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CLASSICAL MECHANICS ON GRASSMANNIAN AND DISC

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Abstract. In these notes, we will discuss from a purely geometric point of view classical mechanics on certain type of Grassmannians and discs. We will briefly discuss a superversion which in some sense combines these two models, and corresponds to the large-N_c limit of $SU(N_c)$ gauge theory with fermionic and bosonic matter fields, both in the fundamental representation, in $1+1$ dimensions [12]. This result is a natural extension of ideas in [16]. There it has been shown that the large-N_c phase space of $1+1$ dimensional QCD is given by an infinite dimensional Grassmannian. The complex scalar field version of this theory is worked out in [18] and it is shown that the phase space is an infinite dimensional disc.

1. Introduction

This is a slightly expanded version of the two talks delivered by T. Turgut at Varna Conference on "Geometry, Integrability and Quantization". Since most of the topics presented had a common theme, which is geometry, we present our notes from this point of view. Indeed the field theory model we will eventually discuss has a rich and interesting geometry and indeed this point of view is most natural. In some sense this is another manifestation of the merits of geometric thinking.

First, we start discussing classical mechanics from geometric terms, this is just to provide the setting for what is to come and establish a language. All of this is standard and we refer the reader to the available excellent sources for a thorough presentation. In the third section we present a self-contained discussion of the finite dimensional cases of interest to us, Grassmannian and disc, this will be a preparation for the infinite dimensions. The next section will extend these results to a certain kind of infinite dimensional generalization. It is not obvious that these systems have any relation to physics, one will see that several interesting ideas exist related to these spaces in the literature, our reason is their relation to large-N_c limits of gauge theories. This point of view is first discussed in [16] using the restricted Grassmannian and later on expanded to a certain kind of discs in [18]. We will present this very interesting physical connection in the last section when we talk about a super version of these spaces. It will unify the two problems worked out before, we will only provide an introduction, and really focus on its purely geometric aspects. The more interesting aspect of it from a physical point of view is the resulting dynamics. This has been worked out in [12] following Rajeev's method, for the details we refer to this short-coming paper.

2. A Brief Discussion of Mechanics

We will describe Hamiltonian dynamics in geometric terms, there are excellent references on this subject [1, 3, 7]. First we define the phase space of the theory as a smooth manifold Γ, and the classical observables will be C^∞-functions on Γ.

To introduce time evolution, we assume that the space has a closed, non-degenerate two form ω. These conditions mean that

$$\mathrm{d}\omega = \frac{\partial \omega_{jk}}{\partial x^i}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = 0\,, \qquad (\omega^{-1})^{jk} \text{ exist.} \tag{2.1}$$

We use this two-form to establish a mapping from the space of non-constant smooth functions to the smooth vector fields on Γ: given $f \in C^\infty(\Gamma)$,

$$-\,\mathrm{d}f = i_{V_f}\omega\,. \tag{2.2}$$

We note that the vector field is uniquely defined due to the invertibility of the two-form, in components;

$$V_f^i = -(\omega^{-1})^{ij}\,\frac{\partial f}{\partial x^j}\,. \tag{2.3}$$

A smooth vector field will have an integral curve, given an initial point there is a unique integral curve passing through this point. If we parametrize this integral curve with respect to "time", we call the corresponding generating function the Hamiltonian of our system and usually denote it by H: the Hamiltonian generates time translations. As we will see "time" may be another more convenient parameter for the system. We can define Poisson brackets of two functions f, g to be

$$\{f, g\} = \omega(V_f, V_g) = (\omega^{-1})^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \,. \tag{2.4}$$

Of course this description is most natural when we are looking at finite dimensional spaces where it is convenient to use explicit coordinates, sometimes it is possible to use coordinate free expressions which we can generalize to infinite dimensions. The above assignment of a vector field provides a natural (anti)homomorphism

$$V_{-\{f,g\}} = [V_f, V_g] \,, \tag{2.5}$$

where the square-brackets refer to the commutator of two vector fields. The Poisson bracket satisfies

$$\{f, g\} = -\{g, f\} \quad \text{and} \quad \{f, \{g, h\}\} + \text{cyclic} = 0 \,. \tag{2.6}$$

Furthermore it is a derivation, multiplication of two functions being compatible in a certain sense with the Poisson brackets, i. e.

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \,, \tag{2.7}$$

the resulting algebraic structure is called a Poisson algebra. In fact it is possible to start with the Poisson algebra properties and build a more general approach, but we will follow the above less general method.

The time evolution of our system means that all the physical observables change in time according to the equations of motion,

$$\frac{\partial f}{\partial t} = \{H, f\} \,. \tag{2.8}$$

Another important point is to introduce the concept of symmetry in classical mechanics. This is not so simple as the historical development shows and it has reached its modern form quite recently [1, 7]. We will always consider a Lie group G and its action on Γ,

$$G \times \Gamma \to \Gamma \,, \qquad (g, p) \mapsto pg \,, \tag{2.9}$$

such that this action is a diffeomorphism. We demand two compatibility conditions with the group structure: $(g_1, (g_2, p)) = (g_1 g_2, p)$ and $(1, p) = p$. At this point we are purely introducing geometric symmetry, there is no relation to the mechanics yet. If this action preserves the symplectic structure, that is if $\phi_g^* \omega = \omega$, we call it a symplectomorphism, and this provides a connection with time evolution. This is not quite enough as we will see. Since the Lie group has an infinitesimal structure, the Lie group action generates vector fields, and there is a homomorphism between the Lie algebra and the vector fields on the phase space which preserves the symplectic structure.

In general, not all vector fields which preserve the symplectic structure are generated by smooth functions, there is a topological obstruction in the first cohomology group. If the Lie group action is in fact generated by a set of smooth functions we will call this a "classical symmetry" (at the kinematic level) and the generating functions are called "moment functions". The correspondence between the Lie algebra elements and these functions is called a "moment map". This implies that the vector fields corresponding to the Lie algebra elements can be obtained from some functions, written explicitly,

$$i_{V_u}\omega = -\,\mathrm{d}f_u \tag{2.10}$$

where V_u is the vector field generated by the Lie algebra element u, and f_u is the moment function assigned to u. It is easy to see that this assignment is linear in u. The moment map provides a Poisson realization of the Lie algebra:

$$\{f_u, f_v\} = f_{[u,v]} + \Sigma(u,v) \tag{2.11}$$

where in general Σ is a non-zero antisymmetric function of the Lie algebra elements which is constant on the phase space. This is easy to see, if we look at the difference $\{f_u, f_v\} - f_{[u,v]}$, this is a constant on the phase space which of course depends on u and v. If we use the Jacobi identity we can see that it is a Lie algebra cocycle:

$$\Sigma(u,[v,w]) + \Sigma(v,[w,u]) + \Sigma(w,[u,v]) = 0\,. \tag{2.12}$$

If it is exact, that is, if it is the trivial element of the Lie algebra cohomology, then we can redefine the moment functions and remove this constant piece. Nevertheless we will see that when this element is non-trivial, the above equation will provide a Poisson realization of a central extension of the original symmetry group and it could be quite important. For compact Lie groups and for finite dimensional semisimple Lie groups this term is always trivial, the interesting cases appear typically for non-semisimple or infinite dimensions.

The true symmetry is a dynamical one, that is if the Hamiltonian is actually invariant under the group action, $H(p) = H(pg)$. In this case we have conserved quantities, that is if V_H has the flow F_t, we get $f_u(F_t(p)) = f_u(p)$, for all the moment maps and the true dynamics takes place on a reduced phase space. Since the system we are eventually interested in does not satisfy this condition we will not consider the reduction.

3. The Disc and the Grassmannian in Finite Dimensions

We are actually interested in the infinite dimensional case, but it is a good practice to look at various aspects of the finite dimensional case and simply generalize most of the properties.

We start with the Disc. It is defined to be the space of $m\times(M-m)$ complex matrices, Z, which satisfy the inequality $1_{(M-m)\times(M-m)} - Z^\dagger Z > 0$. This gives an open region in $\mathbb{C}^{m\times(M-m)}$, which is contractible. We see that $D_M(m)$ is a non-compact complex manifold with a single coordinate chart. We introduce a matrix ϵ given by

$$\epsilon = \begin{pmatrix} -1_{m\times m} & 0 \\ 0 & 1_{(M-m)\times(M-m)} \end{pmatrix}. \tag{3.1}$$

We define the pseudo-unitary group as follows:

$$U(m, M-m) = \{g;\, g \in GL(M,\mathbb{C})\,, \quad g\epsilon g^\dagger = \epsilon\} \tag{3.2}$$

where ϵ is the matrix defined above. Any element g of $U(m, M-m)$ has an action on $D_M(m)$ given by

$$Z \mapsto (aZ+b)(cZ+d)^{-1} \tag{3.3}$$

where

$$g = \begin{pmatrix} a_{m\times m} & b_{m\times(M-m)} \\ c_{(M-m)\times m} & d_{(M-m)\times(M-m)} \end{pmatrix} \tag{3.4}$$

is the decomposition of g into a block form. Let us write down the conditions on the elements explicitly,

$$aa^\dagger - bb^\dagger = 1\,, \qquad ca^\dagger = db^\dagger\,, \qquad dd^\dagger - cc^\dagger = 1\,. \tag{3.5}$$

Similar conditions exist for $g^\dagger \epsilon g = \epsilon$. The action of $U(m, M-m)$ is transitive on the set of matrices with $1_{(M-m)\times(M-m)} - Z^\dagger Z > 0$ and the stability subgroup of $Z = 0$ is given by $U(m)\times U(M-m)$. The last one is obvious from the group action, we show that the action is well-defined and it is transitive: first, we note that the inverse is well-defined at any point, from the group property we get $dd^\dagger = 1 + cc^\dagger$. Hence, d^{-1} exists, and $d^{-1}cZ + 1$ has a norm convergent expansion for the inverse due to $\|d^{-1}cZ\| < 1$. Using $(aZ+b)(cZ+d)^{-1}$ we compute $1 - [(aZ+b)(cZ+d)^{-1}]^\dagger (aZ+b)(cZ+d)^{-1}$, this gives us $(cZ+d)^{-1\dagger}(1 - Z^\dagger Z)(cZ+d)^{-1}$ using the group property. If $A > 0$ then $x^\dagger A x > 0$, this result says that the resulting element satisfies the required inequality. For transitivity, we write down a group element for any given point, such that one can reach this point from the point $Z = 0$. We simply solve the equation $Z = bd^{-1}$ and get $d = (1 - Z^\dagger Z)^{-1/2}U$. Note that the inverse square root is well-defined due to the conditions on Z. We can then solve for the other elements, and have

$$g(Z) = \begin{pmatrix} (1 - ZZ^\dagger)^{-1/2}V & Z(1 - Z^\dagger Z)^{-1/2}U \\ (1 - Z^\dagger Z)^{-1}Z^\dagger(1 - ZZ^\dagger)^{1/2}V & (1 - Z^\dagger Z)^{-1/2}U \end{pmatrix} . \tag{3.6}$$

where $V \in U(m)$ and $U \in U(M-m)$, they represent the ambiguity in choosing these elements. We also note that the inverse square root $(1 - ZZ^\dagger)^{-1/2}$ exists, using $(1 - ZZ^\dagger)^{-1} = 1 + Z(1 - Z^\dagger Z)^{-1}Z^\dagger$ and this element is positive and hence has a well-defined square root.

As a result, $D_M(m)$ is a homogeneous space, written as a quotient

$$D_M(m) = \frac{U(m, M-m)}{U(m)\times U(M-m)} . \tag{3.7}$$

We will provide a map from the Disc to a set of pseudo-Hermitian matrices Φ; if we define

$$\Phi(Z) = 1_{M\times M} - 2\begin{pmatrix} (1_{m\times m} - ZZ^\dagger)^{-1} & -(1_{m\times m} - ZZ^\dagger)^{-1}Z \\ Z^\dagger(1_{m\times m} - ZZ^\dagger)^{-1} & -Z^\dagger(1_{m\times m} - ZZ^\dagger)^{-1}Z \end{pmatrix} . \tag{3.8}$$

In many cases we will ignore the variable Z and write Φ only. The above set of matrices Φ satisfy the properties

$$\Phi^\dagger = \epsilon\Phi\epsilon \qquad \Phi^2 = 1 \qquad \operatorname{Tr}\Phi = M - 2m . \tag{3.9}$$

These are rather straightforward to check, so we skip the details. Under the action of $U(m, M-m)$ the Φ's transform as

$$\Phi \mapsto g\Phi g^{-1} \quad \text{for} \quad Z \mapsto g\circ Z \quad g \in U(m, M-m) . \tag{3.10}$$

This is not so obvious and requires a careful computation. First we note that when $Z \mapsto (aZ + b)(cZ + d)^{-1}$, we have $(1 - Z^\dagger Z)^{-1} \mapsto (cZ + d)(1 - Z^\dagger Z)^{-1}(cZ + d)^{-1}$. Next we rewrite $\Phi(Z)$:

$$\Phi = 1 - 2\begin{pmatrix} K^{-1} & -K^{-1}Z \\ Z^\dagger K^{-1} & -Z^\dagger K^{-1}Z \end{pmatrix} = -1 - 2\begin{pmatrix} Z^\dagger S^{-1}Z & -ZS^{-1} \\ S^{-1}Z^\dagger & -S^{-1} \end{pmatrix} , \tag{3.11}$$

where $K = (1 - ZZ^\dagger)$ and $S = (1 - Z^\dagger Z)$. Using the above observation we see that

$$\begin{aligned} \Phi(g\circ Z) = &-1 \\ &- 2\begin{pmatrix} (aZ+b)S^{-1}(aZ+b)^\dagger & -(aZ+b)S^{-1}(cZ+d)^\dagger \\ (cZ+d)S^{-1}(aZ+b)^\dagger & -(cZ+d)S^{-1}(cZ+d)^\dagger \end{pmatrix} . \end{aligned} \tag{3.12}$$

One can check that the above expression is equal to:

$$\Phi(g \circ Z) = -1 - 2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Z^\dagger S^{-1} Z & -ZS^{-1} \\ S^{-1} Z^\dagger & -S^{-1} \end{pmatrix} \begin{pmatrix} a^\dagger & -c^\dagger \\ -b^\dagger & d^\dagger \end{pmatrix}, \tag{3.13}$$

and this is precisely what we claimed. Next point to check is $\Phi(Z) = g(Z)\epsilon g(Z)^{-1} = g(Z)g(Z)^\dagger\epsilon$ for consistency but this is left to the reader as an exercise.

The reader may find these slightly off the point but we will see that the classical dynamics is most natural in this language. The other advantage is in showing the parallels with the Grassmannian.

Now we turn to the Grassmannian. To be self-contained we will collect together some basic facts about the Grassmannian [6, 10, 24]. For our purposes it is most convenient to define the Grassmannian as a set of Hermitian matrices satisfying a quadratic constraint

$$Gr_M(m) = \{\Phi;\, \Phi^\dagger = \Phi\,,\ \Phi^2 = 1\,,\ \mathrm{Tr}\,\Phi = M - 2m\} \tag{3.14}$$

where Φ is an $M \times M$ matrix. It has eigenvalues $+1$ and -1. Due to the trace condition, m of them are -1 and the remaining $M-m$ are $+1$. Every Hermitian matrix can be diagonalized by some unitary matrix $g \in U(N)$, therefore Φ can be written as $g\epsilon g^\dagger$ where ϵ is the same as before. The Grassmannian $Gr_M(m)$ is thus the orbit of ϵ under $U(M)$. One should note that we will obtain the same matrix Φ by using gh instead of g where h is a unitary matrix which commutes with ϵ. The set of such submatrices is the subgroup $U(m) \times U(M-m)$. Therefore, if we start from ϵ and act on it with $U(M)$, the stability subgroup of ϵ is $U(m) \times U(M-m)$. This defines the orbit of ϵ as a quotient of $U(N)$ with its closed subgroup $U(m) \times U(M-m)$

$$Gr_M(m) = \frac{U(M)}{U(m) \times U(M-m)}\,. \tag{3.15}$$

This also shows that $Gr_M(m)$ is a compact manifold. The action of unitary group on $Gr_M(m)$ will be given by $\Phi \mapsto g\Phi g^\dagger$. One can give a more geometric meaning to $Gr_M(m)$; to each $\Phi \in Gr_M(m)$ there is a subspace of $\mathbb{C}^M$ of dimension m; namely, the eigensubspace of Φ with eigenvalue -1. $Gr_M(m)$ can thus be viewed as the set of m dimensional subspaces of $\mathbb{C}^M$.

This geometric picture provides us with another description of $Gr_M(m)$ as a coset space. The subspace corresponding to ϵ consists of vectors $\binom{v}{0}$, where $v \in \mathbb{C}^m$. The stabilizer of this subspace under the action of $GL(M, \mathbb{C})$ is the

Borel subgroup

$$B_m = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; \begin{array}{l} a \in GL(m, \mathbb{C})\,,\; d \in GL(M-m, \mathbb{C}) \\ \text{and } b \in \mathrm{Hom}(\mathbb{C}^{M-m}, \mathbb{C}^m) \end{array} \right\} . \tag{3.16}$$

Moreover, any m-dimensional subspace can be brought to this form by an action of $GL(M, \mathbb{C})$. Thus we can think of the Grassmannian as a coset space of $GL(M, \mathbb{C})$ as well

$$Gr_M(m) = GL(M, \mathbb{C})/B_m\,. \tag{3.17}$$

This point of view shows that $Gr_M(m)$ is a complex manifold, since it is the quotient of a complex Lie group by a closed complex subgroup.

It is possible to give an explicit coordinate system for the Grassmannian in terms of $m \times (M-m)$ complex matrices Z, given by

$$\Phi(Z) = 1_{M\times M} - 2 \begin{pmatrix} (1_{m\times m} + ZZ^\dagger)^{-1} & (1_{m\times m} + ZZ^\dagger)^{-1} Z \\ Z^\dagger (1_{m\times m} + ZZ^\dagger)^{-1} & Z^\dagger (1_{m\times m} + ZZ^\dagger)^{-1} Z \end{pmatrix} . \tag{3.18}$$

This variable Z comes from the following idea, we write down the complex coordinates of a set of linearly independent vectors $v_1, v_2, \ldots, v_m$ which lies on a given m-dimensional hyperplane in M dimensions. This will be a matrix of the form

$$W = \begin{pmatrix} v_1^1 & v_1^2 & \ldots & v_1^M \\ v_2^1 & v_2^2 & \ldots & v_2^M \\ \vdots & \vdots & \ddots & \vdots \\ v_m^1 & v_m^2 & \ldots & v_m^M \end{pmatrix} . \tag{3.19}$$

Since these vectors are linearly independent the rank of this matrix is m, yet we could use another set of vectors which span the same hyperplane, $r_1, r_2, \ldots, r_m$. These two sets are related by an invertible matrix A, that is we can write the matrix corresponding to r's as AW. When we decompose W into its rank m piece and the rest and arrange it as $[W_1\; W_2]$, we get $[AW_1\; AW_2]$. Hence we can remove the ambiguity in the choice of these vectors by bringing this into a canonical form

$$\left(1_{m\times m} \;\middle|\; Z_{m\times(M-m)}\right) \tag{3.20}$$

where $Z_{m\times M-m} = W_1^{-1} W_2$. This is not a global coordinate system and we need $\binom{M}{m}$ different charts to cover $Gr_M(m)$. Because of that the use of coordinate systems is not efficient. In a given chart, $U(M)$ acts on Z by fractional linear transformations; $Z \mapsto (aZ+b)(cZ+d)^{-1}$ where $g \in U(M)$ is decomposed into the block form

$$g = \begin{pmatrix} a_{m\times m} & b_{m\times(M-m)} \\ c_{(M-m)\times m} & d_{(M-m)\times(M-m)} \end{pmatrix} . \tag{3.21}$$

This is closer to the conventional point of view but not as effective for our purposes, from now on we will think about Grassmannian and Disc in terms of Φ. We will also find it more convenient to think in terms of these when we look at the infinite dimensional case.

We will introduce classical dynamics on $Gr_M(m)$ and $D_M(m)$; there is a symplectic form on each one given by

$$\omega = \frac{\mathrm{i}}{4} \operatorname{Tr} \Phi \, \mathrm{d}\Phi \wedge \mathrm{d}\Phi . \tag{3.22}$$

This is a matrix form equation and it is easier to understand when written in terms of its action on the vector fields. It is invariant under $U(M)$ for the Grassmannian and invariant under $U(m, M-m)$ for the Disc:

$$\operatorname{Tr} g\Phi g^{-1} g \, \mathrm{d}\Phi g^{-1} \wedge g \, \mathrm{d}\Phi g^{-1} = \operatorname{Tr} g^{-1} g \Phi \, \mathrm{d}\Phi \wedge \mathrm{d}\Phi . \tag{3.23}$$

Since the spaces are homogeneous it is enough to check the non-degeneracy at the point ϵ. If we denote the components of a tangent vector at point Φ as $V(\Phi)$, it has to satisfy the equation $[V(\Phi), \Phi]_+ = 0$ which comes from the constraint $\Phi^2 = 1$. We also need to have unitarity and pseudo-unitarity conditions respectively. At ϵ this means that for the Grassmannian; $V(\epsilon) = \begin{pmatrix} 0 & v \\ v^\dagger & 0 \end{pmatrix}$ and for the Disc; $V(\epsilon) = \begin{pmatrix} 0 & v \\ -v^\dagger & 0 \end{pmatrix}$. Contracting with ω at ϵ we get

$$\omega(V_1(\epsilon), V_2(\epsilon)) = \frac{\mathrm{i}}{8} \operatorname{Tr} \epsilon [V_1(\epsilon), V_2(\epsilon)] = -\mathrm{i} \operatorname{Tr}(v_1 v_2^\dagger - v_2 v_1^\dagger) . \tag{3.24}$$

which is clearly non-degenerate. Incidentally, this demonstrates that ω is of type (1,1) with respect to the complex structure. Closedness of ω can be proved using

$$\begin{aligned} \mathrm{d}\omega &= \frac{\mathrm{i}}{4} \operatorname{Tr} \, \mathrm{d}\Phi \wedge \mathrm{d}\Phi \wedge \mathrm{d}\Phi = \frac{\mathrm{i}}{4} \operatorname{Tr} \, \mathrm{d}\Phi \wedge \mathrm{d}\Phi \wedge \mathrm{d}\Phi \Phi^2 \\ &= -\frac{\mathrm{i}}{4} \operatorname{Tr} \Phi \, \mathrm{d}\Phi \wedge \mathrm{d}\Phi \wedge \mathrm{d}\Phi \Phi = -\frac{\mathrm{i}}{4} \operatorname{Tr} \Phi^2 \, \mathrm{d}\Phi \wedge \mathrm{d}\Phi \wedge \mathrm{d}\Phi = -\, \mathrm{d}\omega \end{aligned} \tag{3.25}$$

where we have used $\Phi \, \mathrm{d}\Phi + \mathrm{d}\Phi \Phi = 0$ and the cyclicity of the trace.

Since both of these symplectic manifolds are homogeneous, it is possible to find a generating function for the respective group actions. The infinitesimal group action is given by

$$\delta\Phi = \mathrm{i}t[u,\Phi] \tag{3.26}$$

where t is an infinitesimal parameter, $u = u^\dagger$ for the unitary group and $u = \epsilon u^\dagger \epsilon$ for the pseudo-unitary group. The components of the vector field generating the group action is $V_u(\Phi) = \mathrm{i}[u,\Phi]$. If we insert this to the equation $-\,\mathrm{d}f_u = i_{V_u}\omega$, we get

$$-\,\mathrm{d}f_u = \frac{\mathrm{i}}{8}\,\mathrm{Tr}\,\Phi[\mathrm{i}[u,\Phi],\,\mathrm{d}\Phi] = -\frac{1}{8}\,\mathrm{Tr}[\Phi,[u,\Phi]]\,\mathrm{d}\Phi\,, \tag{3.27}$$

and using $\Phi^2 = 1$, $\mathrm{Tr}[\Phi,[u,\Phi]] = \mathrm{Tr}(\Phi u \Phi\,\mathrm{d}\Phi - u\,\mathrm{d}\Phi)$. Trace is cyclic and $\Phi\,\mathrm{d}\Phi = -\,\mathrm{d}\Phi\Phi$, one can see that $\mathrm{d}f_u = -\frac{1}{2}\,\mathrm{Tr}\,u\,\mathrm{d}\Phi$. An immediate solution for this is given by $f_u = -\frac{1}{2}\,\mathrm{Tr}\,u\Phi$. These are the moment maps for our system. We can calculate the Poisson brackets of the moment maps in both cases and we see that

$$\{f_u, f_v\} = f_{-\mathrm{i}[u,v]}\,. \tag{3.28}$$

As we discuss in our previous section on mechanics they provide a symplectic realization of the respective Lie algebras. We can express them by using the explicit coordinates. Define $\Phi = \Phi^i_j e^j_i$ for the Grassmannian and $\Phi = \tilde{\Phi}^i_j \lambda^j_i$ for the Disc, where e^i_j's are called the Weyl matrices, they have matrix elements $(e^i_j)^k_l = \delta^i_l \delta^k_j$ and $\lambda^i_j = \epsilon^i_k e^k_j$. Note that $\Phi^{*i}_j = \Phi^j_i$ and $\tilde{\Phi}^{*i}_j = \tilde{\Phi}^j_i$ as reality conditions in these basis. For the Grassmannian

$$\{\Phi^i_j, \Phi^k_l\} = -\mathrm{i}(\Phi^i_l \delta^k_j - \Phi^k_j \delta^i_l)\,. \tag{3.29}$$

and as for the Disc

$$\{\tilde{\Phi}^i_j, \tilde{\Phi}^k_l\} = -\mathrm{i}(\tilde{\Phi}^i_l \epsilon^k_j - \tilde{\Phi}^k_j \epsilon^i_l)\,. \tag{3.30}$$

In applications, the Hamiltonians of interest are of the form $E = \mathrm{Tr}(h\Phi + \hat{G}(\Phi)\Phi)$ where h is Hermitian and pseudo-Hermitian respectively. $\hat{G}(e^i_j)^k_l = G^{ik}_{jl}$ will represent the interaction and chosen such that $G^{ij}_{kl} = G^{*kl}_{ij}$ for the Grassmannian and for the Disc, $G^{ij}_{kl} = [(\epsilon\otimes\epsilon)G(\epsilon\otimes\epsilon)]^{*kl}_{ij}$.

For the sake of completeness, we will give the solution to the equations of motion when there is no interaction

$$\frac{\mathrm{d}\Phi^i_j}{\mathrm{d}t} = \mathrm{i}[h,\Phi]^i_j \quad \to \quad \Phi(t) = \mathrm{e}^{\mathrm{i}ht}\Phi(0)\,\mathrm{e}^{-\mathrm{i}ht} \tag{3.31}$$

for the Grassmannian and as for the Disc

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\Phi}\epsilon)^i_j = \mathrm{i}[h, (\tilde{\Phi}\epsilon)]^i_j \quad \to \quad \tilde{\Phi}(t) = \mathrm{e}^{\mathrm{i}ht}\tilde{\Phi}(0)\,\mathrm{e}^{-\mathrm{i}h^\dagger t}\,. \tag{3.32}$$

where we think of $\tilde{\Phi}$ as a Hermitian matrix. Note that h is pseudo-Hermitian and the exponential on the right is in fact the conjugate of the one on the left. This preserves the hermiticity on $\tilde{\Phi}^i_j$ and also the constraint.

At this point we would like to make a digression. The homogeneous symplectic manifolds $Gr_M(m)$ and $D_M(m)$ arise naturally in the theory of group representations and coadjoint orbits. Let us define the coadjoint orbits: Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The vector space dual to $\mathfrak{g}$, is denoted by $\mathfrak{g}^*$. There is an action of G on $\mathfrak{g}$ by conjugation, called the **adjoint action**: $g \in G$ and $u \in \mathfrak{g}$, then $u \to gug^{-1}$. We denote this by Ad_g. One can define an action of G on $\mathfrak{g}^*$ by using the adjoint action:

$$(\mathrm{Ad}^*_g\,\xi, u) \equiv -(\xi, \mathrm{Ad}_g\,u) \tag{3.33}$$

where ξ is in the dual space, and $(.,.)$ denotes the natural pairing. Given any point in the dual space, there is an orbit corresponding to it under the coadjoint action. The remarkable fact is that, these spaces have a symplectic form on them, and the resulting orbits are homogeneous symplectic manifolds of the group G. The infinitesimal actions will lead to tangent vectors on the orbit; and we can think of the vectors corresponding to the Lie algebra elements u, v as ad^*_u and ad^*_v respectively. We define the symplectic form at the point ξ to be:

$$\omega_\xi(\mathrm{ad}^*_u, \mathrm{ad}^*_v) = -\xi([u, v])\,, \tag{3.34}$$

which is well-defined since ξ is in the dual. One can check that this form is closed, non-degenerate, and homogeneous [11]. For semi-simple Lie algebras, one can identify the dual of the Lie algebra with the Lie algebra itself as a vector space, using the Killing form. In this case coadjoint orbits are the same as the orbits of Lie algebra elements under the adjoint action. This, of course, is not true in general. We see that the Grassmannian and the Disc are both coadjoint orbits of the matrix ϵ, under the unitary group $U(M)$ and the pseudo-unitary group $U(m, M-m)$ respectively. The symplectic forms we defined agree with the symplectic form defined on a coadjoint orbit by (3.34) up to a numerical factor.

4. Infinite Dimensional Disc and Grassmannian

We will describe certain infinite dimensional versions of the two homogenous symplectic manifolds we discussed in the previous section: the Disc and the

Grassmannian. Our approach is much influenced by the discussion of the Grassmannian in the book by Pressley and Segal [15].

Let $\mathcal{H}$ be a separable complex Hilbert space; $\mathcal{H}_-$ and $\mathcal{H}_+$ are two orthogonal subspaces with $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. Define the Disc $D_1(\mathcal{H}_-, \mathcal{H}_+)$ to be the set of all operators $Z\colon \mathcal{H}_+ \to \mathcal{H}_-$ such that $1 - Z^\dagger Z > 0$ and Z is Hilbert–Schmidt: $\operatorname{Tr} Z^\dagger Z < \infty$. Note that this is the new feature we have, in finite dimensions this will be automatically true.

Just as in the finite dimensional case, the pseudo-unitary group is defined to be a subset of the invertible operators from $\mathcal{H}$ to $\mathcal{H}$:

$$U_1(\mathcal{H}_-, \mathcal{H}_+) = \left\{ g \,\middle|\, \begin{array}{l} g\epsilon g^\dagger = \epsilon\,,\ g^{-1} \text{ exists,} \\ \text{and } [\epsilon, g] \text{ is Hilbert–Schmidt} \end{array} \right\}. \tag{4.1}$$

Again, the last condition is due to the infinite dimensionality. Here $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. If we decompose the matrix into block forms

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have, $a\colon \mathcal{H}_- \to \mathcal{H}_-$, $b\colon \mathcal{H}_+ \to \mathcal{H}_-$, $c\colon \mathcal{H}_- \to \mathcal{H}_+$ and $d\colon \mathcal{H}_+ \to \mathcal{H}_+$. Then, the off diagonal elements b and c are Hilbert–Schmidt and the diagonal elements a and d are bounded and invertible operators. The space of Hilbert–Schmidt operators form a two-sided ideal (which we will denote by $\mathcal{I}_2$) in the algebra of bounded operators.[1]

Thus the condition on the off-diagonal elements is preserved by multiplication and taking the inverse, to illustrate we write the multiplication

$$\begin{pmatrix} \mathcal{B} & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{B} \end{pmatrix} \begin{pmatrix} \mathcal{B} & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{B}\mathcal{B} + \mathcal{I}_2\mathcal{I}_2 & \mathcal{B}\mathcal{I}_2 + \mathcal{I}_2\mathcal{B} \\ \mathcal{I}_2\mathcal{B} + \mathcal{B}\mathcal{I}_2 & \mathcal{I}_2\mathcal{I}_2 + \mathcal{B}\mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{B} & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{B} \end{pmatrix}. \tag{4.2}$$

(1) Hilbert–Schmidt operators are a subset of compact operators, in fact for a separable Hilbert space compact operators is the only norm closed ideal. Compact operators on a separable Hilbert space can be approximated by finite rank operators in norm. This means that a selfadjoint compact operator has only point spectrum and the multiplicity of each eigenvalue is finite. We may define some subsets of the compact operators, simply by imposing certain summability conditions on the eigenvalues of the absolute value, i. e. on the sequence $\lambda_i((A^\dagger A)^{1/2})$. Hilbert–Schmidt corresponds to $\sum_i \lambda_i^2 < \infty$. We may think of this as a non-commutative version of l^2 sequence space. This space has its own norm coming from the above sum. In fact it is also an ideal inside the bounded operators closed with respect to the above norm. A simple argument will show that the product of two such operators lies in another ideal, the so called trace class operators. They are defined by the condition that the sum of the eigenvalues of the absolute value convergences. In such a class we can define a trace which corresponds to the extension of the ordinary notion of trace to this class. We recommend the book by Simon for all these issues [19].

We see that U_1 is a group. There is also a natural topology on this group which comes from the norm on the Hilbert–Schmidt condition: $|||g||| = \|[\epsilon, g]\|_2 + \|[\epsilon, g]_+\|$, where $\|.\|_2$ denotes $\mathrm{Tr}(A^\dagger A)$ and called the Hilbert–Schmidt norm. The geometric meaning of the condition on $[\epsilon, g]$ is that the linear transformation g does not mix the subspaces $\mathcal{H}_\pm$ by 'too much'.

We define an action of $U_1(\mathcal{H}_-, \mathcal{H}_+)$ on the Disc D_1:

$$Z \mapsto g \circ Z = (aZ + b)(cZ + d)^{-1}\,. \tag{4.3}$$

The condition $1 - Z^\dagger Z > 0$ implies that $cZ + d$ is invertible and bounded, just as in finite dimensions(same proof works), we need to see that the resulting operator is Hilbert–Schmidt, this is also clear because $b \in \mathcal{I}_2$ and the space of Hilbert–Schmidt operators is a two-sided ideal, $(aZ+b)(cZ+d)^{-1}$ is Hilbert–Schmidt. Thus our action is well-defined. We note that the stability subgroup of the point $Z = 0$ is $U(\mathcal{H}_-)\times U(\mathcal{H}_+)$, $U(\mathcal{H}_\pm)$ being the group of *all* unitary operators on $\mathcal{H}_\pm$. Moreover, any point Z is the image of 0 under the action of the group, $g\circ(Z = 0) = bd^{-1}$, just as in finite dimensions. Our formula in finite dimensions works in this case as well, we need to see that the Hilbert–Schmidt condition is satisfied:

$$g(Z) = \begin{pmatrix} (1 - ZZ^\dagger)^{-1/2}V & Z(1 - Z^\dagger Z)^{-1/2}U \\ (1 - Z^\dagger Z)^{-1}Z^\dagger(1 - ZZ^\dagger)^{1/2}V & (1 - Z^\dagger Z)^{-1/2}U \end{pmatrix}\,. \tag{4.4}$$

Here $Z(1-Z^\dagger Z)^{-1/2}U \in \mathcal{I}_2$ since $Z \in \mathcal{I}_2$ and the other terms are all bounded, same for the other off-diagonal term.

We therefore see that D_1 is a homogeneous space and given by the quotient

$$D_1 = \frac{U_1(\mathcal{H}_-, \mathcal{H}_+)}{U(\mathcal{H}_-)\times U(\mathcal{H}_+)}\,. \tag{4.5}$$

We know from finite dimensions that it is convenient to parametrize the Disc by using the operators $\Phi(Z)\colon \mathcal{H} \to \mathcal{H}$,

$$\Phi = 1 - 2\begin{pmatrix} (1 - ZZ^\dagger)^{-1} & -(1 - ZZ^\dagger)^{-1}Z \\ Z^\dagger(1 - ZZ^\dagger)^{-1} & -Z^\dagger(1 - ZZ^\dagger)^{-1}Z \end{pmatrix}\,. \tag{4.6}$$

One can see that under the transformation $Z \mapsto g \circ Z$, $\Phi \mapsto g^{-1}\Phi g$. Φ satisfies $\epsilon\Phi^\dagger\epsilon = \Phi$ and $\Phi^2 = 1$. Also, $\Phi - \epsilon \in \mathcal{I}_2$, so that as an operator Φ does not differ from ϵ by 'too much'. In [16, 18] one can see that many physical quantities are most naturally described as functions of the deviation of Φ from the standard value ϵ, $M = \Phi - \epsilon$. In fact ϵ corresponds to the vacuum state, so this vacuum subtraction is the geometric analogue of normal ordering in quantum field theory. For more details see [16, 18].

Given a complex Hilbert space $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ and orthogonal subspaces $\mathcal{H}_\pm$ as before, we can define another homogenous space, the Grassmannian. We define the Grassmannian to be the following set of operators on $\mathcal{H}$:

$$Gr_1 = \{\Phi;\, \Phi = \Phi^\dagger\,,\ \Phi^2 = 1\,,\ \Phi - \epsilon \in \mathcal{I}_2\}\,. \tag{4.7}$$

This is the same as the restricted Grassmannian of [15]. The reader will find an excellent discussion of several other points in this book.

To each point in the Grassmannian there corresponds a subspace of $\mathcal{H}$, the eigenspace of Φ with eigenvalue -1. In fact the Grassmannian is viewed usually as the set of subspaces of a Hilbert space. We could have taken the Hilbert space finite dimensional then we would have to impose the trace condition. The difference is that without the trace condition in finite dimensions one gets a disconnected union of all possible Grassmannians, so the above set really corresponds to a generalization of $Gr = \cup_{m=0}^{M} Gr_M(m)$.

Gr_1 is homogeneous space of a certain unitary group. In order to have a well-defined action on Gr_1, we must restrict to an appropriate sub-group of $U(\mathcal{H})$. We define

$$U_1(\mathcal{H}) = \{g;\, g^\dagger g = 1\,,\ [\epsilon, g] \in \mathcal{I}_2\}\,. \tag{4.8}$$

Let us split g into 2×2 blocks

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\,. \tag{4.9}$$

The convergence condition on $[\epsilon, g]$ is the statement that the off-diagonal blocks g_{12} and g_{21} are in $\mathcal{I}_2$. It then follows, that g_{11} and g_{22} are Fredholm operators. To see this, we recall that an operator is Fredholm if it is invertible modulo a compact operator. Any operator in $\mathcal{I}_2$ is compact and moreover, g is invertible, the inverse given by the Hermitian conjugate. When we write this condition explicitly we get $g_{11}g_{11}^\dagger + g_{12}g_{12}^\dagger = 1$, here the last term is compact hence g_{11} has an inverse upto a compact operator. The Fredholm index of g_{11} is opposite to that of g_{22}; this integer is a homotopy invariant of g and we can decompose $U_1(\mathcal{H})$ into connected components labeled by this integer.

With the projection $g \to g\epsilon g^\dagger$, we see that Gr_1 is a homogeneous space of $U_1(\mathcal{H})$:

$$Gr_1 = U_1(\mathcal{H})/U(\mathcal{H}_-)\times U(\mathcal{H}_+)\,. \tag{4.10}$$

For, any $\Phi \in Gr_1$ can be diagonalized by an element of $U_1(\mathcal{H})$, $\Phi = g\epsilon g^\dagger$; this g is ambiguous up to right multiplication by an element that commutes with ϵ.

Such elements form the subgroup

$$U(\mathcal{H}_-)\times U(\mathcal{H}_+) = \{h;\, h = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix}\,,\; h_{11}^\dagger h_{11} = h_{22}^\dagger h_{22} = 1\}\,. \qquad (4.11)$$

Each point $\Phi \in Gr_1$ corresponds to a subspace of $\mathcal{H}$: the eigenspace of Φ with eigenvalue -1. Thus Gr_1 consists of all subspaces obtained from $\mathcal{H}_-$ by an action of U_1. We remark that the tangent space to the Grassmannian at ϵ may be identified with the Hilbert space $\mathcal{I}_2(\mathcal{H}_-;\mathcal{H}_+)$.

Although for our discussions it is not essential Gr_1 is a coset space of complex Lie groups at the same time. This defines a complex structure on Gr_1 which is rather useful for geometric quantization [18, 15]. Define the restricted general linear group

$$GL_1 = \{\gamma;\, \gamma \text{ is invertible}\,,\; [\epsilon,\gamma] \in \mathcal{I}_2\}\,. \qquad (4.12)$$

Again if we were to decompose into 2×2 submatrices $\gamma_{12}, \gamma_{21} \in \mathcal{I}_2$ while γ_{11} and γ_{22} are Fredholm. Define the subgroup (Borel subgroup) of matrices which are upper triangular in this decomposition

$$B_1 = \left\{\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{22} \end{pmatrix}\,;\, \beta \in GL_1\right\}\,. \qquad (4.13)$$

This is the stability group of $\mathcal{H}_-$ under the action of GL_1 on $\mathcal{H}$. Thus the Grassmannian (which is the orbit of $\mathcal{H}_-$) is the complex coset space

$$Gr_1 = GL_1/B_1\,. \qquad (4.14)$$

The geometric idea behind this is clear, but it is important to know that the two quotient manifolds are the same in the topology we discussed. This requires one to know that the polar decomposition $\gamma = g|\gamma|$ is continuous in the given topology, this is shown in a note [22] for a more general case.

We have seen that the Grassmannian and Disc have a symplectic structure $\omega = {}^{\mathrm{i}}\!/_4 \operatorname{Tr}\Phi\, \mathrm{d}\Phi\, \mathrm{d}\Phi$. It is not clear that this symplectic form exists in the infinite dimensional case; the trace could diverge. But if we think of the same expression in terms of its contraction with tangent vectors, we see that

$$\omega(V_u, V_v) = -\frac{\mathrm{i}}{8}\operatorname{Tr}\Phi[[u,\Phi],[v,\Phi]] = -\frac{\mathrm{i}}{8}\operatorname{Tr}\epsilon[[\epsilon, g^{-1}ug],[\epsilon, g^{-1}vg]]\,. \quad (4.15)$$

This expression is well-defined, since $[\epsilon, g^{-1}ug] \in \mathcal{I}_2$ for any u in the Lie algebra. Indeed this is why we imposed the convergence conditions. It is possible to weaken the convergence condition (which is interesting for quantum field theories in dimensions greater than two [14]), without changing much of the structure but we will loose the symplectic form.

The above form is invariant under the action of $U_1(\mathcal{H})$ for the Grassmannian and invariant under the action of $U_1(\mathcal{H}_-, \mathcal{H}_+)$ for the Disc. Thus, Gr_1 and D_1 are both homogeneous symplectic manifolds just as in the finite dimensional case. Due to the homogeneity, it is enough to prove that ω is non–degenerate at one point, say $\Phi = \epsilon$, and this is done in the previous section.

We can look for the moment maps, which generate the infinitesimal action of $U_1(\mathcal{H}_-, \mathcal{H}_+)$ and $U_1(\mathcal{H})$ respectively. In the finite dimensional case, this is just the function $-\frac{1}{2}\operatorname{Tr} u\Phi$, where u is a Hermitian matrix for the Grassmannian and a pseudo-Hermitian ($u^\dagger = \epsilon u \epsilon$) matrix for the Disc. We cannot take $f_u = -\frac{1}{2}\operatorname{Tr} u\Phi$ in the infinite dimensional case, because the trace diverges. However, we do a vacuum subtraction from this expression and get instead $-\frac{1}{2}\operatorname{Tr}(\Phi - \epsilon)u$; we are not done yet! The trace now is only conditionally convergent, but we have a possibility to obtain the moment maps.

We may see this as, $u \in \begin{pmatrix} \mathcal{B} & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{B} \end{pmatrix}$, and $\Phi - \epsilon = g\epsilon g^{-1} - \epsilon$, rewrite this as $g[\epsilon, g^{-1}]$, but by definition $[\epsilon, g^{-1}] \in \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix}$. Here $\mathcal{B}$ is the space of bounded operators. Thus the diagonal blocks in $(\Phi - \epsilon)u$ are both trace-class. We now define the conditional trace $\operatorname{Tr}_\epsilon$ of an operator to be the sum of the traces of its diagonal submatrices: $\operatorname{Tr}_\epsilon X = \frac{1}{2}\operatorname{Tr}[X + \epsilon X \epsilon]$. Using this conditional trace we define

$$f_u = -\frac{1}{2}\operatorname{Tr}_\epsilon(\Phi - \epsilon)u\,. \tag{4.16}$$

If we restrict to finite rank matrices u, this function differs by a constant from the previous moment map; therefore it generates the same Hamiltonian vector fields:

$$\omega(V_{f_u}, .) = -\,\mathrm{d}f_u \mapsto V_{f_u} = \mathrm{i}[u, \Phi]\,. \tag{4.17}$$

However, there is an important change in the Poisson bracket relations; they will differ by a constant term from the previous ones:

$$\{f_u, f_v\} = f_{-\mathrm{i}[u,v]} - \frac{\mathrm{i}}{2}\operatorname{Tr}_\epsilon[\epsilon, u]v\,. \tag{4.18}$$

In the finite dimensional case we can remove the extra term by adding a constant term to f_u. However this is not possible in the infinite dimensional case, as the term we must add to f_u will diverge. This is, in fact, the Lie algebra of the non-trivial central extension of GL_1. Its explicit form can be given by

$$\Sigma(u, v) = \mathrm{i}\operatorname{Tr}(u_{12}v_{21} - u_{21}v_{12})\,. \tag{4.19}$$

The actual proof of the above relations is hard, it takes a long and careful computation so we will avoid doing this.[1] If we ignore the issues of convergence and simply write the finite dimensional answer by subtracting ϵ and rewrite it in such a way that the result makes sense the above form is found. We can also verify the cocycle condition $\Sigma([u,v],w)+\Sigma([v,w],u)+\Sigma([w,u],v)=0$ by explicit computation(left as an exercise).

We will conclude our discussion here and move to a next level which will eventually have an application to a physical system.

5. Super-Grassmannian

In this section we will briefly extend the above analysis to the super geometry context, and then in the next section discuss its physical importance in a concise manner. More details will appear in our paper [12], a more complete discussion of an analog of the Disc will appear in [23]. A good reference for super geometry is Berezin's book [4].

Let us define a super-operator Φ which goes from $\mathcal{H}^e|\mathcal{H}^o \to \mathcal{H}^e|\mathcal{H}^o$, where $\mathcal{H}^e$, $\mathcal{H}^o$ correspond to the even and odd spaces respectfully. We use a Z_2 formalism for the super space, and assume that each Hilbert space is separable. We write each Hilbert space as a direct sum of two isomorphic pieces, i. e. $\mathcal{H}^e=\mathcal{H}^e_-\oplus\mathcal{H}^e_+$ and same for the odd part. On each one, there is the same ϵ operator defined with respect to this orthogonal decomposition: $\epsilon=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$.

We impose the following conditions on the set of Φ's:

$$\Phi^2=1\,,\qquad E\Phi^\dagger E=\Phi \tag{5.1}$$

where $E=\begin{pmatrix}1&0\\0&\epsilon\end{pmatrix}$. Furthermore we demand a convergence condition on Φ

$$\Phi-\begin{pmatrix}\epsilon&0\\0&\epsilon\end{pmatrix}\in\mathcal{I}_2 \tag{5.2}$$

where $\mathcal{I}_2$ refers to the Hilbert–Schmidt operators on each space. We could have used $[\epsilon,\Phi]\in\mathcal{I}_2$ for the convergence condition, but we feel more comfortable with this one. One can see that this is a natural extension of our ideas to the

[1] There is actually an indirect method, we know that these are moment maps hence we know from general principles discussed in the first section that the result is true with a cocycle $\Sigma(u,v)$. It is easier to compute this cocycle at a special point, for example at $\Phi=\epsilon$. This gives us exactly the same answer.

super case. It is natural to think of this super-manifold as the restricted super-Grassmannian. This manifold can be thought of as a homogenous manifold as well.

We define the following restricted super-unitary group (of course the super-unitary group could be defined in many ways, this is one possibility):

$$U_1(\mathcal{H}^e\mathcal{H}^o_-, \mathcal{H}^o_+) = \left\{ g;\ gEg^\dagger = E\,,\ \left[\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, g \right] \in \mathcal{I}_2(\mathcal{H}_-, \mathcal{H}_+) \right\}. \tag{5.3}$$

It is not necessary to keep the indices on the Hilbert spaces since the bar already implies even and odd parts. We see that the conditions on Φ will be preserved if we define the group action:

$$\Phi \mapsto g\Phi g^{-1}\,. \tag{5.4}$$

If we look at the orbit of $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ under this super-unitary group, we see that it can be parametrized by Φ.

We conclude that the orbit of $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ is in fact a homogeneous super-symplectic manifold:

$$SGr_1 = \frac{U_1(\mathcal{H}^e|\mathcal{H}^o_-, \mathcal{H}^o_+)}{U(\mathcal{H}^e_-|\mathcal{H}^e_-)\times U(\mathcal{H}^e_+|\mathcal{H}^o_+)}\,. \tag{5.5}$$

The stability subgroup has a natural embedding into the full group.(1)

Notice that a tangent vector at any point on this super-Grassmannian is given by its effect on Φ, $V_u(\Phi) = \mathrm{i}[u, \Phi]_s$, where we use the super-Lie bracket. It is defined to be

$$\begin{aligned} \left[\begin{pmatrix} a_1 & \beta_1 \\ \gamma_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & \beta_2 \\ \gamma_2 & d_2 \end{pmatrix} \right]_s = {} & \left[\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} \right] + \left[\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix} \right] \\ & + \left[\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix} \right]_+ \end{aligned} \tag{5.6}$$

To see that the above homogeneous manifold is the phase space, we formally define a two-form

$$\Omega = \frac{\mathrm{i}}{4}\,\mathrm{Str}\,\Phi\,\mathrm{d}\Phi \wedge \mathrm{d}\Phi\,. \tag{5.7}$$

One can give the symplectic form explicitly via its action on vector fields, and this defines the above two form:

(1) The most obvious generalization of the Grassmannian would be $SGr_1^s = \frac{U_1(\mathcal{H}|\mathcal{H})}{U(\mathcal{H}_-|\mathcal{H}_-)\times U(\mathcal{H}_+|\mathcal{H}_+)}$, but it seems that this has no physical meaning.

$$i_{V_u} i_{V_v} \Omega = \frac{\mathrm{i}}{8} \operatorname{Str} \Phi[[u, \Phi]_s, [v, \Phi]_s]_s \,. \tag{5.8}$$

Using exactly the same methods in the previous section(but much more algebra and care required), we can show that it is closed and non-degenerate. In fact it is also a homogeneous two-form, that is invariant under the group action, as can be verified (this is the whole point of Φ: many things become simple verifications).

Therefore we may introduce a classical dynamical system defined on this super-Grassmannian with this symplectic form once we choose a Hamiltonian for our system. We will see in the last section that there is a natural Hamiltonian which is a quadratic even function on this space. This system corresponds to the large-N_c limit of $1+1$-dimensional gauge theory which is coupled to both bosonic and fermionic matter fields in the fundamental representation.

The group action is generated by moment maps $F_u = -\frac{1}{2} \operatorname{Str}_\epsilon u(\Phi - \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix})$, this is in a way an obvious extension of the previous cases. The conditional super-trace is defined by $\operatorname{Str}_\epsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{Tr}_\epsilon A - \operatorname{Tr}_\epsilon D$, and $\operatorname{Tr}_\epsilon A = \frac{1}{2} \operatorname{Tr}(A + \epsilon A \epsilon)$. Notice that the convergence conditions on Φ guarantees that the conditional trace exists. This can be seen most easily by using, $\Phi - \bar{\epsilon} = g \bar{\epsilon} g^{-1} - \bar{\epsilon} = -[\bar{\epsilon}, g] g^{-1}$. If we explicitly write the commutator

$$[\bar{\epsilon}, g] g^{-1} =$$

$$\begin{pmatrix} \begin{pmatrix} 0 & g_a^{12} \\ g_a^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & g_\beta^{12} \\ g_\beta^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & g_\gamma^{12} \\ g_\gamma^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & g_d^{12} \\ g_d^{21} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} (g_a^{-1})^{11} & (g_a^{-1})^{12} \\ (g_a^{-1})^{21} & (g_a^{-1})^{22} \end{pmatrix} & \begin{pmatrix} (g_\beta^{-1})^{11} & (g_\beta^{-1})^{12} \\ (g_\beta^{-1})^{21} & (g_\beta^{-1})^{22} \end{pmatrix} \\ \begin{pmatrix} (g_\gamma^{-1})^{11} & (g_\gamma^{-1})^{12} \\ (g_\gamma^{-1})^{21} & (g_\gamma^{-1})^{22} \end{pmatrix} & \begin{pmatrix} (g_d^{-1})^{11} & (g_d^{-1})^{12} \\ (g_d^{-1})^{21} & (g_d^{-1})^{22} \end{pmatrix} \end{pmatrix} . \tag{5.9}$$

This shows now that in this decomposition we get

$$\begin{pmatrix} \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix} & \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix} \\ \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix} & \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix} \end{pmatrix} . \tag{5.10}$$

If we multiply this with an element of the Lie algebra we see that the conditional traces exist.

The moment maps provide the following super-Poisson realization of the super-unitary group:

$$\{F_u, F_v\} = F_{-\mathrm{i}[u,v]_s} + \Sigma_s(u, v) \tag{5.11}$$

where,

$$\begin{aligned}\Sigma_s(u,v) &= -\frac{\mathrm{i}}{8}\operatorname{Str}\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}\Big[\Big[\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix},u\Big],\Big[\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix},v\Big]\Big]_s\\ &= -\frac{\mathrm{i}}{2}\Big(\operatorname{Tr}_\epsilon[\epsilon,a(u)]a(v) - \operatorname{Tr}_\epsilon([\epsilon,\beta(u)]\gamma(v)\\ &\qquad + [\epsilon,\beta(v)]\gamma(u)) - \operatorname{Tr}_\epsilon[\epsilon,d(u)]d(v)\Big)\\ &= -\frac{\mathrm{i}}{2}\operatorname{Str}_\epsilon\Big[\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix},u\Big]v\,.\end{aligned} \tag{5.12}$$

Here $[.,.]_s$ again denotes the super-Lie bracket and we use the decomposition of a super-matrix into block form according to the decomposition $\mathcal{H}^e|\mathcal{H}^o$ as $\begin{pmatrix}a & \beta\\ \gamma & d\end{pmatrix}$.

The proof of this relation is long and tedious, so we do not attempt to write it at all.[(1)]

We will leave the discussion of the geometry of the phase space at this point and return to its physical meaning.

6. Gauge Theory in Two Dimensions

In this section we will briefly present the physical reason for the above phase space and refer the reader to our paper to be published for more details (especially the study of the equations of motion and the consequences of them).

Let us write down the action functional of two dimensional QCD in the light-cone gauge that we specify below. In this reference frame the gauge field A_i that can be completely eliminated in favour of static 2D Coulomb potential. We will use the light cone coordinates $x^+ = \frac{1}{\sqrt{2}}(t+x)$, $x^- = \frac{1}{\sqrt{2}}(t-x)$ and choose the $A_+ = 0$ gauge. We have gauge-coupled complex bosons with a quadratic self-interaction term and Dirac fermions both of the matter fields are in the fundamental representation of $SU(N_c)$:

$$\begin{aligned}S = \int \mathrm{d}x^+\,\mathrm{d}x^-\Big[&-\frac{1}{2}\operatorname{Tr}F_{+-}F^{+-} + \mathrm{i}\sqrt{2}\psi_L^{*\alpha}(\partial_- + \mathrm{i}gA_-)_\alpha^\beta\psi_{L\beta}\\ &+ \mathrm{i}\sqrt{2}\psi_R^{*\alpha}\partial_+\psi_{R\alpha} - m_F(\psi_L^{*\alpha}\psi_{R\alpha} + \psi_R^{*\alpha}\psi_{L\alpha})\\ &- 2\phi^{*\alpha}\partial_-\partial_+\phi_\alpha + \mathrm{i}g(\partial_+\phi^{*\alpha}A_{-\,\alpha}^{\;\beta}\phi_\beta - \phi^{*\alpha}A_{-\,\alpha}^{\;\beta}\partial_+\phi_\beta)\\ &- m_B^2\phi^{*\alpha}\phi_\alpha - \frac{\lambda^2}{4}\phi^{*\alpha}\phi_\alpha\phi^{*\beta}\phi_\beta\Big]\,.\end{aligned} \tag{6.1}$$

[(1)] For a quick answer one can apply the previous trick and evaluate it at ϵ.

We can further use the Gauss constraint to eliminate the gauge field A_- and the fermionic equations of motion to eliminate the right moving fermion ψ_R ($\psi_{R\alpha}$). The resulting action is first order in "time direction" x^- so we can pass to Hamiltonian formalism in a straightforward way. One should of course discuss if the theory we have is a Poincare invariant when we choose such a reference frame but this is discussed in the literature and indeed we can see from our Hamiltonian that it is Lorentz invariant directly, so we will not talk about this here.

We introduce the Fourier mode expansions

$$\phi_\alpha(x^+) = \int a_\alpha(p)\,\mathrm{e}^{\mathrm{i}px^+}\frac{\mathrm{d}p}{2\pi(2|p|)^{1/2}}\,, \qquad \psi_{L\alpha}(x^+) = \int \chi_\alpha(p)\,\mathrm{e}^{\mathrm{i}px^+}\frac{\mathrm{d}p}{2\pi 2^{1/4}}\,.$$

The normalization factors are chosen to give the correct classical limits. The commutation/anticommutation relations for the fields in the light cone gauge take the form (see [17] for details)

$$\begin{aligned}[][\chi^\alpha(p),\chi^\dagger_\beta(q)]_+ &= \delta^\alpha_\beta 2\pi\delta(p-q)\,,\\ [a^\alpha(p),a^\dagger_\beta(q)] &= \mathrm{sign}(p)\delta^\alpha_\beta 2\pi\delta(p-q)\,.\end{aligned} \tag{6.2}$$

Notice that each time we have an integral over p it is divided by 2π. So the identity for example is written as $2\pi\delta(p-q)$ in this basis! We also may define $\delta[p-q] = 2\pi\delta(p-q)$, then we get $\int[\,\mathrm{d}p]\delta[p-q] = 1$. One defines a Fock vacuum state $|0\rangle$ by conditions $a^\alpha(p)|0\rangle = \chi^\alpha(p)|0\rangle = 0$ for $p \geq 0$ and $a^\dagger_\alpha(p)|0\rangle = \chi^\dagger_\alpha(p)|0\rangle = 0$ for $p < 0$. The corresponding normal orderings are defined as

$$:\chi^\dagger_\alpha(p)\chi^\beta(q): \;= \begin{cases} -\chi^\beta(q)\chi^\dagger_\alpha(p) & \text{if } p<0, q<0\\ \chi^\dagger_\alpha(p)\chi^\beta(q) & \text{otherwise}\;, \end{cases} \tag{6.3}$$

$$:a^\dagger_\alpha(p)a^\beta(q): \;= \begin{cases} a^\beta(q)a^\dagger_\alpha(p) & \text{if } p<0, q<0\\ a^\dagger_\alpha(p)a^\beta(q) & \text{otherwise}\;. \end{cases} \tag{6.4}$$

Written as quantum operators

$$\psi_{R\alpha} = \frac{m_F}{\mathrm{i}\sqrt{2}\partial_+}\psi_{L\alpha} \tag{6.5}$$

and the Hermitian conjugate of this equation.

Notice that A_- is given in terms of the other fields as

$$A^a_- = -\frac{g}{\partial_+^2} : (\sqrt{2}\psi_L^{*\alpha}(T^a)^\beta_\alpha\psi_{L\beta} + \mathrm{i}[\phi^{*\alpha}(T^a)^\beta_\alpha\partial_+\phi_\beta - \partial_+\phi^{*\alpha}(T^a)^\beta_\alpha\phi_\beta]): \tag{6.6}$$

Here we use Hermitian generators T^a, they are normalized as $\mathrm{Tr}\, T^a T^b = 1/2\delta^{ab}$. Using the expressions (6.6), (6.5), one can express the action functional [16, 17, 5] in terms of fields $\psi_{L\alpha}$, $\phi(x)$, and their first order derivatives in time direction x^- (and Hermitian conjugates thereof). The passage to Hamiltonian formalism is straightforward (see [17] for details).

$$H = H_0 + H_I \tag{6.7}$$

$$H_0 = \left(\frac{m_B^2}{4} - \frac{g^2}{4\pi}\right) \int \frac{[\mathrm{d}p]}{|p|} N(p,p) + \left(\frac{m_F^2}{4} - \frac{g^2}{4\pi}\right) \int \frac{[\mathrm{d}p]}{p} M(p,p) \tag{6.8}$$

$$\begin{aligned} H_I = \int [\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}s\,\mathrm{d}t] \Big(& G_1(p,q;s,t) M(p,q) M(s,t) \\ & + G_2(p,q;s,t) N(p,q) N(s,t) \\ & + G_3(p,q;s,t) Q(p,q) \bar{Q}(s,t) \Big) \,. \end{aligned} \tag{6.9}$$

We are not writing the explicit forms of the integral kernels, they are given in [12], most of the details can be found in this work.

The theory we obtain still possesses a global $SU(N_c)$ invariance. The corresponding color operator is

$$\begin{aligned} \hat{Q}^\beta_\alpha = \int [\,\mathrm{d}p] & \Big(: \chi^\dagger_\alpha(p) \chi^\beta(p) : - \frac{1}{N_c} \delta^\beta_\alpha : \chi^\dagger_\gamma(p) \chi^\gamma(p) : \Big) \\ + \int [\,\mathrm{d}p] \,\mathrm{sign}(p) & \Big(: a^\dagger_\alpha(p) a^\beta(p) : - \frac{1}{N_c} \delta^\beta_\alpha : a^\dagger_\gamma(p) a^\gamma(p) : \Big) \,. \end{aligned} \tag{6.10}$$

We will restrict our model to the color invariant sector. In general for a gauge theory it is expected that in the large N_c limit [8, 9] any gauge invariant correlator splits, i. e. $\langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/N_c)$(an excellent source for large-N_c limit is [25], the use of bilinears in the path integral context is pointed out there and worked out in the linear approximation directly for our problem in [5], this is a particularly good reference to consult). So when the two dimensional theory restricted to the color invariant subspace in the large N_c limit any color invariant correlator should be expressible in terms of correlators of color invariant bilinear operators

$$\hat{M}(p,q) = \frac{2}{N_c} : \chi^{\dagger\alpha}(p) \chi_\alpha(q) : \tag{6.11}$$

$$\hat{N}(p,q) = \frac{2}{N_c} : a^{\dagger\alpha}(p) a_\alpha(q) : \tag{6.12}$$

and their odd counterparts

$$\hat{Q}(p,q) = \frac{2}{N_c} \chi^{\dagger\alpha}(p) a_\alpha(q) \,, \qquad \hat{\bar{Q}}(r,s) = \frac{2}{N_c} a^{\dagger\alpha}(r) \chi_\alpha(s) \,. \tag{6.13}$$

This is really the essential simplification. This simplification will not happen in higher dimensional gauge theories nor in the case of matter fields belonging to the adjoint representation even in two dimensions. Some of these issues are discussed in a serious papers by Lee and Rajeev [13].

Following the philosophy introduced in [16] we will treat these as our coordinates on the phase space of the theory. We can understand the dynamics if we know the Poisson bracket relations among these bilenears. To get this we compute various commutators and anticommutators of these variables, and take the large-N_c limit simply by removing the factors of $\frac{1}{N_c}$ in front. It is straightforward to get the (anti)commutation relations between these bilinears

$$\begin{aligned}
\left[\hat{M}(p,q),\hat{M}(r,s)\right] &= \frac{2}{N_c}\Big[\hat{M}(p,s)\delta[q-r]-\hat{M}(r,q)\delta[p-s] \\
&\qquad -\delta[p-s]\delta[q-r](\operatorname{sign}(p)-\operatorname{sign}(q))\Big] \\
\left[\hat{N}(p,q),\hat{N}(r,s)\right] &= \frac{2}{N_c}\Big[\hat{N}(p,s)\operatorname{sign}(q)\delta[q-r]-\hat{N}(r,q)\operatorname{sign}(p)\delta[p-s] \\
&\qquad +\delta[q-r]\delta[p-s](\operatorname{sign}(p)-\operatorname{sign}(q))\Big]\,, \\
\left[\hat{Q}(p,q),\hat{\hat{Q}}(r,s)\right]_{+} &= \frac{2}{N_c}\Big[\hat{M}(p,s)\operatorname{sign}(q)\delta[q-r]+\hat{N}(r,q)\delta[p-s] \\
&\qquad +\delta[p-s]\delta[q-r](1-\operatorname{sign}(p)\operatorname{sign}(q))\Big] \qquad (6.14) \\
\left[\hat{M}(p,q),\hat{Q}(r,s)\right] &= \frac{2}{N_c}\delta[q-r]\hat{Q}(p,s) \\
\left[\hat{N}(p,q),\hat{Q}(r,s)\right] &= -\frac{2}{N_c}\delta[p-s]\operatorname{sign}(p)\hat{Q}(r,q) \\
\left[\hat{M}(p,q),\hat{\hat{Q}}(r,s)\right] &= -\frac{2}{N_c}\delta[p-s]\hat{\hat{Q}}(r,q) \\
\left[\hat{N}(p,q),\hat{\hat{Q}}(r,s)\right] &= \frac{2}{N_c}\delta[q-r]\operatorname{sign}(q)\hat{\hat{Q}}(p,s)\,.
\end{aligned}$$

All the other (anti)commutators vanish. These (anti)commutation relations define an infinite dimensional Lie superalgebra. Its even part is isomorphic to a direct sum of central extensions of infinite-dimensional unitary and pseudo-unitary groups each one generated by operators $\hat{M}(p,q)$ and $\hat{N}(p,q)$ respectively (see [18] for details). As the right hand sides of (6.14) all contain a factor of $1/N_c$ in the large N_c limit all of the bilinears commute and can be thought of as coordinates on a classical phase space. We denote the classical variables corresponding to $\hat{M}$, $\hat{N}$, $\hat{Q}$, $\hat{\hat{Q}}$ by the same letters with hats removed. This classical phase space is an infinite dimensional supermanifold endowed with a super Poisson structure inherited from the (anti)commutation relations (6.14). Let us write down the Poisson superbrackets obtained from the (anti)commutators in (6.14) by substituting i instead of $1/N_c$ factors. (Note that this brings an extra factor of 2, there is no simple way to decide what factor should be the quantum parameter when we take the classical limit. If one does geometric quantization of this model, the symplectic form should be an integer multiple of the Chern character of the line bundle, the symplectic form we have in the next section is in fact the basic two form):

$$\begin{aligned}
\{M(p,q), M(r,s)\} &= 2\mathrm{i}\Big[M(p,s)\delta[q-r] - M(r,q)\delta[p-s] \\
&\qquad - \delta[p-s]\delta[q-r](\mathrm{sign}(p) - \mathrm{sign}(q))\Big]\,, \\
\{N(p,q), N(r,s)\} &= 2\mathrm{i}\Big[N(p,s)\,\mathrm{sign}(q)\delta[q-r] - N(r,q)\,\mathrm{sign}(p)\delta[p-s] \\
&\qquad + \delta[q-r]\delta[p-s](\mathrm{sign}(p) - \mathrm{sign}(q))\Big]\,, \\
\{Q(p,q), \bar{Q}(r,s)\}_+ &= 2\mathrm{i}\Big[M(p,s)\,\mathrm{sign}(q)\delta[q-r] + N(r,q)\delta[p-s] \\
&\qquad + \delta[p-s]\delta[q-r](1 - \mathrm{sign}(p)\,\mathrm{sign}(q))\Big]\,, \qquad (6.15) \\
\{M(p,q), Q(r,s)\} &= 2\mathrm{i}\delta[q-r]Q(p,s) \\
\{N(p,q), Q(r,s)\} &= -2\mathrm{i}\delta[p-s]\,\mathrm{sign}(p)Q(r,q) \\
\{M(p,q), \bar{Q}(r,s)\} &= -2\mathrm{i}\delta[p-s]\bar{Q}(r,q) \\
\{N(p,q), \bar{Q}(r,s)\} &= 2\mathrm{i}\delta[q-r]\,\mathrm{sign}(q)\bar{Q}(p,s)
\end{aligned}$$

However this super-Poisson structure only gives a local structure of the classical phase space of the theory. In addition to that there are some global constraints on the classical variables assigned to the color invariant bilinears. The constraints emerge in the large N_c limit as consequences of the color invariance condition $\hat{Q}_\alpha^\beta = 0$. It is more convenient to introduce the following operator product convention

$$AB(p,q) = \int [\mathrm{d}r]\, A(p,r)B(r,q)$$

where A, B stand for any of the above (classical) bilinears before we write down these constraints. Let us introduce operators 1 and ϵ as the ones having kernels $\delta[p-q]$ and $-\mathrm{sign}(p)\delta[p-q]$ respectively. Then the constraints can be shown to be the following ones

$$
\begin{aligned}
(M+\epsilon)^2 + Q\epsilon Q^\dagger &= 1 \\
\epsilon Q^\dagger M + \epsilon Q^\dagger \epsilon + \epsilon N \epsilon Q^\dagger + Q^\dagger &= 0 \\
MQ + \epsilon Q + Q\epsilon N + Q\epsilon &= 0 \\
(\epsilon N + \epsilon)^2 + \epsilon Q^\dagger Q &= 1\,.
\end{aligned}
\tag{6.16}
$$

For brevity we will not present a derivation of these constraints and refer the reader to [12].

We notice that if we introduce the operator

$$
\Phi = \begin{pmatrix} M+\epsilon & Q \\ \epsilon\bar{Q} & \epsilon N + \epsilon \end{pmatrix}
\tag{6.17}
$$

the above constraint becomes

$$
\Phi^2 = 1\,.
\tag{6.18}
$$

Furthermore we have hermiticity conditions on these variables coming from the quantum operators, they are $M^\dagger = M$, $N^\dagger = N$ and $Q^\dagger = \bar{Q}$ in our operator notation. This can be written as:

$$
\Phi = \begin{pmatrix} M+\epsilon & Q \\ \epsilon Q^\dagger & \epsilon N + \epsilon \end{pmatrix}, \qquad E\Phi^\dagger E = \Phi
\tag{6.19}
$$

where E is defined in the previous section as $E = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$. We see that the super-Grassmannian we have discussed in the previous section fits exactly to the large-N_c limit of this two dimensional gauge theory. The only ingredient we have not mentioned is the convergence condition, this comes from the finiteness of this two dimensional theory. That is this is a theory in which there is really no renormalization, it is completely finite (the self energy term of the quadratic interaction and the one coming from gauge theory are rather harmless one can remove them by an additive renormalization. Multiplicative renormalization is the one we cannot simply entangle and this requires much more sophisticated techniques. In fact even in these cases the basic large-N_c idea is valid except that the Poisson brackets do not come from the above symplectic form and we need to change the convergence conditions imposed depending on the degree of divergence). These issues are under investigation.

The rest is computing the equations of motion for our variables and see what they imply. Clearly, these are non-linear and rather complicated, we may look at various approximations. One essential step is to look at a linearization, this assumes that all the components M, N, Q are small, so we keep them to linear order and also linearize the constraint. This process is discussed in detail first in [16] and then in [18, 17]. It is shown in these references that one gets the ‘t Hooft equation [9] for fermions and its analog for the bosonic fields [21, 20]. In this situation one can also study the linearization and see that there are a set of bound state equations which contain the ‘t Hooft equation and its analog for the fermionic and bosonic bilinears, and there is also a bound state equation for mesons made up of a bosonic and a fermionic quark. These equations appeared in [2] following the original approach of ‘t Hooft and then from a path integral method in [5]. In the method we discuss there are exotic baryon states given by $\mathrm{Str}(\Phi - \bar{\epsilon})$, it is interesting to study the large fluctuations corresponding to these excitations. For a more complete presentation and going beyond the linear approximation we refer to [12].

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