Preface

The Riemann zeta function $\zeta(s)$ was first introduced and studied by L. Euler (1707–83) as a function of a real variable s greater than 1. On account of the behavior of $\zeta(s)$ as s approaches 1, he showed that the sum of the inverses of prime numbers is divergent (1737). After that, G.F.B. Riemann (1826–66) treated $\zeta(s)$ successfully as a function of a complex variable s and realized its importance in the study of the distribution of prime numbers (1859). For this reason, $\zeta(s)$ is called the Riemann zeta function. $\zeta(s)$ is holomorphic and has no zeros in the half-plane $\operatorname{Re} s^{\dagger 1} > 1$. Riemann proved, moreover, that it has an analytic continuation to the whole complex plane, is meromorphic everywhere, and has a unique pole $s = 1$.

$\zeta(s)$ has no zeros in the closed half-plane $\operatorname{Re} s \geq 1$, and its zeros in the closed half-plane $\operatorname{Re} s \leq 0$ are only negative even integers $-2, -4, -6, \ldots$. Riemann conjectured that all zeros in the strip domain $0 < \operatorname{Re} s < 1$ lie on the line $\operatorname{Re} s = \frac{1}{2}$. This is called the Riemann hypothesis, which remains unsolved and is one of the Millennium Prize Problems posed by the Clay Mathematics Institute in 2000.

About the value-distribution of $\zeta(s)$ on the strip $\frac{1}{2} < \operatorname{Re} s \leq 1$, H. Bohr (1887–1951) showed that for fixed σ with $\frac{1}{2} < \sigma \leq 1$, the set of the values of $\zeta(\sigma + \sqrt{-1}t)$ where "time" $t^{\dagger 2}$ moves on the real line is dense in the complex plane (1914). Moreover he showed a similar result to hold for the log zeta function $\log \zeta$ (1915). Bohr was the first to develop the theory of almost periodic functions for his detailed study of the value-distribution of $\zeta(s)$ (1924). It was in the 1930s that he, together with B. Jessen, arrived at the final result:

> *There exists an asymptotic probability distribution of* $\log \zeta(\sigma + \sqrt{-1}\cdot)$*; roughly speaking, in a certain sense the time mean of the values of* $\log \zeta(\sigma + \sqrt{-1}t)$ *over a long finite time-interval is convergent to a probability distribution as the length of the time-interval goes to infinity.*

This statement is the Bohr-Jessen limit theorem, though it is restated in terms of the modern probability theory. After Bohr-Jessen, alternative proofs of the limit theorem were given by Jessen-Wintner [16], Borchsenius-Jessen [6], Laurinčikas [21, 22, 23], Matsumoto [24, 25] and others. They dealt with this theorem in the framework of probability theory, which originated with Jessen-Wintner and is a standard procedure nowadays.

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†1 For a complex number $s = \sigma + \sqrt{-1}t$, $\operatorname{Re} s$ denotes the real part σ and $\operatorname{Im} s$ the imaginary part t, where $\sqrt{-1}$ is the imaginary unit.

†2 In this Preface only, but not in other places of this monograph, the variable t in $\zeta(\sigma + \sqrt{-1}t)$ as well as $\log \zeta(\sigma + \sqrt{-1}t)$ is regarded as and called "time" so as to be easy to understand what is said.

In this monograph, we revisit this limit theorem and propose an idea to refine their works. We present the limit probability distribution in a suitable form through which we can verify the convergence of the time mean of the values of $\log \zeta(\sigma + \sqrt{-1}t)$. To this end, we introduce a probability space of large volume with a help of the almost periodic functions, and it is on this space that we are to define what is the limit. This passage will need a little more explanation as follows.

In view of the Euler product expression of $\zeta(s)$, $\log \zeta(\sigma + \sqrt{-1}t)$ is expected to be a convergent infinite series summing over all primes. While this infinite series is not convergent in a usual sense when $\frac{1}{2} < \sigma \leq 1$, as we shall see, its Nth partial sum is convergent to $\log \zeta(\sigma + \sqrt{-1}t)$ as N goes to infinity in a certain sense. This convergence is established with a great effort depending on number theory, which in fact is a number-theoretic aspect in the proof of the Bohr-Jessen limit theorem. On the other hand, in order to look at a probability-theoretic aspect, we define a counterpart of the Nth partial sum as a random variable on the above-mentioned probability space to be constructed with the almost periodic functions. Then the so-defined sequence of random variables is almost surely convergent to a random variable as N goes to infinity. This limit random variable is just what is desired and the convergence occurs as a natural consequence of the construction of the probability space. Combining the convergences observed in the number- and probability-theoretic contexts, we will see that the time mean of the values of $\log \zeta(\sigma + \sqrt{-1}t)$ over a long finite time-interval is convergent, as the length of the time-interval goes to infinity, to a probability distribution of this random variable. The emergence of the probabilistic counterpart of the Nth partial sum is a natural outcome from our introduction of the probability space, which is surely adapted to an intention of Bohr, who developed the theory of almost periodic functions to study the value-distribution of $\zeta(s)$. The present monograph introduces this new approach, which reveals the whole story of the proof of the Bohr-Jessen limit theorem. With our method, the reader must be able to understand the essence of the proof in depth but without difficulty. We hope that our approach will be accepted as a natural and effective way for working with similar kinds of problems.

The proofs of several known facts about the Riemann zeta function $\zeta(s)$ mentioned above will be not familiar to many readers. So we are giving detailed proofs, in other words, we provide self-contained proofs accompanied by patient step-by-step arguments, diminishing logical gaps of each step and making thoughtful and scrupulous explanation for various facts and related matters to be needed. With this writing style, motivated graduate students will be able to enjoy the materials.

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