Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 369–381

TWO DIMENSIONAL HAMILTONIAN WITH GENERALIZED SHAPE INVARIANCE SYMMETRY

H. PANAHI-TALEMI[*‡] and M. A. JAFARIZADEH[*†‡]

[*] *Department of Theoretical Physics and Astrophysics, Tabriz University 51664 Tabriz, Iran*

[†] *Institute for Studies in Theoretical Physics and Mathematics 19395-1795 Tehran, Iran*

[‡] *Research Institute for Fundamental Sciences, 51664 Tabriz, Iran*

Abstract. The two dimensional Hamiltonian with generalized shape invariance symmetry over S^2, has been obtained via Fourier transformation over the three coordinates of the $SU(3)$ Casimir operator defined on $SU(3)/SU(2)$ symmetric space. It is shown that the generalized shape invariance is equivalent to $SU(3)$ symmetry and that there is one to one correspondence between the representations of the generalized shape invariance and $SU(3)$ Verma modules. Also the two dimensional Hamiltonian in $\mathbb{R}^2$ space which posseses ordinary shape invariance symmetry with respect to two parameters, has been obtained via Inönü–Wigner contraction over $SU(3)$ manifold.

1. Introduction

Exactly solvable potentials are among the central and fundamental problems of mathematical physics, consequently they have attracted much interest both in theoretical physics and mathematics. They are also extensively applied in the investigation of many physical systems in quantum optics, condensed matter, nuclear physics, and solid state physics. There are many methods of obtaining exactly solvable potentials in quantum mechanics. The most powerful are the algebraic, supersymmetric and shape invariant factorization methods of Schrödinger equation [1–5]. One of the authors has shown the equivalence of these two methods in one [6], two and three dimensional [7–10] exactly solvable models. In all these works it is shown that there is a close connection between the shape invariance symmetry of one or higher dimensional Hamiltonians and some rank one semi-simple Lie algebra or higher rank non semi-simple algebra. The equivalence between the one dimensional shape invariant and rank one semi-simple Lie algebra has been shown in [11].

In this work we introduce two Hamiltonians, one with a new kind of shape invariance symmetry (we call it generalized shape invariance) and another one with an ordinary shape invariance symmetry with respect to two parameters. It is shown that the generalized shape invariance is equivalent to $SU(3)$ symmetry and that there is one to one correspondence between the representations of the generalized shape invariance symmetry and $SU(3)$ Verma modules.

The paper is organized as follows. In Section 2 after introducing the parameterization of $SU(3)$ Lie group we derive its left invariant vector fields and Casimir operator. In Section 3 using the Fourier transformation together with the coset reduction, we obtain two dimensional Hamiltonian $H(m_1, m_2)$ of charged particle on the sphere S^2 in the presence of an electric field. Section 4 is devoted to $SU(3)$ Verma modules [12] and their connection with eigen-spectrum and degeneracy of above Hamiltonian. In Section 5 we discuss the generalized shape invariance of this Hamiltonian. Finally in Section 6 using Inönü–Wigner contraction over the $SU(3)$ manifold, we obtain new 2-dimensional Hamiltonian in $\mathbb{R}^2$ space which posseses ordinary shape invariance symmetry with respect two parameters. The paper ends with a brief conclusion.

2. The Left and Right Invariant Vector Fields of $SU(3)$ Group Manifold

Using the parameterization proposed by Byrd [13], we can write an arbitrary element of $SU(3)$ group manifold in the following form:

$$U(\alpha,\beta,\gamma,\theta,a,\varphi,c,\eta) = \mathrm{e}^{\mathrm{i}\lambda_3\alpha}\,\mathrm{e}^{\mathrm{i}\lambda_2\beta}\,\mathrm{e}^{\mathrm{i}\lambda_3\gamma}\,\mathrm{e}^{\mathrm{i}\lambda_5\theta}\,\mathrm{e}^{\mathrm{i}\lambda_3 a}\,\mathrm{e}^{\mathrm{i}\lambda_2\varphi}\,\mathrm{e}^{\mathrm{i}\lambda_3 c}\,\mathrm{e}^{\mathrm{i}\lambda_8\eta} \tag{2.1}$$

where λ_i, $i = 1,2,\ldots,8$ are 3×3 Hermitian Gell-Mann matrices. Usually the left (right) invariant vector fields of $SU(3)$ group manifold can be obtained from left invariant (right invariant) $su(3)$ Lie algebra valued one-forms $U^{-1}\,\mathrm{d}U(\,\mathrm{d}UU^{-1})$. We consider $U^{-1}\,\mathrm{d}U = \omega^i\lambda_i = \mathrm{e}^i_\mu\,\mathrm{d}\xi^\mu\lambda_i$ where e^i_μ are left invariant fiel-beins and $\xi^\mu = (\alpha,\beta,\gamma,\theta,a,\varphi,c,\eta)$ are coordinates of $SU(3)$ group manifold and ω^i are left invariant one-forms. The left invariant vector fields are defined in terms of the inverses of fiel-beins, that is $L_i = e^\mu_i \dfrac{\partial}{\partial\xi^\mu}$, $i = 1,2,\ldots,8$. After some lengthy and tedious calculation via Maple software we get

$$
\begin{aligned}
L_\pm &= \frac{1}{2}\,\mathrm{e}^{\pm 2\mathrm{i}c}\left(-\frac{\mathrm{i}}{\sin 2\varphi}\frac{\partial}{\partial a} \pm \frac{\partial}{\partial \varphi} + \mathrm{i}\cot 2\varphi\frac{\partial}{\partial c}\right)\,,\\
L_3 &= -\frac{\mathrm{i}}{2}\frac{\partial}{\partial c}\,, \qquad L_8 = -\frac{\mathrm{i}}{2}\frac{\partial}{\partial \eta}\,,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
Y_\pm = {} & \frac{1}{2}\,\mathrm{e}^{\mp\mathrm{i}(2\gamma-\sqrt{3}\eta-c+a)}\frac{\sin\varphi}{\sin\theta}\left(\frac{\mathrm{i}}{\sin 2\beta}\frac{\partial}{\partial \alpha} \pm \frac{\partial}{\partial \beta} - \mathrm{i}\cot 2\beta\frac{\partial}{\partial \gamma}\right)\\
& + \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\sqrt{3}\eta+c+a)}\bigg(-2\mathrm{i}\frac{\cos\varphi}{\sin 2\theta}\frac{\partial}{\partial \gamma} \pm \cos\varphi\frac{\partial}{\partial \theta}\\
& \qquad + \mathrm{i}\frac{1-\sin^2\theta\sin^2\varphi}{\cos\varphi\sin 2\theta}\frac{\partial}{\partial a}\bigg)\\
& + \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\sqrt{3}\eta+c+a)}\bigg(\mp\sin\varphi\cot\theta\frac{\partial}{\partial \varphi} + \mathrm{i}\frac{\sin\varphi}{\sin 2\varphi}\cot\theta\frac{\partial}{\partial c}\\
& \qquad - \mathrm{i}\frac{\sqrt{3}}{2}\tan\theta\cos\varphi\frac{\partial}{\partial \eta}\bigg)\,,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
X_\pm = {} & \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(2\gamma-\sqrt{3}\eta+c+a)}\frac{\cos\varphi}{\sin\theta}\left(-\frac{\mathrm{i}}{\sin 2\beta}\frac{\partial}{\partial \alpha} \pm \frac{\partial}{\partial \beta} + \mathrm{i}\cot 2\beta\frac{\partial}{\partial \gamma}\right)\\
& + \frac{1}{2}\,\mathrm{e}^{\mp\mathrm{i}(\sqrt{3}\eta-c+a)}\bigg(-2\mathrm{i}\frac{\sin\varphi}{\sin 2\theta}\frac{\partial}{\partial \gamma} \mp \sin\varphi\frac{\partial}{\partial \theta}\\
& \qquad + \mathrm{i}\frac{1-\sin^2\theta\cos^2\varphi}{\sin\varphi\sin 2\theta}\frac{\partial}{\partial a}\bigg)\\
& + \frac{1}{2}\,\mathrm{e}^{\mp\mathrm{i}(\sqrt{3}\eta-c+a)}\bigg(\mp\cos\varphi\cot\theta\frac{\partial}{\partial \varphi} - \mathrm{i}\frac{\cos\varphi}{\sin 2\varphi}\cot\theta\frac{\partial}{\partial c}\\
& \qquad - \mathrm{i}\frac{\sqrt{3}}{2}\tan\theta\sin\varphi\frac{\partial}{\partial \eta}\bigg)\,,
\end{aligned}
\tag{2.4}
$$

where $L_\pm = \frac{1}{2}(L_1 \mp \mathrm{i}L_2)$, $Y_\pm = \frac{1}{2}(L_4 \mp \mathrm{i}L_5)$ and $X_\pm = \frac{1}{2}(L_6 \pm \mathrm{i}L_7)$. Also the $su(3)$ quadratic Casimir operator is defined as:

$$
C = \frac{1}{2}(L_+L_- + L_-L_+ + Y_+Y_- + Y_-Y_+ + X_+X_- + X_-X_+) + L_3^2 + L_8^2\,.
$$

After some algebraic calculation one can show that, the above generators satisfy $su(3)$ Lie algebra commutation relations. Similarly we can calculate the $SU(3)$ right invariant vector fields, where its structure constant is minus the left invariant one, but its quadratic Casimir operator has the same form as the left ones.

3. Reduction of the Casimir Operator to 2-Dimensional Hamiltonian of a Charged Particle on S^2 Sphere

In order to reduce the $SU(3)$ Casimir operator together with its left invariant vector fields defined on $SU(3)$ group manifold to the Casimir operator and the generators to be defined on the coset manifold $SU(3)/SU(2)$, it is sufficient to eliminate their α, β, γ dependence simply by considering the representations which are independent of these coordinates. We see that the inverse of left invariant fiel-beins are function of the coordinates (a, η) through the combination $\sqrt{3}\eta + a$, hence by making the change of the variables from (a, η) to (x, y) defined as $x = \sqrt{3}\eta + a$, $y = \sqrt{3}\eta - a$, the inverse of left invariant fiel-beins will be independent of y coordinate. Therefore, we can eliminate the coordinate y simply by considering the y-independent representations. Also it is convenient to do the similarity transformation $f^{-1}(\theta, \varphi) L_j f(\theta, \varphi)$, $j = 1, \ldots, 8$ over the generators with the similarity function $f^{-1}(\theta, \varphi) = \sin\theta\sqrt{\cos\theta \sin 2\varphi}$.

Finally after the elimination of the coordinates α, β, γ and y via the above explained prescription and doing the similarity transformation, the generators (2.2), (2.3) and (2.4) take the following form:

$$
\begin{gathered}
L_{\pm} = \frac{1}{2}\, \mathrm{e}^{\pm 2\mathrm{i}c} \left(-\frac{\mathrm{i}}{\sin 2\varphi} \frac{\partial}{\partial x} \pm \frac{\partial}{\partial \varphi} + \mathrm{i} \cot 2\varphi \frac{\partial}{\partial c} \mp \cot 2\varphi \right) \\
L_3 = -\frac{\mathrm{i}}{2} \frac{\partial}{\partial c}\,, \qquad L_8 = -\mathrm{i} \frac{\sqrt{3}}{2} \frac{\partial}{\partial x}\,,
\end{gathered}
\tag{3.1}
$$

$$
\begin{aligned}
Y_{\pm} = \frac{1}{2}\, \mathrm{e}^{\pm \mathrm{i}(c+x)} \Bigg(& \pm \cos\varphi \frac{\partial}{\partial \theta} + \mathrm{i} \frac{\cos^2\theta - 2\sin^2\theta \cos^2\varphi}{\cos\varphi \sin 2\theta} \frac{\partial}{\partial x} \\
& \mp \sin\varphi \cot\theta \frac{\partial}{\partial \varphi} \Bigg) \\
+ \frac{1}{2}\, \mathrm{e}^{\pm \mathrm{i}(c+x)} & \left(\mathrm{i} \frac{\sin\varphi}{\sin 2\varphi} \cot\theta \frac{\partial}{\partial c} \pm \frac{\cos^2\varphi \sin^2\theta - \cos^2\theta}{\sin 2\theta \cos\varphi} \right),
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
X_{\pm} = \frac{1}{2}\, \mathrm{e}^{\pm \mathrm{i}(c-x)} \Bigg(& \mp \sin\varphi \frac{\partial}{\partial \theta} + \mathrm{i} \frac{\cos^2\theta - 2\sin^2\theta \sin^2\varphi}{\sin\varphi \sin 2\theta} \frac{\partial}{\partial x} \\
& \mp \cos\varphi \cot\theta \frac{\partial}{\partial \varphi} \Bigg) \\
- \frac{1}{2}\, \mathrm{e}^{\pm \mathrm{i}(c-x)} & \left(\mathrm{i} \frac{\cos\varphi}{\sin 2\varphi} \cot\theta \frac{\partial}{\partial c} \pm \frac{\sin^2\varphi \sin^2\theta \cos^2\theta}{\sin 2\theta \sin\varphi} \right),
\end{aligned}
\tag{3.3}
$$

and the Casimir operator reduces to:

$$\begin{aligned} C = & -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - \cot^2(\theta) - \frac{1}{4}\tan^2\theta \\ & - \frac{\cot^2(2\varphi)}{\sin^2\theta} - \frac{1}{\sin^2\theta\sin^2 2\varphi}\left(\frac{\partial^2}{\partial c^2} - 2\cos 2\varphi\frac{\partial^2}{\partial c\partial x}\right. \\ & \left. + \frac{\cos^2\theta + \sin^2\theta\sin^2 2\varphi}{\cos^2\theta}\frac{\partial^2}{\partial x^2}\right). \end{aligned} \tag{3.4}$$

Now we can eliminate another two coordinates through Fourier transformation over the coordinates c and x, with the usual Fourier transformation of the wave function:

$$\psi^t_{m_1,m_2}(\theta,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\psi(c,x,\theta,\varphi)\,\mathrm{e}^{\mathrm{i}(m_1+m_2)c}\,\mathrm{e}^{\mathrm{i}(m_1-m_2)x}\,\mathrm{d}c\,\mathrm{d}x\,,$$

and the Casimir operator C reduces to:

$$\begin{aligned} H(m_1,m_2) = & -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - \cot^2(\theta) - \frac{1}{4}\tan^2\theta \\ & - \frac{\cot^2(2\varphi)}{\sin^2\theta} + \frac{1}{\sin^2\theta\sin^2 2\varphi}\Big(2(m_1^2+m_2^2) \\ & - 2(m_1^2-m_2^2)\cos 2\varphi + (m_1-m_2)^2\tan^2\theta\sin^2 2\varphi\Big)\,. \end{aligned} \tag{3.5}$$

This Casimir operator can be interpreted as the Hamiltonian of a charged particle on the sphere S^2 in the presence of an electric field with scalar potential V

$$\begin{aligned} V = & \frac{1}{\sin^2\theta\sin^2 2\varphi}\Big(2(m_1^2+m_2^2) - 2(m_1^2-m_2^2)\cos 2\varphi \\ & + (m_1-m_2)^2\tan^2\theta\sin^2 2\varphi\Big) - \cot^2(\theta) - \frac{1}{4}\tan^2\theta - \frac{\cot^2(2\varphi)}{\sin^2\theta}\,. \end{aligned}$$

4. The Algebraic Solution of the Hamiltonian via Verma Modulae

Here in this section we shall solve our Hamiltonian algebraically, that is, we shall obtain its eigen spectrum by using the Verma module of $su(3)$ or A_2 Lie algebra. According to [12], Verma bases of the irreducible representation space $V(\Lambda)$ corresponding to the highest weight $\Lambda = (p,q)$ of A_2 Lie algebra over C, consist of all vectors $f_1^{a_3} f_2^{a_2} f_1^{a_1}|p,q\rangle$, such that $0 \le a_1 \le p,\ 0 \le a_2 \le q + a_1,$

$0 \leq a_3 \leq \min[q, a_2]$, and e_i, f_i, h_i, $(i = 1, 2)$ are bases of A_2 Lie algebra satisfying the commutation relations:

$$[e_i, f_i] = h_i\,, \qquad [h_i, e_i] = 2e_i\,, \qquad [h_i, f_i] = -2f_i\,, \qquad i = 1, 2 \tag{4.1}$$

for each simple roots $\alpha_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\alpha_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$.
Now, by comparing the our commutation relations with (4.1) we have

$$h_1 = L_3 + \sqrt{3}L_8\,, \qquad h_2 = L_3 - \sqrt{3}L_8\,, \qquad e_1 = Y_+\,, \qquad e_2 = X_+\,,$$
$$f_1 = Y_-\,, \qquad f_2 = X_-\,.$$

In an arbitrary representation of A_2 with highest weight $\Lambda(p, q)$, the highest weight satisfies

$$e_i|p, q\rangle = 0\,, \qquad i = 1, 2\,, \tag{4.2}$$

on the other hand, highest weight is the eigenstate of Cartan subalgebra, either with its Gell-Mann basis [14, 15], L_3 and L_8 or h_{α_1} and h_{α_2} basis which are associated with simple roots α_1 and α_2 and they can be written in terms of these bases in the form $h_{\alpha_1} = L_3 + \sqrt{3}L_8$, $h_{\alpha_2} = L_3 - \sqrt{3}L_8$.
Writing the highest weight in terms of fundamental weights, that is $\mu = p\mu^1 + q\mu^2$ and considering the relations between the simple roots and fundamental weights

$$2\frac{(\mu^i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta^i_j\,, \qquad i, j = 1, 2$$

we can write

$$(L_3 + \sqrt{3}L_8)\psi^{p,q}(\theta, \varphi, c, x) = p\psi^{p,q}(\theta, \varphi, c, x)$$
$$(L_3 - \sqrt{3}L_8)\psi^{p,q}(\theta, \varphi, c, x) = q\psi^{p,q}(\theta, \varphi, c, x)$$

where $\psi^{p,q}(\theta, \varphi, c, x) = \langle x, c, \varphi, \theta|p, q\rangle$.
Integrating the above differential equations and using the integrability condition we obtain $p = q$, namely real representation of $SU(3)$ Lie group are relevant to the spectrum of our Hamiltonian. Therefore we have

$$\psi^{p,p}(\theta, \varphi, c, x) = \mathrm{e}^{2ipc}\sin^{2p+1}(\theta)\sin^p(2\varphi)\sqrt{\cos\theta\sin 2\varphi}\,. \tag{4.3}$$

Now we can obtain the lower eigen weights or Verma basis

$$\psi^{(a_3,a_2,a_1)}(\theta, \varphi, c, x) = Y_-^{a_3}X_-^{a_2}Y_-^{a_1}\psi^{p,p}(\theta, \varphi, c, x) \tag{4.4}$$

with

$$0 \leq a_1 \leq p\,, \qquad 0 \leq a_2 \leq p + a_1\,, \qquad 0 \leq a_3 \leq \min[p, a_2]\,. \tag{4.5}$$

In order to obtain the spectrum of the Hamiltonian (3.5), it is sufficient to eliminate x and c coordinate dependence of $SU(3)$ Verma basis by Fourier integrating over them. Also it is straightforward to see that an arbitrary Verma basis is proportional to $\mathrm{e}^{\mathrm{i}(m_1-m_2)x}\,\mathrm{e}^{\mathrm{i}(m_1+m_2)c}$ or we can write $\psi^p_{m_1,m_2}(\theta,\varphi,c,x) = \mathrm{e}^{\mathrm{i}(m_1+m_2)c}\,\mathrm{e}^{\mathrm{i}(m_1-m_2)x}\chi^p_{m_1,m_2}(\theta,\varphi)$. Therefore the general eigenfunctions of the Hamiltonian $H(m_1,m_2)$ with eigenvalue $p(p+2)$ can be written as:

$$\begin{aligned}\chi^p_{m_1,m_2}(\theta,\varphi) &= \prod_{i=1}^{a_3} Y_-(m_1+a_3-i+1,m_2)\prod_{i=1}^{p-m_2} X_-(m_1+a_3,p-i+1)\\ &\quad\times\prod_{i=1}^{p-m_1-a_3} Y_-(p-i+1,m_2)\\ &\quad\times\left(\sin^{2p+1}\theta\sin^p 2\varphi\sqrt{\cos\theta\sin 2\varphi}\right)\end{aligned}\tag{4.6}$$

where the operators appear in the products as: $\prod_{i=1}^m K(i) := K(m)K(m-1)\cdots K(1)$ and the Fourier transformed operators have the following form:

$$L_\pm(m_1,m_2) = \frac{1}{2}\left(\pm\frac{\partial}{\partial\varphi} + \frac{(m_1-m_2)-(m_1+m_2\pm 1)\cos 2\varphi}{\sin 2\varphi}\right)\tag{4.7}$$

$$\begin{aligned}Y_\pm(m_1,m_2) &= \frac{1}{2}\left(\pm\cos\varphi\frac{\partial}{\partial\theta}\mp\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\\ &\quad+\frac{(m_1-m_2\pm\frac{1}{2})\sin^2\theta\cos^2\varphi-(m_1\pm\frac{1}{2})\cos^2\theta}{\cos\varphi\sin 2\theta},\end{aligned}\tag{4.8}$$

$$\begin{aligned}X_\pm(m_1,m_2) &= \frac{1}{2}\left(\mp\sin\varphi\frac{\partial}{\partial\theta}\mp\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\\ &\quad+\frac{(m_1-m_2\mp\frac{1}{2})\sin^2\theta\sin^2\varphi+(m_2\pm\frac{1}{2})\cos^2\theta}{\sin\varphi\sin 2\theta},\end{aligned}\tag{4.9}$$

$$L_3(m_1,m_2) = \frac{m_1+m_2}{2}, \qquad L_8(m_1,m_2) = \frac{\sqrt{3}(m_1-m_2)}{2},\tag{4.10}$$

whith $m_1 = p-a_1-a_3$ and $m_2 = p-a_2$ and we have $-p\le m_1\le p$ and $-p\le m_2\le p$. In order to determine the degeneracy of the Hamiltonian $H(m_1,m_2)$ for given of integer-valued parameters m_1 and m_2, we should determine the range of variation of integer a_3 by imposing the inequalities (4.5). For $0\le m_2\le p$ we have the following three different regions for the parameter m_1:

I: $-m_1\le a_3\le p-m_2,\quad -p\le m_1<0,\quad m_2-m_1\le p,$
degeneracy $= p-m_2+m_1+1$

II: $0\le a_3\le p-m_2,\quad 0\le m_1\le m_2,$
degeneracy $= p-m_2+1$

III: $0 \leq a_3 \leq p - m_1, \quad m_2 < m_1 \leq p,$
degeneracy $= p - m_1 + 1$

while for $-p \leq m_2 < 0$ the corresponding regions for m_1 are

IV: $0 \leq a_3 \leq p - m_1 + m_2, \quad 0 \leq m_1, \quad m_1 - m_2 \leq p,$
degeneracy $= p - m_1 + m_2 + 1$

V: $-m_1 \leq a_3 \leq p - m_1 + m_2, \quad m_2 < m_1 < 0,$
degeneracy $= p + m_2 + 1$

VI: $-m_1 \leq a_3 \leq p, \quad -p \leq m_1 < m_2,$
degeneracy $= p + m_1 + 1.$

For the given integer-valued parameters p, m_1 and m_2 the spectrum of the Hamiltonian $H(m_1, m_2)$ exists in above six regions of (m_1, m_2) plane (see Fig. 1). Therefore for the given values of m_1 and m_2 the eigenspectrum can be simply obtained by consecutive application of the lowering operators over the highest weight according to the path shown in Fig. 1. Horizontal lines means application the lowering operator Y_- while the vertical line indicate application the lowering operator X_-.

5. Generalized Shape Invariance Symmetry

In this section we show that the Hamiltonian $H(m_1, m_2)$ posses a new kind of shape invariance symmetry, we call it generalized shape invariance. Obviously $SU(3)$ symmetry of the Casimir operator before the reduction generate this special shape invariance symmetry. Using this symmetry we will obtain below the spectrum of the Hamiltonian $H(m_1, m_2)$, that is the eigen functions $\chi^p_{m_1,m_2}(\theta, \varphi)$ corresponding to eigen value $p(p+2)$ by consecutive application of raising (lowering) operators over the ground (highest) state. We will also obtain its degeneracy for the given values of p , where it is the same as the one which can be obtained by using the inequalities (4.5) corresponding to Fig. 1.

First we write the Hamiltonian $H(m_1, m_2)$ operator in terms of the Fourier transformed operators given in (4.7), (4.8), (4.9) and (4.10),

$$\begin{aligned} H(m_1, m_2) = \frac{1}{2}\Big(&L_+(m_1 - 1, m_2 - 1)L_-(m_1, m_2) \\ &+ L_-(m_1 + 1, m_2 + 1)L_+(m_1, m_2)\Big) \\ + \frac{1}{2}\Big(&Y_+(m_1 - 1, m_2)Y_-(m_1, m_2) \\ &+ Y_-(m_1 + 1, m_2)Y_+(m_1, m_2)\Big) \end{aligned}$$

$$+\frac{1}{2}\Big(X_+(m_1,m_2-1)X_-(m_1,m_2)$$
$$+X_-(m_1,m_2+1)X_+(m_1,m_2)\Big)$$
$$+\frac{(m_1+m_2)^2}{4}+\frac{3(m_1-m_2)^2}{4}\,.$$

Using the commutation relations and through the left action of the operators $Y_\pm(m_1,m_2)$ and $X_\pm(m_1,m_2)$ on both sides of the following eigenvalue equations

$$H(m_1,m_2)\chi^p_{m_1,m_2}(\theta,\varphi)=p(p+2)\chi^p_{m_1,m_2}(\theta,\varphi),$$
$$\Big(L_3(m_1,m_2)+\sqrt{3}L_8(m_1,m_2)\Big)\chi^p_{m_1,m_2}(\theta,\varphi)=(2m_1-m_2)\chi^p_{m_1,m_2}(\theta,\varphi),$$
$$\Big(L_3(m_1,m_2)-\sqrt{3}L_8(m_1,m_2)\Big)\chi^p_{m_1,m_2}(\theta,\varphi)=(-m_1+2m_2)\chi^p_{m_1,m_2}(\theta,\varphi)\,,$$

we get

$$\chi^p_{m_1\pm1,m_2}(\theta,\varphi)=Y_\pm(m_1,m_2)\chi^p_{m_1,m_2}(\theta,\varphi), \tag{5.1}$$
$$\chi^p_{m_1,m_2\pm1}(\theta,\varphi)=X_\pm(m_1,m_2)\chi^p_{m_1,m_2}(\theta,\varphi)\,. \tag{5.2}$$

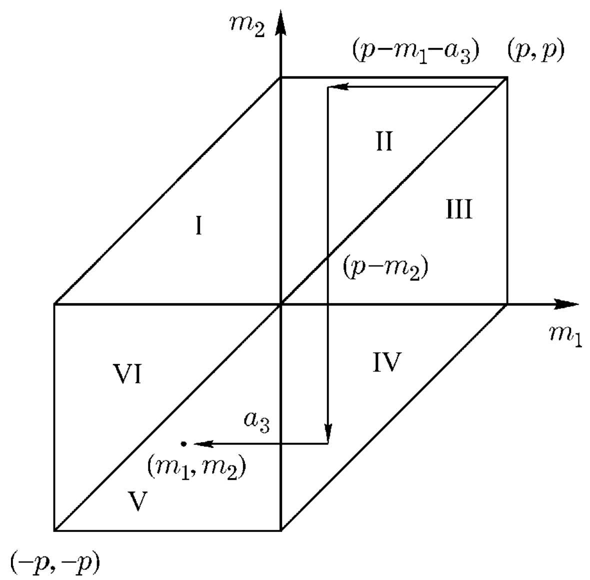

Figure 1

Therefore, the operators $Y_\pm(m_1,m_2)$ shift the parameter m_1 by one unit or they push the unrenormalized eigenfunctions horizontally in diagram Fig. 1, while the operators $X_\pm(m_1,m_2)$ shift the parameter m_2 by one unit or they push the eigenfunctions vertically in diagram. Obviously the eigenfunctions vanish in the forbidden regions of the diagram. Using the relations (5.1) and (5.2), we obtain the following relations

$$\begin{aligned} Y_+(m_1-1,m_2)Y_-(m_1,m_2)\chi^p_{m_1,m_2}(\theta,\varphi) &\simeq \chi^p_{m_1,m_2}(\theta,\varphi)\,, \\ Y_-(m_1,m_2)Y_+(m_1-1,m_2)\chi^p_{m_1-1,m_2}(\theta,\varphi) &\simeq \chi^p_{m_1-1,m_2}(\theta,\varphi)\,. \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} X_+(m_1,m_2-1)X_-(m_1,m_2)\chi^p_{m_1,m_2}(\theta,\varphi) &\simeq \chi^p_{m_1,m_2}(\theta,\varphi)\,, \\ X_-(m_1,m_2)X_+(m_1,m_2-1)\chi^p_{m_1,m_2-1}(\theta,\varphi) &\simeq \chi^p_{m_1,m_2-1}(\theta,\varphi)\,. \end{aligned} \tag{5.4}$$

which indicate that the Hamiltonian (3.5) posseses the shape invariance symmetry. Actually the first pair of equation given in (5.3) implies the horizontal shape invariance while the second pair of equations given in (5.4) implies the vertical shape invariance symmetry in diagram, respectively. Therefore, using this symmetry we can obtain the eigenfunctions of the isospectral Hamiltonians $H(m_1,m_2)$ with the eigenvalue $p(p+2)$ simply by applying the lowering operators X_- and Y_- over the highest weight $\chi^p_{p,p}(\theta,\varphi)$, namely we obtain all the eigenstates for the values of parameters m_1 and m_2 given in the allowed regions of the diagram such that the eigenfunctions vanish for the values of the parameters corresponding the forbidden regions. Also one can show that in this way we obtain exactly the same eigenspectrum that we have obtained in Section 4 by using the Verma basis.

6. Two Dimensional Hamiltonian with Ordinary Shape Invariance Symmetry

Here in this section we first make Inönü-Wigner contraction [16] over the generators of $su(3)$ Lie algebra given in (3.1), (3.2) and (3.3), simply by making the change of coordinate $\theta = \frac{r}{R}$ and relating the new contracted generators to the old ones by $L^c_\pm = L_\pm$, $L^c_3 = L_3$, $X^c_\pm = \frac{1}{R}X_\pm$, $Y^c_\pm = \frac{1}{R}Y_\pm$, $L^c_8 = L_8$. Then in the limit of $R \to \infty$ the set of $su(3)$ bases reduces to

$$\begin{aligned} Y^c_\pm &= \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(c+x)}\left(\pm\cos\varphi\frac{\partial}{\partial r} \mp \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi} + \frac{\mathrm{i}}{2r\cos\varphi}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial c} \pm \mathrm{i}\right)\right)\,, \\ X^c_\pm &= \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(c-x)}\left(\mp\sin\varphi\frac{\partial}{\partial r} \mp \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi} + \frac{\mathrm{i}}{2r\sin\varphi}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial c} \mp \mathrm{i}\right)\right)\,. \end{aligned}$$

The generators X^c_-, X^c_+, Y^c_- and Y^c_+ commute with each other and the quadratic Casimir operator (3.4) reduces to

$$
\begin{aligned}
C^c &= \frac{1}{2}(X_-^c X_+^c + X_+^c X_-^c + Y_-^c Y_+^c + Y_+^c Y_-^c) \\
&= -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \\
&\quad + \frac{1}{r^2 \sin^2 2\varphi}\left[\frac{\partial^2}{\partial c^2} - 2\cos 2\varphi \frac{\partial^2}{\partial c \partial x} + \frac{\partial^2}{\partial x^2} + 1\right].
\end{aligned}
$$

Now, by Fourier transformation over the coordinates c and x with the kernel $\exp\left(\mathrm{i}(m_1+m_2)c + \mathrm{i}(m_1-m_2)x\right)$, the above Casimir operator reduces to

$$
\begin{aligned}
H^c(m_1, m_2) &= -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \\
&\quad + \frac{1}{r^2 \sin^2 2\varphi}\left(2(m_1^2+m_2^2) - 2(m_1^2-m_2^2)\cos 2\varphi - 1\right).
\end{aligned}
$$

Also after Inönü–Wigner contraction together with the Fourier transformation, the Casimir eigenvalue equation $\frac{C}{R^2}\psi = \frac{p(p+2)}{R^2}\psi$ reduces to $H^c(m_1, m_2)\psi^{(k)}_{m_1,m_2} = k^2 \psi^{(k)}_{m_1,m_2}$ provided that, we let $p \to \infty$, $R \to \infty$ such that $\frac{p}{R} = \text{finite} = k$. Therefore we have a hierarchy of isospectral Hamiltonian labeled by the parameters m_1 and m_2 and one can show that this isospectral symmetry comes from the shape invariance symmetry of these Hamiltonians. To see this we first write the Hamiltonians $H^c(m_1, m_2)$ in terms of the lowering and rising operators

$$
\begin{aligned}
Y_\pm^c(m) &= \frac{1}{2}\left(\pm\cos\varphi \frac{\partial}{\partial r} \mp \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi} - \frac{1}{2r\cos\varphi}(2m-1)\right) \\
X_\pm^c(m) &= \frac{1}{2}\left(\mp\sin\varphi \frac{\partial}{\partial r} \mp \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi} + \frac{1}{2r\sin\varphi}(2m-1)\right)
\end{aligned}
$$

in the following form

$$
\begin{aligned}
H^c(m_1, m_2) &= X_+^c(m_2) X_-^c(m_2) + Y_+^c(m_1) Y_-^c(m_1) \\
H^c(m_1, m_2 - 1) &= X_-^c(m_2) X_+^c(m_2) + Y_+^c(m_1) Y_-^c(m_1) \\
H^c(m_1 - 1, m_2) &= X_+^c(m_2) X_-^c(m_2) + Y_-^c(m_1) Y_+^c(m_1).
\end{aligned}
$$

Now, multiplying the eigevalue equation $H^c(m_1, m_2)\psi^{(k)}_{m_1,m_2} = k^2 \psi^{(k)}_{m_1,m_2}$ from the left hand side by the operator $X_-^c(m_2)$ and using the fact that $X_-^c(m_2)$ commutes with $Y_\pm^c(m_1)$, we obtain

$$
H^c(m_1, m_2 - 1)\left(X_-^c(m_2)\psi^{(k)}_{m_1,m_2}\right) = k^2 (X_-^c(m_2)\psi^{(k)}_{m_1,m_2}),
$$

therefore $X_-^c(m_2)\psi_{m_1,m_2}^{(k)}$ is an eigenfunction of $H^c(m_1, m_2-1)$ with the same eigenvalue k^2, hence the operator $X_-^c(m_2)$ lower the index m_2 by one unit. Similarity one show that $X_+^c(m_2)$ raises m_2 by one unit and $Y_+^c(m_1)$ ($Y_-^c(m_1)$) raises (lowers) the parameter m_1 by one units, respectively. Therefore, the Hamiltonian $H^c(m_1, m_2)$ posseses ordinary shape invariance symmetry with respect to the two parameters m_1 and m_2.

For half-integer values of the parameters m_1 and m_2 we can obtain the continuous eigen-spectrum of these Hamiltonians simply by acting these lowering and rasing operators over the eigen-function of free particle as follows.

Since for $m_1 = m_2 = \frac{1}{2}$ the Hamiltonian $H^c(m_1, m_2)$ reduces to $H^c(\frac{1}{2}, \frac{1}{2}) = -\nabla^2$ with an eigenvalue $E = k^2$ and eigenfunction $\psi_{\frac{1}{2},\frac{1}{2}}^{(\vec{k})} = \mathrm{e}^{\mathrm{i}\vec{k}.\vec{x}}$ and for Hamiltonian $H(n_1 + \frac{1}{2}, n_2 + \frac{1}{2})$ with non-negative integers n_1 and n_2, we have $H(n_1 + \frac{1}{2}, n_2 + \frac{1}{2})\psi_{n_1,n_2}^{(\vec{k})} = k^2\psi_{n_1,n_2}^{(\vec{k})}$ with

$$\psi_{n_1,n_2}^{(\vec{k})} = \prod_{l=1}^{n_1} Y_+^c\left(l + \frac{1}{2}\right) \prod_{j=1}^{n_2} X_+^c\left(j + \frac{1}{2}\right) \mathrm{e}^{\mathrm{i}\vec{k}.\vec{x}} .$$

7. Conclusion

In this work we have generalized the ordinary exactly solvable shape invariance Hamiltonian to the Hamiltonian with a non-ableian type of shape invariance symmetry and also shape invariance with respect to two parameters. Again it is shown that the new kind of shape invariance symmetry has its origin in group theory or better to say, the exact solvability of the Hamiltonians are related in some way to Lie algebras or Lie groups.

References

[1] Cooper F., Khare A. and Sukhatme U., *Supersymmetry and Quantum Mechanics*, Phys. Rep. **251** (1995) 297–385.

[2] Fakhri H. and Jafarizadeh M., *Calculation of the Determinent of Shape Invariant Operators*, Phys. Lett. A **230** (1997) 157–163.

[3] Fakhri H. and Jafarizadeh M., *Supersymmetry and Shape Invariance in Differential Equations of Mathematical Physics*, Phys. Lett. A **230** (1997) 164–170.

[4] Fakhri H. and Jafarizadeh M., *Parasupersymmetry and Shape Invariance in Differential Equations of Mathematical Physics and Quantum Mechanics*, Ann. Phys. (N.Y) **262** (1998) 260–276.

[5] Esfandyari A., Jafarizadeh M. and Panahi-Talemi H., *Isospectral Deformation of Some Shape Invariant Potentials*, J. Math. Phys. **41** (2000) 675–700.

[6] Fakhri H. and Jafarizadeh M., *The Embedding of Parasupersymmetry and Dynamical Stmmetry into $GL(2,\mathbb{C})$ Group*, Ann. Phys. (N.Y) **266** (1998) 178–206.

[7] Fakhri H., Jafarizadeh M. and Moayedi S., *Solutions of (2 + 1)-dimensional Einstein-Maxwell Theory with Para-supersymmetry*, Phys. Rev. D **60** (1999) 084026-1-6.

[8] Jafarizadeh M. and Moayedi S., *Degeneracy and Para-supersymmetry of Dirac Hamiltonian in (2 + 1)-space-time*, J. Math. Phys. **40** (1999) 4274-4289.

[9] Fakhri H. and Jafarizadeh M., *Master Function Approuch to Quantum Solvable Models on* $SL(2,\mathbb{C})$ *and* $SL(2,\mathbb{C})/GL(1,\mathbb{C})$ *Manifold*, J. Math. Phys. **41** (2000) 504-546.

[10] Faizi E., Jafarizadeh M. and Panahi-Talemi H., *Two- and Three-dimentionals Hamiltonians with Shape Invariance Symmetry*, J. Math. Phys. **42** (2001) 4122-4137.

[11] Balantekin A., *Algebraic Approach to Shape Invariance*, Phys. Rev. A **57** (1998) 4188-4191.

[12] Li S., Moody R., Nicolescu M. and Patera J., J. Math. Phys. **27** (1986) 668-677.

[13] Byrd M., *The Geometry of* $SU(3)$, Physics/9708015.

[14] Georgi H., *Lie Algebras in Particle Physics*, The Benjamin/Cummings Publishing Company Inc., 1982.

[15] Bauerle G. and de Kerf E., *Finite and Infinite Dimensional Lie Algebras and Applications in Physics*, Part 1, Elsevier Science Publishers, 1990.

[16] Gilmore R., *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley, New York 1974.