INEQUALITIES FOR THE JOINT SPECTRUM OF SIMULTANEOUSLY TRIANGULARIZABLE MATRICES

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1. INTRODUCTION

Let $A = (A_1, ..., A_m)$ be an m-tuple of n by n matrices. We say that A is *triangularizable* if there is an invertible matrix Q such that $Q^{-1}A_jQ$ is (upper) triangular for each $j = 1, ..., m$. In this case, for $1 \le k \le n$, let $\alpha_j^{(k)} = (Q^{-1}A_jQ)_{kk}$ the (k, k) element of $Q^{-1}A_jQ$, and set $\alpha^{(k)} = (\alpha_1^{(k)}, ..., \alpha_m^{(k)}) \in \mathbb{C}^m$. The set

$$\sigma(A) = \{\alpha^{(k)} : 1 \le k \le n\} \tag{1.1}$$

is called the *joint spectrum* of A. For a discussion of this spectrum see Pryde [16].

In particular $\sigma(A)$ has an important subset $\sigma_{pt}(A)$, the *joint point spectrum*, whose elements $\lambda = (\lambda_1, ..., \lambda_m)$ satisfy $A_jx = \lambda_jx$ for all j and some non-zero $x \in \mathbb{C}^n$. We say that λ is a *joint eigenvalue* of A with corresponding *joint eigenvector* x. If the A_j commute then $\sigma(A) = \sigma_{pt}(A)$, though this is not the case in general. However, by a theorem of Lie, if A is triangularizable then $\sigma_{pt}(A)$ is non-empty.

Our aim in this paper is to investigate perturbation inequalities for the joint spectra of triangularizable m-tuples. For this purpose we define the function S(K, L) on compact subsets K and L of $\mathbb{C}^m$ by

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$$(1.2) \qquad S(K, L) = \max_{\alpha \in K} \min_{\beta \in L} |\alpha - \beta|.$$

As an example of a perturbation inequality it follows from theorem 3.1 below that

$$(1.3) \qquad S(\sigma(A), \sigma(B)) \leq |A - B|$$

for all m-tuples A and B of commuting self-adjoint matrices. The norm here is defined by $|T| = \|\mathrm{Cliff}(T)\|$, where $\mathrm{Cliff}(T)$ is the Clifford operator associated with an m-tuple T. These are defined in section 2 below using Clifford algebras.

It may be noted that (1.3) is an analogue for m-tuples of the well-known Bauer-Fike theorem for single matrices. Generalizations of theorems of Henrici appear in section 4 and of Bhatia-Friedland in section 5. We conclude with some inequalities for normal matrices in section 6.

The use of Clifford algebras as a tool for studying joint spectra began with McIntosh and Pryde [10], [11]. It was further developed by Pryde [14], [15], [16] and by Bhatia and Bhattacharyya [3], [4].

It is often convenient to have m-tuples (or 2m-tuples) of matrices with real spectra. For this purpose we use the following construction, initiated in McIntosh and Pryde [11]. If $A = (A_1, \ldots, A_m)$ is an m-tuple of n by n matrices then we can always decompose the A_j in the form $A_j = A_{1j} + iA_{2j}$ where the A_{kj} all have real spectra. We write $\pi(A) = (A_{11}, \ldots, A_{1m}, A_{21}, \ldots, A_{2m})$ and call $\pi(A)$ a *partition* of A. If the A_{kj} all commute we say that $\pi(A)$ is a *commuting partition*, and if the A_{kj} are simultaneously triangularizable $\pi(A)$ is a *triangularizable partition*. If the A_{kj} are all semisimple (diagonalizable) then $\pi(A)$ is called a *semisimple partition*.

It is proved in [11] that if the A_j commute then A has a commuting partition. It is obvious that if the A_j are simultaneously triangularizable, then A has a triangularizable partition $\pi(A)$. Moreover, then

$$\sigma(A) = p(\sigma(\pi(A))) \tag{1.1}$$

where $p : \mathbb{R}^{2m} \to \mathbb{C}^m$ is defined by $p(\xi, \eta) = \xi + i\eta$ for $\xi, \eta \in \mathbb{R}^m$. Finally, it is proved in [14] that if the A_j are commuting semisimple matrices then A has a unique commuting semisimple partition.

2. CLIFFORD ALGEBRAS

Let $\mathbb{R}_{(m)}$ denote the Clifford algebra over $\mathbb{R}$ with generators $e_1, \ldots, e_m$ and relations $e_ie_j = -e_je_i$ for $i \neq j$ and $e_i^2 = -1$. Then $\mathbb{R}_{(m)}$ is an associative algebra of dimension 2^m. Let $S(m)$ denote the set of subsets of $\{1, \ldots, m\}$. Then the elements e_S for $S \in S(m)$ form a basis of $\mathbb{R}_{(m)}$ if we define $e_\varnothing = 1$ and $e_S = e_{s_1} \ldots e_{s_k}$ when $S = \{s_1, \ldots, s_k\}$ and $1 \leq s_1 < s_2 < \ldots < s_k \leq m$. Elements of $\mathbb{R}_{(m)}$ are denoted by $\lambda = \sum_S \lambda_S e_S$ where $\lambda_S \in \mathbb{R}$. Under the inner product $\langle\lambda, \mu\rangle = \sum_S \lambda_S \mu_S$, $\mathbb{R}_{(m)}$ becomes a Hilbert space with orthonormal basis $\{e_s\}$.

Let M_n denote the space of n by n complex matrices equipped with the operator bound norm and $L(X)$ the space of linear operators on a vector space X. The *Clifford operator* of an m-tuple $A = (A_1, \ldots, A_m) \in M_n^m$ is the operator $\mathrm{Cliff}(A) \in M_n \otimes \mathbb{R}_{(m)}$ defined by

$$\mathrm{Cliff}(A) = i \sum_{j=1}^{m} A_j \otimes e_j. \tag{2.1}$$

Each element $T = \sum_S T_S \otimes e_S \in M_n \otimes \mathbb{R}_{(m)}$ acts on elements $x = \sum_S x_S \otimes e_S \in \mathbb{C}^n \otimes \mathbb{R}_{(m)}$ by $T(x) = \sum_{S,S'} T_S(x_{S'}) \otimes e_S e_{S'}$.

So $\mathrm{Cliff}(A) \in M_n \otimes \mathbb{R}_{(m)} \subseteq L(\mathbb{C}^n \otimes \mathbb{R}_{(m)})$.

The space $\mathbb{C}^n \otimes \mathbb{R}_{(m)}$ is a Hilbert space with inner product given by $\langle x, y\rangle = \sum_S \langle x_S, y_S\rangle$ where $\langle x_S, y_S\rangle$ is the standard inner product on $\mathbb{C}^n$. By $\|\mathrm{Cliff}(A)\|$ we will mean the operator bound norm of Cliff(A) as an element of $L(\mathbb{C}^n \otimes \mathbb{R}_{(m)})$. Its spectrum is denoted $\sigma(\mathrm{Cliff}(A))$.

As an indication of the close relationship between A and Cliff(A) we state the following result, which appeared in [16].

PROPOSITION 2.2. *If* A *is a triangularizable* m-*tuple of matrices with real spectra then*

(a) $\sigma(\mathrm{Cliff}(A)) = \{\pm|\alpha| : \alpha \in \sigma(A)\}$;

(b) *an element* α *of* $\mathbb{R}^m$ *belongs to* $\sigma(A)$ *if and only if* $\mathrm{Cliff}(A - \alpha I)$ *is not invertible.*

3. GENERALIZATIONS OF THE BAUER-FIKE THEOREM

The following theorem and an analogue for matrices with arbitrary complex spectra appear in Pryde [14].

THEOREM 3.1. *Let* $A = (A_1, ..., A_m)$ *and* $B = (B_1, ..., B_m)$ *be* m-*tuples of commuting* n *by* n *matrices with real spectra. Suppose there is an invertible matrix* Q *such that* $Q^{-1}B_jQ$ *is diagonal for each* $j = 1, ..., m$. *Then*

$$S(\sigma(A), \sigma(B)) \leq \|Q\| \; \|Q^{-1}\| \; \|\mathrm{Cliff}(A - B)\|.$$

The corresponding conclusion for single matrices A and B, proved by Bauer and Fike [1], is

$$S(\sigma(A), \sigma(B)) \leq \|Q\| \; \|Q^{-1}\| \; \|A - B\|. \tag{3.2}$$

As observed in Stewart and Sun [18], and in Bhatia and Bhattacharyya [3], Bauer and Fike actually proved a stronger result. The following theorem, with the extra assumption that the matrices commute, is the corresponding strengthening of theorem 3.1. The commuting case of theorem 3.2 was proved by Bhatia and Bhattacharyya [3]. Their proof uses the fact that each $\alpha \in \sigma(A)$ is a joint eigenvalue if the A_j commute. So that proof fails under the weaker assumption that A is triangularizable.

THEOREM 3.2. *Let* $A = (A_1, ..., A_m)$ *and* $B = (B_1, ..., B_m)$ *be triangularizable* m*-tuples of* n *by* n *matrices with real spectra. Let* $\alpha \in \sigma(A) \setminus \sigma(B)$. *Then for each invertible* n *by* n *matrix* Q,

$$\|Q^{-1}\mathrm{Cliff}(B - \alpha I)^{-1}Q\|^{-1} \le \|Q^{-1}\mathrm{Cliff}(A - B)Q\|.$$

Proof. Note firstly that by proposition 2.2 $\mathrm{Cliff}(B - \alpha I)$ is invertible since $\alpha \in \mathbb{R}^m \setminus \sigma(B)$. Hence

$$\begin{aligned}
&\mathrm{Cliff}(A - \alpha I)\\
&\qquad = \mathrm{Cliff}(B - \alpha I) + \mathrm{Cliff}(A - B)\\
&\qquad = \mathrm{Cliff}(B - \alpha I)(I + \mathrm{Cliff}(B - \alpha I)^{-1}\mathrm{Cliff}(A - B))\\
&\qquad = \mathrm{Cliff}(B - \alpha I)Q(I + Q^{-1}\mathrm{Cliff}(B - \alpha I)^{-1}QQ^{-1}\mathrm{Cliff}(A - B)Q)Q^{-1}.
\end{aligned}$$

Since $\alpha \in \sigma(A)$, $\mathrm{Cliff}(A - \alpha I)$ is not invertible. Hence

$$\begin{aligned}
1 &\le \|Q^{-1}\mathrm{Cliff}(B - \alpha I)^{-1}QQ^{-1}\mathrm{Cliff}(A - B)Q\|\\
&\le \|Q^{-1}\mathrm{Cliff}(B - \alpha I)^{-1}Q\| \, \|Q^{-1}\mathrm{Cliff}(A - B)Q\|
\end{aligned}$$

from which comes the result. ∎

4. GENERALIZATIONS OF THE HENRICI THEOREM

In [3] Bhatia and Bhattacharyya defined a *measure of non-normality* of an m-tuple $A = (A_1, ..., A_m)$ of commuting matrices. The same construction is valid for triangularizable m-tuples. Firstly, if $A = (A_1, \ldots, A_m)$ is triangularizable then there is a unitary matrix U such that $U^* A_j U = T_j$ for all j, where the T_j are triangular. As is also stated in [14], the simplest proof of this fact is perhaps by induction on n, using the fact that the A_j have a common unit eigenvector, and a result of Radjavi [17] that a semigroup Σ of matrices is triangularizable if and only if $\mathrm{trace}(S_1S_2S_3 - S_2S_1S_3) = 0$ for all $S_1, S_2, S_3 \in \Sigma$.

Now write $T_j = \Lambda_j + N_j$ where Λ_j is diagonal and N_j is nilpotent triangular. Let $N = (N_1, \ldots, N_m)$. The *measure of non-normality* of A is given by

$$(4.1) \qquad \Delta(A) = \inf\|\mathrm{Cliff}(N)\|$$

where the infimum is taken over all choices of unitary U for which each $U^* A_j U$ is triangular.

THEOREM 4.2. *Let* $A = (A_1, \ldots, A_m)$ *and* $B = (B_1, \ldots, B_m)$ *be triangularizable* m*-tuples of matrices with real spectra. Let* $\alpha \in \sigma(A) \setminus \sigma(B)$ *and set*

$\delta = \min\{|\alpha - \beta| : \beta \in \sigma(B)\}$. *Then*

$$\delta \le \left[\sum_{k=0}^{n-1} (\Delta(B)/\delta)^k\right] \|\mathrm{Cliff}(A - B)\|.$$

This theorem is due to Henrici [8] for the case $m = 1$. It is due to Bhatia and Bhattacharyya [3] for the case of commuting m-tuples. Their proof uses the commuting case of theorem 3.2 with $Q = U$, a unitary matrix for which the infimum $\Delta(B) = \inf\|\mathrm{Cliff}(N)\|$ is achieved. Using theorem 3.2 for triangularizable m-tuples, their proof remains valid and we do not repeat it.

5. GENERALIZATIONS OF THE BHATIA-FRIEDLAND INEQUALITY

Let A and B be n by n matrices and set $M = \max(\|A\|, \|B\|)$. Bhatia and

Friedland [5] proved that

(5.1) $$S(\sigma(A), \sigma(B)) \leq n^{1/n}(2M)^{1-1/n}\|A - B\|^{1/n}.$$

Their proof used characteristic polynomials and Grassman powers. Elsner [7] gave another proof using Henrici's theorem. Bhatia and Bhattacharyya [4], having extended Henrici's theorem to commuting m-tuples, used it to obtain:

THEOREM 5.2. *Let* A *and* B *be* m*-tuples of commuting* n *by* n *matrices with commuting partitions* $\pi(A)$ *and* $\pi(B)$ *respectively. If*
$M = \max(\|\text{Cliff}(\pi(A))\|, \|\text{Cliff}(\pi(B))\|)$ *then*
$S(\sigma(A), \sigma(B)) \leq n^{1/n}(2M)^{1-1/n}\|\text{Cliff}(\pi(A) - \pi(B))\|^{1/n}$.

Recently, Pryde [16] has constructed joint characteristic polynomials $\phi_A(\zeta)$ for triangularizable m-tuples. In fact, $\phi_A(\zeta) = \det(\text{Cliff}(A - \zeta I))$ for $\zeta \in \mathbb{C}^m$. Using these polynomials and Grassman products, he obtained the following:

THEOREM 5.3. *Let* A *and* B *be triangularizable* m*-tuples of* n *by* n *matrices with partitions* $\pi(A)$ *and* $\pi(B)$ *respectively. If*
$M = \max(\|\text{Cliff}(\pi(A))\|, \|\text{Cliff}(\pi(B))\|)$ and $N = 4^m n$ *then*
$S(\sigma(A), \sigma(B)) \leq N^{1/N}(2M)^{1-1/N}\|\text{Cliff}(\pi(A) - \pi(B))\|^{1/N}$.

As remarked in [16], the Bhatia-Bhattacharyya proof of theorem 5.2 also shows that for triangularizable m-tuples

(5.4) $$S(\sigma_{pt}(A), \sigma(B)) \leq n^{1/n}(2M)^{1-1/n}\|\text{Cliff}(\pi(A) - \pi(B))\|^{1/n}.$$

As a final contribution, we can now use the Bhatia-Bhattacharyya proof, noting that the Henrici theorem 4.2 is valid for triangularizable m-tuples, not just commuting ones, to obtain

THEOREM 5.5. *Let* A *and* B *be triangularizable* m-*tuples of* n *by* n *matrices with partitions* $\pi(A)$ *and* $\pi(B)$ *respectively. If*
$M = \max(\|\mathrm{Cliff}(\pi(A))\|,\ \|\mathrm{Cliff}(\ (B))\|)$ *then*
$S(\sigma(A),\ \sigma(B)) \le n^{1/n}(2M)^{1-1/n}\|\mathrm{Cliff}(\pi(A) - \pi(B))\|^{1/n}$.

We remark that neither of theorems 5.3 and 5.5 follows from the other.

6. INEQUALITIES FOR NORMAL MATRICES

In this section we consider m-tuples of normal matrices. In view of the following proposition, we assume the matrices commute.

PROPOSITION 6.1. *If* $A = (A_1, \ldots, A_m)$ *is a triangularizable* m-*tuple of normal matrices* A_j *then the* A_j *commute.*

Proof. As commented at the beginning of section 4, there is a unitary matrix U such that, for all j, $U^* A_j U = T_j$ a triangular matrix. Since A_j is normal, so is T_j. A triangular normal matrix is diagonal. So the T_j are diagonal and hence commute. So the A_j also commute. ∎

We will be concerned with norms $|\cdot|$ on M_n^m which satisfy the following condition:

(6.2) $$\|\Sigma\alpha_j T_j\| \le |\alpha|\ |T| \quad \text{for all } \alpha \in \mathbb{C}^m,\ T \in M_n^m.$$

The following theorem was proved in McIntosh, Pryde and Ricker [12].

THEOREM 6.3. *Let* A *and* B *be* m-*tuples of commuting* n *by* n *normal matrices with joint spectra* $\sigma(A) = \{\alpha^{(k)} : 1 \le k \le n\}$ *and* $\sigma(B) = \{\beta^{(k)} : 1 \le k \le n\}$. *There exists a permutation* τ *of the index set* $\{1, \ldots, n\}$ *such that*

$$(6.4) \qquad |\alpha^{(k)} - \beta^{(\tau(k))}| \le c_m |A - B|$$

for all k and all norms $|\cdot|$ satisfying (6.2).

In this theorem, c_m is a constant depending only on m. In fact, $c_m = \dot{e}_{m,0}$ an explicit constant defined in [12, (2.4)].

It should be mentioned that somewhat weaker results are stated in [10, corollary 2.2], where the theorem was initially announced, and in [12, remark 7]. In fact, in these references A and B are m-tuples of self-adjoint matrices and the norm $|\cdot|$ is given by

$$(6.5) \qquad |T| = \sup\{(\sum_1^m \|T_j x\|^2)^{1/2} : x \in \mathbb{C}^n, \ |x| < 1\}.$$

Moreover, $c_m = c_{m,0}$ a constant defined in [12, remark 1]. The stronger result for normal matrices and more general norms is mentioned in [12, remarks 1 and 9].

In addition to the norm defined by (6.5) we consider two other norms satisfying (6.2). Firstly we have:

PROPOSITION 6.6. *The norm* $|\cdot|$ *on* M_n^m *defined by* $|T| = \|\mathrm{Cliff}(T)\|$ *satisfies (6.2).*

Proof. Let $\alpha \in \mathbb{C}^m$ and $T \in M_n^m$. Choose a unit vector $x \in \mathbb{C}^n$ such that $\|\sum_1^m \alpha_j T_j\| = \|\sum_1^m \alpha_j T_j x\|$ and set $y = \sum_1^m \alpha_j x \otimes e_j \in \mathbb{C}^n \otimes \mathbb{R}_{(m)}$. Then $\|y\| = |\alpha|$ and $\mathrm{Cliff}(T)y = i\sum_{j,k} (T_j \otimes e_j)(\alpha_k x \otimes e_k) = -i \sum_j \alpha_j T_j x + i\sum_{j<k} (\alpha_k T_j x - \alpha_j T_k x) \otimes e_j e_k$. Recalling that the e_S form an orthonormal basis of $\mathbb{R}_{(m)}$ we obtain $\|\sum_1^m \alpha_j T_j\| = \|\sum_1^m \alpha_j T_j x\| \le \|\mathrm{Cliff}(T)y\| \le \|\mathrm{Cliff}(T)\| \ \|y\| = |\alpha| \ |T|$ as required. ∎

Secondly, let $\|S\|_F$ denote the Frobenius norm (or Schatten 2-norm, or Hilbert-Schmidt norm, or Schur norm, or Euclidean norm) of a matrix $S \in M_n$. So $\|S\|_F = (\text{trace } S^*S)^{1/2} = (\sum_{k,l} |S_{kl}|^2)^{1/2}$ where $S = (S_{kl})$. We define the *Frobenius norm* $\|T\|_F$ of an m-tuple $T = (T_1, \ldots, T_m) \in M_n^m$ by

$$(6.7) \qquad \|T\|_F = (\sum_j \|T_j\|_F^2)^{1/2} = (\sum_{j,k,l} |(T_j)_{kl})^2)^{1/2}.$$

For $\alpha \in \mathbb{C}^m$, $\|\Sigma\alpha_j T_j\| \le |\alpha|(\Sigma\|T_j\|^2)^{1/2} \le |\alpha|(\Sigma\|T_j\|_F^2)^{1/2} = |\alpha|\ \|T\|_F$. So $|T| = \|T\|_F$ defines a norm satisfying (6.2).

Recall that the norm $\|\text{Cliff}(A - B)\|$ appeared in sections 3 and 4 and the norm $\|\text{Cliff}(\pi(A) - \pi(B))\|$ in section 5. The Frobenius norm $\|A - B\|_F$ appears in the following theorem of Bhatia and Bhattacharyya [4]. When $m = 1$ this is the well-known theorem of Hoffman and Wielandt [9].

THEOREM 6.8. *Let* A *and* B *be* m*-tuples of commuting* n *by* n *normal matrices with joint spectra* $\sigma(A) = \{\alpha^{(k)} : 1 \le k \le n\}$ *and* $\sigma(B) = \{\beta^{(k)} : 1 \le k \le n\}$. *There exists a permutation* τ *of the index set* $\{1, \ldots, n\}$ *such that* $\sum_k |\alpha^{(k)} - \beta^{(\tau(k))}|^2 \le \|A - B\|_F^2$.

Again let A and B be m-tuples of commuting normal matrices. Let $\pi(A)$ and $\pi(B)$ be their commuting semisimple partitions. From theorem 6.3 and the comments following it, and from proposition 6.6 we obtain

$$(6.9) \qquad S(\sigma(A), \sigma(B)) \le e_{m,0}\|\text{Cliff}(A - B)\|$$

and

$$(6.10) \qquad S(\sigma(A), \sigma(B)) \le e_{m,0}\|A - B\|_F.$$

We can also apply theorem 6.3 to the 2m-tuples of commuting self-adjoint matrices $\pi(A)$ and $\pi(B)$ to obtain

(6.11) $S(\sigma(A), \sigma(B)) \leq c_{2m,0}\|\mathrm{Cliff}(\pi(A) - \pi(B))\|$

and

(6.12) $S(\sigma(A), \sigma(B)) \leq c_{2m,0}\|\pi(A) - \pi(B)\|_F.$

From theorem 3.1 we obtain inequality (6.11) with the constant $c_{2m,0}$ replaced by 1. Similarly, from theorem 6.8 we obtain inequalities (6.10) and (6.12) with $e_{m,0}$ and $c_{2m,0}$ replaced by 1.

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