Zil'ber's Trichotomy and o-minimal Structures

(Extended abstract) ⋆

Ya'acov Peterzil

Department of Mathematics
University of Haifa
Haifa, Israel
kobi@mathcs2.haifa.ac.il

A first-order structure $\mathcal{M}$ is called *a geometric structure* (see [2]) if it has the following properties:
(i) $acl(-)$ satisfies the Exchange principle.
Namely, given $a, b, \bar{c}$ from M, if $a \in acl(b, \bar{c}) \setminus acl(\bar{c})$ then $b \in acl(a, \bar{c})$.
(ii) For any formula $\varphi(\bar{x}, \bar{y})$ there is $n \in \mathbb{N}$ such that for any $\bar{b}$ in M, $\varphi(\bar{x}, \bar{b})$ has either less than n solutions in M or infinitely many.

O-minimal and strongly minimal structures are geometric structures. The field of p-adics and pseudo-finite fields are geometric structures as well.

Given a geometric structure $\mathcal{M}$ one can assign a dimension to definable sets in a natural way which in all the field cases mentioned above is just the algebro-geometric dimension of the Zariski closure. *A curve* is any definable 1-dimensional subset of M^2 and a definable (or interpretable) family $\mathcal{F}$ of curves is called *normal* if any two curves from $\mathcal{F}$ which are given by different parameters intersect at most finitely many times. If $\mathcal{F}$ is normal its dimension is taken to be the dimension of the parameter set.

Given a geometric structure, one and only one of the following holds.
Z1. Every interpretable normal family of curves $\mathcal{F}$ is of dimension at most 1 and for all but finitely many curves $\mathcal{C} \in \mathcal{F}$, for all but finitely many points $\langle a, b\rangle \in \mathcal{C}$, either $\dim(\mathcal{C} \cap (\{a\} \times M)) = 1$ or $\dim(\mathcal{C} \cap (M \times \{b\})) = 1$.
Z2. Every interpretable normal family of curves is of dimension at most 1, but Z1 does not hold.
Z3. There is an interpretable normal family of curves of dimension greater than 1.

In the early 1980's Boris Zil'ber (see [6]), in his analysis of $\aleph_1$-categorical structures, suggested that the above trichotomy corresponds to the interpretability (or the lack of which) of certain algebraic structures in $\mathcal{M}$. He called Z2 and Z3 the module-like and field-like cases, respectively, and conjectured that if a strongly minimal structure satisfies Z3 then it can interpret a field. We formulate this correspondence as follows:

Definition 1 A class $\mathcal{K}$ of geometric structures is said to satisfy the Zil'ber Principle, ZP, if for every $\mathcal{M} \in \mathcal{K}$,

⋆ This abstract discusses a joint work of the author and Sergei Starchenko. For the proof of the main theorem see [4]. An expanded version of this abstract has appeared in [5].

(i) $\mathcal{M}$ satisfies Z1 if and only if no group is interpretable in $\mathcal{M}$.
(ii) $\mathcal{M}$ satisfies Z2 if and only if all definable sets arise from an interpretable vector space (or more generally a module) structure.
(iii) $\mathcal{M}$ satisfies Z3 if and only if a field can be interpreted in $\mathcal{M}$.

Hrushovski (see [1]) disproved Zil'ber's conjecture by constructing a strongly minimal structure satisfying Z3, with no group interpretable in it. Later, Hrushovski and Zil'ber showed that a more restrictive class of strongly minimal structures called Zariski geometries (see [3]) satisfies the Zil'ber Principle.

The Trichotomy Theorem below implies that the class of o-minimal structures satisfies the Zil'ber Principle (with the exception that that in the Z2 case it is only locally, around each nontrivial point, that the definable sets arise from a vector space). The theorem seems to reduce in many cases model theoretic questions on abstract o-minimal structures to algebraic and differential questions on o-minimal expansions of real closed fields.

We call a point a in an o-minimal structure *nontrivial* if there is an open interval I containing a and a definable function $F(x,y)$ on $I \times I$ which is continuous and strictly monotone in each variable.

The Trichotomy Theorem[4] *Let $\mathcal{M}$ be an o-minimal, ω^+-saturated structure. Given $a \in M$, one and only one of the following hold:*
T1. *a is trivial;*
T2. *The structure that $\mathcal{M}$ induces on some convex neighborhood of a is an ordered vector space over an ordered division ring;*
T3. *The structure that $\mathcal{M}$ induces on some open interval around a is an o-minimal expansion of a real closed field.*

References

1. E. Hrushovski, *A new strongly minimal set*, Annals of Pure and Applied Logic 62 (1993), pp. 147-166.
2. —— and A. Pillay, *Groups definable in local fields and pseudo-finite fields*, Israel Journal of Mathematics 85 (1994), pp. 203-262.
3. —— and B. Zil'ber, *Zariski Geometries*, Bulletin (New Series) of the AMS 28, #2, April 1993.
4. Y. Peterzil and S. Starchenko *A trichotomy theorem for o-minimal structures*, preprint.
5. ————, *Geometry, calculus and Zil'ber's conjecture*, Bulletin of Symbolic Logic.
6. B. Zil'ber, *Structural properties of models of $\aleph_1$-categorical theories*, in: R. Barcon Marcus et al. eds., Logic Methodology and Philosophy of Science VII (North Holland, Amsterdam, 1986) pp. 115-128.