INVARIANT MEASURES ON PRODUCT SPACES

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1. Introduction

Let G be a locally compact group (which we always assume to satisfy the second axiom of countability), and let X be a standard Borel space on which G acts as a Borel transformation group (see [7], p. 628). That is, we have a homomorphism of G into the group of Borel automorphisms of X such that if $x \rightarrow g \cdot x$ is the automorphism corresponding to $g \in G$, then $(x, g) \rightarrow g \cdot x$ is a Borel function from $G \times X$ into X where G is endowed with the σ-field of sets generated by the open sets, and $G \times X$ is given the product σ-field. Further, let μ be a σ-finite measure on X (all measures henceforth will be understood to be σ-finite) which is quasi-invariant under G; that is, for every $g \in G$, $g \cdot \mu$ and μ are equivalent in the sense of mutual absolute continuity. (Here $g \cdot \mu$ is the transform of μ by g defined by $(g \cdot \mu)(\sigma) = \mu(g^{-1} \cdot \sigma)$ for Borel sets σ of X.) One says that μ is ergodic under G if for every Borel σ in G such that $\mu(\sigma \Delta g \cdot \sigma) = 0$ for all $g \in G$, we have $\mu(\sigma) = 0$ or $\mu(X - \sigma) = 0$. It is clear that these two properties of μ depend not on μ, but only on the equivalence class $C(\mu)$ of μ. We shall say, following [8], that $C(\mu)$ is a quasi-orbit of G if μ is quasi-invariant and ergodic. Note that each orbit of G on X carries a unique equivalence class $C(\mu)$ of such measures (see [8], p. 295). One calls these classes transitive quasi-orbits or simply orbits.

We shall say that a measure ν is invariant if $g \cdot \nu = \nu$ for all $g \in G$. In this note we are going to discuss a special case of the following circle of questions: given a quasi-orbit $C(\mu)$ on X, when does it contain an invariant measure ν or more specifically an invariant ν with specified properties? We note that if $\nu \in C(\mu)$ is invariant, it is unique up to multiplication by positive scalars. This is an immediate consequence of ergodicity. Furthermore, any $\lambda \in C(\mu)$ is either atomic (consists of point masses) or nonatomic (no point masses).

The systems $(G, X, C(\mu))$ which we will discuss will be such that G is countable and acts freely on X in the sense that $\{x: g \cdot x = x \text{ for some } g \neq e\}$ is a μ-null set. When these conditions are satisfied, von Neumann in [10] has shown how to construct a certain factor von Neumann algebra associated with $(G, X, C(\mu))$. The type of this factor is determined by the measure theoretic properties of $C(\mu)$ discussed in the previous paragraph (see [10], theorem IX). One can show

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that these kinds of factors arise as the rings generated by certain representations of restricted products of groups. Our results will be applied to this situation in a subsequent paper.

Our attention will be restricted to those systems $(G, X, C(\mu))$ which arise in a very particular fashion. Let A be a countable index set, and let X_α, $\alpha \in A$ be countable spaces with the σ-field of all subsets, and let G_α be a discrete group acting transitively on X_α. Let μ_α be a probability measure on X_α, quasi-invariant under G_α. (This simply means, of course, that each point of X_α gets positive mass.) Let X be the full Cartesian product of the X_α given the product Borel structure (which is a standard Borel space), and let μ be the infinite product of the measures μ_α (see [4]; recall that $\mu_\alpha(X_\alpha) = 1$). Finally let G be the direct sum of G_α; $G = \{(g_\alpha) : g_\alpha \in G_\alpha, g_\alpha = e \text{ for almost all } \alpha\}$. ('Almost everywhere' with respect to sets of indices α will mean 'except for a finite number'; there should be no confusion with the usual measure theoretic usage of the term.) Then G is countable and acts on X in the obvious way. Moreover it is clear (see below) that $C(\mu)$ is a quasi orbit. Our results concern conditions under which $C(\mu)$ contains an invariant measure.

Von Neumann raised this question in the case when X_α is a two-point space, and implicitly states a theorem but gives no details (see [10], p. 95, last sentence of the third paragraph of 4) and compare with ([9], pp. 66–77). Pukansky treated a special case of this, ([11], p. 140, lemma 6), and recently J. D. C. Bures [2] has given a systematic treatment of some aspects of this including a proof of von Neumann's assertion. See also the paper of Araki [1]. Bures' results are unfortunately not strong enough for our purposes. Bures works in the context of the von Neumann algebra associated to the system, whereas we shall work just with the measure theory. We wish to thank R. V. Kadison for calling our attention to this paper of Bures.

2. Statement of the results

Let $(G, X, C(\mu))$ be a triple as discussed above, not necessarily arising from the product construction, with G countable and acting freely where $C(\mu)$ is a quasi-orbit.

DEFINITIONS. (1) *We shall say that $C(\mu)$ is type* I *if it is transitive (or equivalently that $C(\mu)$ consists of atomic measures, or that $C(\mu)$ contains an atomic invariant measure).*

(2) *We shall say that $C(\mu)$ is type* II_1 *if it is nontransitive (that is, nonatomic), and contains a finite invariant measure.*

(3) *We shall say that $C(\mu)$ is type* II_∞ *if it is nontransitive, and contains an infinite invariant measure.*

(4) *We shall say that $C(\mu)$ is type* III *if it contains no invariant measure.*

We observe that these possibilities are mutually exclusive and exhaustive. One could, of course, formulate these definitions for an arbitrary G, but we shall not need this, and moreover, if G does not act freely, this classification does not seem to be too appropriate (see [8]). We note that the von Neumann algebra constructed from $(G, X, C(\mu))$ mentioned in section 1 is a factor whose type is exactly that of $C(\mu)$ ([10], theorem IX). Our approach here is to fix G and let $C(\mu)$ vary. However, the recent result of Tulcea [13] concerns the opposite. This says roughly that if $C(\mu)$ is fixed and G is fixed to be the integers, but its action is allowed to vary, then in a suitable categorical sense $C(\mu)$ is almost always type III. Our results indicate the same, namely that the type III case is generic.

Let us now return to the specific case which we shall consider. That is, $X = \Pi_\alpha X_\alpha$, $\mu = \Pi \mu_\alpha$ where the X_α are countable and the μ_α are probability measures on X_α giving each point positive mass, and G is the direct sum of countable groups G_α with G_α transitive and free on X_α.

LEMMA 2.1. *The class $C(\mu)$ is a quasi-orbit for G.*

PROOF. This is essentially known, but we include a proof for completeness. If $\alpha \in A$, then $X = X_\alpha \times X_\alpha^*$ where X_α^* is the product of the X_β, $\beta \neq \alpha$. Also, $\mu = \mu_\alpha \times \mu_\alpha^*$ where μ_α^* is the product of the remaining μ_β, $\beta \neq \alpha$. Now let us identify G_α with the subgroup of G, all of whose coordinates except the α-th are the identity element. Then G_α acts on X by its usual action on X_α and trivially on X_α^*. Clearly μ is quasi-invariant under G_α since μ_α is. Since G is generated by the G_α, μ is quasi-invariant under G. Finally, to see that $C(\mu)$ is ergodic, we simply note that this assertion is the Borel zero-one law of probability theory (see [3], p. 102).

We next want to note that the type of $C(\mu)$ does not depend on any precise information about the G_α, since one can characterize G-invariant measures on X independently of the G_α.

LEMMA 2.2. *Let ν be a Borel measure on X. Then ν is invariant under G if and only if for each α, $\nu = \nu_\alpha \times \nu_\alpha^*$, where ν_α^* is some Borel measure on X_α^* and ν_α is the measure on X_α, giving mass one to each point.*

PROOF. If our condition is satisfied, then $\nu = \nu_\alpha \times \nu_\alpha^*$, and since ν_α is invariant under G_α, ν is invariant under G_α since G_α acts trivially on X_α^*. Thus again, as G is generated by the G_α, ν is G-invariant. Conversely, let ν be G-invariant, and write $X = X_\alpha \times X_\alpha^*$. Let $x \in X_\alpha$ and define a measure ν_α^* on X_α^* by $\nu_\alpha^*(\sigma) = \nu(\{x\} \times \sigma)$ for $\sigma \subset X_\alpha^*$. It is easy to see that ν_α^* is σ-finite so that it is a measure under our conventions. Since G_α is transitive on X_α and ν is invariant under G_α, one sees that ν_α^* is independent of the choice of x. As X_α is countable, it is clear that $\nu = \nu_\alpha \times \nu_\alpha^*$ on all 'rectangles' in the product. Therefore, these two measures are the same. This completes the proof.

We note indeed that the condition of the lemma is independent of the G_α. We shall simply say that ν on X is invariant if it satisfies this condition. Thus we can speak of the type of a measure class $C(\mu)$ with μ constructed as above without reference to G (just as long as G is built from some groups G_α). This also means that we are free to take any G_α we want in our proofs. For instance if X_α has n_α points ($n_\alpha = \infty$, possibly), then we can identify X_α and G_α with the integers mod n_α with G_α acting on itself by translation.

In general it will be convenient notationally to number the points of each X_α

by the integers from 1 to n_α. The choice of the numbering is, however, at our disposal. The data we are given then essentially are the measures μ_α, which in turn are determined by $p_\alpha^i = \mu_\alpha(\{i\})$ for the points i of X_α. Since $\sum_i p_\alpha^i = 1$ for every α, we can number the points of X_α so that p_α^i is decreasing as a function of i.

We shall now state the main results.

THEOREM 1. (1) *$C(\mu)$ is type* I *if and only if $\sum_\alpha (1 - p_\alpha^1)$ converges.*

(2) *$C(\mu)$ is type* II_1 *if and only if $n_\alpha < \infty$ for all α and $n_\alpha > 1$ for infinitely many α, and*

$$\sum_{\alpha,i} (n_\alpha)^{-1}|1 - (p_\alpha^i n_\alpha)^{1/2}|^2 \tag{1}$$

converges.

Bures [2] has obtained part (1) which is rather easy, and has obtained something equivalent to (2) under the additional hypothesis that the n_α are uniformly bounded. Our proof does not need this side condition. It is somewhat more subtle to distinguish the II_∞ from the III case, and we must impose a side condition similar but weaker than Bures' condition (but not 'essentially' weaker).

Let us establish the notation $|x|_c = \inf(|x|, c)$ for $c > 0$.

THEOREM 2. *Suppose that $p_\alpha^1 \geq a > 0$ for some a and all α. Then $C(\mu)$ is type* III *if and only if*

$$\sum_{\alpha,i} p_\alpha^i|(p_\alpha^1/p_\alpha^i) - 1|_c^2 \tag{2}$$

diverges for some (and hence all) positive c.

Therefore, of course, we can distinguish the II_∞ case at least under our side condition. Notice that $p_\alpha^1 \geq a > 0$ is automatically satisfied if the n_α are uniformly bounded by some K, for we can take $a = 1/K$. It is an easy matter to see that the convergence or divergence of the series in the theorem is independent of the value of c. We have stated our results in terms of absolutely convergent double series. These criteria can be rephrased in terms of conditionally convergent double series as in [2], but we feel they are more convenient in the present form.

PROOF OF THEOREM 1. We include the trivial proof of part 1, theorem 1 for completeness. Recall that $C(\mu)$ is type I if and only if μ gives positive measure to some point. If $x \in X$ with $x = (j_\alpha)$, $1 \leq j_\alpha \leq n_\alpha$, then $\mu(\{x\}) = \prod \mu_\alpha(\{j_\alpha\})$, since μ is a product measure. Now it is clear that this product is as large as possible when $j_\alpha = 1$ for all α. Thus $C(\mu)$ is type I if and only if $\prod_\alpha p_\alpha^1 > 0$, and this is true if and only if $\sum(1 - p_\alpha^1)$ is convergent.

We pass now to part (2). Suppose that $C(\mu)$ is type II_1. Then by lemma 2.2, it is clear that n_α must be finite. Furthermore, if $n_\alpha = 1$ for almost all α, X is finite, and we are in the type I case. We must prove now that the series in question converges.

Let λ be a finite invariant measure on X. As we mentioned before, we may as well take X_α to the integers mod n_α and $G_\alpha = X_\alpha$ acting on itself. In that case X is a compact abelian group, and G may be viewed as a dense subgroup. Since λ is finite and invariant under translation by G, it is invariant under X. (To see this, look at the Fourier-Stieltjes transform of ν.) Therefore λ is the Haar measure on X, and we may take $\lambda(X) = 1$. Then if λ_α is the measure on X_α giving mass $(n_\alpha)^{-1}$ to each point, it is clear that λ is the infinite product of the λ_α. Now since $\lambda \sim \mu = \Pi\, \mu_\alpha$, we may apply a criterion of Kakutani [5] for equivalence of infinite product measures. His result says that $\lambda \sim \mu$ if and only if $\sum_\alpha |1 - (d\mu_\alpha/d\lambda_\alpha)^{1/2}|^2$ converges, where $|\cdot|$ indicates the norm in $L_2(\lambda_\alpha)$. But now this series can be seen by substituting the p_α^i, to be precisely the series in the statement of the theorem, and so we are done. (We are indebted to Kakutani for calling our attention to his paper.)

Conversely, let the conditions of the theorem be satisfied; then in the same notation, $\lambda \sim \mu$ so that $\lambda \in C(\mu)$, using Kakutani's result the other way. Thus as λ is invariant, $C(\mu)$ is either type I or II_1. The condition $n_\alpha > 1$ for infinitely many α assures us that we are in the II_1 case. This completes the proof.

3. Proof of first part of theorem 2

We come now to the proof of theorem 2. We first prove the sufficiency of our condition which is the more difficult part. If (a_n) and (b_n) are sequences of non-negative numbers, we say that $a_n \sim b_n$ if there exist positive constants k_1 and k_2 such that $k_1 a_n \leq b_n \leq k_2 a_n$. If $a_n \sim b_n$, then clearly $\sum a_n$ converges or diverges with $\sum b_n$.

Suppose now that $C(\mu)$ contains an invariant measure ν, and let $f = d\nu/d\mu$ so that f is a Borel function on X, unique up to μ null sets and > 0 almost everywhere. According to lemma 2.2, we have $\nu = \nu_\alpha \times \nu_\alpha^*$, and since $\mu = \mu_\alpha \times \mu_\alpha^*$, we see that $d\nu/d\mu = d\nu_\alpha/d\mu_\alpha \times d\nu_\alpha^*/d\mu_\alpha^*$. To be precise, if $x \in X$, $x = (x_\alpha x_\alpha^*)$, then $f(x) = f_\alpha(x_\alpha) f_\alpha^*(x_\alpha^*)$ for almost all pairs (x_α, x_α^*) where $f_\alpha = d\nu_\alpha/d\mu_\alpha$ and $f_\alpha^* = d\nu_\alpha^*/d\mu_\alpha^*$. Since X_α is discrete, this holds for almost all x_α^* for each $x_\alpha \in X_\alpha$. Moreover, since ν_α^* is unique in this decomposition, it follows that ν_α^* is an invariant measure on X_α^* in the class $C(\mu_\alpha^*)$. We can repeat this argument to find the following.

LEMMA 3.1. *Let F be a finite subset of A, and let $X_F = \Pi_{\alpha \in F} X_\alpha$ and $X_F^* = \Pi_{\alpha \notin F} X_\alpha$. Then $X = X_F \times X_F^*$ where $x = (x_F, x_F^*)$ are the coordinates of a point x. Then $\nu = (\Pi_{\alpha \in F} \nu_\alpha) \times \nu_F^*$ for some invariant measure ν_F^* on X_F^* and $f(x) = (\Pi_{\alpha \in F} f_\alpha(x_\alpha)) \cdot f_F^*(x_F^*)$ for almost all x_F^* where $f_F^* = d\nu_F^*/d\mu_F^*$ and $\mu_F^* = \Pi_{\alpha \notin F} \mu_\alpha$.*

Now let g^λ, g_α^λ, $g_F^{*\lambda}$ denote respectively the functions $\exp(2\pi i\lambda \log(\cdot))$, where $(\cdot)$ is f, f_α, f_F^* for real λ. These functions are of modulus one and

$$g^\lambda(x) = \prod_{\alpha \in F} g_\alpha^\lambda(x_\alpha) g_F^{*\lambda}(x_F^*). \tag{3}$$

We view each of these functions as an element of $L_2(X, \mu)$ by making them constant in the variables on which they do not depend. Since $\mu(X) = 1$, they are all unit vectors.

Now denote by P_F the orthogonal projection of $L_2(X)$ onto $L_2(X_F)$ where we view $L_2(X_F)$ as the subspace of $L_2(X)$ of all functions which depend only on the coordinates x_α, $\alpha \in F$, of the point $x \in X$. It is immediate that

$$P_F(g^\lambda) = \Big(\prod_{\alpha\in F} g_\alpha^\lambda\Big)\cdot (g_F^{*\lambda}, 1) \tag{4}$$

where 1 denotes the function equal to one everywhere. Then it is well known that $P_F(g^\lambda) \to g^\lambda$ in the norm topology on $L_2(X)$. Therefore,

$$|P_F(g^\lambda)| = |(g_F^{*\lambda}, 1)| \to 1,$$

and in particular $(g_F^{*\lambda}, 1) \neq 0$ for large F. It follows at once that $(g_\alpha^\lambda, 1) = 0$ can hold for only a finite number of α. Now let us choose constants $a_\alpha^\lambda = \exp(2\pi i\lambda b_\alpha^\lambda)$ of modulus one such that $(a_\alpha^\lambda g_\alpha^\lambda, 1) \geq 0$. Of course b_α^λ is not determined at all for a finite set of α, and in general is only determined modulo $1/\lambda$.

Now we write

$$P_F(g^\lambda) = \Big(\prod_{\alpha\in F} a_\alpha^\lambda g_\alpha^\lambda\Big)\cdot\Big(\prod_{\alpha\in F} a_\alpha^{-\lambda}\Big)(g_F^{*\lambda}, 1), \tag{5}$$

and denote by γ_F the product of the last two terms. We observe that by our construction, for large F, the argument of γ_F independent of F. If we put $d = \bar{\gamma}_F/|\gamma_F|$ for large F, then

$$P_F(dg^\lambda) = \prod_{\alpha\in F} a_\alpha^\lambda g_\alpha^\lambda |\gamma_F|. \tag{6}$$

We have observed before that $|\gamma_F| \to 1$ as $F \to \infty$, and we immediately conclude that $\prod_{\alpha\in F} a_\alpha^\lambda g_\alpha^\lambda$ converges to dg^λ in $L_2(X)$.

On the other hand, one knows precisely when this happens.

LEMMA 3.2. *Let $h_\alpha \in L_2(X_\alpha)$ with $|h_\alpha| = 1$, and view them as elements of $L_2(X)$. Suppose that $(h_\alpha, 1) \geq 0$. Then $\prod_{\alpha\in F} h_\alpha$ converges in $L_2(X)$ if and only if $\sum |h_\alpha - 1|^2$ converges.*

PROOF. This is well known (cf. [10], lemma 3.3.4, p. 24).

We apply this to the situation at hand to conclude that $\sum_\alpha |a_\alpha^\lambda g_\alpha^\lambda - 1|^2$ converges for every λ. Now we know what f_α, and hence g_α^λ, are; indeed $f_\alpha(i) = (d\nu_\alpha/d\mu_\alpha)(i) = (p_\alpha^j)^{-1}$ for $1 \leq j \leq n_\alpha$. We find then by putting in these values for g_α^λ that the series of nonnegative terms

$$\sum_{\alpha,i} p_\alpha^i |\exp(2\pi i\lambda(b_\alpha^\lambda - \log p_\alpha^i)) - 1|^2 \tag{7}$$

converges for all real λ. In particular, we may rearrange at will and conclude that the sum of the terms with $i = 1$ converges. Since p_α^1 is assumed bounded below, we conclude that the series

$$\sum_\alpha |\exp(2\pi i\lambda(b_\alpha^\lambda - \log p_\alpha^1)) - 1|^2 \tag{8}$$

converges for all λ. In particular the individual terms of the series approach zero. Now let $\|x\|$ for a real number x denote the distance from x to the nearest integer. Then it is clear that for large α, the series (8) will dominate term by term the series

$$\sum_\alpha \|\lambda(b_\alpha^\lambda - \log p_\alpha^1)\|^2, \tag{9}$$

and hence the latter series converges. As we remarked before, the b_α^λ are determined only modulo $1/\lambda$, and now we make use of this freedom of choice to choose b_α^λ so that

$$\sum_\alpha |b_\alpha^\lambda - \log p_\alpha^1|^2 \tag{10}$$

converges. This says that the b_α^λ are in some way almost independent of λ in the sense that they do not differ very much from a sequence which is independent of λ. Note that the hypothesis $p_\alpha^1 \geq a > 0$ was used in a very crucial way to accomplish this.

Now if $s_\alpha^\lambda = \exp(-2\pi i\lambda(b_\alpha^\lambda - \log p_\alpha^1))$, then by (10), $\sum_\alpha |1 - s_\alpha^\lambda|^2$ converges.

LEMMA 3.3. *Let the unit vectors h_α be as in lemma 3.2 with $(h_\alpha, 1) \geq 0$ and $\sum |h_\alpha - 1|^2 < \infty$. If $\sum |s_\alpha - 1|^2$ converges with $|s_\alpha| = 1$, then $\sum |s_\alpha h_\alpha - 1|^2$ converges.*

PROOF. This is quite obvious by the inequality $|x + y|^2 \leq 2(|x|^2 + |y|^2)$ for vectors x and y.

We now apply lemma 3.3 to the vectors $a_\alpha^\lambda g_\alpha^\lambda$ and to s_α^λ. We then conclude the following, using the fact that $|\exp(ix) - 1|^2 = 2(1 - \cos x)$.

LEMMA 3.4. *The series*

$$\sum_{\alpha,i} p_\alpha^i(1 - \cos(2\pi\lambda \log(p_\alpha^1) - \log(p_\alpha^i))) \tag{11}$$

converges for all real λ.

The following lemma now essentially completes the proof of the first part of theorem 2.

Recall that $|x|_c = \inf(|x|, c)$ for real x and $c > 0$.

LEMMA 3.5. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers. Then the following are equivalent:*

(1) $\sum a_n(1 - \cos(\lambda b_n))$ *converges for all λ.*

(2) $\sum a_n(1 - \cos(\lambda b_n))$ *converges for λ in a set of positive Lebesgue measure.*

(3) $\sum a_n|b_n|_c^2$ *converges for some $c > 0$.*

(4) $\sum a_n|b_n|_c^2$ *converges for all $c > 0$.*

Assuming this, let us complete the proof of the first part of theorem 2. According to lemmas 3.4 and 3.5, we see that $\sum_{\alpha,i} p_\alpha^i|\log p_\alpha^1/p_\alpha^i|_c^2$ converges for some, and hence all $c > 0$. Now observe that $|\log(p_\alpha^1/p_\alpha^i)|_c \sim |(p_\alpha^1/p_\alpha^i) - 1|_c$, and hence we see that

$$\sum_{\alpha,i} p_\alpha^i|(p_\alpha^1/p_\alpha^i) - 1|_c^2 \tag{12}$$

converges for some, and hence all $c > 0$. This proves the sufficiency of the condition in theorem 2.

PROOF OF LEMMA 3.5. Clearly (1) $\Rightarrow$ (2), and also (3) $\Rightarrow$ (4), for it is obvious that $a_n|b_n|_c^2 \sim a_n|b_n|_d^2$ for any two $c, d > 0$. Therefore, $\sum a_n|b_n|_c^2$ converges if and only if $\sum a_n|b_n|_d^2$ converges. Furthermore, (4) $\Rightarrow$ (1) is clear since, for fixed λ and c, a constant multiple of $a_n|b_n|_c^2$ dominates $a_n(1 - \cos(\lambda b_n))$ term by term. It remains then to prove that (2) $\Rightarrow$ (3).

Let S be a set of positive Lebesgue measure where the series in (2) converges, and let $M(\lambda)$ be its sum for $\lambda \in S$. Clearly $M(\lambda)$ is a Borel function of λ, and consequently, by regularity of Lebesgue measure, we can find a compact set T of positive measure contained in S such that $M(\lambda) \leq K$ for some K and all λ in T. Then

$$\int_T M(\lambda)\, d\lambda \leq K|T| < \infty, \tag{13}$$

where $|T|$ is the Lebesgue measure of T. Since we are dealing with a series of positive terms, we can invert the order of summation and integration to find that

$$\sum a_n \int_T (1 - \cos(\lambda b_n))\, d\lambda \tag{14}$$

converges. Now we claim that the function $f(t) = \int_T (1 - \cos(\lambda t))\, d\lambda$ has a lower bound $m(c)$ on the set $|t| \geq c$. In fact $f(t)$ is continuous in t, and positive except for $t = 0$. It suffices to show then that it is bounded from zero at infinity. But this is clear for as $|t| \to \infty$, $f(t) \to |T|$ by the Riemann-Lebesgue lemma.

Therefore, if $L(c) = \{n: |b_n| \geq c\}$, we see that $\sum_{n \in L(c)} a_n \leq (m(c))^{-1}K|T| < \infty$. On the other hand, by assumption $\sum a_n(1 - \cos \lambda b_n)$ converges for some $\lambda > 0$. Then choose c so small that $1 - \cos x \geq x^2/4$ if $|x| < c\lambda$. Then

$$\sum a_n |b_n|_c^2 = \sum_{n \notin L(c)} a_n |b_n|_c^2 + \sum_{n \in L(c)} a_n, \tag{15}$$

and the first series is dominated term by term by $4 \sum a_n(1 - \cos \lambda b_n)$, $(n \notin L_c)$, and the second series is convergent as we have shown above. This completes the proof.

4. Restricted products of measures

We now have proved in theorem 2 that if $C(\mu)$ is not of type III, then the series of the theorem converges. To show the converse, we must be able to construct invariant measures λ on $X = \Pi X_\alpha$, provided that the infinite series of the theorem converges. This section is then devoted to a short digression on a method of construction of measures on product spaces which will have applications beyond the immediate scope of this paper.

Suppose that S_α is a standard Borel space for each $\alpha \in A$, a countable index set, and let K_α be a Borel subset of S_α. We let S be the full Cartesian product of the S_α with the product Borel structure, and we let $E = \{(s_\alpha): s_\alpha \in K_\alpha$ almost everywhere$\}$. We call E the restricted product of the spaces S_α relative to the subsets K_α.

LEMMA 4.1. *The restricted product E is a Borel subset of S, and hence a standard Borel space; moreover, $E = S_\alpha \times E_\alpha^*$ where E_α^* is the restricted product of the S_β, $\beta \neq \alpha$.*

PROOF. If F is a finite subset of A, let $S^F = \Pi_{\alpha \in F} S_\alpha \times \Pi_{\alpha \notin F} K_\alpha$. Then S^F is clearly a Borel subset of F. Moreover, $E = \bigcup_F S^F$ so that E is also a Borel subset of S. The second statement is obvious.

We note that if σ is a subset of E, then σ is a Borel subset of E if and only if

$\sigma \cap S^F$ is a Borel subset of S^F for every finite set F. Thus the Borel structure on E is the 'inductive limit' of the Borel structures on S^F. This method could also be used to define the Borel structure on E independently of its structure as a subset of S. Suppose further that λ_α is a family of measures on S_α such that $\lambda_\alpha(K_\alpha) = 1$. We want to define the restricted product measure $\lambda = \Pi\, \lambda_\alpha$ as a measure on E.

Indeed, let F be a finite set of indices, and let λ^F be the product of the measures λ_α $(\alpha \in F)$ and the measures λ_α restricted to K_α, $\alpha \notin F$ on $S^F = \Pi_{\alpha \in F} S_\alpha \times \Pi_{\alpha \notin F} K_\alpha$. This is well defined since $\lambda_\alpha(K_\alpha) = 1$, and is a ($\sigma$-finite) measure on S^F. Then if σ is a Borel subset of E, define $\lambda(\sigma) = \lim \lambda^F(\sigma \cap S^F)$. The limit exists as the sequence on the right increases as F increases.

LEMMA 4.2. *The restricted product measure λ is a (σ-finite) measure on E and $\lambda = \lambda_\alpha \times \lambda_\alpha^*$ (product measure), where $E = S_\alpha \times E_\alpha^*$ and λ_α^* is the restricted product of the λ_β, $\beta \neq \alpha$ on E_α^*.*

PROOF. The assertions are completely obvious.

Since E is a subset of S, we may always regard any measure λ on E as a measure on S by $\lambda(\sigma) = \lambda(\sigma \cap E)$ for Borel subsets σ of S. With this convention the following is clear.

COROLLARY. *Let X_α $(= S_\alpha)$ be discrete, and let K_α be a finite subset of X_α consisting of m_α elements. Let λ_α be the measure on X_α giving mass $(m_\alpha)^{-1}$ to each point, so that $\lambda_\alpha(K_\alpha) = 1$. Then the restricted product measure λ on $X = \Pi\, X_\alpha$ is invariant (cf. lemma 2.2) and ergodic.*

PROOF. This follows from the definition of invariance and the second statement of the previous lemma. The ergodicity follows by an easy modification of lemma 2.1.

5. Completion of the proof of theorem 2

We now return to the second half of the proof of theorem 2. Thus we assume that the series

$$\sum_{\alpha,i} p_\alpha^i |(p_\alpha^1/p_\alpha^i) - 1|_c^2 \tag{16}$$

converges for all $c > 0$, and that $p_\alpha^1 \geq a > 0$ for some a. It is convenient to extend the definition of p_α^i for $i > n_\alpha$ (= number of points in X_α) by making it zero for $i > n_\alpha$.

We first notice a trivial remark. As before, if F is a finite set of indices, let $X_F = \Pi_{\alpha \in F} X_\alpha$ and $X_F^* = \Pi_{\alpha \notin F} X_\alpha$, and similarly for μ_F and μ_F^*.

LEMMA 5.1. *The class $C(\mu)$ on X is type* III *if and only if μ_F^* on X_F^* is type* III.

PROOF. If $\nu \in C(\mu)$ is invariant, it follows by repeated application of lemma 2.2 that $\nu = \nu_F \times \nu_F^*$, where ν_F gives each point of X_F mass one and $\nu_F^* \in C(\mu_F^*)$ is invariant. Conversely, if $\nu_F^* \in C(\mu_F^*)$ is invariant, then ν defined by the above formula is in $C(\mu)$, and is invariant. This completes the proof.

The meaning of this is that we are free to delete any finite set of indices from A

as far as the question of type III is concerned. Since the convergence or divergence of the series in theorem 2 clearly does not depend on any finite set F of indices from A, we may ignore finite sets of indices when proving theorem 2.

Now if $b > 0$ is fixed and k is a positive integer, define a subset $L(b, k)$ of A by $L(b, k) = \{\alpha\colon p_\alpha^k \leq b\}$. Since $p_\alpha^{k+1} \leq p_\alpha^k$, it is clear that $L(b, k) \subset L(b, k + 1)$. If we take $b = a/2$ where $p_\alpha^1 \geq a$, then

$$\sum{}' p_\alpha^i |(p_\alpha^1/p_\alpha^i) - 1|_1^2 = \sum{}' p_\alpha^i < \infty, \tag{17}$$

where the primed summation is over all $\alpha \in L(b, k)$ and all $i \geq k$. On the other hand, if $\alpha \notin L(b, k)$, and $i \leq k$, then $p_\alpha^i \geq b > 0$, and so

$$\sum |(p_\alpha^1/p_\alpha^i) - 1|_c^2, \qquad (\alpha \notin L(b, k), i \leq k) \tag{18}$$

converges for any $c > 0$. This of course immediately implies that

$$\sum |p_\alpha^1/p_\alpha^i - 1|^2, \qquad (\alpha \notin L(b, k), i \leq k) \tag{19}$$

converges since p_α^1/p_α^i is bounded on this range of indices. Now suppose $1/n < a$. Then the convergence of (19) implies that except for a finite number of $\alpha \notin L(b, k)$, we have $p_\alpha^i \geq 1/n$ for all $i \leq k$. Therefore, if $A - L(b, k)$ is infinite, there exists an α such that $p_\alpha^i \geq 1/n$ for all $i \leq k$. It follows then that $1 \geq \sum_{i \leq k} p_\alpha^i \geq k/n$. In other words, if $k > n$, then $A - L(b, k)$ must be a finite set. By our previous remarks, we are free to assume that $A = L(b, n + 1)$.

We now define a partition of A into n disjoint sets $A(k)$, $1 \leq k \leq n$ by $A(k) = L(b, k + 1) - L(b, k)$ so that $\alpha \in A(k)$ if and only if $p_\alpha^k > b$, and $p_\alpha^{k+1} \leq b$. It is clear that if $\alpha \in A(k)$, then X_α has at least k points. We deduce then from (17) and (19) the convergence of

$$\sum p_\alpha^i, \qquad (\alpha \in A(k), i > k) \tag{20}$$

and of

$$\sum |(p_\alpha^1/p_\alpha^i) - 1|^2, \qquad (\alpha \in A(k), i \leq k). \tag{21}$$

Now let K_α be the subset of X_α consisting of the first k points when $\alpha \in S(k)$. We let ν_α be the measure on X_α giving mass $1/k$ to each point when $\alpha \in S(k)$, and we let ν be the restricted product of the ν_α using the sets K_α. Then, according to the corollary to lemma 4.2, ν is an invariant measure on X. The following completes the proof of theorem 2.

LEMMA 5.2. *Let μ and ν be as above, then $\nu \in C(\mu)$.*

PROOF. Since ν and μ are both ergodic under some group G acting on X, they are either equivalent or mutually singular. Thus it suffices to show that μ and ν are not mutually singular. Let K be the full Cartesian product of the sets K_α. Then $\nu(K) = 1$ by construction. On the other hand, if $a_\alpha = \sum_{i > k(\alpha)} p_\alpha^i$ where $k(\alpha) = k$ if $\alpha \in S(k)$, then by (20), $\sum a_\alpha$ converges. But now $\mu(K) = \prod_\alpha \mu_\alpha(K_\alpha) = \prod_\alpha (1 - a_\alpha)$, and this infinite product is therefore positive. Now let λ_α be the measure on K_α which is $(1 - a_\alpha)^{-1}$ times the restriction of μ_α to K_α. Then $\lambda_\alpha(K_\alpha) = 1$ and the infinite product of the λ_α, say λ, is a multiple of μ restricted to K. Thus to complete the proof, it is enough to show that

$\lambda \sim \nu$ as measures on K. Observe that λ_α gives mass $(1 - a_\alpha)^{-1}p_\alpha^i = q_\alpha^i$ to the i-th point of K_α, $i \leq k(\alpha)$. Then by (21),

$$\sum |(q_\alpha^1/q_\alpha^i) - 1|^2 \tag{22}$$

converges where the sum extends over all α and all $i \leq k(\alpha)$.

Now since q_α^i has a lower bound, b, over the range of summation, we see from the above that

$$\sum (q_\alpha^1 - q_\alpha^i)^2, \qquad (\alpha, i \leq k(\alpha)) \tag{23}$$

converges. Now using the fact that $\sum_{j=1}^{k(\alpha)} q_\alpha^j = 1$, and the Schwarz inequality, we deduce immediately that

$$k(\alpha)(q_\alpha^1 - k(\alpha)^{-1})^2 \leq \sum_{i=1}^{k(\alpha)} (q_\alpha^1 - q_\alpha^i)^2. \tag{24}$$

Now summing on α, and using (23) and the inequality $(s + t)^2 \leq 2(s^2 + t^2)$, we see that

$$\sum ((q_\alpha^1/q_\alpha^i) - (k(\alpha)q_\alpha^i)^{-1})^2, \qquad (\alpha, i \leq k(\alpha)) \tag{25}$$

is dominated term by term by a convergent series, and so also converges. Now combining (22) and (25) and the same numerical inequality, we see that

$$\sum ((q_\alpha^i k(\alpha))^{-1} - 1)^2, \qquad (\alpha, i \leq k(\alpha)) \tag{26}$$

is dominated term by term by twice the sum of (22) and (25), and thus converges. Finally we observe that $(q_\alpha^i k(\alpha))$ is bounded above and below so that

$$((q_\alpha^i k(\alpha))^{-1} - 1)^2 \sim ((q_\alpha^i k(\alpha))^{1/2} - 1)^2; \tag{27}$$

thus,

$$\sum_{\alpha, j \leq k(\alpha)} ((q_\alpha^i k(\alpha))^{1/2} - 1)^2 \tag{28}$$

is convergent. Now by Kakutani's criterion [5] for equivalence of infinite product measures used in section 2, we find that $\lambda \sim \nu$, and this completes the proof of the lemma and hence of theorem 2.

6. Complements

We would like to add a few words about the meaning of the series $\sum p_\alpha^i(p_\alpha^1/p_\alpha^i - 1)^2$ of theorem 2. For this to converge, either p_α^i must be small, or the second term must be small. In other words, the convergence of this series means that asymptotically the measure μ_α gives approximately equal mass to some set of points in X_α (those i such that the second factor is small) and very small mass to the rest. Note that both factors cannot both be simultaneously small; this is the meaning of the hypothesis $p_\alpha^1 \geq a > 0$. We can interpret the condition for $C(\mu)$ to be type II_1 as simply that, all the points of X_α get approximately the same mass, and the type I condition can be interpreted to say that the first point in X_α carries nearly all the mass with the other points having very small mass.

We do not know what happens when the condition $p_\alpha^1 \geq a > 0$ is dropped. However, one can show by the same arguments as used in the second part of the proof of theorem 2, that if there exist integers $k(\alpha)$ such that

$$\sum_{\alpha,i>k(\alpha)} p_\alpha^i \tag{29}$$

and

$$\sum_{\alpha,i\leq k(\alpha)} |(k(\alpha))^{-1/2} - (p_\alpha^i)^{1/2}|^2 \tag{30}$$

converge, then $C(\mu)$ is not type III.

Let us consider for a moment an example of theorem 2 which was our original motivation. Let A be the set of rational primes, and let X_q be infinite for each prime q with μ_q defined by $p_q^i = q^{1-i}(1 - q^{-1})$ for $i = 1, 2, \cdots$. Then since $p_q^1 \geq \frac{1}{2}$ for all q, theorem 2 applies. We observe that terms of the series in theorem 2 for $i = 2$ are asymptotic to q^{-1}, and since $\sum q^{-1}$ diverges, we are in the type III case. It is amusing to note that we do not need lemma 3.5 to do this. Indeed if we look at the series in lemma 3.4 instead, and take the terms with $i = 2$, we see that it is termwise asymptotic to

$$\sum q^{-1}(1 - \cos(2\pi\lambda \log q), \tag{31}$$

and if we were in the type I or II case, this would converge for all real λ. But it is well known that

$$\sum q^{-1} \cos(2\pi\lambda \log q) \tag{32}$$

is (conditionally) convergent for real $\lambda \neq 0$ (see [12], p. 56); this is equivalent to the fact that the Dirichlet series for $\log |\zeta(s)|$ converges for Re $(s) = 1$, $s \neq 1$, and hence that the zeta-function, $\zeta(s)$, has no zeroes on this line). Now by addition, we see that $\sum q^{-1}$ would converge which is of course nonsense.

We note that this example is not covered by Bures' criterion ([2], p. 170) for the type III case. This example will be used elsewhere to show that the adele group of the '$ax + b$' group [14] has as regular representation a type III factor representation.

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