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QUANTIZATION OF LOCALLY SYMMETRIC KÄHLER MANIFOLDS

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Abstract. We introduce noncommutative deformations of locally symmetric Kähler manifolds. A Kähler manifold M is said to be a locally symmetric Kähler manifold if the covariant derivative of the curvature tensor is vanishing . An algebraic derivation process to construct a locally symmetric Kähler manifold is given. As examples, star products for noncommutative Riemann surfaces and noncommutative $\mathbb{CP}^N$ are constructed.

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1. Review of the Deformation Quantization with Separation of Variables

In this section, we review the deformation quantization with separation of variables to construct noncommutative Kähler manifolds.

An N-dimensional Kähler manifold M is described by using a Kähler potential. Let Φ be a Kähler potential and ω be a Kähler two-form

$$\omega := \mathrm{i} g_{k\bar{l}} \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^l, \qquad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l} \tag{1}$$

where $z^i, \bar{z}^i$ $(i = 1, 2, \ldots, N)$ are complex local coordinates.

In this article, we use the Einstein summation convention over repeated indices. The $g^{\bar{k}l}$ is the inverse of the Kähler metric tensor $g_{k\bar{l}}$. That means $g^{\bar{k}l} g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}$.

In the following, we use

$$\partial_k = \frac{\partial}{\partial z^k}, \qquad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \tag{2}$$

Deformation quantization is defined as follows.

Definition 1 (Deformation quantization)**.** *Deformation quantization of Poisson manifolds is defined as follows. $\mathcal{F}$ is defined as a set of formal power series:* $\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k;\, f_k \in C^\infty(M) \right\}$. *A star product is defined as*

$$f * g = \sum_k C_k(f, g)\hbar^k \tag{3}$$

such that the product satisfies the following conditions

1. $(\mathcal{F}, +, *)$ *is a (noncommutative) algebra.*
2. $C_k(\cdot, \cdot)$ *is a bidifferential operator.*
3. C_0 *and* C_1 *are defined as* $C_0(f, g) = fg,\ C_1(f, g) - C_1(g, f) = \{f, g\}$ *where* $\{f, g\}$ *is the Poisson bracket.*
4. $f * 1 = 1 * f = f$.

Karabegov introduced a method to obtain a deformation quantization of a Kähler manifold in [6]. His deformation quantization is called deformation quantizations with separation of variables

Definition 2 (A star product with separation of variables)**.** *The operation $*$ is called a star product with separation of variables on a Kähler manifold when $a * f = af$ for an arbitrary holomorphic function a and $f * b = fb$ for an arbitrary anti-holomorphic function b.*

We use

$$D^{\bar{l}} = g^{\bar{l}k}\partial_k$$

and introduce $\mathcal{S} := \left\{ A;\ A = \sum_\alpha a_\alpha D^\alpha,\ \ a_\alpha \in C^\infty(M) \right\}$, where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

In this article, we also use the Einstein summation convention over repeated multi-indices and $a_\alpha D^\alpha := \sum_\alpha a_\alpha D^\alpha$.

There are some useful formulae. $D^{\bar{l}}$ satisfies the following equations.

$$[D^{\bar{l}}, D^{\bar{m}}] = 0, \qquad [D^{\bar{l}}, \partial_{\bar{m}}\Phi] = \delta^{\bar{l}}{}_{\bar{m}}, \qquad \text{for all} \quad l, m \tag{4}$$

where $[A,\ B] = AB - BA$. Using them, one can construct a star product as a differential operator L_f such that $f * g = L_f g$.

Theorem 1. [Karabegov [6]]. For an arbitrary Kähler form ω, there exist a star product with separation of variables $*$ and it is constructed as follows. Let f be an element of $\mathcal{F}$ and $A_n \in \mathcal{S}$ be a differential operator whose coefficients depend on f, i.e.,

$$A_n = a_{n,\alpha}(f) D^{\alpha}, \qquad D^{\alpha} = \prod_{i=1}^{n} (D^{\bar{i}})^{\alpha_i}, \qquad (D^{\bar{i}}) = g^{\bar{i}l}\partial_l \tag{5}$$

where α is an multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n \tag{6}$$

is uniquely determined such that it satisfies the following conditions.

1. For $R_{\partial_{\bar{l}}\Phi} = \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}$

$$\left[L_f, R_{\partial_{\bar{l}}\Phi}\right] = 0\,. \tag{7}$$

2.

$$L_f 1 \ = \ f * 1 = f. \tag{8}$$

Then the star products are given by

$$L_f g \ := \ f * g \tag{9}$$

and the star products satisfy the associativity

$$L_h(L_g f) \ = \ h * (g * f) = (h * g) * f = L_{L_h g} f. \tag{10}$$

Recall that each two of $D^{\bar{i}}$ commute each other, so if a multi index α is fixed then the A_n is uniquely determined. The equations (8)-(10) imply that $L_f g = f * g$ gives deformation quantization.

Definition 3. *A map from differential operators to formal polynomials is defined as*

$$\sigma\left(A; \xi\right) := \sum_{\alpha} a_{\alpha} \xi^{\alpha}$$

where

$$A = \sum_{\alpha} a_{\alpha} D^{\alpha}.$$

This map is called "twisted symbol". It becomes easier to calculate commutators by using the following theorem.

Proposition 2 (Karabegov [6])**.** Let $a(\xi)$ be a twisted symbol of an operator A. Then the twisted symbol of the operator $[A, \partial_{\bar{i}}\Phi]$ is equal to $\partial a/\partial \xi^{\bar{i}}$

$$\sigma\left([A, \partial_{\bar{i}}\Phi]\right) = \frac{\partial}{\partial \xi^{\bar{i}}}\sigma\left(A\right).$$

This proposition follows from (4), i.e.,

$$\sigma\left([D^{\bar{l}}, \partial_{\bar{i}}\Phi]\right) = \delta^{\bar{l}}_{\bar{i}}.$$

2. Deformation Quantization with Separation of Variables for a Locally Symmetric Kähler Manifold

In this section, explicit formulas to obtain star products on local symmetric Kähler manifolds are constructed. A method of Karabegov in Section 1 is used for the constructing.

Operators $D^{\vec{\alpha_n}}$ and $D^{\vec{\beta^*_n}}$ are defined by using $D^k = g^{k\bar{m}}\partial_{\bar{m}}$ and $D^{\bar{j}} = g^{\bar{j}l}\partial_l$ as

$$D^{\vec{\alpha_n}} := D^{\alpha_1^n} D^{\alpha_2^n} \cdots D^{\alpha_N^n}, \qquad D^{\vec{\beta_n}} := D^{\beta_1} D^{\beta_2} \cdots D^{\beta_N}$$

where

$$D^{\alpha_k} := \left(D^k\right)^{\alpha_k}, \qquad D^{\beta_j} := \left(D^{\bar{j}}\right)^{\beta_j}$$

and $\vec{\alpha_n}$ and $\vec{\beta^*_n}$ are N-dimensional vectors whose summation of their all elements are set to be n

$$\vec{\alpha}_n \in \left\{ (\gamma_1^n, \gamma_2^n, \cdots, \gamma_N^n) \in \mathbb{Z}^N \ ; \ \sum_{k=1}^{N} \gamma_k^n = n \right\}$$

$$\vec{\beta^*_n} \in \left\{ (\gamma_1^n, \gamma_2^n, \cdots, \gamma_N^n)^* \in \mathbb{Z}^N \ ; \ \sum_{k=1}^{N} \gamma_k^n = n \right\}$$

i.e.,

$$\vec{\alpha_n} := \left(\alpha_1^n, \alpha_2^n, \cdots, \alpha_N^n\right), \ \ |\vec{\alpha}_n| := \sum_{k=1}^{N} \alpha_k^n = n$$

$$\vec{\beta^*_n} := \left(\beta_1^n, \beta_2^n, \cdots, \beta_N^n\right)^*, \ \ |\vec{\beta^*_n}| := \sum_{k=1}^{N} \beta_k^n = n.$$

For $\vec{\alpha_n} \notin \mathbb{Z}^N_{\geq 0}$ we define $D^{\vec{\alpha_n}} := 0$.

For example, $D^{(1,2,3)} = D^1 \left(D^2\right)^2 \left(D^3\right)^3$, $D^{(2,4,0)^*} = \left(D^{\bar{1}}\right)^2 \left(D^{\bar{2}}\right)^4$ and $D^{(5,-2,3)} = 0$ for a three-dimensional manifolds case with $n = 6$.

$\vec{e_i}$ is used as a N-dimensional vector

$$\vec{e_i} = (\delta_{1i}, \delta_{2i}, \cdots, \delta_{Ni}). \tag{11}$$

A Riemannian (Kähler) manifold (M, g) is called a locally symmetric Riemannian (Kähler) manifold when $\nabla_m R_{ijk}{}^{l} = 0$ for all i, j, k, l, m. Only locally symmetric Kähler manifolds are disscussed.

We assume that a star product with separation of variables for smooth functions f and g on a locally symmetric Kähler manifold M has a form

$$L_f g = f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n} \vec{\beta_n^*}} T^n_{\vec{\alpha_n} \vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f \right) \left(D^{\vec{\beta_n^*}} g \right) \tag{12}$$

where $T^n_{\vec{\alpha_n} \vec{\beta_n^*}}$ are covariantly constants. If $\vec{\alpha_n} \notin \mathbb{Z}^N_{\geq 0}$ or $\vec{\beta_n} \notin \mathbb{Z}^N_{\geq 0}$ then we define $T^n_{\vec{\alpha_n} \vec{\beta_n^*}} := 0$. $\sum_{\vec{\alpha_n} \vec{\beta_n^*}}$ is defined by the summation over all $\vec{\alpha_n^*}$ and $\vec{\beta_n^*}$ satisfying $\left| \vec{\alpha_n^*} \right| = \left| \vec{\beta_n^*} \right| = n$. In brief

$$n = \left| \vec{\alpha_n^*} \right| := \sum_{i=1}^{N} \alpha_i^n, \qquad n = \left| \vec{\beta_n^*} \right| := \sum_{i=1}^{N} \beta_i^n, \qquad \sum_{\vec{\alpha_n} \vec{\beta_n^*}} := \sum_{\left| \vec{\alpha_n} \right| = \left| \vec{\beta_n^*} \right| = n} .$$

Theorem 3. When the star product with separation of variables for smooth functions f and g on a local symmetric Kähler manifold is given as

$$f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n} \vec{\beta_n^*}} T^n_{\vec{\alpha_n} \vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f \right) \left(D^{\vec{\beta_n^*}} g \right)$$

these smooth functions $T^n_{\vec{\alpha_n} \vec{\beta_n^*}}$, which are covariantly constants, are determined by the following recurrence relations for all i

$$\begin{aligned}
&\sum_{d=1}^{N} \hbar g_{\bar{i} d} T^{n-1}_{\vec{\alpha_n} - \vec{e_d} \vec{\beta_n^*} - \vec{e_i}} \\
&= \beta_i T^n_{\vec{\alpha_n} \vec{\beta_n^*}} + \sum_{k=1}^{N} \sum_{p=1}^{N} \frac{\hbar \left(\beta_k^n - \delta_{kp} - \delta_{ik} + 1 \right) \left(\beta_k^n - \delta_{kp} - \delta_{ik} + 2 \right)}{2} \\
&\quad \times R_{\bar{p}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n} \vec{\beta_n^*} - \vec{e_p} + 2\vec{e_k} - \vec{e_i}} + \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{p=1}^{N} \hbar \left(\beta_k^n - \delta_{kp} - \delta_{ik} + 1 \right) \\
&\quad \times \left(\beta_{k+l}^n - \delta_{(k+l),p} - \delta_{i,(k+l)} + 1 \right) R_{\bar{p}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n} \vec{\beta_n^*} - \vec{e_p} + \vec{e_k} + \vec{e_{k+l}} - \vec{e_i}} .
\end{aligned}$$

Outline of Proof. *Let f and g be smooth functions on a Kähler manifold $M$and L_f be a left star product by f given as* (12). *Then*

$$\begin{aligned}\sigma\left([L_f, \partial_{\bar{i}}\Phi]\right) &= \frac{\partial \sigma\left(L_f\right)}{\partial \xi^{\bar{i}}} \\ &= \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(\xi^{\bar{1}^{\beta_1^n}} \cdots \xi^{\bar{i}^{\beta_i^n-1}} \cdots \xi^{\bar{N}^{\beta_N^n}}\right)\end{aligned}$$

or equivalently,

$$[L_f, \partial_{\bar{i}}\Phi]\, g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*}-\vec{e_i}} g\right). \tag{13}$$

The following formulas are given in [10]. *For smooth functions f and g on a locally symmetric Kähler manifold, the following formulas are given.*

$$\begin{aligned}\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f &= g_{l_1\bar{j}_1} \cdots g_{l_n\bar{j}_n} D^{l_1} \cdots D^{l_n} f \\ \nabla_{k_1} \cdots \nabla_{k_n} g &= g_{\bar{m}_1 k_1} \cdots g_{\bar{m}_n k_n} D^{\bar{m}_1} \cdots D^{\bar{m}_n} g \\ D^{l_1} \cdots D^{l_n} f &= g^{l_1\bar{j}_1} \cdots g^{l_n\bar{j}_n} \nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f \\ D^{\bar{m}_1} \cdots D^{\bar{m}_n} g &= g^{\bar{m}_1 k_1} \cdots g^{\bar{m}_n k_n} \nabla_{k_1} \cdots \nabla_{k_n} g.\end{aligned}$$

If M is a locally symmetric Kähler manifold, these formulas derive

$$\begin{aligned}&[L_f, \hbar\partial_{\bar{i}}]g \\ &= \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \sum_{k=1}^{N} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \frac{\beta_k^n\left(\beta_k^n-1\right)}{2} R_{\bar{\rho}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*}+\vec{e_\rho}-\vec{e_k}} g\right) \\ &\quad + \hbar \sum_{n=0}^{\infty} \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_k^n \beta_{k+l}^n R_{\bar{\rho}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*}+\vec{e_\rho}-\vec{e_k}} g\right) \\ &\quad - \hbar \sum_{n=1}^{\infty} \sum_{\vec{\alpha_{n-1}}\vec{\beta_{n-1}^*}} \sum_{d=1}^{N} g_{\bar{i}d} T^{n-1}_{\vec{\alpha_{n-1}}\vec{\beta_{n-1}^*}} \left(D^{\vec{\alpha_{n-1}}+\vec{e_d}} f\right) \left(D^{\vec{\beta_{n-1}^*}} g\right).\end{aligned}$$

■

Details of this proof are given in [5].

3. $*-$Products for Riemann Surfaces

$*-$products for Riemann surfaces are studied in this section for arbitrary Riemann surfaces regarded as locally symmetric Kähler manifold. Applying Theorem 3 for complex 1 dimensional case, $*-$products for Riemann surfaces are obtained concretely. A formal discussions are given in [11], and star products are studied in [9].

The Scalar curvature R is defined as

$$R = g^{i\bar{j}} R_{i\bar{j}} = R_{\bar{l}}{}^{\bar{j}\bar{l}}{}_{\bar{j}}.$$

Theorem 4. Let M be a one-dimensional locally symmetric Kähler manifold $(N = 1)$ and f and g be smooth functions on M. The star product with separation of variables for f and g can be described as [1]

$$f*g = \sum_{n=0}^{\infty} \left[\left(g^{1\bar{1}}\right)^n \left\{ \prod_{k=1}^{n} \frac{2\hbar}{2k + \hbar k\,(k-1)\,R} \right\} \left\{ \left(g^{1\bar{1}} \frac{\partial}{\partial z}\right)^n f \right\} \left\{ \left(g^{1\bar{1}} \frac{\partial}{\partial \bar{z}}\right)^n g \right\} \right].$$

Example 1. Let $(\mathbb{C}, g)$ be a complex plane as a one-dimensional locally symmetric Kähler manifold. The star product with separation of variables for f and g can be described as

$$f * g = \sum_{n=0}^{\infty} \left[\frac{\hbar^n}{n!} \left\{ \left(\frac{\partial}{\partial z}\right)^n f \right\} \left\{ \left(\frac{\partial}{\partial \bar{z}}\right)^n g \right\} \right].$$

Example 2. Wellknown flat torus embedding $X : S^1 \times S^1 \to \mathbb{R}^4$

$$X\,(u, v) = (\cos u, \sin u, \cos v, \sin v)\,,\ u = \mathrm{Re}\,(z)\,,\ v = \mathrm{Im}\,(z)$$
$$\Longrightarrow R = \frac{-1}{\sqrt{EG}} \left\{ \frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right\} = 0$$

where first fundamental forms are

$$E = \frac{\partial X}{\partial u} \cdot \frac{\partial X}{\partial u} = 1, \qquad F = \frac{\partial X}{\partial u} \cdot \frac{\partial X}{\partial v} = 0, \qquad G = \frac{\partial X}{\partial v} \cdot \frac{\partial X}{\partial v} = 1$$

hence u, v are isothermal coordinates on a torus and the pullback metric is defined as

$$\tilde{g}_{1\bar{1}} = E = G = 1.$$

If $(M, g) = \left(S^1 \times S^1, \tilde{g}\right)$ then $R = R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} = 0$. Hence the star product with separation of variables for f and g can be described as also

$$f * g = \sum_{n=0}^{\infty} \left[\frac{\hbar^n}{n!} \left\{ \left(\frac{\partial}{\partial z}\right)^n f \right\} \left\{ \left(\frac{\partial}{\partial \bar{z}}\right)^n g \right\} \right].$$

[1]Here we correct the typos in page 562 in [5]. $\prod_{k=1}^{n-1}$ in [5] should be $\prod_{k=1}^{n}$.

4. Projective Space Cases

In this section, we calculate star products of $\mathbb{CP}^N$. These star products are also equal to the ones given in [1, 4, 10]. A projective space is a special Grassmann manifold and a Grassmann manifold is a special flag manifold. Deformation quantization of flag manifolds and Grassmann manifolds were studied in [2, 3, 7, 8].
At first, a function similar to the determinant is defined on the matrix space.

Definition 4 (permanent)**.** *Let $C = (C_{k,l})_{1\leq k\leq n, 1\leq l\leq n}$ be a $n \times n$ matrix. We define $|\cdot|^+$ as a $\mathbb{C}$-valued function on $M(n, n; \mathbb{C})$ such that*

$$|C|^+ := \sum_{\sigma_n \in S_n} \prod_{k=1}^{n} C_{k,\sigma_n(k)}.$$

This is called "permanent".

Definition 5. *A matrix $G^{\vec{\alpha_n},\vec{\beta_n^*}}$ is defined by using the Hermitian metrics on M. Its elements are metrics on M and are located as follows. $\vec{\alpha_n}$ and $\vec{\beta_n}$ are elements of $\mathbb{Z}^N$*

$$G^{\vec{\alpha_n},\vec{\beta_n^*}} = \begin{pmatrix} \tilde{G}_{11} & \cdots & \tilde{G}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{n1} & \cdots & \tilde{G}_{nn} \end{pmatrix}$$

where

$$\tilde{G}_{pq} =: g_{p\bar{q}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in M\left(\alpha_p^n, \beta_q^n; \mathbb{C}\right).$$

Theorem 5. Let f and g be smooth functions on a projective space $\mathbb{CP}^N$. A star product with separation of variables on a projective space $\mathbb{CP}^N$ is given as

$$f * g = f \cdot g + \sum_{n=1}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \left| G^{\vec{\alpha_n},\vec{\beta_n^*}} \right|^+ \left(\prod_{l=1}^{N} \frac{1}{\alpha_l^n! \beta_l^n!} \right) \prod_{k=1}^{n} \frac{\hbar}{(1 + \hbar - \hbar k)} \left(D^{\vec{\alpha_n}} f \right) \left(D^{\vec{\beta_n^*}} g \right). \tag{14}$$

Here, we correct the typos in (5.4) in [5].

Proof. *We show that*

$$T^n_{\vec{\alpha_n}\vec{\beta_n^*}} = \left| G^{\vec{\alpha_n},\vec{\beta_n^*}} \right|^+ \left(\prod_{l=1}^{N} \frac{1}{\alpha_l^n! \beta_l^n!} \right) \prod_{k=1}^{n} \frac{\hbar}{(1 + \hbar - \hbar k)}$$

satisfies (3)

$$\sum_{d=1}^{N} \frac{\hbar g_{\bar{i}d}}{(1+\hbar-\hbar n)\,\beta_i^n} T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta_n^*}-\vec{e_i}}$$

$$= \sum_{d=1}^{N} g_{\bar{i}d}\alpha_d^n \left| G^{\vec{\alpha_n}-\vec{e_d},\vec{\beta_n^*}-\vec{e_i}} \right|^{+} \frac{\hbar}{(1+\hbar-\hbar n)} \left(\prod_{l=1}^{N} \frac{1}{\alpha_l^n!\beta_l^n!} \right) \prod_{k=1}^{n} \frac{\hbar}{(1+\hbar-\hbar k)}.$$

Using cofactor expansion of permanent, the R.H.S. of the above is rewritten as

$$\left| G^{\vec{\alpha_n},\vec{\beta_n^*}} \right|^{+} \left(\prod_{l=1}^{N} \frac{1}{\alpha_l^n!\beta_l^n!} \right) \prod_{k=1}^{n} \frac{\hbar}{(1+\hbar-\hbar k)}.$$

This shows the given $T^n_{\vec{\alpha_n}\vec{\beta_n^*}}$ *satisfies the recurrence relation* (3). ■

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