversely, if $T \in \{S^{(p)}_{\chi_E} : E \in \mathcal{B}_p\}^c$, then T commutes with $S^{(p)}_{\varphi}$ for all $\varphi \in \overline{\text{sim}}(\mathcal{B}_p)$. But, it was noted above that $\{e_s : s \in \mathbf{R}\} \subseteq \overline{\text{sim}}(\mathcal{B}_p)$ and $S^{(p)}_{e_s} = \tau_s$, for $s \in \mathbf{R}$. Accordingly, $T \in \{\tau_s : s \in \mathbf{R}\}^c = Op(\mathcal{M}^{(p)})$.

The operator algebra $Op(\overline{\text{sim}}(\mathcal{B}_p))$ contains *all* spectral (= scalar-type spectral) multiplier operators, for $1 < p < \infty$, [[29]; Proposition 7]. Because of the underlying Banach space being a reflexive L^p-space, it turns out that the spectral multiplier operators form a *subalgebra* of $Op(\overline{\text{sim}}(\mathcal{B}_p))$, $1 < p < \infty$, [[29]; p.401]. Moreover, given any compact subset $K \subseteq \mathbf{C}$ and $1 < p < \infty$ there exists a scalar-type spectral multiplier operator $S^{(p)}_{\varphi}$ with $\varphi \in \overline{\text{sim}}(\mathcal{B}_p)$ such that $\sigma(S^{(p)}_{\varphi}) = K$, [[29]; Proposition 8]. Not all elements of $Op(\overline{\text{sim}}(\mathcal{B}_p))$ are scalar-type spectral. This follows from the inclusion $\{\tau_s : s \in \mathbf{R}\} \subseteq Op(\overline{\text{sim}}(\mathcal{B}_p))$ noted above and a result of T.A. Gillespie stating that for $p \neq 2$ the operator τ_s is spectral iff $s = 0$, [[20]; Theorem 2]. In fact, "most" of the functions listed in Proposition 4.1 (which all belong to $\overline{\text{sim}}(\mathcal{B}_p)$) yield multiplier operators which *fail* to be scalar-type spectral. This is a consequence of the fact (for $p \neq 2$) that no non-constant function from $C^2(\mathbf{R}) \cap \mathcal{M}^{(p)}$ can induce a scalar-type spectral multiplier operator, [[4]; Proposition 2.2]. Actually more is true. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be any p-multiplier for which there exists some point $u \in \mathbf{R}$ such that φ is continuous and strictly monotone in a neighbourhood of u. Then $S^{(p)}_{\varphi}$ is not scalar-type spectral for any $p \in (1, 2) \cup (2, \infty)$, [[4]; Proposition 2.4]. In particular, if φ belongs to $\hat{L}^1$ or to $(BM^{(d)}_r)\hat{}$, $1 < p < \infty$, in which case φ is necessarily continuous, then $S^{(p)}_{\varphi}$ is almost never spectral.

Qu.16. *Does there exist a non-constant $\varphi \in \overline{\text{sim}}(\mathcal{B}_p)$, $p \neq 2$, which is continuous in some interval and such that $S^{(p)}_{\varphi}$ is scalar-type spectral?*

The following fact (which provides further evidence for hoping for a positive answer to Qu.1(a)) does *not* follow from Proposition 3.2(i) and (ii).

Proposition 4.2. *Let $\varphi \in \mathcal{D}^{(p)}$ and $\psi \in \overline{\text{sim}}(\mathcal{B}_p)$, $1 \leq p < \infty$. Then both $\varphi + \psi$ and $\varphi\psi$ belong to $\mathcal{D}^{(p)}$.*

Proof. There exist functions $s_n \in \text{sim}(\mathcal{B}_p)$ such that $\|S^{(p)}_{\psi} - S^{(p)}_{s_n}\| \to 0$ as $n \to \infty$. Hence, also the left-hand-side of the inequality

$$\sup\{|\lambda| : \lambda \in \sigma(S^{(p)}_{\varphi\psi} - S^{(p)}_{\varphi s_n})\} \leq \|S^{(p)}_{\varphi}\| \cdot \|S^{(p)}_{\psi} - S^{(p)}_{s_n}\|, \qquad n \in \mathbf{N},$$

tends to 0 as $n \to \infty$. Since $S^{(p)}_{s_n}$ is obviously a scalar-type spectral operator it follows that $\varphi s_n \in \mathcal{D}^{(p)}$, for all $n \in \mathbf{N}$, [[8]; Proposition 2.10]. Then Proposition 3.3 implies that $\varphi\psi \in \mathcal{D}^{(p)}$.

Concerning the sum $\varphi + \psi$, let G be an open disc in $\mathbf{C}$ containing $\sigma(-S^{(p)}_{\psi})$ and let $\lambda \in \rho(-S^{(p)}_{\psi}) \cap \overline{G}$. Then both $S^{(p)}_{\varphi} - \lambda I$ and $(S^{(p)}_{\psi} + \lambda I)^{-1}$ belong to $Op(\mathcal{D}^{(p)})$, [[13]; p.36]. Actually, $(S^{(p)}_{\psi} + \lambda I)^{-1} \in Op(\overline{\text{sim}}(\mathcal{B}_p))$; this is immediate from the easily verified fact that for any $s \in \mathrm{sim}(\mathcal{B}_p)$ the operator $S_s^{(p)}$ is invertible in $\mathcal{L}(L^p(\mathbf{R}))$ iff $s(x) \neq 0$ for a.e. $x \in \mathbf{R}$ (in which case $\frac{1}{s} \in \mathrm{sim}(\mathcal{B}_p)$), combined with the fact that the set of all invertible elements in $\mathcal{L}(L^p(\mathbf{R}))$ is an open set for the operator norm topology and $T \mapsto T^{-1}$ is continuous on this set. Accordingly, from the products case established above we deduce that $I + (S_\varphi^{(p)} - \lambda I)(S_\psi^{(p)} + \lambda I)^{-1}$ is a decomposable multiplier operator. Obviously $(S_\psi^{(p)} + \lambda I) \in Op(\overline{\mathrm{sim}}(\mathcal{B}_p))$ and so again the result for products, together with the formula

$$S_\varphi^{(p)} + S_\psi^{(p)} = (S_\psi^{(p)} + \lambda I) \cdot [I + (S_\varphi^{(p)} - \lambda I)(S_\psi^{(p)} + \lambda I)^{-1}],$$

shows that $(\varphi + \psi) \in \mathcal{D}^{(p)}$. □

It may be worth recording explicitly the fact (established in the proof of Proposition 4.2) that $\overline{\mathrm{sim}}(\mathcal{B}_p)$ is an *inverse closed* subalgebra of $\mathcal{M}^{(p)}$, $1 < p < \infty$, that is, if $\varphi \in \overline{\mathrm{sim}}(\mathcal{B}_p)$ has an inverse in $\mathcal{M}^{(p)}$ (necessarily equal to $\frac{1}{\varphi}$), then actually $\frac{1}{\varphi} \in \overline{\mathrm{sim}}(\mathcal{B}_p)$; see also [[23]; Lemma 1].

The class of well bounded operators (necessarily of type (B) in the reflexive spaces $L^p(\mathbf{R})$, $1 < p < \infty$), which arose out of a need to extend the class of scalar-type spectral operators (with real spectrum), has been intensively studied in recent decades. For *multiplier* operators of this kind more detailed results are available.

Proposition 4.3. *Let $1 < p < \infty$ and $\varphi \in \mathcal{M}^{(p)}$ be $\mathbf{R}$-valued.*
(i) *If $S_\varphi^{(p)}$ is well bounded of type (B), then $S_\varphi^{(p)}$ is a regular generalized scalar operator and so, in particular, $S_\varphi^{(p)} \in \mathcal{D}^{(p)}$ and $\sigma(S_\varphi^{(p)}) =$ ess range(φ). Moreover, the map $g \mapsto S_{g\circ\varphi}^{(p)}$, for $g \in C^\infty(\mathbf{R}^2)$, is a regular spectral distribution for $S_\varphi^{(p)}$ taking all of its values in $\{S_\varphi^{(p)}\}^{cc} \subseteq Op(\mathcal{M}^{(p)})$.*
(ii) *If $S_\varphi^{(p)}$ is well bounded of type (B) and J is any interval containing* ess range(φ), *then the $AC(J)$-functional calculus for $S_\varphi^{(p)}$ is given by $g \mapsto S_{g\circ\varphi}^{(p)}$, for $g \in AC(J)$, and assumes all of its values in $\{S_\varphi^{(p)}\}^{cc}$.*

For each $\mathbf{R}$-valued, piecewise monotone function $g \in AC(J)$ the operator $S_{g\circ\varphi}^{(p)}$ is again well bounded of type (B). In particular, $S_{|\varphi|}^{(p)}$ is well bounded of type (B).
(iii) *$S_\varphi^{(p)}$ is well bounded of type (B) iff $\chi_{(-\infty,\lambda]} \circ \varphi \in \mathcal{M}^{(p)}$, for each $\lambda \in \mathbf{R}$, and*

$$\sup\{|||\chi_{(-\infty,\lambda]} \circ \varphi|||_p : \lambda \in \mathbf{R}\} < \infty.$$

In this case the spectral family $E : \mathbf{R} \to \mathcal{L}(L^p(\mathbf{R}))$ is given by the multiplier projections $E(\lambda) = S_{\chi_{(-\infty,\lambda]}\circ\varphi}^{(p)}$, for $\lambda \in \mathbf{R}$, and $\{S_\varphi^{(p)}\}^c = \{E(\lambda) : \lambda \in \mathbf{R}\}^c$.

Proof. Let $S_\varphi^{(p)}$ be well bounded of type (B). It was noted in the Introduction that $S_\varphi^{(p)}$ is generalized scalar and so $\sigma(S_\varphi^{(p)}) =$ ess range(φ). The first claim of part (ii) is Proposition 3.2.2 of [27], from which the regularity of $g \mapsto S_{g\circ\varphi}^{(p)}$ in part (i) then follows. The claim about well boundedness of $S_{g\circ\varphi}^{(p)}$ when $g \in AC(J)$ is $\mathbf{R}$-valued and piecewise monotone then follows from [[10]; Lemma 6]. The characterization of well boundedness given in (iii) and the formula for the spectral family E is given in

Theorem 3.2.4 of [27]. The statement about the commutants in (iii) is a well known general fact; see [[14]; Part 5], for example. $\square$

Parts (i) and (ii) of Proposition 4.3 are also valid for $p = 1$; see the appropriate results in [27] which were cited above.

It is known that there exist $\mathbf{R}$-valued multipliers $\varphi \in \mathcal{M}^{(p)}$, $p \neq 2$, such that $|\varphi| \notin \mathcal{M}^{(p)}$. For such a multiplier φ, Proposition 4.3(ii) shows that the operator $S_\varphi^{(p)}$ cannot be well bounded of type (B).

Which multiplier operators are known to be well bounded of type (B)? The condition in Proposition 4.3(iii) can be difficult to check. Let us consider an example.

Let $\varphi(x) = \cos x$, for $x \in \mathbf{R}$. If $\lambda \leq -1$ then $\chi_{(-\infty,\lambda]} \circ \varphi = 0$ and if $\lambda \geq 1$ then $\chi_{(-\infty,\lambda]} \circ \varphi = \mathbb{1}$. For $|\lambda| < 1$, let $0 \leq \theta_1 < \theta_2 \leq 2\pi$ be the unique points satisfying $\varphi(\theta_1) = \lambda = \varphi(\theta_2)$. If $K_\lambda := \{e^{i\theta} : \theta \in [\theta_1, \theta_2]\} \subseteq \mathbf{T}$, then it follows from [[20]; Lemma 6] that there exists $\alpha_p > 0$ (independent of λ) such that $x \mapsto \chi_{K_\lambda}(e^{ix})$, for $x \in \mathbf{R}$, belongs to $\mathcal{M}^{(p)}$, $1 < p < \infty$, and $\sup_\lambda |||\chi_{K_\lambda} \circ e^{i(\cdot)}|||_p \leq \alpha_p$. Since $\chi_{(-\infty,\lambda]} \circ \varphi = \chi_{K_\lambda} \circ e^{i(\cdot)}$, for each $\lambda \in \mathbf{R}$, the criterion of Proposition 4.3(iii) is satisfied. So, the multiplier operator $S_{\cos x}^{(p)}$ (and $S_{\sin x}^{(p)}$ by a similar argument) is well bounded for all $1 < p < \infty$. Actually, it is not difficult to show that the same is true of $S_{\cos \beta x}^{(p)}$ and $S_{\sin \beta x}^{(p)}$, for all $\beta \in \mathbf{R}$. It then follows from Proposition 4.3(ii) that $S_\psi^{(p)}$ is well bounded, where $\psi(x) = g(\sin \beta x)$ or $\psi(x) = g(\cos \beta x)$, for $x \in \mathbf{R}$, and $\beta \in \mathbf{R}$ and $g : [-1,1] \to \mathbf{R}$ is piecewise monotone and belongs to $AC([-1,1])$.

Examples of a different kind are also known. A function $\varphi : \mathbf{R} \to \mathbf{R}$ is called *piecewise monotone* if there exist finitely many points $t_1 < \ldots < t_n$ such that φ is monotone on $(-\infty, t_1)$, on (t_n, ∞) and on (t_j, t_{j+1}) for each $1 \leq j < n$. If such a function φ is bounded, then it belongs to BV and hence to $\mathcal{M}^{(p)}$, $1 < p < \infty$. The following result, [[27]; Corollary 3.2.5], shows that many multiplier operators with real spectrum which fail to be scalar-type spectral are nevertheless well bounded of type (B).

Proposition 4.4. *Let $\varphi : \mathbf{R} \to \mathbf{R}$ be bounded and piecewise monotone. Then the multiplier operator $S_\varphi^{(p)}$ is well bounded of type (B), for every $1 < p < \infty$.*

Of course, every multiplier operator (with real spectrum) which is scalar-type spectral is also well bounded of type (B). Accordingly, every operator which is a finite real linear combination of projections from $\{S_{\chi_E}^{(p)} : E \in \mathcal{B}_p\}$ is well bounded of type (B). Typically such operators provide examples which fail to satisfy the assumptions of Proposition 4.4, already in the simplest cases such as $S_{\chi_E}^{(p)}$ for certain sets $E \in \mathcal{B}_p$; see the discussion in Section 2 concerning examples of sets belonging to $\mathcal{B}_p$ which are generated by Hadamard sequences.

It is also possible to construct new well bounded operators of type (B) from known ones. Indeed, suppose that $S_\varphi^{(p)}$ and $S_\psi^{(p)}$ are both well bounded of type (B) and that there exist disjoint sets $E, F \in \mathcal{B}_p$ with the property that $\varphi^{-1}(\mathbf{R}\backslash\{0\}) \subseteq E$ and $\psi^{-1}(\mathbf{R}\backslash\{0\} \subseteq F$. Then the formula

$$\chi_{(-\infty,\lambda]} \circ (\varphi + \psi) = (\chi_{(-\infty,\lambda]} \circ \varphi)\chi_E + (\chi_{(-\infty,\lambda]} \circ \psi)\chi_F + \chi_{(-\infty,\lambda]}(0)\chi_{(E \cup F)^c}, \quad (4.1)$$

valid for each $\lambda \in \mathbf{R}$, together with Proposition 4.3(iii) shows that $S^{(p)}_{\varphi+\psi}$ is also well bounded of type (B). For instance, if $1 < p < \infty$, then the operator $S^{(p)}_g$, where

$$g : x \mapsto e^{-x^2}\chi_{(-\infty,0]}(x) + \Sigma_{j=1}^{\infty}\chi_{E(j)}(x), \qquad x \in \mathbf{R},$$

and $E(j) = [2^{2j-1}, 2^{2j})$ for $j \in \mathbf{N}$, is well bounded of type (B), $1 < p < \infty$. This follows from (4.1) with $\varphi(x) = e^{-x^2}\chi_{(-\infty,0]}(x)$ and $\psi(x) = \chi_{\cup_{j=1}^{\infty} E(j)}$, after noting that $S^{(p)}_{\varphi}$ is well bounded by Proposition 4.4 and $S^{(p)}_{\psi}$ is a scalar-type spectral multiplier operator as $\cup_{j=1}^{\infty} E(j) \in \mathcal{B}_p$. Still further examples can then be exhibited by applying the second statement in Proposition 4.3(ii).

Let T be a well bounded operator of type (B) in an arbitrary Banach space X and J be an interval in $\mathbf{R}$ containing $\sigma(T)$. Then there exists a unique spectral family (concentrated on J) $E : \mathbf{R} \to \mathcal{L}(X)$ satisfying $T = \int_J^{\oplus} \lambda dE(\lambda)$, where the integral exists as a limit in the strong operator topology of Riemann-Stieltjes sums. For each $\lambda \in \mathbf{R}$, it is a consequence of being of type (B) that $E(\lambda^-) := \lim_{\alpha\to\lambda^-} E(\alpha)$ exists in the strong operator topology.

Qu.17. *Does T always belong to the uniform operator closed algebra generated by $\{E(\lambda) : \lambda \in \mathbf{R}\}$ or generated by $\{E(\lambda) : \lambda \in \mathbf{R}\} \cup \{E(\lambda^-) : \lambda \in \mathbf{R}\}$?*

It is known that Qu.17 has a positive answer whenever $T \in \mathcal{L}(X)$ is well bounded of type (B) and $\sigma(T)$ is a countable set with at most one limit point. This follows from an examination of the proof of Theorem 3.4 in [12], where the limit point is required to be 0, and the fact that $T - \alpha I$ is also well bounded of type (B) for each $\alpha \in \mathbf{R}$.

For the particular setting of $X = L^p(\mathbf{R})$, $1 < p < \infty$, and T a well bounded *multiplier* operator of type (B) it was noted previously that the spectral family E of T necessarily satisfies $\{E(\lambda) : \lambda \in \mathbf{R}\} \subseteq Op(\mathcal{M}^{(p)})$ and so also $\{E(\lambda^-) : \lambda \in \mathbf{R}\} \subseteq Op(\mathcal{M}^{(p)})$. In particular, $\{E(\lambda), E(\lambda^-) : \lambda \in \mathbf{R}\} \subseteq Op(\overline{\mathrm{sim}}(\mathcal{B}_p)) \subseteq Op(\mathcal{D}^{(p)})$. So, even if the answer in the general setting of Qu.17 turns out to be negative, it does not necessarily negate the following more specialized question.

Qu.18. *Let $1 < p < \infty$ and let $T \in Op(\mathcal{M}^{(p)})$ be well bounded of type (B). Is $T \in Op(\overline{\mathrm{sim}}(\mathcal{B}_p))$?*

There is certainly a case to expect a positive answer to Qu.18. Indeed, combining Propositions 4.1, 4.3 & 4.4 we have seen it is possible to exhibit many examples of well bounded multiplier operators of type (B) all of which belong to $Op(\overline{\mathrm{sim}}(\mathcal{B}_p))$. However, not every operator from $Op(\overline{\mathrm{sim}}(\mathcal{B}_p))$ with real spectrum is necessarily well bounded of type (B).

Proposition 4.5.
(i) *Let $1 < p < \infty$, $p \neq 2$. Then there exists a $\mathbf{R}$-valued function $\varphi \in \overline{\mathrm{sim}}(\mathcal{B}_p)$ such that $S^{(p)}_{\varphi}$ is not well bounded of type (B).*
(ii) *The set of well bounded operators of type (B) from $Op(\mathcal{M}^{(p)}), p \notin \{1,2\}$, is not closed for the operator norm topology.*

Proof. (i) Fix $p \neq 2$. By Proposition 2.1(ii) there exists an open set $V \subseteq [0,1]$ such that $\chi_V \notin \mathcal{M}^{(p)}$. Write V as a union $\cup_{n=1}^{\infty}(a_n, b_n)$ of disjoint open intervals. Define $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(x) = 0$ if $x \notin V$, $\varphi(\frac{a_n+b_n}{2}) = \frac{1}{n^2}$ and make φ affine in $[a_n, \frac{a_n+b_n}{2}]$ (resp. $[\frac{a_n+b_n}{2}, b_n]$) by joining up the points $(a_n, 0) \in \mathbf{R}^2$ (resp. $(b_n, 0) \in \mathbf{R}^2$) with the points $(\frac{a_n+b_n}{2}, \frac{1}{n^2}) \in \mathbf{R}^2$ (resp. $(\frac{a_n+b_n}{2}, \frac{1}{n^2}) \in \mathbf{R}^2$) via a line segment. Then $\varphi \in BV$, [[27]; Theorem 3.3.8], and it is easily checked that φ has zero continuous singular part and satisfies $\varphi(-\infty) = 0$. Accordingly, Proposition 4.1(iii) implies that $\varphi \in \overline{\mathrm{sim}}(\mathcal{B}_p)$, $1 < p < \infty$. The fact that $\chi_V \notin \mathcal{M}^{(p)}$ together with Proposition 4.3(iii) imply that $S_\varphi^{(p)}$ is not well bounded of type (B); see the proof of [[27]; Theorem 3.3.8].

(ii) For each $n \in \mathbf{N}$, let $\varphi_N = \varphi\chi_N$ where χ_N is the characteristic function of $\cup_{n=1}^{N}(a_n, b_n)$. Then $\|\varphi - \varphi_N\|_{BV} \to 0$ as $N \to \infty$ (see the proof of [[27]; Theorem 3.3.8]) and so $S_{\varphi_N}^{(p)} \to S_\varphi^{(p)}$ in $\mathcal{L}(L^p(\mathbf{R}))$. Since $S_{\varphi_N}^{(p)}$ is well bounded of type (B), for each $N \in \mathbf{N}$ (c.f. Proposition 4.4) the conclusion follows from (i). □

The proof of part (ii) even shows that the operator norm limit of a sequence of well bounded operators of type (B) from $Op(\overline{\mathrm{sim}}(\mathcal{B}_p))$ need not be well bounded of type (B). This is because each operator $\varphi_N \in \overline{\mathrm{sim}}(\mathcal{B}_p)$, for $N \in \mathbf{N}$; see Proposition 4.1(iii).

5. *AC*-multiplier operators. Let us begin with an important subclass of the AC-operators. An operator $T \in \mathcal{L}(X)$, with X a Banach space, is called *trigonometrically well bounded* if there exists a well bounded operator $A \in \mathcal{L}(X)$ of type (B) such that $T = e^{iA}$, [11]. This is equivalent to the existence of an operator norm continuous homomorphism $\Phi : AC(\mathbf{T}) \to \mathcal{L}(X)$ such that $\Phi(\mathbb{1}) = I$ and $\Phi(id_{\mathbf{T}}) = T$, and $\Phi(B) \subseteq \mathcal{L}(X)$ is relatively compact for the weak operator topology for each bounded set $B \subseteq AC(\mathbf{T})$, [[11]; Theorem 2.3]. It is known that $\sigma(T) \subseteq \mathbf{T}$, [[9]; Theorem 3.2.3], and that T has a functional calculus of the form

$$\Phi(\varphi) := \int_{[0,2\pi]}^{\oplus} \varphi(e^{i\theta})dE(\theta), \qquad \varphi \in AC(\mathbf{T}), \tag{5.1}$$

where $E : \mathbf{R} \to \mathcal{L}(X)$ is a spectral family concentrated on $[0, 2\pi]$ and the integral is defined as a limit of Riemann-Stieltjes sums with respect to the strong operator topology. The well bounded operator A of type (B) which satisfies $T = e^{iA}$ is *unique* with respect to the properties $\sigma(A) \subseteq [0, 2\pi]$ and 2π is not an eigenvalue of A. The spectral family E in (5.1) is the spectral family of this unique operator A. A further characterization is that $T \in \mathcal{L}(X)$ is trigonometrically well bounded iff there exist commuting well bounded operators A, B of type (B) such that $T = A + iB$ and $A^2 + B^2 = I$, [11]. The importance of trigonometrically well bounded operators (from the point of view of this article) stems from the following result of T.A. Gillespie, [20]; it is valid for arbitrary LCA groups but we only formulate it for the line group $\mathbf{R}$.

Proposition 5.1. *Let $1 < p < \infty$. For each $s \in \mathbf{R}$, the translation operator $\tau_s \in Op(\mathcal{M}^{(p)})$ is trigonometrically well bounded and the unique well bounded operator A_s of type (B) such that $\tau_s = e^{iA_s}$ (and $\sigma(A_s) \subseteq [0, 2\pi]$ with 2π not an eigenvalue of A_s) is a p-multiplier operator.*

It was noted in Section 4 that τ_s, $s \neq 0$ is never a scalar-type spectral operator for $p \neq 2$. The spectral family E which satisfies (5.1) with $T = \tau_s$ is (essentially) identified in [[9]; p.456].

Let us indicate how Proposition 5.1 can be used to generate other trigonometrically well bounded operators. Fix $1 < p < \infty$ and, for the case of simplicity, let $s = 1$. By Proposition 5.1 we are guaranteed a functional calculus $\Phi_1 : AC(\mathbf{T}) \to \mathcal{L}(L^p(\mathbf{R}))$ for the translation operator τ_1 given by

$$\Phi_1(\varphi) := \int_{[0,2\pi]}^{\oplus} \varphi(e^{i\theta}) dE_{(1)}(\theta), \qquad \varphi \in AC(\mathbf{T}), \tag{5.2}$$

where $E_{(1)}$ is the spectral family for the corresponding unique well bounded (*multiplier*) operator A_1. That is,

$$A_1 = \int_{[0,2\pi]}^{\oplus} t\, dE_{(1)}(t).$$

It was noted in Section 4 that all the projections in the spectral family of A_1 belong to $\{S_{\chi_E}^{(p)} : E \in \mathcal{B}_p\}$. Accordingly, each Riemann-Stieltjes sum approximating the integral (5.2) is a p-multiplier operator and hence, also the strong operator limit $\Phi_1(\varphi)$ of these sums is a p-multiplier operator. Putting $s = -1$ we are also guaranteed by Proposition 5.1 a functional calculus $\Phi_{-1} : AC(\mathbf{T}) \to \mathcal{L}(L^p(\mathbf{R}))$ for the translation operator τ_{-1} given by

$$\Phi_{-1}(\varphi) = \int_{[0,2\pi]}^{\oplus} \varphi(e^{i\theta}) dE_{(-1)}(\theta), \qquad \varphi \in AC(\mathbf{T}), \tag{5.3}$$

where $E_{(-1)}$ is the spectral family for the corresponding unique well bounded (*multiplier*) operator A_{-1} (as given by Proposition 5.1). A similar argument as for $s = 1$ shows that $\{\Phi_{-1}(\varphi) : \varphi \in AC(\mathbf{T})\} \subseteq Op(\mathcal{M}^{(p)})$. In particular,

$$\Phi_1(\varphi)\Phi_{-1}(\psi) = \Phi_{-1}(\psi)\Phi_1(\varphi), \qquad \varphi, \psi \in AC(\mathbf{T}).$$

Using this commutativity property it is routine to verify that the map $\Phi : AC(\mathbf{T}) \to \mathcal{L}(L^p(\mathbf{R}))$ defined by

$$\Phi(\varphi) := S_{\chi_{(-\infty,0)}}^{(p)} \Phi_{-1}(\varphi) + S_{\chi_{[0,\infty)}}^{(p)} \Phi_1(\varphi), \qquad \varphi \in AC(\mathbf{T}),$$

is a homomorphism satisfying $\Phi(\mathbb{1}) = I$ and $\Phi(id_{\mathbf{T}}) = S_g^{(p)}$, where $g(x) := e^{i|x|}$, for $x \in \mathbf{R}$. The inequality

$$\|\Phi(\varphi)\|_{\mathcal{L}(L^p(\mathbf{R}))} \leq |||\chi_{(-\infty,0)}|||_p \|\Phi_{-1}(\varphi)\|_{\mathcal{L}(L^p(\mathbf{R}))} + |||\chi_{[0,\infty)}|||_p \|\Phi_1(\varphi)\|_{\mathcal{L}(L^p(\mathbf{R}))},$$

valid for all $\varphi \in AC(\mathbf{T})$, shows that Φ is also continuous. Accordingly, the p-multiplier operator $S_g^{(p)}$ corresponding to $g(x) = e^{i|x|} = \chi_{(-\infty,0)}(x)e^{-ix} + \chi_{[0,\infty)}(x)e^{ix}$ is trigonometrically well bounded. The same is true of $S_{g_s}^{(p)}$, for each $s \in \mathbf{R}$, where $g_s(x) = e^{is|x|}$, for $x \in \mathbf{R}$.

A more extensive class of operators is the AC-operators. Since we are only concerned with the reflexive spaces $L^p(\mathbf{R})$, $1 < p < \infty$, for an operator $T \in \mathcal{L}(L^p(\mathbf{R}))$ to be an *AC-operator* means that there exist commuting well bounded operators A, B of type (B) such that $T = A + iB$, [[10]; §4]. Moreover, in this case A, B are *unique*, [[10]; p.318], and satisfy

$$\{T\}^c = \{A\}^c \cap \{B\}^c; \tag{5.4}$$

see [[10]; Lemma 4]. Equivalently, there exist compact intervals J and K in $\mathbb{R}$ and an operator norm continuous functional calculus $\Phi : AC(J\times K) \to \mathcal{L}(L^p(\mathbb{R}))$ satisfying $\Phi(\mathbb{1}) = I$ and $\Phi(id_{J\times K}) = T$. If $u(x,y) := x$ and $v(x,y) := y$, then the unique commuting well bounded operators A, B of type (B) which satisfy $T = A + iB$ are given by $\Phi(u) = A$ and $\Phi(v) = B$. Moreover, the calculus Φ is unique. Since every trigonometrically well bounded operator T has such a cartesian decomposition $T = A + iB$ it is clear that trigonometric well bounded operators are AC-operators.

Proposition 5.2. *Let $1 < p < \infty$ and $S_\varphi^{(p)}$ be an AC-operator for some $\varphi \in \mathcal{M}^{(p)}$. Let $\Phi : AC(J\times K) \to \mathcal{L}(L^p(\mathbb{R}))$ be the unique functional calculus for $S_\varphi^{(p)}$.*
(i) *The range of Φ is contained in $Op(\mathcal{M}^{(p)})$.*
(ii) *Both $S^{(p)}_{\mathrm{Re}(\varphi)}$ and $S^{(p)}_{\mathrm{Im}(\varphi)}$ are well bounded of type (B) and satisfy $S_\varphi^{(p)} = S^{(p)}_{\mathrm{Re}(\varphi)} + iS^{(p)}_{\mathrm{Im}(\varphi)}$. Moreover, $\Phi(h) = S^{(p)}_{h\circ\varphi}$ for each $h \in AC(J\times K)$.*
(iii) *$S_\varphi^{(p)}$ is a regular generalized scalar operator. In particular, $\varphi \in \mathcal{D}^{(p)}$.*

Proof. (i) Let A, B be the (unique) commuting well bounded operators of type (B) such that $S_\varphi^{(p)} = A + iB$. Since $S_\varphi^{(p)}$ commutes with all translation operators so do both A and B (c.f. (5.4) with $T = S_\varphi^{(p)}$). Hence, both $A, B \in Op(\mathcal{M}^{(p)})$. Since $A = \Phi(u)$ and $B = \Phi(v)$, it follows that $\Phi(q) = q(A,B)$ for every polynomial q. Accordingly, $\Phi(q) \in Op(\mathcal{M}^{(p)})$. Then the continuity of Φ and the density of polynomials in $AC(J\times K)$ implies that $\Phi(g) \in Op(\mathcal{M}^{(p)})$ for each $g \in AC(J\times K)$.

(ii) From $S_\varphi^{(p)} = A + iB$ and $S_\varphi^{(p)} = S^{(p)}_{\mathrm{Re}(\varphi)} + iS^{(p)}_{\mathrm{Im}(\varphi)}$ we deduce that

$$A - S^{(p)}_{\mathrm{Re}(\varphi)} = i(S^{(p)}_{\mathrm{Im}(\varphi)} - B). \tag{5.5}$$

But, $A \in Op(\mathcal{M}^{(p)})$ and so commutes with $S^{(p)}_{\mathrm{Re}(\varphi)}$ from which it follows that $\sigma(A - S^{(p)}_{\mathrm{Re}(\varphi)}) \subseteq \mathbb{R}$; see the discussion after Qu.8. Similarly, $\sigma(S^{(p)}_{\mathrm{Im}(\varphi)} - B) \subseteq \mathbb{R}$ and so $\sigma(S^{(p)}_{\mathrm{Im}(\varphi)} - B) = \{0\} = \sigma(A - S^{(p)}_{\mathrm{Re}(\varphi)})$; see (5.5). As noted before $Op(\mathcal{M}^{(p)})$ contains no non-zero quasinilpotent operators and so $A = S^{(p)}_{\mathrm{Re}(\varphi)}$ and $B = S^{(p)}_{\mathrm{Im}(\varphi)}$.

For each polynomial q we have $\Phi(q) = q(A,B) = q(S^{(p)}_{\mathrm{Re}(\varphi)}, S^{(p)}_{\mathrm{Im}(\varphi)}) = S^{(p)}_{q\circ\varphi}$. Let $h \in AC(J\times K)$ and choose polynomials $\{q_n\}_{n=1}^\infty$ such that $q_n \to h$ in $AC(J\times K)$ as $n \to \infty$. In particular, since $S_\varphi^{(p)}$ is generalized scalar (see the Introduction), we have that ess range$(\varphi) = \sigma(S_\varphi^{(p)}) \subseteq J\times K$ and so $q_n\circ\varphi \to h\circ\varphi$ pointwise a.e. on $\mathbb{R}$. Moreover,

$$\sup\{|||q_n\circ\varphi|||_p : n \in \mathbb{N}\} = \sup\{\|\Phi(q_n)\|_{\mathcal{L}(L^p(\mathbb{R}))} : n \in \mathbb{N}\} < \infty$$

since the continuity of Φ implies that $\Phi(q_n) \to \Phi(h)$ in $\mathcal{L}(L^p(\mathbb{R}))$. It follows from standard multiplier theorems that $h\circ\varphi \in \mathcal{M}^{(p)}$ and $S^{(p)}_{q_n\circ\varphi} \to S^{(p)}_{h\circ\varphi}$ in the weak operator topology. But also $S^{(p)}_{q_n\circ\varphi} = \Phi(q_n) \to \Phi(h)$ in $\mathcal{L}(L^p(\mathbb{R}))$ and we deduce that $\Phi(h) = S^{(p)}_{h\circ\varphi}$.

(iii) Let $R \in \mathcal{L}(L^p(\mathbb{R}))$ be any operator commuting with $S_\varphi^{(p)}$. By (5.4) and part (ii) see that $RS^{(p)}_{\mathrm{Re}(\varphi)} = S^{(p)}_{\mathrm{Re}(\varphi)}R$ and $RS^{(p)}_{\mathrm{Im}(\varphi)} = S^{(p)}_{\mathrm{Im}(\varphi)}R$. Hence, R commutes with $\Phi(q) = q(S^{(p)}_{\mathrm{Re}(\varphi)}, S^{(p)}_{\mathrm{Im}(\varphi)})$ for each polynomial q. By the density of polynomials in

$AC(J\times K)$ and the continuity of Φ we have $R\Phi(g) = \Phi(g)R$ for each $g \in AC(J\times K)$, i.e.

$$\{\Phi(g) : g \in AC(J\times K)\} \subseteq \{S_\varphi^{(p)}\}^{cc}. \tag{5.6}$$

Since $\tilde{\Phi} : h \mapsto \Phi(h|_{J\times K})$, for $h \in C^\infty(\mathbf{R}^2)$, is a spectral distribution for $S_\varphi^{(p)}$ (see the Introduction) it follows from (5.6) that $\tilde{\Phi}$ is a regular spectral distribution for $S_\varphi^{(p)}$. □

It follows immediately that a multiplier operator $S_\varphi^{(p)}$, $1 < p < \infty$ is an AC-operator if and only if both $S_{\mathrm{Re}(\varphi)}^{(p)}$ and $S_{\mathrm{Im}(\varphi)}^{(p)}$ are well bounded of type (B). Since a multiplier operator A is well bounded of type (B) iff $(-A)$ is well bounded of type (B)– see Proposition 4.3(ii) – it is clear that a multiplier operator $S_\varphi^{(p)}$ is an AC-operator iff $S_{\overline{\varphi}}^{(p)}$ is an AC-operator. In regard to Qu.4, it follows from (5.4) and Proposition 5.2 that $\{S_\varphi^{(p)}\}^c = \{S_{\mathrm{Re}(\varphi)}^{(p)}\}^c \cap \{S_{\mathrm{Im}(\varphi)}^{(p)}\}^c$ whenever $S_\varphi^{(p)}$ is an AC-operator.

In Section 4 we recorded some results and constructions for producing well bounded multiplier operators of type (B). Given two such operators A, B (in which case $AB = BA$ is automatic as $Op(\mathcal{M}^{(p)})$ is commutative) it follows from the definition that $A+iB$ is an AC-multiplier operator. The following result produces AC-multiplier operators from a single well bounded multiplier operator, [[10]; Theorem 10].

Proposition 5.3. *Let $1 < p < \infty$, $S_\varphi^{(p)} \in Op(\mathcal{M}^{(p)})$ be well bounded of type (B) and $\Phi : AC(J) \to \mathcal{L}(L^p(\mathbf{R}))$ be the unique functional calculus for $S_\varphi^{(p)}$ given by $\Phi(g) = S_{g\circ\varphi}^{(p)}$, for $g \in AC(J)$. If $h \in AC(J)$ has the property that both $Re(h)$ and $Im(h)$ are piecewise monotone, then $S_{h\circ\varphi}^{(p)}$ is an AC-operator.*

We conclude with some comments about polar multiplier operators. Recall from the Introduction that $S \in \mathcal{L}(L^p(\mathbf{R}))$ is a *polar operator* if there exist commuting well bounded operators R, A of type (B) such that $S = Re^{iA}$.

Proposition 5.4. *Let $1 < p < \infty$, $\varphi \in \mathcal{M}^{(p)}$ and suppose that $S_\varphi^{(p)}$ is polar. Then,*

(i) $|\varphi| \in \mathcal{M}^{(p)}$ *with $S_{|\varphi|}^{(p)}$ well bounded of type (B),*

(ii) *there exists $g \in \mathcal{M}^{(p)}$ with $S_g^{(p)}$ well bounded of type (B) and $S_{e^{ig}}^{(p)}$ an AC-multiplier operator such that* ess range$(g) = [0, 2\pi] \cap \overline{\arg(\text{ess range}(\varphi)\backslash\{0\})}$, *where* arg: $\mathbf{C}\backslash\{0\} \to \mathbf{R}$ *is the branch of argument function with* $0 \le \arg(z) < 2\pi$, *and*

(iii) $S_\varphi^{(p)} = S_{|\varphi|}^{(p)} S_{e^{ig}}^{(p)}$.

Proof. By [[9]; Theorem 3.16] there exist (unique) commuting well bounded operators R, A of type (B) such that $S_\varphi^{(p)} = Re^{iA}$, the spectra of R and A satisfy $\sigma(R) \subseteq [0, \infty)$ and $\sigma(A) \subseteq [0, 2\pi]$ with 2π not an eigenvalue of A, and $F(0)e^{iA} = F(0)$ where $F : \mathbf{R} \to \mathcal{L}(L^p(\mathbf{R}))$ is the spectral family of R. It is known, [[9]; Theorem 3.18], that

$$\{S_\varphi^{(p)}\}^c = \{R\}^c \cap \{A\}^c$$

from which it follows that both $R, A \in Op(\mathcal{M}^{(p)})$. Let $f, g \in \mathcal{M}^{(p)}$ satisfy $S_f^{(p)} = R$ and $S_g^{(p)} = A$. Since both A, R are well bounded of type (B), they are generalized scalar and so ess range$(f) = \sigma(S_f^{(p)}) = \sigma(R) \subseteq [0, \infty)$ and ess range$(g) = \sigma(S_g^{(p)}) = \sigma(A) \subseteq [0, 2\pi]$. The analytic functional calculus operates in the Banach algebra $\mathcal{M}^{(p)}$ and so $e^{ig} \in \mathcal{M}^{(p)}$ with $S_{e^{ig}}^{(p)} = e^{iA}$. Accordingly, $S_\varphi^{(p)} = \mathrm{Re}^{iA} = S_f^{(p)} S_{e^{ig}}^{(p)} = S_{fe^{ig}}^{(p)}$ from which we deduce that $\varphi = fe^{ig}$ (equality in $\mathcal{M}^{(p)}$). Taking absolute values and recalling that $f \geq 0$ a.e. gives $|\varphi| = f \in \mathcal{M}^{(p)}$. In particular, $S_{|\varphi|}^{(p)} = S_f^{(p)} = R$ is well bounded of type (B). The operator $S_{e^{ig}}^{(p)} = \exp(iS_g^{(p)})$ is an AC-multiplier operator by Proposition 5.3. The formula for ess range(g) in part (ii) follows from the formula (3.20) of [[9]; p.442] and the inclusion ess range$(g) \subseteq [0, 2\pi]$. □

REMARKS 5.1. (i) It is known that there exist multipliers $\varphi \in \mathcal{M}^{(p)}$, $p \neq 2$, such that $|\varphi| \notin \mathcal{M}^{(p)}$. For such a multiplier φ, Proposition 5.4(i) shows that $S_\varphi^{(p)}$ is *not* a polar operator.
(ii) Let $S_\varphi^{(p)}$ be a polar operator. Proposition 5.4 implies that $S_\varphi^{(p)} = S_{|\varphi|}^{(p)} S_{e^{ig}}^{(p)} = S_{|\varphi|}^{(p)} e^{iS_g^{(p)}}$ with $S_g^{(p)}$ well bounded of type (B). As observed before $-S_g^{(p)}$ is also well bounded of type (B) and so $S_{\overline{\varphi}}^{(p)} = S_{|\varphi|}^{(p)} e^{-iS_g^{(p)}} = S_{|\varphi|}^{(p)} S_{e^{-ig}}^{(p)}$ is polar. Accordingly, a multiplier operator $S_\varphi^{(p)}$ is polar iff $S_{\overline{\varphi}}^{(p)}$ is polar.
(iii) In the notation of Proposition 5.4, let $S_\varphi^{(p)} = S_{|\varphi|}^{(p)} S_{e^{ig}}^{(p)}$ be a polar multiplier operator. By Proposition 5.2 the real and imaginary parts $S_{\cos(g)}^{(p)}$ and $S_{\sin(g)}^{(p)}$ of the AC-multiplier operator $S_{e^{ig}}^{(p)} = e^{iS_g^{(p)}}$ are both well bounded multiplier operators of type (B). Then the identity $S_\varphi^{(p)} = (S_{|\varphi|}^{(p)} S_{\cos(g)}^{(p)}) + i(S_{|\varphi|}^{(p)} S_{\sin(g)}^{(p)})$ raises the following points.

Qu.19. *Are the sum and product of well bounded multiplier operators of type (B) in $L^p(\mathbf{R})$, $1 < p < \infty$, again well bounded of type (B)?*

Actually, it suffices to know the answer to Qu.19 for sums since the result for products would then follow from the identity $AB = \frac{1}{2}((A+B)^2 - A^2 - B^2)$ together with the fact that αT (for any $\alpha \in \mathbf{R}$), T^2 and $-T$ are all well bounded of type (B) whenever T has this property.

It is known that Qu.19 has a negative answer for arbitrary commuting well bounded operators of type (B), even in Hibert spaces, [[14]; p.362]. For arbitrary Banach spaces (even reflexive ones) there does not seem to be any obvious connection between polar operators and AC-operators. Perhaps in the setting of $L^p(\mathbf{R})$ and for the special class of multiplier operators there may be some relationship.

Qu.20. *Let $1 < p < \infty$. Is every polar (resp.* AC*) operator from $Op(\mathcal{M}^{(p)})$ necessarily an* AC *(resp. polar) operator? Are the sum and product of AC (resp. polar) multiplier operators again* AC *(resp. polar) operators? Is the product of trigonometrically well bounded multiplier operators again trigonometrically well bounded?*

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