THE RADON-NIKODÝM DERIVATIVE OF A CORRESPONDENCE

GERARD DEBREU
and
DAVID SCHMEIDLER
UNIVERSITY OF CALIFORNIA, BERKELEY

1. Introduction

Let $(A, \mathscr{A}, \nu)$ be a complete, totally σ-finite, positive measure space and S be an ordered finite dimensional real vector space with its usual topology and the Borel σ-field $\mathscr{S}$ generated by this topology. Given a function γ from A to $\mathscr{P}(S)$, the set of subsets of S, we define its integral over $E \in \mathscr{A}$ by

$$(1.1) \qquad \int_E \gamma d\nu = \{x \in S \mid \text{there is an integrable function } f \text{ from } E \text{ to } S \text{ such that } x = \int_E f d\nu \text{ and a.e. in } E, f(a) \in \gamma(a)\}.$$

And given a function Γ from $\mathscr{A}$ to $\mathscr{P}(S)$, we say that a function γ from A to $\mathscr{P}(S)$ is a Radon-Nikodým derivative of Γ if

$$(1.2) \qquad \text{for every } E \in \mathscr{A}, \ \Gamma(E) = \int_E \gamma d\nu.$$

When $\Gamma(E)$ is nonempty for every $E \in \mathscr{A}$, we call Γ a correspondence from $\mathscr{A}$ to $\mathscr{S}$. In this article we characterize the correspondences from $\mathscr{A}$ to S, having a measurable, positive, closed, convex valued Radon-Nikodým derivative, where a function γ from A to $\mathscr{P}(S)$ is defined as measurable if its graph

$$(1.3) \qquad G(\gamma) = \{(a, x) \in A \times S \mid x \in \gamma(a)\}$$

belongs to the product σ-field $\mathscr{A} \otimes \mathscr{S}$.

The need for such a characterization arose in the theory of economic systems in which certain sets of negligible agents are not negligible. To describe this situation mathematically one introduces a set A of agents, a σ-field $\mathscr{A}$ of subsets of A (the σ-field of coalitions), and a positive measure ν defined on $\mathscr{A}$. Now the

The work of G. Debreu was done partly at the University of California, Berkeley under a research grant of the National Science Foundation, partly while he was visiting the Institute of Mathematics of the Hebrew University of Jerusalem, and partly while he was visiting the Center for Operations Research and Econometrics of the Catholic University of Louvain as a Guggenheim Fellow. The work of D. Schmeidler was done at the Institute of Mathematics of the Hebrew University of Jerusalem.

primitive concepts of the economic theory under discussion can be presented either in terms of functions and correspondences defined on the set of agents (this is the "individual" point of view of R. J. Aumann [1]) or in terms of functions and correspondences defined on the set of coalitions (this is the "collective" point of view of K. Vind [15]). The study of the equivalence of these two viewpoints requires a theory of the Radon-Nikodým derivatives of correspondences.

2. Statement of results

The positive cone P of the space S is a closed, convex cone with vertex 0 such that $[x \in P$ and $-x \in P]$ implies $[x = 0]$. A subset of S is said to be positive if it is contained in P. An element v of the dual S' of S is said to be strictly positive if $[x \in P$ and $x \neq 0]$ implies $[v(x) > 0]$.

Occasionally it will be convenient to use a norm on S. The norm of $x \in S$ will then be denoted by $|x|$.

Given a sequence $\{X_i\}$ of subsets of S, we define

$$\sum_i X_i = \{x \in S \mid \text{there is an absolutely convergent series } (x_i) \text{ such that } x = \sum_i x_i \text{ and for every } i,\ x_i \in X_i\}. \tag{2.1}$$

A function Γ from $\mathscr{A}$ to $\mathscr{P}(S)$ is said to be (i) *countably additive* if for every sequence $\{E_i\}$ of pairwise disjoint elements of $\mathscr{A}$, $\Gamma(\cup_i E_i) = \Sigma_i \Gamma(E_i)$; (ii) *continuous* if $[E \in \mathscr{A}$ and $v(E) = 0]$ implies $[\Gamma(E) = \{0\}]$.

The characterization of correspondences from $\mathscr{A}$ to S having a measurable, positive, closed, convex valued Radon-Nikodým derivative will be in terms of the following concepts. For two correspondences Ψ^1, Ψ^2 from $\mathscr{A}$ to S, the ordering relation $\Psi^1 \subset \Psi^2$ is defined by $\Psi^1(E) \subset \Psi^2(E)$ for every $E \in \mathscr{A}$. Given a correspondence Φ from $\mathscr{A}$ to S, the correspondence $\bar{\Phi}$ from $\mathscr{A}$ to S is defined by $\bar{\Phi}(E) = \overline{\Phi(E)}$ for every $E \in \mathscr{A}$. Consider then a countably additive correspondence Φ from $\mathscr{A}$ to S and let $\mathscr{M}$ be the set of countably additive correspondences Ψ from $\mathscr{A}$ to S such that $\Psi \subset \bar{\Phi}$.

THEOREM 1. *If Φ is a countably additive positive valued correspondence from $\mathscr{A}$ to S such that $\Phi(\varnothing) = \{0\}$, then $\mathscr{M}$ has a greatest element $\hat{\Phi}$. If, in addition, Φ is convex valued, then $\hat{\Phi}$ is convex valued.*

Our main result is:

THEOREM 2. *A countably additive, v continuous, positive, convex valued correspondence Φ from $\mathscr{A}$ to S has a measurable, positive, closed, convex valued Radon-Nikodým derivative if and only if $\Phi = \hat{\Phi}$.*

In the particular case in which Φ is compact valued (and therefore trivially satisfies the equality $\Phi = \hat{\Phi}$) the proof of Section 4 admits of a considerable simplification. This case can also be treated by an entirely different technique. Since the set of nonempty, compact, convex subsets of S can be embedded in a Banach space in the manner of H. Rådström [11], Φ can be considered as a function from $\mathscr{A}$ to a Banach space to which the Radon-Nikodým theorem of M. Rieffel [12] (see also M. Métivier [10]) is applied. Indeed this remark was the point of departure of the present study. We also notice that when Φ is compact convex valued, generalizations to infinite dimensional spaces S are possible (C. Castaing [3], M. Valadier [14]).

3. Lemmas and proof of Theorem 1

LEMMA 1. *If γ is a measurable correspondence from A to S and v is a linear form on S, then the function $a \twoheadrightarrow \sup v(\gamma(a))$ from A to $\bar{R}$ is measurable.*

PROOF. We repeat the proof of (4.5) of [5]. Let c be a real number. The set

$$A_c = \{a \in A \mid \sup v(\gamma(a)) > c\} \tag{3.1}$$

is the projection on A of the set $\{(a, x) \in G(\gamma) \mid v(x) > c\}$ which belongs to $\mathscr{A} \otimes \mathscr{S}$. Therefore, A_c belongs to $\mathscr{A}$ (see, for instance, (3.4) of [5]). *Q.E.D.*

The following lemma and its proof are borrowed from W. Hildenbrand [9].

LEMMA 2. *If γ is a measurable correspondence from A to S such that $\int_A \gamma dv \neq \emptyset$ and v is a linear form on S, then*

$$\sup v\left(\int_A \gamma dv\right) = \int_A \sup v(\gamma(a))\, dv(a). \tag{3.2}$$

PROOF. The left side is clearly at most equal to the right side.

To prove the reverse inequality we first remark that we lose no generality in assuming that in A, $0 \in \gamma(a)$, since $\int_A \gamma dv \neq \emptyset$. Let r be a strictly positive integrable function from A to R, and for every positive integer n let $B_n(a)$ be the closed ball with center 0, radius $nr(a)$. Let $\gamma_n(a) = \gamma(a) \cap B_n(a)$. For every $a \in A$, one has $\gamma_n(a) \neq \emptyset$. Moreover, the correspondence B_n from A to S is measurable by (5.10) of [5]. As $G(\gamma_n) = G(\gamma) \cap G(B_n)$, the correspondence γ_n is measurable. Now let $s_n(a) = \sup v(\gamma_n(a))$ and $s(a) = \sup v(\gamma(a))$. By Lemma 1, the functions s_n and s are measurable. They are positive and for every $a \in A$, $s_n(a) \uparrow s(a)$. Hence $\int_A s_n dv$ converges to $\int_A s dv$ (by [7], p. 112).

Consider a real number $\alpha < \int_A s dv$. For some n, $\alpha < \int_A s_n dv$. There is an integrable function g from A to R such that $\alpha < \int_A g dv$ and in A, $g(a) < s_n(a)$. Let

$$\psi(a) = \{x \in \gamma_n(a) \mid v(x) > g(a)\}. \tag{3.3}$$

For every $a \in A$, $\psi(a) \neq \emptyset$. Moreover, the graph of the correspondence ψ is clearly measurable. Therefore, by a measurable selection theorem of Aumann [2], there is a measurable function f from A to S such that in A, $f(a) \in \psi(a)$. As $|f(a)| \leqq nr(a)$, the function f is integrable. Since in A, $g(a) < v(f(a))$, one has $\alpha < v(\int_A f dv)$. Thus, $\alpha < \sup v(\int_A \gamma dv)$, and consequently, $\int_A s dv \leqq \sup v(\int_A \gamma dv)$.

COROLLARY 1. *If $\{X_i\}$ is a sequence of subsets of S having a nonempty sum and v is a linear form on S, then* $\sup v(\Sigma_i X_i) = \Sigma_i \sup v(X_i)$.

LEMMA 3. *If in S, C is a closed cone with vertex 0 and L is a straight line such that $C \cap L = \{0\}$, then $C + L$ is closed.*

PROOF. Let L_1 and L_2 be the two closed half lines with origin 0 whose union is L. Consider a sequence $\{x_i\}$ in $C + L$ converging to x. For every i, there are $c_i \in C$ and $\ell_i \in L$ such that $x_i = c_i + \ell_i$. We wish to prove that the sequence $\{\ell_i\}$ is bounded. Assume that it has in L_1 a subsequence $\{\ell'_i\}$ such that $|\ell'_i|$ tends to $+\infty$, and let $\{x'_i\}$ and $\{c'_i\}$ be the corresponding subsequences of $\{x_i\}$ and $\{c_i\}$. Since x'_i converges to x and $c'_i = x'_i - \ell'_i$, the closed half line with origin 0 containing c'_i converges to L_2 which would therethore be contained in C, a contradiction of $C \cap L = \{0\}$.

Thus the sequence $\{\ell_i\}$ is bounded and so is the sequence $\{c_i\}$. Extract from the sequence $\{(c_i, \ell_i)\}$ a subsequence converging to $(c, \ell) \in C \times L$. The equality $x = c + \ell$ shows that $x \in C + L$.

LEMMA 4. *If* $\dim S > 0$ *and* H *is a hyperplane through* 0, *then there is in* S *a straight line* L *through* 0 *such that the projection of* P *into* H *parallel to* L *is a closed, convex cone with vertex* 0 *containing no straight line.*

PROOF. The assertion of the lemma is trivially true if $\dim S = 1$. In the remainder of the proof, we shall assume that $\dim S > 1$. Since P contains no straight line, its polar has a nonempty interior and there is a hyperplane M different from H, supporting for P, and such that $M \cap P = \{0\}$. Select a straight line L through 0, contained in M but not contained in H. Denote the projection of a subset X of S into H parallel to L by $\dot{X}$ and notice that $\dot{X} = (X + L) \cap H$. Clearly $\dot{P}$ is a convex cone with vertex 0. By Lemma 3, $P + L$ is closed. Therefore, $\dot{P}$ is closed. Finally, $\dot{P} \cap \dot{M} = \{0\}$ because

$$(3.4) \qquad [(P + L) \cap H] \cap [(M + L) \cap H] = (P + L) \cap M \cap H = \{0\}.$$

The first equality follows from the fact that $M + L = M$ and the second from the fact that $(P + L) \cap M = L$ and $L \cap H = \{0\}$. In H, $\dot{M}$ is a hyperplane supporting for $\dot{P}$. Thus, $\dot{P} \cap \dot{M} = \{0\}$ implies that $\dot{P}$ contains no straight line. *Q.E.D.*

The convex hull of a subset X of S is denoted by co X.

LEMMA 5. *If* $\{X_i\}$ *is a sequence of subsets of* P, *then* $\operatorname{co}(\Sigma_i X_i) = \Sigma_i \operatorname{co} X_i$.

PROOF. If $\Sigma_i \operatorname{co} X_i = \emptyset$, then $\Sigma_i X_i = \emptyset$ and $\operatorname{co}(\Sigma_i X_i) = \emptyset$.

Assume now that $\Sigma_i \operatorname{co} X_i \neq \emptyset$. For every i, there is $x_i \in \operatorname{co} X_i$ such that the series (x_i) converges. Let v^0 be a strictly positive linear form on S. For every i, there is $x'_i \in X_i$ such that $v^0(x'_i) \leqq v^0(x_i)$. The series $(v^0(x'_i))$ converges. So does the series (x'_i). Thus, $\Sigma_i X_i \neq \emptyset$.

Therefore, for every linear form v on S, one has

$$(3.5) \qquad \begin{aligned} \sup v\Big(\sum_i \operatorname{co} X_i\Big) = \sum_i \sup v(\operatorname{co} X_i) = \sum_i \sup v(X_i) &= \sup v\Big(\sum_i X_i\Big) \\ &= \sup v\Big(\operatorname{co} \sum_i X_i\Big), \end{aligned}$$

the first and the third equalities resulting from the corollary of Lemma 2. Consequently, $\Sigma_i \operatorname{co} X_i$ and $\operatorname{co}(\Sigma_i X_i)$ have the same closure.

Given a nonempty subset X of S and a linear form v on S, let

$$(3.6) \qquad X^v = \{x \in X \mid v(x) = \sup v(X)\}.$$

It is immediately seen that (i) if $\{Y_i\}$ is a sequence of subsets of S having a nonempty sum, then $(\Sigma_i Y_i)^v = \Sigma_i Y_i^v$, and (ii) if Y is a nonempty subset of S, then $(\text{co } Y)^v = \text{co } Y^v$.

We complete the proof by showing that for every $v \neq 0$, one has $(\Sigma_i \text{co } X_i)^v = (\text{co } \Sigma_i X_i)^v$. This follows from the chain of equalities (the first and the fourth by (i); the second and the fifth by (ii); and the third by induction on dim S as we prove below):

$$(3.7) \qquad \left(\sum_i \text{co } X_i\right)^v = \sum_i (\text{co } X_i)^v = \sum_i \text{co } X_i^v = \text{co} \sum_i X_i^v = \text{co} \left(\sum_i X_i\right)^v$$
$$= \left(\text{co} \sum_i X_i\right)^v.$$

Let $H = \{x \in S \mid v(x) = 0\}$ and let L be a straight line in S through 0 as in Lemma 4. Project (that is, in this case, translate) the sets X_i^v into H parallel to L. The induction assumption according to which the lemma is true in H establishes the third equality.

Proof of Theorem 1. For every $E \in \mathscr{A}$, let $\hat{\Phi}(E) = \bigcup_{\Psi \in \mathscr{M}} \Psi(E)$. Clearly $\Phi(E) \subset \hat{\Phi}(E) \subset \bar{\Phi}(E)$. Thus, $\hat{\Phi}$ is a correspondence from $\mathscr{A}$ to P included in $\bar{\Phi}$. It is also clear that $\Psi \in \mathscr{M}$ implies $\Psi \subset \hat{\Phi}$. To establish that $\hat{\Phi}$ is the greatest element of $\mathscr{M}$ it will therefore suffice to prove that $\hat{\Phi}$ is countably additive. To this end consider a sequence $\{E_i\}$ of pairwise disjoint elements of $\mathscr{A}$ and their union E.

Let x be an element of $\hat{\Phi}(E)$. There is $\Psi \in \mathscr{M}$ such that $x \in \Psi(E)$. Since Ψ is countably additive, there is a sequence $\{x_i\}$ in P such that $x = \Sigma_i x_i$ and for every i, $x_i \in \Psi(E_i)$, which is contained in $\hat{\Phi}(E_i)$.

Conversely, let $\{x_i\}$ be a sequence in P such that $x = \Sigma_i x_i$ and for every i, $x_i \in \hat{\Phi}(E_i)$. For every i, there is $\Psi_i \in \mathscr{M}$ such that $x_i \in \Psi_i(E_i)$. For every $F \in \mathscr{A}$, let

$$(3.8) \qquad \Psi(F) = \sum_i \Psi_i(E_i \cap F).$$

We wish to prove that $\Psi \in \mathscr{M}$. Since $\Psi(E_i) = \Psi_i(E_i)$, this will establish that x belongs to $\hat{\Phi}(E)$.

First we notice that

$$(3.9) \qquad \Psi(F) = \sum_i \Psi_i(E_i \cap F) \subset \sum_i \overline{\Phi(E_i \cap F)} \subset \overline{\sum_i \Phi(E_i \cap F)} = \overline{\Phi(F)},$$

the second inclusion following from the fact that the sum of the closures of a sequence of subsets of S is contained in the closure of their sum. Thus, $\Psi(F) \subset \bar{\Phi}(F)$.

Next we show that Ψ is countably additive. Consider a sequence $\{F_j\}$ of pairwise disjoint elements of $\mathscr{A}$ and their union F.

Let $\{y_j\}$ be a sequence of elements of P such that $y = \Sigma_j y_j$ and for every j, $y_j \in \Psi(F_j)$. For every j, there is a sequence $\{y_{i,j}\}$ in P such that $y_j = \Sigma_i y_{i,j}$ and for every i, $y_{i,j} \in \Psi_i(E_i \cap F_j)$. Let $y_i' = \Sigma_j y_{i,j}$. Then $y = \Sigma_i y_i'$ and for every i, $y_i' \in \Psi_i(E_i \cap F)$. Thus, $y \in \Psi(F)$.

Conversely, let y be an element of $\Psi(F)$. There is a sequence $\{y_i'\}$ in P such that $y = \Sigma_i y_i'$ and for every i, $y_i' \in \Psi_i(E_i \cap F)$. For every i, there is a sequence

$\{y_{i,j}\}$ in P such that $y'_i = \Sigma_j y_{i,j}$ and for every j, $y_{i,j} \in \Psi_i(E_i \cap F_j)$. Let $y_j = \Sigma_i y_{i,j}$. Then $y = \Sigma_j y_j$ and for every j, $y_j \in \Psi(F_j)$.

Finally, we remark that for every $B \in \mathscr{A}$, $\Psi(B)$ is not empty because

$$\Psi(A) = \Psi(B) + \Psi(A \backslash B) \tag{3.10}$$

and $x \in \Psi(A)$.

There remains to prove that if Φ is convex valued, then so is $\hat{\Phi}$. Define the correspondence $\text{co}\,\hat{\Phi}$ from $\mathscr{A}$ to S by $\text{co}\,\hat{\Phi}(E) = \text{co}\,(\hat{\Phi}(E))$ for every $E \in \mathscr{A}$. Clearly $\text{co}\,\hat{\Phi} \subset \Phi$. Moreover, by Lemma 5, $\text{co}\,\hat{\Phi}$ is countably additive. Therefore, $\text{co}\,\hat{\Phi} \in \mathscr{M}$. Hence, $\hat{\Phi} = \text{co}\,\hat{\Phi}$.

4. Lemmas and proof of Theorem 2

LEMMA 6. *If X is a closed, convex subset of S containing no straight line and $0 \notin X$, then the smallest closed cone C with vertex 0 containing X contains no straight line.*

PROOF. The assertion of the lemma is trivially true if X is empty. We exclude this case in the remainder of the proof. If L_0 is a ray contained in C and such that $L_0 \cap X = \emptyset$, then there is a sequence $\{x^q\}$ in X such that $|x^q|$ tends to $+\infty$ and the ray through x^q tends to L_0. Let x be a point of X. The closed half line with origin x through x^q tends to $\{x\} + L_0$. Therefore $\{x\} + L_0 \subset X$.

Suppose now that C contains a straight line L through 0 and let L_1, L_2 be the rays whose union is L.

If $L_1 \cap X \neq \emptyset$ and $L_2 \cap X \neq \emptyset$, then $0 \in X$, a contradiction.

If $L_1 \cap X = \emptyset$ and $L_2 \cap X \neq \emptyset$, select a point x' in $L_2 \cap X$. According to the first paragraph, $\{x'\} + L_1 \subset X$. Therefore, again $0 \in X$.

If $L_1 \cap X = \emptyset$ and $L_2 \cap X = \emptyset$, select a point x in X. According to the first paragraph, $\{x\} + L_1 \subset X$ and $\{x\} + L_2 \subset X$. Therefore, X contains a straight line, also a contradiction.

COROLLARY 2. *If K is a nonempty, closed, convex subset of S containing no straight line and x is a point of S not belonging to K, then there is a nonempty open set of elements of the dual of S strictly separating x and K.*

PROOF. The closed cone C with vertex 0 generated by $K - \{x\}$, the translate of K by $-x$, contains no straight line by Lemma 6. Therefore, the polar of C has a nonempty interior. Since $0 \notin K - \{x\}$, every element of this interior strictly separates 0 and $K - \{x\}$, therefore, also x and K.

LEMMA 7. *Let ψ_1 be a measurable correspondence from A to S having a nonempty integral over A, and let ψ_2 be a measurable, positive, closed, convex valued correspondence from A to S. If, for every $E \in \mathscr{A}$, $\int_E \psi_1\, d\nu \subset \int_E \psi_2\, d\nu$, then a.e., $\psi_1(a) \subset \psi_2(a)$.*

PROOF. Let $V = \{v_i\}$ be a countable dense subset of the dual of S. For every i, for every $E \in \mathscr{A}$,

$$\sup v_i\left(\int_E \psi_1\, d\nu\right) \leqq \sup v_i\left(\int_E \psi_2\, d\nu\right). \tag{4.1}$$

By Lemma 2,

$$\int_E \sup v_i(\psi_1(a))\, dv(a) \leqq \int_E \sup v_i(\psi_2(a))\, dv(a). \tag{4.2}$$

Therefore, for every i, a.e.,

$$\sup v_i(\psi_1(a)) \leqq \sup v_i(\psi_2(a)). \tag{4.3}$$

Let a be an element of A for which this inequality holds for every i and consider a point x of S not in $\psi_2(a)$. The set $\psi_2(a)$ is closed, convex, and contains no straight line. Consequently, Corollary 2 applies. For some element v_j of V one has

$$\sup v_j(\psi_2(a)) < v_j(x). \tag{4.4}$$

Hence, $x \notin \psi_1(a)$.

LEMMA 8. *Let $\{X_i\}$ be a sequence of subsets of P having a nonempty sum X, and let v be a strictly positive linear form on S. If*

$$Y_i = \{y \in \bar{X}_i \mid v(y) = \inf v(X_i)\} \tag{4.5}$$

and

$$Y = \{y \in \bar{X} \mid v(y) = \inf v(X)\}, \tag{4.6}$$

then $Y = \Sigma_i Y_i$.

PROOF. Let $\{x_i\}$ be a sequence of points of P such that $x = \Sigma_i x_i$ and for every i, $x_i \in Y_i$. Thus, $x_i \in \bar{X}_i$ and consequently, $x \in \bar{X}$. Moreover, by Corollary 1, $\inf v(X) = \Sigma_i \inf v(X_i)$. Therefore, $v(x) = \inf v(X)$ and $x \in Y$.

Conversely, let x be an element of Y. There is a sequence $\{x^q\}$ of elements of X converging to x. For every q, there is a sequence $\{x_i^q\}$ of elements of P such that $x^q = \Sigma_i x_i^q$ and for every i, $x_i^q \in X_i$. Since for every i and q, $v(x_i^q) \leqq v(x^q)$ and $v(x^q)$ converges to $v(x)$, for some well-chosen positive real number c, all the x_i^q belong to the compact set $\{y \in P \mid v(y) \leqq c\}$. Thus, one can extract from the sequence $\{s^q\}$ (where $s^q = \{x_i^q\}$) a subsequence $\{t^q\}$ (where $t^q = \{y_i^q\}$) converging pointwise to $t = \{y_i\}$. Since $y_i^q \in X_i$ and y_i^q converges to y_i, one has $y_i \in \bar{X}_i$. Moreover, letting $y^q = \Sigma_i y_i^q$, one has

$$v(y_i^q) - \inf v(X_i) \leqq v(y^q) - \inf v(X). \tag{4.7}$$

Therefore, $v(y_i^q)$ converges to $\inf v(X_i)$. Hence, $v(y_i) = \inf v(X_i)$. Summing up, $y_i \in Y_i$. There remains to prove that $x = \Sigma_i y_i$.

Given a real number $\alpha \geqq 0$, the diameter of the compact set $\{y \in P \mid v(y) \leqq \alpha\}$ is proportional to α. Let $k > 0$ be the proportionality factor. For every i and q, $|y_i^q - y_i| \leqq kv(y_i^q)$. Thus, for any positive integer $\bar{\imath}$,

$$\begin{aligned} \sum_{i>\bar{\imath}} |y_i^q - y_i| \leqq k \sum_{i>\bar{\imath}} v(y_i^q) &= k \sum_{i>\bar{\imath}} [v(y_i^q) - v(y_i)] + k \sum_{i>\bar{\imath}} v(y_i) \\ &\leqq k(v(y^q) - v(x)) + k \sum_{i>\bar{\imath}} v(y_i). \end{aligned} \tag{4.8}$$

Consequently,

$$\sum_{i} |y_i^q - y_i| \leqq \sum_{i \leqq \bar{i}} |y_i^q - y_i| + k(v(y^q) - v(x)) + k \sum_{i > \bar{i}} v(y_i). \tag{4.9}$$

Given a real number $\varepsilon > 0$, choose $\bar{i}$ so that $k \Sigma_{i > \bar{i}} v(y_i) < \frac{1}{3}\varepsilon$. There is q' such that for every $q > q'$, $k(v(y^q) - v(x)) < \frac{1}{3}\varepsilon$ and there is q'' such that for every $q > q''$, $\Sigma_{i \leqq \bar{i}} |y_i^q - y_i| < \frac{1}{3}\varepsilon$. Therefore, for every q greater than q' and q'', $\Sigma_i |y_i^q - y_i| < \varepsilon$. This proves that $\Sigma_i y_i^q$ converges to $\Sigma_i y_i$. In other words, $x = \Sigma_i y_i$.

LEMMA 9. *For every i in a finite set I, let f_i be a measurable function from $E \in \mathscr{A}$ to R, bounded below by an integrable function, and let v_i be a linear form on S such that for every $a \in E$, the set*

$$L^I(a) = \{x \in S \,|\, \text{for every } i \in I,\ v_i(x) = f_i(a)\} \tag{4.10}$$

is not empty. Let K^I be the correspondence from E to S defined by

$$K^I(a) = \{x \in S \,|\, \text{for every } i \in I,\ v_i(x) \leqq f_i(a)\}. \tag{4.11}$$

If $\nu(E) > 0$, then

$$\int_E K^I \, d\nu = \left\{x \in S \,\middle|\, \text{for every } i \in I,\ v_i(x) \leqq \int_E f_i \, d\nu\right\}. \tag{4.12}$$

PROOF. Obviously, $\int_E K^I \, d\nu \subset \{x \in S \,|\, \text{for every } i \in I,\ v_i(x) \leqq \int_E f_i \, d\nu\}$. To prove the converse inclusion, let

$$D^I = \{x \in S \,|\, \text{for every } i \in I,\ v_i(x) = 0\} \tag{4.13}$$

and

$$C^I = \{x \in S \,|\, \text{for every } i \in I,\ v_i(x) \leqq 0\}; \tag{4.14}$$

let $J = \{i \in I \,|\, f_i \text{ is not integrable over } E\}$ and $J' = I \backslash J$. Denoting the interior of C^J by int C^J, we first observe that

(i) Int $C^J \neq \varnothing$.

Let us prove (i). If Int $C^J = \varnothing$, the polar of C^J contains a straight line. Therefore, for $i \in J$, there are real numbers $\lambda_i \geqq 0$, not all equal to zero, and such that $\Sigma_{i \in J} \lambda_i v_i = 0$. For every $a \in E$, select an element $e(a)$ of $L^J(a)$. One has for every $a \in E$,

$$\sum_{i \in J} \lambda_i f_i(a) = \sum_{i \in J} \lambda_i v_i(e(a)) = 0. \tag{4.15}$$

Consider $j \in J$ for which $\lambda_j > 0$. One has for every $a \in E$,

$$\lambda_j f_j(a) = - \sum_{i \in J,\, i \neq j} \lambda_i f_i(a). \tag{4.16}$$

For every $i \in J$, f_i is bounded below by a function integrable over E. Therefore, f_j is bounded above by a function integrable over E, and consequently, is integrable over E, a contradiction. *Q.E.D.*

Our second observation is that

(ii) $\int_E K^J \, dv \neq \varnothing$.

Proving (ii) in S is clearly equivalent to proving it in the quotient space S/D^J. In the proof of (ii) we shall, therefore, assume, without loss of generality, that for every $a \in E$, $L^J(a)$ has exactly one element.

For every $i \in J$, f_i is bounded below by a function f_i' integrable over E. For every $a \in E$, let

$$X(a) = \{x \in S \mid \text{for every } i \in J,\ v_i(x) \leqq f_i'(a)\}. \tag{4.17}$$

The set $X(a)$ is contained in $K^J(a)$. According to (i), $X(a)$ is not empty. Since $L^J(a)$ has exactly one element, $X(a)$ has a nonempty set $\hat{X}(a)$ of extreme points. Each such extreme point is the intersection of a family of hyperplanes of the form

$$H_i(a) = \{x \in S \mid v_i(x) = f_i'(a)\} \tag{4.18}$$

with $i \in J$. Therefore, one can easily obtain a measurable selector s for $\hat{X}$. The function s is clearly integrable over E. *Q.E.D.*

Now observe that

(iii) $\int_E K^J \, dv = S$.

Proving (iii) for K^J is equivalent to proving it for the correspondence $a \twoheadrightarrow K^J(a) - \{s(a)\}$ from E to S. In the proof of (iii), we shall therefore assume, without loss of generality, that for every $a \in E$, one has $0 \in K^J(a)$. Thus, for every $a \in E$, for every $i \in J, f_i(a) \geqq 0$. As in the proof of (ii), we shall also assume in the proof of (iii) that for every $a \in E$, $L^J(a)$ has exactly one element $\ell^J(a)$.

Consider a point $x \in S$. Let r be a strictly positive integrable real valued function on E, and for every positive integer n, let

$$M_n = \{a \in E \mid \text{for every } i \in J, f_i(a) \leqq nr(a)\}. \tag{4.19}$$

The set M_n belongs to $\mathscr{A}$. For every $i \in J, f_i$ is integrable over M_n. Since $M_n \uparrow E$, one has for every $i \in J$, $\int_{M_n} f_i \, dv \to +\infty$. Choose $\bar{n}$ such that for every $i \in J$, $v_i(x) \leqq \int_{M_{\bar{n}}} f_i \, dv$. Define the function g from E to S as follows: for every $a \in M_{\bar{n}}$, $g(a) \in \ell^J(a)$; for every $a \in E \backslash M_{\bar{n}}$, $g(a) = 0$. Clearly, for every $a \in E$, $g(a) \in K^J(a)$ and, letting $y = \int_E g dv$, for every $i \in J$, $v_i(y) = \int_{M_{\bar{n}}} f_i \, dv \geqq v_i(x)$. Choose now an integrable nonnegative function t from E to R such that $\int_E t dv = 1$ and let $h(a) = g(a) + (x - y)t(a)$. For every $a \in E$, $h(a) \in K^J(a)$ and $\int_E h dv = x$. *Q.E.D.*

By considering the quotient space $S/D^{J'}$, one immediately obtains an integrable function q from E to S such that for every $a \in E$, $q(a) \in L^{J'}(a)$. Since proving the lemma for K^I is equivalent to proving it for the correspondence $a \twoheadrightarrow K^I(a) - \{q(a)\}$ from E to S, without loss of generality, we shall assume until the end of the proof of Lemma 9 that for every $a \in E$, $0 \in L^{J'}(a)$. That is to say, for every $a \in E$, $L^{J'}(a) = D^{J'}$, or for every $a \in E$, for every $i \in J'$, $f_i(a) = 0$.

Finally observe that

(iv) $D^{J'} \cap \operatorname{Int} C^J \neq \varnothing$.

If this intersection is empty, there is a hyperplane H containing $D^{J'}$ and supporting for C^J. In other words, there is $v \neq 0$ in the polar of C^J, vanishing on

$D^{J'}$. Thus, $v = \Sigma_{i \in J} \lambda_i v_i$ where the λ_i are nonnegative and not all zero, and the condition [for every $i \in J'$, $v_i(x) = 0$] implies [$v(x) = 0$]. For every $a \in E$, select an element $e(a)$ of $L^I(a)$. One has for every $a \in E$, for every $i \in J'$, $v_i(e(a)) = f_i(a) = 0$, hence, $v(e(a)) = 0$, hence,

$$\sum_{i \in J} \lambda_i f_i(a) = \sum_{i \in J} \lambda_i v_i(e(a)) = 0. \tag{4.20}$$

As in the proof of (i), the equality $\Sigma_{i \in J} \lambda_i f_i(a) = 0$ for every $a \in E$ leads to a contradiction. *Q.E.D.*

We now consider a point $x \in S$ such that for every $i \in J'$, $v_i(x) \leqq 0$. Because of (iv), $(\{x\} + D^{J'}) \cap C^J \neq \varnothing$. Select a point y in that intersection. Clearly, $y \in C^I$ and $z = x - y$ belongs to $D^{J'}$. Because of (ii) applied to the space $D^{J'}$, there is an integrable function g from E to $D^{J'}$ such that for every $a \in E$,

$$g(a) \in K^J(a) \cap D^{J'} = K^I(a) \cap D^{J'} \tag{4.21}$$

and that $z = \int_E g dv$. Choose now an integrable nonnegative function t from E to R such that $\int_E t dv = 1$ and let $h(a) = g(a) + yt(a)$. For every $a \in E$, $h(a) \in K^I(a)$ and $\int_E h dv = x$, *which completes the proof of Lemma 9.*

We recall the definition of the asymptotic cone $\mathbf{A}M$ of a subset M of S. For every positive integer k, let $M^k = \{x \in M \mid |x| \geqq k\}$ and let C_k be the smallest closed cone with vertex 0 containing M^k. By definition $\mathbf{A}M = \bigcap_k C_k$. About the properties of asymptotic cones that we shall use, we refer to W. Fenchel [6], to R. T. Rockafellar [13], and to [4].

Given a correspondence ψ from A to S, we denote by $\mathbf{A}\psi$ the correspondence $a \twoheadrightarrow \mathbf{A}(\psi(a))$ from A to S.

LEMMA 10. *If ψ is a measurable correspondence from A to S, then so is $\mathbf{A}\psi$.*

PROOF. For every positive integer k, we define the correspondence ψ^k from A to S by

$$\psi^k(a) = \begin{cases} \{x \in \psi(a) \mid |x| \geqq k\} & \text{if this set is not empty,} \\ \{0\} & \text{otherwise.} \end{cases} \tag{4.22}$$

The set $A^k = \{a \in A \mid \psi^k(a) \neq \{0\}\}$ is the projection on A of the set $\{(a, x) \in G(\psi) \mid |x| \geqq k\}$ which belongs to $\mathscr{A} \otimes \mathscr{S}$. By (3.4) of [5], $A^k \in \mathscr{A}$. Therefore,

$$G(\psi^k) = \{(a, x) \in G(\psi) \mid |x| \geqq k\} \cup [(A \backslash A^k) \times \{0\}] \tag{4.23}$$

belongs to $\mathscr{A} \otimes \mathscr{S}$. Consequently, the correspondence $r\psi^k$ from A to S is measurable for every $r \in Q^+$, the set of positive rationals. So is the correspondence γ^k defined by $\gamma^k(a) = \bigcup_{r \in Q^+} r\psi^k(a)$. The smallest closed cone with vertex 0 containing $\psi^k(a)$ is $\overline{\gamma^k(a)}$ and the correspondence $\overline{\gamma^k}$ is measurable by 4.3 of [5] or Lemma 3 of [8]. Finally, $\mathbf{A}\psi(a) = \bigcap_k \overline{\gamma^k(a)}$. Therefore, the graph of $\mathbf{A}\psi$, which is the intersection of the graphs of the $\overline{\gamma^k}$, belongs to $\mathscr{A} \otimes \mathscr{S}$.

LEMMA 11. *Let E be an element of $\mathscr{A}$, H be a hyperplane through 0 in S, and ψ be a measurable, closed, convex valued correspondence from E to S such that $\int_E \psi dv \neq \varnothing$. If $H \cap \mathbf{A} \int_E \psi dv = \{0\}$, then a.e. in E, $H \cap \mathbf{A}\psi(a) = \{0\}$.*

PROOF. Since $\int_E \psi dv \neq \emptyset$ and since the asymptotic cone of a subset of S is invariant under translations of this subset, there is no loss of generality in assuming that in E, $0 \in \psi(a)$. Let $\Sigma = \{x \in H \mid |x| = 1\}$ and let

$$E' = \{a \in E \mid \sum \cap \mathbf{A}\psi(a) \neq \emptyset\}. \tag{4.24}$$

The latter set is the projection on E of $(E \times \Sigma) \cap G(\mathbf{A}\psi)$ which belongs to $\mathscr{A} \otimes \mathscr{S}$ by Lemma 10. Therefore, by (3.4) of [5], E' belongs to $\mathscr{A}$. And by a measurable selection theorem of Aumann [2], there is a measurable function f from E' to S such that in E', $f(a) \in \Sigma \cap \mathbf{A}\psi(a)$. In E', $|f(a)| = 1$. Therefore, if E' is not null, there is a subset E'' of E' belonging to $\mathscr{A}$, of finite strictly positive measure, such that $x = \int_{E''} f dv \neq 0$. For every real number $t \geqq 0$, and every $a \in E''$, $tf(a) \in \mathbf{A}\psi(a) \subset \psi(a)$, this last inclusion following from the fact that 0 belongs to the closed, convex set $\psi(a)$. Therefore, $tx \in \int_{E''} \psi dv$. Hence, $x \in \mathbf{A}\int_{E''} \psi dv \subset \mathbf{A}\int_E \psi dv$. Since $x \neq 0$ and $x \in H$, a contradiction of the assumption that E' is not null has been obtained.

LEMMA 12. *If $\{X_i\}$ is a family of closed, convex subsets of S having a nonempty intersection, then $\mathbf{A}(\bigcap_i X_i) = \bigcap_i \mathbf{A}X_i$.*

PROOF. For every j, $\bigcap_i X_i \subset X_j$, hence, $\mathbf{A}(\bigcap_i X_i) \subset \mathbf{A}X_j$. Therefore, $\mathbf{A}(\bigcap_i X_i) \subset \bigcap_i \mathbf{A}X_i$.

To prove the reverse inclusion, notice that there is no loss of generality in assuming that $0 \in \bigcap_i X_i$. Then for every j, $\bigcap_i \mathbf{A}X_i \subset \mathbf{A}X_j \subset X_j$. Therefore, $\bigcap_i \mathbf{A}X_i \subset \bigcap_i X_i$. Hence, $\bigcap_i \mathbf{A}X_i \subset \mathbf{A}(\bigcap_i X_i)$.

LEMMA 13. *If $\{X_i\}$ is a decreasing sequence of closed, convex subsets of S having a nonempty intersection, H is a hyperplane through 0 in S such that $H \cap \mathbf{A}(\bigcap_i X_i) = \{0\}$, H' is a hyperplane parallel to H in S such that $H' \cap (\bigcap_i X_i) = \emptyset$, then there is j such that $H' \cap X_j = \emptyset$.*

PROOF. By Lemma 12, $\mathbf{A}(\bigcap_i X_i) = \bigcap_i \mathbf{A}X_i$. Therefore, $H \cap (\bigcap_i \mathbf{A}X_i) = \{0\}$.

Let $\Sigma = \{x \in H \mid |x| = 1\}$ be the unit sphere in H and let $K_i = \Sigma \cap \mathbf{A}X_i$. The decreasing sequence of compact sets $\{K_i\}$ has an empty intersection. Therefore, for some n, $K_n = \emptyset$; hence, $H \cap \mathbf{A}X_n = \{0\}$, and $H' \cap X_n$ is compact. The decreasing sequence of sets $\{H' \cap X_i\}$ has an empty intersection and for $i \geqq n$, these sets are compact. Therefore, for some j, $H' \cap X_j = \emptyset$.

PROOF OF THEOREM 2. In the first part of the proof, we assume that Φ is a countably additive, v continuous, positive, convex valued correspondence from $\mathscr{A}$ to S satisfying $\Phi = \hat{\Phi}$.

Let $V = \{v_i\}$ be a countable dense subset of the dual of S. For every i and every $E \in \mathscr{A}$, define $F_i(E) = \sup v_i(\Phi(E))$. The function F_i is from $\mathscr{A}$ to $]-\infty, +\infty]$ is clearly v continuous. By the corollary of Lemma 2, it is also countably additive. It then follows from P. R. Halmos ([7], p. 131, Ex. 7) that there is a measurable function f_i from A to $]-\infty, +\infty]$ such that for every $E \in \mathscr{A}$, $F_i(E) = \int_E f_i\, dv$.

Define $\psi_i(a) = \{x \in S \mid v_i(x) \leqq f_i(a)\}$ and $\varphi(a) = \bigcap_i \psi_i(a)$.

Clearly, for every $a \in A$, $\varphi(a)$ is closed and convex. Moreover, the function φ from A to $\mathscr{P}(S)$ is measurable, for its graph $G(\varphi)$ equals $\bigcap_i G(\psi_i)$ and every $G(\psi_i)$ belongs to $\mathscr{A} \otimes \mathscr{S}$.

We shall prove below that (i) $\int_A \varphi dv \neq \emptyset$ and (ii) for every $E \in \mathscr{A}$, $\int_E \varphi dv \subset \overline{\Phi(E)}$, which implies that a.e. $\varphi(a) \subset P$. To see this, notice that for every $E \in \mathscr{A}$, $\overline{\Phi(E)}$ is contained in the integral over E of the correspondence from A to S which is constant and equal to P. The assertion follows from Lemma 7.

(i) $\int_A \varphi dv \neq \emptyset$.

To prove (i), let v be a strictly positive linear form on S. For every $E \in \mathscr{A}$, define

$$\Gamma(E) = \{x \in \overline{\Phi(E)} \,|\, v(x) = \inf v(\Phi(E))\}. \tag{4.25}$$

The set $\Gamma(E)$ is nonempty and convex. By Lemma 8, the correspondence Γ from $\mathscr{A}$ to S is countably additive. It is clearly ν continuous. Since Γ is compact valued, it trivially satisfies $\Gamma = \hat{\Gamma}$. Now let $H = \{x \in S \,|\, v(x) = 0\}$ and select in S a straight line L through 0 as in Lemma 4. Denote the projection (that is, in this case, the translate) of $\Gamma(E)$ into H parallel to L by $\dot{\Gamma}(E)$. The correspondence $\dot{\Gamma}$ from $\mathscr{A}$ to H has all the properties that have been assumed about the correspondence Φ from $\mathscr{A}$ to S, and a reasoning by induction on dim S establishes that $\dot{\Gamma}$ has a measurable Radon-Nikodým derivative $\dot{\gamma}$. Thus, Γ has a measurable Radon-Nikodým derivative γ, which can be assumed to be a correspondence. For every $E \in \mathscr{A}$, $\int_E \gamma dv = \Gamma(E) \subset \overline{\Phi(E)}$. Therefore, for every i, for every $E \in \mathscr{A}$, $\sup v_i(\int_E \gamma dv) \leqq \sup v_i(\Phi(E))$. By Lemma 2, the left side equals $\int_E \sup v_i(\gamma(a))\, dv(a)$ while the right side equals $\int_E f_i\, dv$. Consequently, for every i, a.e. in A, $\sup v_i(\gamma(a)) \leqq f_i(a)$, hence, $\gamma(a) \subset \psi_i(a)$. Therefore, a.e. in A, $\gamma(a) \subset \bigcap_i \psi_i(a)$, which yields $\Gamma(A) = \int_A \gamma dv \subset \int_A \varphi dv$. *Q.E.D.*

Denoting by A^* the set $\{a \in A \,|\, \varphi(a) \neq \emptyset\}$, we obtain as an immediate consequence of (i) that $A \backslash A^*$ is null.

(ii) For every $E \in \mathscr{A}$, $\int_E \varphi dv \subset \overline{\Phi(E)}$.

To prove (ii), note one has $\sup v_i(\varphi(a)) \leqq f_i(a)$ for every i and every $a \in A^*$. Therefore, for every i, for every $E \in \mathscr{A}$,

$$\int_E \sup v_i(\varphi(a))\, dv(a) \leqq F_i(E). \tag{4.26}$$

By Lemma 2, $\sup v_i(\int_E \varphi dv) \leqq \sup v_i(\Phi(E))$. Consider now a point x of S not belonging to $\overline{\Phi(E)}$. The set $\overline{\Phi(E)}$ is closed, convex and contains no straight line. Consequently, Corollary 2 applies. For some element $v_i \in V$ one has $\sup v_i(\Phi(E)) < v_i(x)$. Therefore, $x \notin \int_E \varphi dv$. *Q.E.D.*

(iii) For every n, for every $E \in \mathscr{A}$, $\Phi(E)$ is contained in the closure of $\int_E [\bigcap_{i=1}^n \psi_i(a)]\, dv(a)$.

The proof of (iii) is by induction on n. Assume that for every set I of indices such that card $I < n$, and for every $E \in \mathscr{A}$,

$$\Phi(E) \text{ is contained in the closure of } \int_E [\bigcap_{i \in I} \psi_i(a)]\, dv(a). \tag{4.27}$$

Consider an index $j \leqq n$. According to (4.27), for every $E \in \mathscr{A}$,

$$\text{(4.28)} \qquad \sup v_j(\Phi(E)) \leqq \sup v_j \int_E \Big[\bigcap_{i \leqq n,\, i \neq j} \psi_i(a)\Big]\, d\nu(a)$$

$$= \int_E \sup v_j\Big[\bigcap_{i \leqq n,\, i \neq j} \psi_i(a)\Big]\, d\nu(a).$$

Therefore, a.e. in A^*, $f_j(a) \leqq \sup v_j[\bigcap_{i \leqq n, i \neq j} \psi_i(a)]$. Since in A^*, $\bigcap_{i \leqq n} \psi_i(a) \neq \emptyset$, one has,

$$\text{(4.29)} \qquad \text{a.e. in } A^*, f_j(a) = \sup v_j\Big[\bigcap_{i \leqq n} \psi_i(a)\Big].$$

Given $a \in A^*$, let $t(a) = \{i \leqq n \mid f_i(a) < +\infty\}$. For a subset T of $\{1, \cdots, n\}$, define $A_T = \{a \in A^* \mid t(a) = T\}$. The sets of the family $\{A_T\}$ clearly form a finite measurable partition of A^*. Since $\Phi(E)$ and $\int_E [\bigcap_{i \leqq n} \psi_i(a)]\, d\nu(a)$ are finitely additive relative to $E \in \mathscr{A}$, it suffices to prove the inclusion in (iii) for each T, for every E belonging to $\mathscr{A}$ and contained in A_T.

Therefore, we consider now a fixed T. Given $a \in A_T$ and $I \subset T$, we define $K^I(a)$ to be the cone

$$\text{(4.30)} \qquad \{x \in S \mid \text{for every } i \in I,\ v_i(x) \leqq f_i(a)\}$$

if the constraints $v_i(x) = f_i(a)$ for every $i \in I$ are compatible, $K^I(a)$ to be the empty set otherwise.

For every $a \in A_T$, the sets $\bigcap_{i \in T} \psi_i(a)$ have the same asymptotic cone

$$\text{(4.31)} \qquad C^T = \{x \in S \mid \text{for every } i \in T,\ v_i(x) \leqq 0\}.$$

Let v be a linear form on S such that $v(C^T) \leqq 0$. Given $a \in A_T$, we maximize v on $\bigcap_{i \in T} \psi_i(a)$. Let x^0 be a maximizer and let $I = \{i \in T \mid v_i(x^0) = f_i(a)\}$. Clearly,

$$\text{(4.32)} \qquad \max v\Big[\bigcap_{i \in T} \psi_i(a)\Big] = \max v[K^I(a)].$$

The correspondence $a \twoheadrightarrow \bigcap_{i \in T} \psi_i(a)$ from A_T to S is measurable since its graph is the intersection of the graphs of the ψ_i each of which is measurable. Therefore, by Lemma 1, the function $a \twoheadrightarrow \max v[\bigcap_{i \in T} \psi_i(a)]$ is measurable. On the other hand, given I, one has

$$\text{(4.33)} \qquad \{a \in A_T \mid K^I(a) \neq \emptyset\} = \operatorname{proj}_{A_T} \bigcap_{i \in I} \{(a, x) \in A_T \times S \mid v_i(x) = f_i(a)\}.$$

Each set in this intersection is measurable; so is their intersection and, by (3.4) of [5], so is the projection on A_T of their intersection. Clearly, the graph of the correspondence K^I from $\{a \in A_T \mid K^I(a) \neq \emptyset\}$ to S is measurable by a repetition of the reasoning of the first sentence of this paragraph. And, by a new application of Lemma 1, the function $a \twoheadrightarrow \max v[K^I(a)]$ is measurable.

Summing up, given I, the set of $a \in A_T$ for which the equality (4.32) holds is measurable. Therefore, A_T can be partitioned into finitely many measurable sets $\{A_T^k\}$ such that for every k, for every $a \in A_T^k$, the same set I^k of indices satisfying (4.32) can be chosen.

Consider now E belonging to $\mathscr{A}$ and contained in A_T. Let $E^k = E \cap A_T^k$. The E^k form a finite measurable partition of E. According to Lemma 9, if $\nu(E^k) > 0$, then

$$\int_{E^k} K^{I^k}\, d\nu = \{x \in S \mid \text{for every } j \in I^k,\ v_j(x) \leqq F_j(E^k)\}, \tag{4.34}$$

hence, $\Phi(E^k) \subset \int_{E^k} K^{I^k}\, d\nu$. Clearly, this inclusion also holds if $\nu(E^k) = 0$. Consequently,

$$\begin{aligned} \sup v(\Phi(E^k)) \leqq \sup v\left(\int_{E^k} K^{I^k}\, d\nu\right) &= \int_{E^k} \max v[K^{I^k}(a)]\, d\nu(a) \\ &= \int_{E^k} \max v[\bigcap_{i \in T} \psi_i(a)]\, d\nu(a) = \sup v \int_{E^k} [\bigcap_{i \in T} \psi_i(a)]\, d\nu(a), \end{aligned} \tag{4.35}$$

the first and the third equalities following from Lemma 2, and the second from (4.32). The first term being at most equal to the fifth, we obtain by summation over k,

$$\sup v(\Phi(E)) \leqq \sup v\left(\int_E [\bigcap_{i \in T} \psi_i(a)]\, d\nu(a)\right). \tag{4.36}$$

This inequality implies

$$\Phi(E) \text{ is contained in the closure of } \int_E [\bigcap_{i \in T} \psi_i(a)]\, d\nu(a), \tag{4.37}$$

as we now show. Let x be a point of S not in the right set M. There is a linear form v on S such that $\sup v(M) < v(x)$. If $\nu(E) > 0$, then clearly, $v(C^T) \leqq 0$, hence, by (4.36), $\sup v(\Phi(E)) \leqq \sup v(M)$. If $\nu(E) = 0$, then $\sup v(\Phi(E)) = \sup v(M) = 0$. In either case, $\sup v(\Phi(E)) < v(x)$. Therefore, $x \notin \Phi(E)$.

(iv) For every $E \in \mathscr{A}$, $\Phi(E) \subset \overline{\int_E \varphi d\nu}$.

To prove (iv), consider a point x of S not belonging to $\overline{\int_E \varphi d\nu}$. According to (ii), $\overline{\int_E \varphi d\nu}$ is contained in P and consequently, contains no straight line. By Lemma 6 neither does the closed cone C with vertex 0 generated by $\overline{\int_E \varphi d\nu} - \{x\}$. Therefore, we can select a linear form v in the nonempty interior of the polar of C. Let $H = \{y \in S \mid v(y) = 0\}$. We have

$$\sup v\left(\int_E \varphi d\nu\right) < v(x), \tag{4.38}$$

$$H \cap \mathbf{A} \int_E \varphi d\nu = \{0\}. \tag{4.39}$$

According to Lemma 11, equation (4.39) implies a.e. in E,

$$H \cap \mathbf{A}\varphi(a) = \{0\}. \tag{4.40}$$

Then let f be an integrable function from E to R such that a.e. in E,

$$\sup v(\varphi(a)) < f(a), \qquad \int_E f dv < v(x). \tag{4.41}$$

Also let E_0 be the null subset of E in which (4.40) does not hold, or (4.41) does not hold, or $\bigcap_i \psi_i(a) = \varnothing$. Given $a \in E \backslash E_0$, by Lemma 13, there is n such that

$$\max v\left(\bigcap_{i=1}^{n} \psi_i(a)\right) < f(a). \tag{4.42}$$

Denote by E'_n the subset of $E \backslash E_0$ in which this inequality is satisfied. As we have seen in the proof of (iii), the function $a \twoheadrightarrow \max v(\bigcap_{i=1}^{n} \psi_i(a))$ from $E \backslash E_0$ to R is measurable. Therefore, $E'_n \in \mathscr{A}$. Clearly, $E'_n \subset E'_{n+1}$. Define $E_1 = E'_1$ and for $n > 1$, $E_n = E'_n \backslash E'_{n-1}$. The sets $\{E_n\}_{n \geqq 0}$ form a countable measurable partition of E. By (iii), for every $n \geqq 1$,

$$\sup v(\Phi(E_n)) \leqq \sup v\left(\int_{E_n} [\bigcap_{i=1}^{n} \psi_i(a)]\, dv(a)\right). \tag{4.43}$$

However, by Lemma 2 and (4.42), the right side is at most equal to $\int_{E_n} f dv$. Therefore, summing over n, we obtain $\sup v(\Phi(E)) \leqq \int_E f dv < v(x)$. Hence, $x \notin \Phi(E)$.

(v) For every $E \in \mathscr{A}$, $\Phi(E) \subset \int_E \varphi dv$.

To prove (v), note $\Phi(E) \subset \overline{\int_E \varphi dv}$, by (iv). Therefore, it suffices to prove that if a point x of $\Phi(E)$ is in the boundary of $\Phi(E)$, then $x \in \int_E \varphi dv$.

Let $v \neq 0$ be a linear form on S such that $\sup v(\Phi(E)) = v(x)$ and let $\mathscr{B} = \{B \in \mathscr{A} \mid B \subset E\}$. For every $B \in \mathscr{B}$, we define

$$\Gamma(B) = \{y \in \Phi(B) \mid v(y) = \sup v(\Phi(B))\}. \tag{4.44}$$

According to (i) in the proof of Lemma 5, Γ is countably additive. Since $\Gamma(E) \neq \varnothing$, for every $B \in \mathscr{B}$, $\Gamma(B) \neq \varnothing$. The correspondence Γ from $\mathscr{B}$ to S is clearly v continuous, positive, convex valued. It also satisfies $\Gamma = \hat{\Gamma}$ because if Ψ is a countably additive correspondence from $\mathscr{B}$ to S such that $\Psi \subset \bar{\Gamma}$, then $\Psi \subset \bar{\Phi}|_{\mathscr{B}}$, the restriction of $\bar{\Phi}$ to $\mathscr{B}$. Hence, Φ being the greatest element of $\mathscr{M}$, $\Psi \subset \Phi|_{\mathscr{B}}$. This inclusion with $\Psi \subset \bar{\Gamma}$ implies $\Psi \subset \Gamma$. How project $\Gamma(B)$ into $H = \{y \in S \mid v(y) = 0\}$ parallel to a straight line L chosen as in Lemma 4. An induction assumption on dim S yields a measurable Radon-Nikodým derivative γ for Γ. One has for every $B \in \mathscr{B}$,

$$\int_B \gamma dv \subset \Phi(B) \subset \overline{\int_B \varphi dv}, \tag{4.45}$$

the last inclusion by (iv). By Lemma 7, a.e. in E, $\gamma(a) \subset \varphi(a)$. From $x \in \int_E \gamma dv$, we obtain $x \in \int_E \varphi dv$.

(vi) For every $E \in \mathscr{A}$, $\Phi(E) = \int_E \varphi dv$.

For proof of (vi), clearly, by (ii), the correspondence $E \twoheadrightarrow \int_E \varphi dv$ from $\mathscr{A}$ to S belongs to $\mathscr{M}$. Since Φ is the greatest element of $\mathscr{M}$, one has $\int_E \varphi dv \subset \Phi(E)$ which, with (v) yields the conclusion. *Q.E.D.*

To complete the proof of Theorem 2, we consider a measurable positive, closed, convex valued function ψ from A to $\mathscr{P}(S)$ such that $\int_A \psi dv \neq \varnothing$. Denote $\int_E \psi dv$ by $\Phi(E)$. Clearly, Φ is a countably additive, v continuous, positive, convex valued correspondence from $\mathscr{A}$ to S. The greatest element $\hat{\Phi}$ of $\mathscr{M}$ is obviously v continuous and positive valued. According to the second assertion of Theorem 1 it is convex valued. By the first part of the present proof, $\hat{\Phi}$ has a measurable positive, closed, convex valued Radon-Nikodým derivative φ. For every $E \in \mathscr{A}$, $\hat{\Phi}(E) \subset \overline{\Phi(E)}$, hence, $\int_E \varphi dv \subset \overline{\int_E \psi dv}$. By Lemma 7, a.e., $\varphi(a) \subset \psi(a)$. Therefore, $\hat{\Phi} \subset \Phi$. Hence, $\Phi = \hat{\Phi}$.

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