I. THE ALGEBRA OF FUNCTIONS OF ONE VARIABLE

1. The Classical Foundation of the Theory of Functions.

The classical theory starts with the assumption that a field of numbers is given. That is to say, it starts with a system N of things, called numbers, which we add and multiply according to the well-known laws of a field.

Next, the theory of functions of one variable explicitely defines a function f as the association of a number f(x) with each number x of some subset D_f of N. This set D_f is called the domain of f. The set R_f of all numbers f(x) which f associates with the numbers x of D_f, is called the range of f. A function whose range consists of exactly one number is called constant *).

The definition of the concept of functions is followed by explicit definitions of the concept of equality of functions and of three binary operations: addition, multiplication, and substitution. We define: $f = g$ if and only if $D_f = D_g$ and $f(x) = g(x)$ for each number x of $D_f = D_g$. (The concept of equality of numbers of the given field is assumed to be known). If $D_f = D_g$, we call sum of f and g [product of f and g] the function which associates the number $f(x) + g(x)$ [the number $f(x) \cdot g(x)$] with each number x of $D_f = D_g$. If R_g is a subset

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*) Analyzing the somewhat vague concept of "association" we see that f is a set of ordered pairs of numbers (d,r) such that each element of D_f occurs as the first element of exactly one pair. The second element, r, of the pair (d,r) is that element f(d) of R_f which f "associates" with d.

of D_f, we call f of g the function which associates the number $f(g(x))$ with each number x of D_g.

It is an odd fact that in the classical calculus no symbols for the sum of f and g, the product of f and g, and f of g, are introduced. Using the notation of the calculus of operators, we shall denote the results of the three operations by

$$f + g, \qquad f\cdot g, \qquad fg,$$

respectively, so that the numbers associated by these functions with the number x are

$$(f+g)(x), \qquad (f\cdot g)(x), \qquad (fg)(x),$$

respectively. We shall never omit the dot symbolizing multiplication in order to avoid a confusion of multiplication with substitution. In this notation the classical definitions of $f+g$, $f\cdot g$, fg read

$$(f+g)(x) = f(x) + g(x), \quad (f\cdot g)(x) = f(x)\cdot g(x), \quad (fg)(x) = f(g(x)).$$

From these explicit definitions of the classical theory, one deduces properties of the defined concepts. As two examples we mention the commutative law for multiplication and the associative law for substitution.

In order to prove $f\cdot g = g\cdot f$, by virtue of the equality concept, we have to show that $(f\cdot g)(x) = (g\cdot f)(x)$ for each x. From the definition of the product of functions we obtain $(f\cdot g)(x) = f(x)\cdot g(x)$ and $(g\cdot f)(x) = g(x)\cdot f(x)$. From the commutativeness of multiplication in the given field of numbers, it follows that $f(x)\cdot g(x) = g(x)\cdot f(x)$ which completes the proof.

In order to prove $f(gh) = (fg)h$, we have to prove $[f(gh)](x) = [(fg)h](x)$ for each x. Now from the definition of substitution it follows that

$$[f(gh)](x) = f[(gh)(x)] = f[g(h(x))]$$
$$[(fg)h](x) = (fg)(h(x)) = f[g(h(x))]$$

which completes the proof.

The commutative law for substitution does not generally hold, as we see, if we set $f(x) = 1 + x$ and $g(x) = x^2$. We have $f(g(x)) = 1 + x^2$ and $g(fx)) = (1 + x)^2$.

2. Algebra of Functions (Tri-Operational Algebra).

In contrast to the classical approach we do not start with a given field of numbers and do not define functions in terms of numbers. In fact, we shall not give any explicit definition of functions, and we shall abolish the dualism between numbers and functions*).

We start out with a system of things, called functions and denoted by small letters $f, g, \ldots$, and three binary operations: addition denoted by $+$, multiplication denoted by $\cdot$, and substitution denoted by juxtaposition. For these operations we postulate the following laws which in the classical theory (as we have seen in two examples) are deduced from the explicit definitions:

Operation:	Addition	Multiplication	Substitution
Commutative Law:	$f+g=g+f$	$f\cdot g=g\cdot f$	-----------
Associative Law:	$(f+g)+h=f+(g+h)$	$(f\cdot g)\cdot h=f\cdot(g\cdot h)$	$(fg)h=f(gh)$

*) At the same time we rid the foundations of analysis of set theoretical elements which are contained in the explicit definition of functions (see footnote p.4).

Distributive Law:	$(f\cdot g)h = fh\cdot gh$	$(f+g)h = fh+gh$	$(f+g)\cdot h = f\cdot h+g\cdot h$
Neutral Elements:	$f+0 = f$	$f\cdot 1 = f$	$fj = jf = f$
	$0 \neq 1 \neq j \neq 0$		$10 = 1$
Opposite Elements:	$f+(-f) = 0$		

Commutativeness of substitution is not postulated because it does not hold in the classical theory. The three distributive laws listed under

addition multiplication substitution

will be called

multiplicative-substitutive, additive-substitutive, additive-multiplicative, distributive laws, respectively, or briefly,

m.s.d. law a.s.d. law a.m.d. law.

From the commutative law of multiplication and the a.m.d. law it follows that $h\cdot(f+g) = h\cdot f + h\cdot g$. In absence of a commutative law for substitution the a.s.d. law and the m.s.d. law do not imply

$$h(f+g) = hf+hg \quad \text{and} \quad h(f\cdot g) = hf\cdot hg.$$

In fact, in the classical theory these formulae are not generally true. Each of them represents a functional equality characterizing a special class of functions h.

The neutral elements 0 and 1 in our algebra correspond to the classical functions associating with each x the numbers 0 and 1, respectively. From the commutativity of addition and multiplication in conjunction with the postulates concerning 0 and 1, it follows that $0+f = f$ and $1\cdot f = f$. For the neutral element of substitution, j, both $jf = f$ and $fj = f$ must be postulated. The element j corresponds to the classical function associating with each number x the number x. Oddly enough, the classical calculus does not introduce a symbol for this fundamental function.

In the same way as one assumes $0 \neq 1$ in defining a field, we assume that the three neutral elements 0, 1, and j are mutually different. Furthermore, we had to postulate that 0 substituted in 1 yields 1 because we shall have to make use of this assumption and, as Rev. F. L. Brown proved by an example, it is independent of the other postulates. (Under certain conditions, the postulate $10 = 1$ could be replaced by the simpler formula $10 \neq 0$).

The three neutral elements can easily be shown to be unique. For instance, if we had also $fj' = f$ for each f, then by applying this formula to $f = j$ we should obtain $jj' = j$. Now $jj' = j'$ by virtue of the neutrality of j. Hence $j = j'$.

With regard to addition we assume the existence of an opposite element to each f. We denote by $-f$ the function which, added to f, yields the sum 0. The formulae which we assumed with regard to addition and multiplication are just those valid in a commutative ring with a unit, i.e., a neutral element of multiplication*).

In the classical theory the three operations are not universal (i.e., applicable to each pair of functions). E.g., we can form fg only if R_g is part of D_f. Hence we shall not

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*) It should be noted that among the formulae concerning addition and substitution, one which would be valid in a non-commutative ring has not been postulated, namely, $h(f+g) = hf + hg$.

postulate in our algebra that the three operations are universal. We shall interpret our postulates as <u>formulae which are valid if all terms are meaningful</u>. The situation is the same as in a grupoid in which $f+(g+h) = (f+g)+h$ is true provided that $g+h$, $f+g$, $f+(g+h)$, and $(f+g)+h$ exist.

However, it is worth mentioning that our postulates are consistent even in presence of the additional assumption that the three operations are universal. A model satisfying all these assumptions is the system of all polynomials $p = c_o + c_1\cdot j + c_2\cdot j^2 + \cdots + c_m\cdot j^m$ with coefficients c_k belonging to a given ring (where j^k is an abbreviation for a product of k factors j) if sum, product, and substitution are defined in the ordinary way. That is, if

$$q = d_o + d_1\cdot j + d_2\cdot j + \cdots + d_n\cdot j^n, \text{ then}$$
$$p + q = (c_o + d_o) + (c_1 + d_1)\cdot j + (c_2 + d_2)\cdot j^2 + \cdots$$
$$p\cdot q = c_o d_o + (c_o\cdot d_1 + c_1\cdot d_o)\cdot j + (c_o\cdot d_2 + c_1\cdot d_1 + c_2\cdot d_o)\cdot j^2 + \cdots$$
$$pq = c_o + c_1\cdot q + c_2\cdot q^2 + \cdots + c_m\cdot q^m.$$

A simple model consisting of four functions 0,1,j,f, is obtained if the three operations are defined by the following tables:

+	0	1	j	f
0	0	1	j	f
1	1	0	f	j
j	j	f	0	1
f	f	j	1	0

•	0	1	j	f
0	0	0	0	0
1	0	1	j	f
j	0	j	j	0
f	0	f	0	f

	0	1	j	f
0	0	0	0	0
1	1	1	1	1
j	0	1	j	f
f	1	0	f	j

In the classical theory we obtain the above system*) by considering the field consisting of the numbers 0 and 1 (modulo 2) and by calling 0,1,j,f the functions associating with the numbers 0 and 1 the numbers 0,0; 1,1; 0,1; 1,0, respectively. As Rev. F. L. Brown has recently shown, this system is the simplest one satisfying all the postulates since no system consisting of 0,1, and j only satisfies all postulates. However, Father Brown did find a system consisting of the three elements 0,1,j, satisfying all the postulates except the one concerning the existence of a negative element.

3. <u>The Theory of Constant Functions</u>.

We shall now single out a class of functions which we will call constant functions or, briefly, constants. The definition will be in terms of the fundamental operations. From the postulates concerning these operations, it will follow that our constants enjoy the main properties of the constant functions of the classical theory.

We call a function f <u>constant</u> if $f = f0$. If we know of a function that it is constant, then we shall usually denote it by letters c,d,...

From the postulates it follows that $0f = (0+0)f = 0f + 0f$. Adding $-(0f)$ to this equality we obtain the formula

$$0f = 0.$$

In particular $00 = 0$. Hence 0 is a constant. That 1 is a constant, is the content of the postulate $10 = 1$. Since $j0 = 0 \neq j$, we see that j is not a constant. We shall prove

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*) It can also be described as the system of all polynomials modulo $j + j^2$ with coefficients belonging to the field 0,1 modulo 2.

the following theorem:

<u>The constants form a ring</u>*) <u>which is closed with respect to substitution</u>. If c_1 and c_2 are constants, then

$$c_1 + c_2 = c_1 0 + c_2 0 = (c_1 + c_2)0.$$

Similarly, $c_1 \cdot c_2 = (c_1 \cdot c_2)0$. Thus the sum and the product of two constants are constants. From the fact that 0 is a constant, it readily follows that the negative of a constant is a constant. Thus the constants form a ring. Now let c be a constant, and f any function. We have

$$fc = f(c0) = (fc)0.$$

Thus fc is a constant. Using the formula $0f = 0$ we obtain

$$cf = (c0)f = c(0f) = c0 = c.$$

Thus cf is a constant, more specifically, cf is the constant c. This completes the proof of our theorem.

The last formula can also be expressed by saying that if c is a constant, then not only $c0 = c$ but $cf = c$ for each f.

If for a constant c there exists a function c' such that $c \cdot c' = 1$, then this "reciprocal" c' is a constant. For

$$c'0 = 1 \cdot c'0 = (c' \cdot c) \cdot c'0 = c' \cdot (c \cdot c'0) = c' \cdot (c0 \cdot c'0) =$$
$$c' \cdot (c \cdot c')0 = c' \cdot 10 = c' \cdot 1 = c'.$$

We see: if for each constant $c \neq 0$ there exists a reciprocal, then the constants form a field which is closed with respect to substitution. However, the roots of an algebraic

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*) Quite accurately, we should say: For the constants all formulae postulated in a commutative ring are valid if they are meaningful. Under additional assumptions and with a sharper definition of constants, we could prove them to form a ring. We should have to call constant a function c such that c0 is defined and = 0. We should have to assume that if f0 and g0 are defined, then (-f)0, (f + g)0, and (f·g)0 are defined.

equation with constant coefficients need not be constants. For instance, each of the four functions $0, 1, j, 1+j$, studied at the end of the last section, satisfies the algebraic equation with constant coefficients $f + f^2 = 0$. The functions j and $1+j$ are not constant.

The definition of equality of two functions in the classical theory is reflected in the following fundamental proposition of our algebra: If $fc = gc$ for each constant c, then $f = g$. If this proposition holds, then we shall speak of a <u>tri-operational algebra with a base of constants.</u>

Clearly, in such an algebra we have $f = g$ if and only if $fc = gc$ for each constant c. Moreover, in order that f be a constant it is necessary and sufficient that $fc = f0$ for each constant c. For from $fc = f0$ it follows that $fc = f0 = f(0c) = (f0)c$. Applying the equality criterion to f and $f0$ we see that f is equal to the constant function $f0$.

A consequence of this last remark is the following first theorem: If for two constants c_0 and c_1 we have $f(c_0 + j) = c_1$, then $f = c_1$. In fact, for each constant c the assumption implies that

$$fc = f(c_0 + (c - c_0)) = f(c_0 + j)(c - c_0) = c_1(c - c_0) = c_1.$$

Another consequence is the following translation theorem: If $f(j + c) = f$ for each constant c, then f is constant. For from the assumption it follows that

$$fc = f(0 + c) = f(j0 + c0) = f[(j + c)0] = [f(j + c)]0 = f0.$$

An Algebra of Functions admits a representation by functions in the classical sense. With each function of our algebra we can associate a function in the classical sense whose domain and whose range are two sets of constants. With the function f of our algebra we can associate the function f^* in the classical sense whose domain consists of those constants which admit substitution into f, and which associates with each such constant c the constant fc. This association of functions in the classical sense with functions of our algebra is readily seen to be a homomorphism. That is to say, we have

$$(f+g)^* = f^*+g^*, \quad (f,g)^* = f^*\cdot g^*, \quad (fg)^* = f^*g^*$$

where addition, multiplication, and substitution of the classical functions on the right sides of these equalities are to be performed in the classical sense. The postulate of a base of constants implies that the above homomorphism is an isomorphism, that is to say, that $f \neq g$ implies $f^* \neq g^*$.

It is to be noted that even in an algebra in which the three fundamental operations are universal and the constants form a field, the postulate of the base of constants need not be satisfied. We obtain an example of the independence of this postulate by considering all polynomials

$$c_0 + c_1\cdot j + c_2\cdot j^2 + \cdots + c_m\cdot j^m$$

with coefficients 0 and 1 if addition, multiplication, and substitution are defined in the ordinary sense but modulo 2. There are infinitely many such polynomials but only two constants, viz., 0 and 1. For each polynomial the substitution of 0 and 1 yields either 0 and 0, or 0 and 1, or 1 and 0, or 1 and 1. If we write a polynomial in the form

$$p = c_o + p^{k1} + p^{k2} + \cdots + p^{kn}$$

where c_o is 0 or 1, then clearly p belongs to one of the following four classes:

Either $c_o = 0$ and n is even. Then $p0 = 0$, $p1 = 0$,
Or, $c_o = 0$ and n is odd. Then $p0 = 0$, $p1 = 1$.
Or, $c_o = 1$ and n is odd. Then $p0 = 1$, $p1 = 0$.
Or, $c_o = 1$ and n is even. Then $p0 = 1$, $p1 = 1$.

If p_1 and p_2 are two of the infinitely many polynomials belonging to the same class, then for each constant (that is, for $c = 0$ and $c = 1$) we have $p_1c = p_2c$ and yet $p_1 \neq p_2$. The homomorphic representation of the functions of our algebra by functions in the classical sense which we described above, would lead to the four functions defined in the field modulo 2 mentioned at the end of the preceding section. Each function of the first class would be mapped onto the function representing 0, each function of the second class on the function representing j, each function of the third class on the function representing $1 + j$, and each function of the fourth class on the function representing 1.

The following finite example for the same situation may be omitted in a first reading. We consider the polynomials of the preceding example modulo $j + j^4$. That is to say, we set $j + j^4 = 0$. We thus retain a model consisting of 16 polynomials $c_o + c_1 \cdot j + c_2 \cdot j^2 + c_3 \cdot j^3$ with coefficients $c_k = 0,1$. There are only two constants, 0 and 1, and hence only four possibilities for p0 and p1, as before. Each possibility is realized for a class of four polynomials. E.g., we have $p0 = 0$ and $p1 = 0$ for $p_1 = 0$, $p_2 = j + j^2$, $p_3 = j + j^3$, $p_4 = j + j^4$.

This is another example in which each function p, each constant function as well as each of the 14 non-constant functions, satisfies an algebraic equation with constant coefficients, namely, $p + p^4 = 0$, as we see by substituting p into the equality $j + j^4 = 0$.

4. The Lytic Operations.

While we did not postulate universality of the three fundamental operations, we saw that a postulate to this effect would be compatible with our assumptions. Now we are going to introduce a function of a special kind whose very nature, in presence of the other postulates, is incompatible with universality of substitution. We shall call this function rec (an abbreviation for reciprocal) and define it by the equality

$$\text{rec} \cdot j = 1.$$

If we substitute 0 in this equality we obtain $(\text{rec} \cdot j)0 = 10$ from which, in view of $0 \cdot f = 0$, it follows that

$$0 = \text{rec } 0 \cdot 0 = \text{rec } 0 \cdot j0 = (\text{rec} \cdot j)0 = 10 = 1.$$

This contradicts the assumption $0 \neq 1$. We see that in presence of the definition of rec we must give up some of our postulates or abandon the universality of substitution by forbidding the substitution of the function 0 into the function rec. We shall follow the latter course.

If the constants form a field, then 0 is the only constant which cannot be substituted into rec. Into rec f we cannot substitute those constants c for which $fc = 0$. For instance, 1 cannot be substituted into $\text{rec}(j - 1)$.

It goes without saying that, in the classical notation, rec is the function associating with each number $x \neq 0$ the number $\frac{1}{x}$. From the point of view of domains, the function rec·j is not identical with the function 1. The latter is an extension of the former. For the domain of 1 comprises all numbers; that of rec, and hence of rec·j, all numbers $\neq 0$.

We shall disregard this difference and thus from now on be compelled to interpret our postulates as formulae which are valid whenever their terms are meaningful, and we shall have to interpret each result as a formula admitting those substitutions which are admissible in all terms involved in the derivation of the result from the postulates.

To make the analogy between rec f and -f more conspicuous we shall frequently write neg f instead of -f. This notation is justified since there exists a function neg such that we obtain -f by substituting f into the function neg. This function neg is -j or -1·j. In the classical notation it is the function associating with each number x the number -x.

Instead of postulating the existence of -f for each f it would be sufficient to postulate the existence of a function neg such that j + neg = 0. In view of 0f = 0, jf = f, and the a.s.d. law this postulate implies f + neg f = 0 for each f. We tabulate some analogous facts of the algebra of the functions neg and rec which we shall call <u>the lytic functions</u> with regard to addition and multiplication, respectively.

j + neg = 0	j·rec = j
f + neg f = 0	f·rec f = j
neg (f + g) = neg f + neg g	rec(f·g) = rec f·rec g
neg neg = j	rec rec = j

rec neg = neg rec.

In fact, we have

$\mathrm{rec}(f\cdot g) = \mathrm{rec}(f\cdot g)\cdot 1 = \mathrm{rec}(f\cdot g)(f\cdot \mathrm{rec}\ f\cdot g\cdot \mathrm{rec}\ g) =$
$\mathrm{rec}(f\cdot g)\cdot(f\cdot g)\cdot(\mathrm{rec}\ f\cdot \mathrm{rec}\ g) = 1\cdot\ \mathrm{rec}\ f\cdot \mathrm{rec}\ g = \mathrm{rec}\ f\cdot \mathrm{rec}\ g.$

Using the formula rec•rec rec = 1 obtained by substituting rec into j•rec = 1 we see

$$\mathrm{rec\ rec} = (j\cdot \mathrm{rec})\cdot \mathrm{rec\ rec} = j\cdot(\mathrm{rec}\cdot \mathrm{rec\ rec}) = j\cdot 1 = j.$$

The proof of neg neg = j is similar.

From neg f = -1•f it follows that neg f•neg g = f•g. Using this formula we obtain

$\mathrm{rec\ neg} = \mathrm{rec\ neg}\cdot(j\cdot \mathrm{rec}) = \mathrm{rec\ neg}\cdot(\mathrm{neg}\ j\cdot \mathrm{neg\ rec}) =$
$\mathrm{rec\ neg}\cdot(\mathrm{neg}\cdot \mathrm{neg\ rec})=(\mathrm{rec\ neg}\cdot \mathrm{neg})\cdot \mathrm{neg\ rec}=1\cdot \mathrm{neg\ rec} = \mathrm{neg\ rec}.$

We define: f is <u>even</u> if and only if f neg = neg f, and f is <u>odd</u> if and only if f neg = f. The last of the tabulated formulae can be expressed by saying that rec is odd. Clearly, the product of two even or of two odd functions is even, the product of an odd and even function is odd.

Concerning the lysis of substitution, we mention that if for two functions f and g we have fg = j, then we shall call g the right inverse of f, and f the left inverse of g. For instance, the function j is its own right and left inverse since we have jj = j. If c_0 and c_1 belong to the ring of constant functions, and c_1 has a reciprocal c_1', then the function $c_1\cdot j + c_0$ has the function $c_1'\cdot(j - c_0)$ as inverse on either side. In the classical notation, in view of $j(x) = x$, the definition of a pair of inverse functions reads

$$f(g(x)) = x.$$

In other words, f and g are pairs of inverse functions in the classical sense, as log and exp or arctan and tan.

We shall postulate the existence of inverse functions only for special functions f. While there are functions neg and rec satisfying the equations j + neg = 0 and $j\cdot$rec = 1 and such that we obtain the negative and the reciprocal of f by substituting f into neg and rec, respectively, there certainly does not exist a function inv satisfying the equation j inv = j and such that we could obtain the inverse of f by substituting f into inv. For by virtue of the definition of j we should have j inv = inv, so that from j inv = j it would follow that inv = j. But by substituting f into j we obtain f which in general is not the inverse of f. Or we can say: By substituting f into the equality j inv = j we obtain j(inv f) = f rather than f(inv f) = j.

A constant function c clearly does not have inverse functions on either side. For, whatever function f may be, cf and fc are constants, thus $\neq j$ since j is not a constant.

If g is a right inverse of f and has itself at least one right inverse, h, then g has only one right inverse, namely, f, and only one left inverse, namely, f. And f has only one right and one left inverse, namely, g. In fact, from fg = j and gh = j it follows that h = jh = (fg)h = f(gh) = fj = f, a situation familiar from the axiomatics of group theory.

5. Exponential Functions.

We call the function f an exponential function if and only if

$$f(g+h) = fg\cdot fh \quad \text{and} \quad f \neq 0.$$

We shall denote exponential functions by exp. Thus exp(g + h) = exp g$\cdot$exp h. Substituting a constant c_0 for g, and j for h we see that

$\exp(c_0 + j) = \exp c_0 \cdot \exp j = \exp c_0 \cdot \exp$. If $\exp c_0 = 0$, then it follows that $\exp(c_0 + j) = 0$. If our algebra has a base of constants, then the last formula, by virtue of the first theorem of Section 3, implies that $\exp = 0$ in contradiction to the assumption $\exp \neq 0$. We thus see that $\exp c_0 \neq 0$ for each constant c_0. In further consequence, $\exp f \neq 0$ for each f. For if we had $\exp f = 0$, then we should obtain

$$\exp(fc_0) = (\exp f)c_0 = 0c_0 = 0$$

which, in view of the fact that fc_0 is a constant, contradicts the preceding remark.

If $\exp c_1 = 1$, then $\exp(f + c_1) = \exp c_1 \cdot \exp f = \exp f$ for each f. Conversely, if the constants form a field, from $\exp(c + c_1) = \exp c$ in view of $\exp c \neq 0$ it follows that $\exp c_1 = 1$. Now $\exp(c + 0) = \exp c$. Thus $\exp 0 = 1$. Consequently, $1 = \exp(j + \text{neg}) = \exp \cdot \exp \text{neg}$ and hence $\exp \text{neg} = \text{rec} \exp$.

Obviously, in each ring the function 1 is an exponential function. From $\exp 0 = 1$ it follows that 1 is the only constant exponential function. From now on, when talking about exponential functions we shall always mean exponential functions $\neq 1$.

If the constants form a finite field, then no exponential function exists. Let indeed $p \neq 0$ be the characteristic of the field of constants. Since $p - 1$ is the sum of $p - 1$ summands 1, for an exponential function we should have

$$\exp(p - 1) = (\exp 1)^{p-1}.$$

Now in a field of characteristic p we have $c^{p-1} = 1$ for each c. Hence $\exp(p - 1) = 1$. From $(p - 1) + 1 = 0$ it would follow that

$\exp(p-1)\cdot\exp 1 = \exp 0$. Since $\exp 0 = \exp(p-1) = 1$ we should have $\exp 1 = 1$. But then $\exp 2 = \exp 1\cdot\exp 1 = 1$, $\exp 3 = \exp 2\cdot\exp 1 = 1$, etc., hence $\exp c = 1$ for each c.

However, exponential functions do exist in finite rings. For instance, one readily verifies that in the ring of residues modulo 9 the function which under substitution

of 0,1,2,3,4,5,6,7,8

yields 1,4,7,1,4,7,1,4,7, respectively,

is an exponential function. In the infinite ring without divisors of 0 consisting of the numbers $m + n\cdot i$ where m and n are integers and $i^2 = -1$, the function which under substitution of $m + n\cdot i$ yields i^{m+n} is an exponential function. If the constants form the ring of all integers or the field of all rational numbers, then no exponential functions defined for all constants, exist.

6. The Logarithmic Functions.

We shall now take a step towards the algebra of real functions by assuming, in addition to a base of constants, three postulates about the ring of constants. For the sake of brevity we shall call a constant c a square if there exists a constant $c_1 \neq 0$ such that $c = c_1^2$. Now we postulate for each constant c which is not a divisor of 0:

1. If c is a square, then $-c$ is not a square.
2. If c is not a square, then $-c$ is a square.
3. There exists a constant $1/2$ such that $1/2 + 1/2 = 1$ (and consequently for each c a constant $c/2$ such that $c/2 + c/2 = c$, namely, $c\cdot 1/2$).

Clearly, the product of two squares, as well as the product of two negative squares, is a square. The product of a square and a negative square is not a square. It follows that if a square has a reciprocal, the reciprocal is a square.

Postulate 3 is satisfied in each field of characteristic $\neq 2$. Postulate 1 can be expressed by saying that $c_1^2 + c_2^2 = 0$ implies $c_1 = c_2 = 0$, a weaker form of the postulate for a real field. Postulates 1 and 2 are sufficient to establish in the ring what may be called a multiplicative order: If we call each square "positive", then for each element c of the ring either c is a divisor of 0, or c is positive, or -c is positive, and the product of two positive elements is positive. However, even if the ring is a field, postulates 1 and 2 are not sufficient to order the field (i.e., to guarantee that also the sum of two positive elements is positive) as the field of residues modulo 7 shows if we call 1,2,4 positive. Neither does each ordered field satisfy postulate 2 as the field of all rational numbers shows.

We shall now assume that an exponential function which admits the substitution of each constant, has an inverse on both sides which admits the substitution of each square. We shall call such a function a logarithmic function and denote it by log.

For each constant d, from $\exp d = \exp(d/2 + d/2) = \exp(d/2)\cdot\exp(d/2)$ it follows that each constant exp d is a square. This fact implies that log c admits only the substitution of squares. For if log c is defined, then

$$c = jc = (\exp \log)c = \exp(\log c)$$

and exp d is a square for each d. The same reasoning, in view of exp d $\neq 0$, shows that log does not admit the substitution of 0. Consequently, the function log $(j \cdot j)$ admits the substitution of each constant $\neq 0$.

Moreover, we have log 1 = log (exp 0) = (log exp)0 = j0 = 0.

Now let log c_1 and log c_2 be defined. That is, c_1 and c_2 are squares which implies that also $c_1 \cdot c_2$ is a square and $\log(c_1 \cdot c_2)$ is defined. We have

$$\log c_1 + \log c_2 = j(\log c_1 + \log c_2) = \log \exp(\log c_1 + \log c_2)$$
$$= \log(\exp \log c_1 \cdot \log \exp c_2) = \log(c_1 \cdot c_2).$$

It follows that

$$0 = \log 1 = \log(j \cdot j \cdot \mathrm{rec} \cdot \mathrm{rec}) = \log(j \cdot j) + \log(\mathrm{rec} \cdot \mathrm{rec})$$

and hence $\log(\mathrm{rec} \cdot \mathrm{rec}) = \mathrm{neg} \log(j \cdot j)$, formulae which admit the substitution of each constant $\neq 0$.

Similarly,

$$0 = \log 1 = \log(j \cdot \mathrm{rec}) = \log j + \log \mathrm{rec}.$$

However, the last equality admits only the substitution of squares (and of all squares since if c is a square, rec c is a square). Thus the same holds for

$$\log \mathrm{rec} = \mathrm{neg} \log.$$

7. The Absolute and the Signum.

Under the assumptions of the preceding section one can introduce a function which we shall call the absolute value or, briefly, the absolute, and which we shall denote by abs. We define

$$\mathrm{abs} = \exp[\tfrac{1}{2} \cdot \log(j \cdot j)] \text{ and } \mathrm{abs}\ 0 = 0.$$

In the classical theory the function corresponding with abs associates with each x the number $|x|$. The function abs admits the substitution of each constant and is readily seen to enjoy the following properties:

1. $\text{abs } c_1 \cdot \text{abs } c_2 = \text{abs}(c_1 \cdot c_2)$
2. $\text{abs}^2 = j^2$
3. abs neg = abs
4. $\text{abs} \neq 0$.

It is easily seen that abs rec = rec abs.

We further definr a signum function, denoted by sgn, in the following way:

$$\text{sgn} = \text{abs} \cdot \text{rec} \quad \text{and} \quad \text{sgn } 0 = 0.$$

In the classical theory the function corresponding with sgn associates 0 with 0, 1 with each positive, -1 with each negative number.

The function sgn has the following properties:

1. $\text{sgn } c_1 \cdot \text{sgn } c_2 = \text{sgn}(c_1 \cdot c_2)$
2. $\text{sgn}^3 = \text{sgn}$
3. sgn neg = neg sgn
4. $\text{sgn} \neq 0$.

One readily verifies that sgn yields 1 or -1 according to whether a square or the negative of a square is substituted. On this fact one can base another introduction of the assumptions of the preceding section, an introduction which is more in line with the Algebra of Functions than the postulates 1 and 2 concerning squares. We can postulate the existence of a function abs or a function sgn with the four properties mentioned above and define: a constant c which is not divisor of 0, is positive or negative according to whether

$$\text{abs } c = c \quad \text{or} \quad \text{abs } c = -c \ (\text{sgn } c = 1 \text{ or sgn } c = -1).$$

We remark that the four postulates for sgn are independent. In the field of residues modulo 3 the function s which admits the substitution of all three constants 0,1,-1 and (like the function $-j$) yields $s0 = 0$, $s1 = -1$, $s(-1) = 1$ satisfies all postulates except the first. In the field of residues modulo 5 the function s which admits the substitution of $0, \pm 1, \pm 2$ and (like the function j) yields $sc = c$ for each c, satisfies all postulates except the second. In the same field the function s which (like j^2) yields $s0 = 0$, $s1 = s(-1)$, $s2 = s(-2) = -1$ satisfies all postulates except the third. In each field the function 0 satisfies all postulates except the fourth.

We have

$$\text{abs exp} = \exp\,[\tfrac{1}{2}\cdot\log(j\cdot j)]\exp = \exp[\tfrac{1}{2}\cdot\log(\exp\cdot\exp)]$$
$$= \exp[\tfrac{1}{2}\cdot 2\cdot\log\,\exp] = \exp\,j = \exp.$$

For the function log abs, on account of its importance, we shall introduce a special symbol. We shall denote it by logabs. We have

$$\text{logabs} = \log\,\exp[\tfrac{1}{2}\cdot\log(j\cdot j)] = \tfrac{1}{2}\cdot\log(j\cdot j).$$

The function logabs admits the substitution of each constant $\neq 0$. It corresponds to the function associating $\log|x|$ with each $x \neq 0$ in the classical theory. We have

$$\text{logabs exp} = (\log\,\text{abs})\exp = \log(\text{abs exp}) = \log\,\exp = j,$$
$$\exp\,\text{logabs} = \exp\,\log\,\text{abs} = j\,\text{abs} = \text{abs},$$
$$\exp(\text{logabs } f + \text{logabs } g) = \exp\,\text{logabs } f\cdot\exp\,\text{logabs } g = \text{abs } f\cdot\text{abs } g.$$

By virtue of $j = \text{sgn}\cdot\text{abs}$ and $\text{sgn}(c_1\cdot c_2) = \text{sgn } c_1\cdot\text{sgn } c_2$ it follows that

$$f\cdot g = \text{sgn } f\cdot\text{sgn } g\cdot\exp(\text{logabs } f + \text{logabs } g).$$

The last formula could be used as a definition of multiplication in terms of addition and substitution, in conjunction with the exponential and the signum functions. Algebra of Functions might be developed from postulates about two operations and two particular functions, possibly one particular function.

8. <u>The Power Functions</u>.

We shall now for each constant c introduce a function called the c-th power and denoted by c-po. We define c-po in the same way in which it is defined in the theory of complex functions:

$$c - po = \exp(c \cdot \log).$$

From this definition it follows that c - po admits the substitution of all squares and only of squares. More accurately, we should call the above function the c-th power based on the function exp. However, in some cases we shall see that, for algebraic reasons, power functions are independent of the particular choice of the exponential function used in defining them. For instance

$0 - po = \exp(0 \cdot \log) = \exp 0 = 1$ and $1 - po = \exp(1 \cdot \log) = \exp \log = j$.

If exp' is another exponential function, log' the inverse of exp', and if we define

$$c - po' = \exp'(c \cdot \log'),$$

then, as before, we have

$0\text{-}po' = \exp'(0 \cdot \log') = \exp' 0 = 1$ and $1\text{-}po' = \exp'(1 \cdot \log') = \exp' \log' = j$.

Moreover, we obtain the following three functional equations for the power functions

$$c_1 - po \cdot c_2 - po = (c_1 + c_2) - po$$

$$c_1 - po \; c_2 - po = (c_1 \cdot c_2) - po$$

$$c - po \; f \cdot c - po \; g = c - po \; (f \cdot g).$$

Proof:

$$c_1\text{-po}\cdot c_2\text{-po} = \exp(c_1\cdot\log)\cdot\exp(c_2\cdot\log) = \exp(c_1\cdot\log + c_2\cdot\log)$$
$$= \exp[(c_1+c_2)\cdot\log] = (c_1+c_2)\text{-po},$$

$$c_1\text{-po}\ c_2\text{-po} = \exp(c_1\cdot\log)\ \exp(c_2\cdot\log) = \exp[c_1\cdot\log\ \exp(c_2\cdot\log)]$$
$$= \exp[c_1\cdot(c_2\cdot\log)] = \exp[(c_1\cdot c_2)\log] = (c_1\cdot c_2)\text{-po},$$

$$c\text{-po}\ f\cdot c\text{-po}\ g = \exp(c\cdot\log f)\cdot\exp(c\cdot\log g) = \exp(c\cdot\log f + c\cdot\log g)$$
$$= \exp[c\cdot(\log f + \log g)] = \exp[c\cdot\log(f\cdot g)] = c\text{-po}(f\cdot g).$$

From the first of these functional equations it follows that

$$1 = 0\text{-po} = (c + (-c))\text{-po} = c\text{-po}\cdot(-c)\text{-po}, \text{ and hence}$$
$$(-c)\text{-po} = \text{rec } c\text{-po}, \text{ in particular, } (-1)\text{-po} = \text{rec}.$$

Moreover, we have $2\text{-po} = 1\text{-po}\cdot 1\text{-po} = j\cdot j$ and, by induction, we see that for each positive integer n, the n-th power is the product of n factors j. This statement as well as the functional equations for $c\text{-po}$ are independent of the choice of the exponential function exp used in defining the power functions.

From the second functional equation it follows that

$$c\text{-po}\ \frac{1}{c}\text{-po} = \exp(c\cdot\log)\exp(\frac{1}{c}\cdot\log) = \exp[c\cdot\log\ \exp(\frac{1}{c}\cdot\log)]$$
$$= \exp[c\cdot j(\frac{1}{c}\cdot\log)] = \exp(c\cdot\frac{1}{c}\cdot\log) = \exp(1\cdot\log) = \exp\ \log = j.$$

Thus, $c\text{-po}$ and $\frac{1}{c}\text{-po}$ are inverse functions.

In the equation $2\text{-po} = j\cdot j$, the right side admits the substitution of any constant, while the left side admits only the substitution of squares. However, we may consistently extend the definition of $c\text{-po}$ by the following stipulations:

1. $c\text{-po}\ 0 = 0$

2. If c is a rational number n_1/n_2 with an odd denominator n_2 and a numerator n_1 which is relatively prime to n_2, then the function $c\text{-po}$ is even if the number n_1 is even, and odd if n_1 is odd.

We do not permit the substitution of negative squares into c - po in the remaining cases, that is, if c is a rational number with an even denominator or not rational. We remark that in case that c is a rational number n_1/n_2 whose denominator is even, we have not only

$$n_2\text{- po } c\text{-po} = n_1\text{-po} \quad \text{but also} \quad n_2\text{-po neg } c\text{-po} = n_1\text{-po}.$$

After the above extension our definition includes all the cases covered by the classical theory of power functions.

In case that c is a positive integer we readily see that the extended function c - po is identical with the product of c factors j. For $c = \frac{2m}{2n+1}$ or $= \frac{2m+1}{2n+1}$ (m and n integers) it is easily seen that the extended c-th powers can be written

$$\frac{2m}{2n+1}\text{ - po} = \exp[\frac{2m}{2n+1}\cdot\text{logabs}] = \exp[\frac{m}{2n+1}\cdot\log(j\cdot j)]$$

$$\frac{2m+1}{2n+1}\text{ - po} = \text{sgn}\cdot\exp[\frac{2m+1}{2n+1}\cdot\text{logabs}].$$

In operating with integers we have omitted and shall omit the multiplication dot. 2m+1 stands, of course, for $2\cdot m+1$.

The reader can easily check to which extent our supplementary stipulations concerning the definition of c - po are compatible with the three functional equations for power functions. For instance, the equation c_1-po c_2-po $= (c_1\cdot c_2)$-po can not be upheld after the extension; in other words, it does not permit the substitution of negative constants. As an example, we mention

$$\frac{1}{2}\text{ - po 2-po} = \exp[\frac{1}{2}\cdot\log\ \exp(2\cdot\text{logabs})] = \exp\ \text{logabs} = \text{abs}.$$

Thus the extended function 2 - po and the function $\frac{1}{2}$ - po are not inverse. The function $\frac{1}{2}$ - po 2 - po is not = j but = abs which, in fact, is Cauchy's representation of the function abs.

9. The Trigonometric Functions.

We call f a tangential function if and only if

$$f(g+h) = \frac{fg+fh}{1-fg\cdot fh} \quad \text{and} \quad f \neq 0.$$

We shall denote a tangential function by tan. In the second chapter we shall single out among the tangential functions the ordinary tangent function.

From the definition it follows: If tan g = 0, then tan(f + g) = tan f. Moreover,

$$\tan 0 = \tan(0+0) = \frac{2\cdot\tan 0}{1-\tan 0\cdot\tan 0}.$$

Consequently, if the constant tan 0 is to be real, we have tan 0 = 0. Furthermore, it readily follows from the definition that tan is an odd function.

In this section we shall assume that the constants form the field of real numbers. Moreover, we shall postulate the existence of a smallest constant c > 0 such that tan c = 0. Then tan does not admit the substitution of the constants c/2 and -c/2. For if, say, tan (c/2) were defined, then we should have

$$0 = \tan c = \frac{2\cdot\tan(c/2)}{1-\tan(c/2)\cdot\tan(c/2)}.$$

This equality would imply tan (c/2) = 0 in contradiction to our assumption that c is the smallest number 0 for which tan c = 0. It further follows that tan (c/4) = 1 or -1 since, by virtue of the definition of tan, every other value of tan (c/4) would entail a value for tan (c/2). We shall only admit the substitution into tan of constants between $-c/2$ and $c/2$.

Relative to each tangential function we define a sine and a cosine function in the following way:

$$\sin(2\cdot j) = \frac{2\cdot\tan}{1+\tan^2} \quad \text{and} \quad \cos(2\cdot j) = \frac{1-\tan^2}{1+\tan^2}$$

We obviously have

$$\frac{\sin(2\cdot j)}{\cos(2\cdot j)} = \frac{2\cdot\tan}{1-\tan^2} = \tan(2\cdot j).$$

Substituting $\frac{1}{2}\cdot j$ into the equality, we see that $\tan = \sin/\cos$.

Other useful identities are

$$\sin^2 + \cos^2 = 1 \quad \text{and} \quad 1+\tan^2 = \text{rec}\ \cos^2.$$

We postulate an inverse of tan on both sides and call it arctan.