Chapter 0

HISTORICAL STRANDS OF GEOMETRY

All people by nature desire knowledge.
— Aristotle (384 B.C.–322 B.C.), *Metaphysics*

History is the witness that testifies to the passing of time; it illumines reality, vitalizes memory, provides guidance in daily life and brings us tidings of antiquity.
— Cicero (106 B.C.–43 B.C.), *Pro Publio Sestio*

Inherited ideas are a curious thing, and interesting to observe and examine.
— Mark Twain (1835–1910), *A Connecticut Yankee in King Arthur's Court*

The ways in which different ideas have become abstract geometric concepts depend on the ways in which they have been explored. H. Graham Flegg wrote in his book *From Geometry to Topology* (Dover, 2001, p.168):

> New branches of mathematics come into being, not because they are created overnight out of nothing by some individual genius, but because the soil has been prepared over the previous decades (or even centuries) and because some internal or external stress (or perhaps a combination of both) provides the appropriate impetus and motivation at the crucial point in time. More often than not, it is the case that several minds produce independently and almost simultaneously the germs of what subsequently develops into a new theatre of mathematical investigation. For this reason, it is usually ill-advised to point to any one man as being the founder or inventor of any particular branch of mathematics.

This chapter gives suggestions of how certain geometric ideas might have come into existence—we really do not have the ability to go back in time and trace the road of knowledge again. There are many unanswered questions related to the origins of geometry. However, it is helpful to think of the main aspects of geometry today as emerging from four strands of early human activity that seem to have occurred in most cultures:

- art/pattern strand,
- building/structures strand,
- navigation/stargazing strand, and
- motion/machines strand.

These strands developed more or less independently into varying studies and practices that eventually from the 19th century on were woven into what we now call *geometry*.

ART/PATTERN STRAND

Geometric patterns in Stone Age rock engraving and in ochre from Blombos Cave (South Africa)

The English architect and designer Owen Jones (1809– 1874) argued that humans could first learn about the symmetries of their own bodies, and those of animals, from reflections in the water and other observations in the surrounding world. The formation of patterns by the equal division of similar lines, as in weaving, would give to early humans other notions of symmetry and repeating patterns. To produce decorations for their weaving, pottery, and other objects, early artists experimented with many symmetries and repeating patterns. The simplest geometric elements, such as line segments and triangles, would be joined by curvilinear figures for use in ornaments, for example, to decorate tools and weapons. Ancient drawings, paintings, sculptures, and ornaments are less than 100,000 years old—some found in Africa, Australia, Middle East, and Europe.

Over time, patterns possibly first used to mark the property of a certain family or tribe became more culturally charged—they were not solely geometric symbols arranged in some order, but they also reflected the environment where a particular culture was created. For example, strip decorations in Maori patterns reflect the waves of the sea surrounding New Zealand. The Inca civilization did not have writing but did have different ways of preserving and conveying important information through collections of various, mostly geometric, symbols.

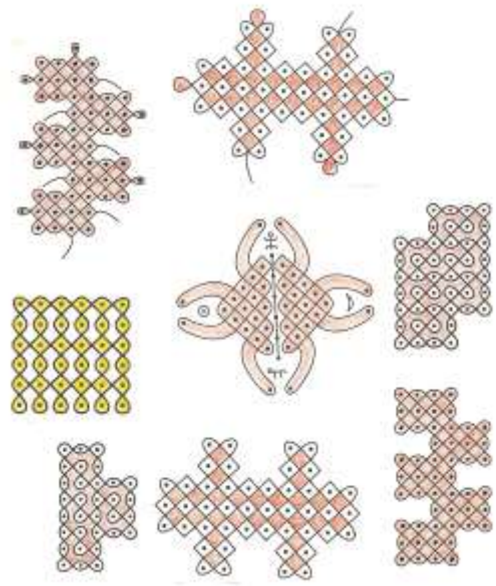
Stone Age artists carved interesting geometric designs such as spirals on large stones in their passage graves. Such passage graves have been found in Newgrange, in Ireland, and to the north of Scotland in the Orkney Islands. The significance of these spirals is not entirely clear: some historians think that they were likely associated with the seasonal decline and subsequent rise of the sun; the other speculation about the Newgrange spirals is that it may be an ancient map.

Neolithic ball with spirals (National Museum of Scotland) and spirals carved in stone at the entrance in a Neolithic period monument Newgrange, Ireland (around 3200 BC).

Spirals were used to decorate Mycenaean Greek jars as early as the fifteenth century BC. Later, arcs and circles were used to decorate amphora (or storage vessels) and kraters (or two-handled jars for mixing wine with water) in the tenth century BC, known as the Proto-geometric Period. Then, more elaborate geometrical patterns, as well as animal and human figures, were used to decorate kraters in the ninth and eighth centuries BC in the full Geometric Period.

Terracota jugs, Greek, 8th century BC and Late Cipriot ca 1600-1450 BC

The Tchokwe people of northeast Angola are famous for their decorative arts, including beautiful woven mats and baskets, pottery, and wood sculptures, and for the striking geometric designs they use to decorate walls of their homes. Beaded masks and decorated basketry have interesting traditions elsewhere in Africa, as well. The Tchokwe people have the curious ancient tradition of using drawings in sand to illustrate their stories. These drawings are called the *sona*.



Tchokwe sona patterns (drawing after Paulus Gerdes).

For ancient artisans, decorating different shapes was also the experience of geometry on different surfaces. For example, we can see curvilinear triangles on Neolithic Chinese pottery and the Neolithic people in Scotland created interesting spherical shapes with ornaments.

Earthware from China, ca. 3500–2800 BC (H. Johnson Museum of Art, Cornell University).

Altar fragment c. 800 BC (Pergamon Museum, Berlin)

Mosaics can be seen in many ancient cultures. The earliest examples of mosaics are found in Sumerian architecture of the third millennium BC as decorations of columns. Pebble pavements with random patterns had appeared as early as the eighth century BC in Asia Minor. But the first ordered patterns, and the first representation of figures and animals in mosaic, appeared around the late fifth and early fourth centuries BC in cities of the Ancient Greek world.

Tilings in Islamic art (Pergamon Museum, Berlin)

Islamic art is strongly based on various geometric figures such as equilateral triangles, squares, and many different regular polygons with sides ranging in number from 5 to 24. Geometric patterns can be found on diverse materials: tiles, bricks, wood, brass, paper, plaster, and glass. They were used on carpets, windows, doors, screens, railings, bowls, and furniture.

The most frequent shapes in Islamic geometric patterns are stars (most often with 5, 6, 8, 10, 12, and 16 rays) and rosette shapes. Some tessellations are based also on other numbers, particularly on the multiples of eight up to 96. Splendid examples of Islamic tessellations can be found in Spain, for example, in the Alhambra in Granada, and elsewhere in the Muslim world.

Alhambra, Granada, Spain.

Later, the study of symmetries of patterns led to such mathematical concepts as tilings, group theory, crystallography, and finite geometries. See examples in Chapters 1, 11, 18, and 24.

The early artists also explored various methods of representing physical objects and living things. In some Roman mosaics we can already notice attempts to represent three-dimensional objects.

Mosaics from Piazzo Armerina, Sicily.

The mathematical development of the geometry of perspective originated in the works of Renaissance sculptor, painters, and architects, like Filippo Brunelleschi (1377–1446), Battista Alberti (1404–1472), Paolo Uccello (1397–1475), Piero della Francesca (c. 1412–1492), Leonardo da Vinci (1452–1519), and Raphael (1483–1520). Piero della Francesca wrote a book on perspective (*De Prospectica pingendi*, c. 1474) and a book on the five regular or Platonic solids and five of thirteen Archimedean solids that are truncated Platonic solids (*Libbellus de quinque corporibus regularibus*, c. 1480). German artist Albrecht Dürer (1471–1528) was very interested in mathematics, like the Italian artists before him, and wrote a book on geometry (*Underweysung der Messung*, c. 1525), treating the subject of measurement with only compass and straightedge as well as other geometric topics. Dürer also wrote a book in four parts on human proportion from a truly geometric point of view (*Vier Bucher von menschlicher proportion*, 1523).

Artistic explorations in the Renaissance led to the study of perspective and then later to projective geometry and descriptive geometry in mathematics. See examples in Problems **17.6** and **20.6**.

Geometry had an impact on modern art. Interplay between art and geometry happened in the late nineteenth century and the early twentieth century when ideas about non-Euclidean space and higher-dimensional spaces gradually began to appear in non-mathematical literature. For artists, the possibility of non-Euclidean space meant that the great achievement of the Renaissance—linear perspective—would be invalid, and that became an inspiration for artistic experiments. Ultimately, the idea of higher-dimensional spaces became far more popular than the notion of curved non-Euclidean space. The Cubists saw the relationship of non-Euclidean geometries to Euclidean geometry as parallel to their own situation in the history of art. In the last two centuries, this art/pattern strand has led to security codes, digital image compression, computer aided graphics, the study of computer vision in robotics, and computer-generated movies.

For more details on the Art/Pattern Strand, see [**AD**: Albarn], [**GC**: Bain], [**EM**: Devlin], [**GC**: Eglash], [**AD**: Field], [**GC**: Gerdes], [**AD**: Ghyka], [**AD**: Gombrich], [**SG**: Hargittai], [**AD**: Ivins], [**AD**: Kappraf].

BUILDING/STRUCTURES STRAND

Swinside stone circle, Lake District, England (Wikimedia Commons)

As humans built shelters, altars, bridges, and other structures, they discovered ways to make circles of various radii, and various polygonal/polyhedral structures. In the process they devised systems of measurement and tools for measuring. The massive stones in Stonehenge and in Northern Scotland were assembled in circles so accurately that they have survived for thousands of years without significant movement. They testify to the mathematical understanding of the stresses and strains in the megalithic construction by the Neolithic engineers who designed it.

The (2000–600 B.C.) *Sulbasutram* [**AT**: Baudhayana] is written for altar builders and contains at the beginning a geometry handbook with proofs of some theorems and a clear general statement of the "Pythagorean" Theorem. This manuscript also described a problem of converting a rectangular shape to the square with the same area. (see Chapter 13). Building upon geometric knowledge from Babylonian, Egyptian, and early Greek builders and scholars, Euclid (325–265 B.C.) wrote his *Elements*, which became the most used mathematics textbook in the world for the next 2300 years and codified what we now call Euclidean geometry.

The Ishtar Gate to the inner city of Babylon 575BCE (Pergamon Museum, Berlin)

Building upon geometric knowledge from Babylonian, Egyptian, and early Greek builders and scholars, Euclid (325–265 BC) wrote his *Elements*, which became the most used mathematics textbook in the world for the next 2300 years and codified what we now call *Euclidean geometry*. Using the *Elements* as a basis, in the period 300 BC to about 1000 CE, Greek and Islamic mathematicians extended Euclid's results, refined postulates, and developed the study of conic sections and geometric algebra. The first full mathematical theory following Euclid's tradition was Apollonius' (ca. 262 BC–ca. 190 BC) *Conics*. Within Euclidean geometry, analytic geometry, vector geometry (linear algebra and affine geometry), and algebraic geometry developed later. The *Elements* also started what became known as the axiomatic method in mathematics—a method in which a few basic facts are given to be true and then other statements are proved (or disproved) based on those facts. Euclid's traditions were so strong that even Sir Isaac Newton wrote his famous *Principia Mathematica* (1687) in the language of Euclidean geometry. Eighteenth-century French mathematician, physicist, and philosopher Jean le Rond d'Alembert (1717–1783) wrote that the *Principia* is "the most extensive, the most admirable, and the happiest application of geometry to physics which has ever been made." For the next 2000 years, mathematicians attempted to prove Euclid's Fifth

(Parallel) Postulate as a theorem (based on the first four postulates); these attempts culminated around 1825 with the discovery of hyperbolic geometry. See Chapter 10 for a discussion of Euclid's postulate and various other parallel postulates. Chapters 3, 6, 9, 12, 15, 16, 19, and 23 contain many of the basic topics of Euclidean geometry.

Pergamon Altar, Greece, first half of 2nd century BC (Pergamon Museum, Berlin)

Further developments with axiomatic methods in geometry led to the axiomatic theories of the real numbers and analysis and to elliptic geometries, axiomatic projective geometry, and other axiomatic geometries. See the three sections at the end of Chapter 10 for more discussion.

For more detail on the Building Structures Strand, see [**AT**: Baudhayana], [**AD**: Blackwell], [**HI**: Burkert], [**GC**: Datta], [**ME**: DeCamp], [**TX**: Hartshorne], [**HI**: Heilbron], [**HI**: Seidenberg].

NAVIGATION/STARGAZING STRAND

Gnomon (Ancient Observatory, Beijing, China)

Life happens in cycles. Early humans must have noticed that all these cycles, whether they are human life, animal, plants, solar, lunar, or seasonal, have different characteristics, but they have a common theme: birth, growth, death. Early humans exercising their reasoning must have noticed that they can get more control over their environment if they can predict certain things happening, like season changes. Predicting lunar and solar eclipses was central to early spiritual practices. But predicting meant being able to record, measure, and compare intervals of time, angles, and distances. Recording the passage of time was needed for humans to prepare for natural events like tides, rains, floods, growing season, and hunting season. Since time is related to the motion of heavenly bodies, astronomy and timekeeping should have developed at the same time. The easiest way to mark time was with shadows. In Ancient Egypt, special shadow boards were used for measuring the time taken to perform tasks or for timing the distribution of water for irrigation. Another method of measuring the apparent position of the sun was to use a shadow stick or gnomon. This primitive device was used to see how much daylight remains in order to set up some communal events, such as mealtimes. Sundials use a gnomon to cast shadows on a marked plate. For political, religious, agricultural, and other purposes, ancient humans attempted to understand the movement of heavenly bodies (stars, planets, sun, and moon) in the apparently hemispherical sky. Early humans used the stars and planets as they started navigating over long distances on land and on the sea. They used this understanding to solve problems in navigation and in attempts to understand the shape of the earth.

Seventh century BC Babylonian map (British Museum)

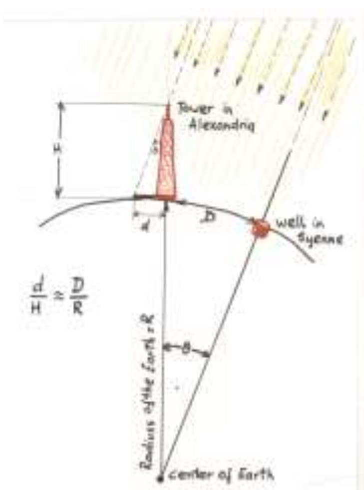
Nearly 4000 years ago, the Babylonians developed the hexadecimal system of angular measurement defining the full circle as 360 degrees (with 1 degree = 60 minutes and 1 minute = 60 seconds), corresponding approximately to the angular movement of the earth during one day in its orbit around the sun. The Babylonians had adopted the vernal equinox, marking spring as the start of their year, and the zodiac. The astronomical developments in Babylon were led from the temple and were interlinked with religion and several gods of the time.

Babylon was at its zenith between 1900 and 1600 BC, but for the following thousand years, Mesopotamia was like a battlefield. Finally, in 539 BC it fell to the Persians, who established the greatest empire then known through most of the Middle East. The other great early civilizations, such as those in Egypt, India, and China, also conducted astronomical studies that were driven by practical, astrological, and religious motives.

Early Chinese cosmology assumed the universe was a rotating sphere with fixed stars.

Aristarchus (ca. 310–230 BC) became a great scholar in Alexandria. People before him had asked questions like how far is it to the moon? To the sun? To the stars? But Aristarchus was the first one to devise geometrical methods to answer them. Aristarchus, for example, noticed that in a lunar eclipse, the sun, earth, and moon are in a line, with the moon appearing full. But when the moon passes into the shadow of the earth, the relative sizes of the bodies can be estimated from the curvature of the moon's bright disk and the curvature of the earth's dark shadow on it. Aristarchus concluded that the Earth's diameter was three times larger than the moon's. Actually, it is nearly four times larger, but Aristarchus' result is quite reasonable for the methods available to him. One of the difficulties of obtaining more accurate results was that the Earth's diameter was unknown.

About 250 BC, a young mathematician, Eratosthenes (ca. 273–195 BC), arrived in Alexandria. He knew that the Earth was round—that had been proposed already by the Pythagoreans. Eratosthenes came up with a method to estimate the radius of the Earth. He had noticed that at Syenne (present-day Aswan) the sun was directly overhead at noon on the midsummer solstice. It means that a vertical pole in Syenne would cast no shadow. At the same time a vertical pole erected in Alexandria was casting a shadow that was one fiftieth of the height of the pole. The angle involved here (about 7 degrees) represents an angle that would be the angle between poles in Alexandria and Syenne if they were to be extended to the center of the Earth. Eratosthenes measured the distance between Alexandria and Syenne to be 5000 stadia, which led him to estimate the circumference of the Earth to be 250,000 stadia. It is believed now that one Egyptian stadium was about 160 m. This means that Eratosthenes' result was 40,000 km (the present value is 39,940 km)—pretty remarkable!



Eratosthenes' measurement of the radius of the earth

Ideas of trigonometry apparently were first developed by the Babylonians in their studies of the motions of heavenly bodies. About 70 Babylonian tablets have been found, originally from the second millennium BC, that refer to the appearances of the sun, moon, and planets, as well as meteorological phenomena. Because of the importance of celestial phenomena for the understanding of events in Babylonian society, the Babylonian temple astronomers had been observing the skies for centuries and had recorded their observations in so-called astronomical diaries, astronomical catalogs of stars and other texts from the seventh century until the first century BC. This is by far the longest continuous scientific record that has ever existed. Compare that with our modern science, which has existed for only half as long. The final, mathematical, phase of Babylonian astronomy dates mainly from the third to the first centuries BC. From this period, we have the ephemerides: tablets containing tables of the computed positions of the sun, moon, or planets, day by day, or over longer periods, such as month by month. There are also tablets called procedure texts, which give schematically the rules for computing ephemerides, much like a modern computer program. Our Zodiac was also developed in Babylon.

Islamic astrolabe (Whipple Museum, Cambridge)

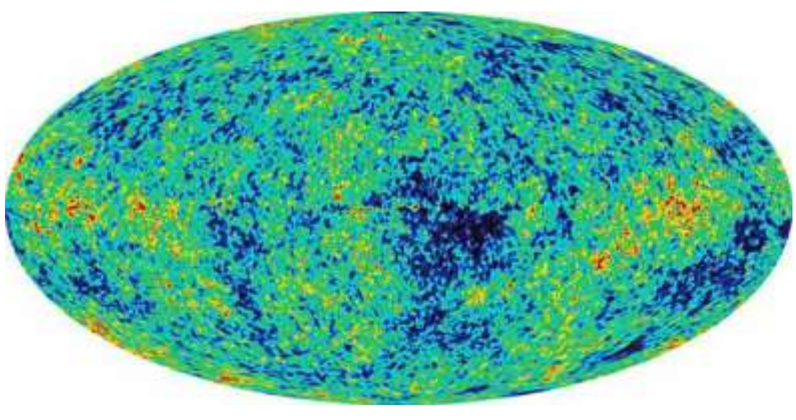
In China, a calendar had been developed by the fourth century BC. A Chinese astronomer, Shih Shen, drew up what may be the earliest star catalog, listing about 800 stars. Chinese records mention comets, meteors, large sunspots, and novas, which mean they did extensive observations and data collecting.

Observations of heavenly bodies were carried out in ancient Egypt and Babylon, mainly for astrological purposes and for making a calendar, which was important for organizing society. Claudius Ptolemy (ca. 100–178 CE), in his Almagest, cites Babylonian observations of eclipses and stars dating back to the eighth century BC. The Babylonians originated the notion of dividing a circle into 360 degrees—speculations as to why they chose this number include that it was close to the number of days in a year, it was convenient to use in their sexagesimal system of counting, and 360 is the number of ways that seven points can be placed on a circle without regard to orientation (for the ancients, there were seven “wandering bodies”—sun, moon, Mercury, Venus, Mars, Saturn, and Jupiter).

Ptolemy's system from a medieval Islamic manuscript and a model explaining Ptolemy's system (Whipple Museum, Cambridge)

The ancient Greeks became familiar with Babylonian astronomy around the fourth century BC and developed spherical geometry. Even Euclid wrote an astronomical work, *Phaenomena*, [**AT**: Berggren], in which he studied properties of curves on a sphere, using spherical geometry, which is different from the geometry on a plane that we now call Euclidean geometry.

Navigation and large-scale surveying developed over the centuries around the world and along with it cartography, trigonometry, and spherical geometry. This strand is represented wherever intrinsic geometry of surfaces is discussed, especially Chapters 4, 7, 8, and 18. Map making is discussed in Chapter 16 and trigonometry in Chapter 20. Examples most closely associated with this strand in the last two centuries are the study of surfaces and manifolds, which led to many modern spatial theories in physics and cosmology. See Chapters 2, 5, 18, 22, and 24.



Picture of the cosmic background radiation taken by NASA's WMAP satellite, released February 11, 2003

For more details on the Navigation/Stargazing Strand, see [**CE**: Bagrow], [**HI**: Burkert], [**UN**: Ferguson], [**UN**: Osserman], [**SP**: Todhunter].

MOTION/MACHINES STRAND

The first uses of some kind of mechanical device probably were log rollers placed beneath a load to be moved, as in the Paleolithic era (15,000–75,000 years ago). Based on a diagram found on ancient clay tablets, the earliest known use of the wheel was a potter's wheel that was used at Ur in Mesopotamia as early as 3500 BC. It is possible that there was an independent discovery of the wheel in China around 2800 BC, but there has been less historical evidence for this.

Etruscans pots produced on a potter's wheel

The first use of the wheel in transportation was in Mesopotamian chariots around 3200 BC. A wheel with spokes first appeared in Egyptian chariots around 2000 BC, and wheels seem to have developed in Europe around Sundial (Ancient Observatory., Beijing, China)

Wagon driven by bulls, early Bronze 2400–2000 BC,
(Metropolitan Museum of Art, New York)

1400 BC without any influence from the Middle East. Celtic chariots introduced an iron rim around the spiked wheel, and this design, still unchanged, is used in horse carriages today! Despite the overwhelming utility of the wheel, some civilizations failed to discover it, for example, those of sub-Saharan Africa, Australia, and the Americas. Archeologists have found some children's toys from the Incan civilization that suggest that this society was at least familiar with wheel-like shapes, but they apparently were not used for utilitarian purposes.

Scholars from Harvard University and the Max Planck Institute for the History of Science in Berlin analyzed technical treatises and literary sources dating back to the fifth century BC and found several mechanisms in use among practitioners with limited theoretical knowledge. For example, they found that the steelyard—a balance with unequal arms—was in use as early as the fourth and fifth centuries BC, before Archimedes and other thinkers of the Hellenistic era gave a mathematical explanation of its use, using the law of the lever.

In ancient Greece, Archimedes, Heron, and other geometers used linkages (straight sticks pinned together in a way that they can move) and gears (wheels with pins) to solve geometric problems, such as trisecting an angle, duplicating a cube, and squaring a circle (finding a square with the same area as a given circle). These solutions were not accepted in the building/structures strand, which leads to a common misconception that these problems are unsolvable and/or that Greeks did not allow motion in geometry. The truth is that one cannot solve these problems using only a compass and unmarked straightedge sequence. See Problem **15.4**.

Why did solvability of these three problems become so important in geometry? They really did not have such a big importance in practical applications. They are also not fundamental problems—there is no particular theory based on them. These three ancient problems became famous just because so many people tried to solve them, and these attempts actually led to many new methods in mathematics. See [**TX**: Martin] and [**HI**: Katz].

It seems that motion was first explored in connection with astronomy (the geometry of the heavens), where planetary motion was translated into geometrical terms so that techniques of Euclid's *Elements* could be applied. About 365 BC, the Greek scholar Eudoxus visited Egypt, where he acquired from the priests of Heliopolis knowledge of planetary motions and Chaldean astrology. Later he completed his book *On Speeds* about motions within our solar system (perhaps his greatest, but lost, writings). Eudoxus became the first mathematician seriously to attempt to describe the intricate motions of celestial bodies using a mathematical model based on spherical geometry. Geometry and motion came closer together for ancient engineers. According to some ancient references, one of the first mechanical solutions to the three famous problems was offered by Menaechmus (ca. 380 BC–320 BC), math tutor to Alexander the Great, but there are no actual accounts of it available. Around 420 BC Hippias devised the mechanism that would draw a curve, called a quadratrix, which can be used to trisect angles and square circles.

Plato criticized this mechanistic approach and called instead for a purely theoretical solution. "Motion" would involve mechanics and experiments; it means manual work, but in ancient Greece that meant "fit only for slaves." As Aristotle (384 BC–322 BC) wrote: "These inferior persons should never be admitted to citizenship because no man can practice virtue that is living the life of a mechanic or laborer." Still, the oldest known engineering textbook is attributed to Aristotle, though some authors think that it was written by his student Straton. In this book, we find the first mention of gear wheels. The Romans and Greeks made wise use of gearing in clocks and astronomical devices. Gears were also used to measure distance or speed.

Antikythera mechanism (Wikimedia Commons)

One of the most interesting relics from ancient Greece is the Antikythera mechanism, which is an astronomical computer. It had many gears in it, some of which were planetary gears. Just before the opening of the 2008 Olympic Games, scientists announced a discovery that this mechanism was a complex clock that combines calendars and also showed the four-year cycle of the Ancient Greek games.16 The mechanism was housed in a wooden case. Like a clock it had a large circular face with at least seven rotating hands which would represent the Sun, the Moon, and planets visible by naked eye (Mercury, Venus, Jupiter, Mars, Saturn). It had a handle and as it was turned, trains of interlocking gears drove rotating hands at various speeds displaying celestial time instead of hours and minutes. James Evans, a historian of astronomy, thinks that the eclipse cycle represented on the Antikythera mechanism is Babylonian in origin and begins in 205 BC. He speculates that maybe it was Hipparchus who worked out mathematics behind this device because Hipparchus is known for combining numerical Babylonian prediction traditions with Greek geometry.

Contemporary auger is the same Archimedean screw still used after more than 2000 years

The greatest geometer and engineer of antiquity was Archimedes (ca. 282 BC–ca. 212 BC). Plutarch wrote about him: He would not deign to leave behind him any writings on his mechanical discoveries. He regarded the business of engineering, and indeed of every art which misters to the material needs of life, as an ignoble and sordid activity, and he concentrated his ambition exclusively upon those speculations whose beauty and subtlety are untainted by the claims of necessity…. Certainly in the whole science of geometry it is impossible to find more difficult and intricate problems handled in simpler and purer terms than in his works. In fact, however, there is evidence that Archimedes did write on certain mechanical subjects, for example, his book *On Sphere Making* or *The Method*—a discovery by mechanics of many important results about areas and volumes.

Two geometers – David Hendersona and Archimedes, Royal Palace, Palermo, Sicily

Since ancient times, mechanisms were used for drawing curves. For example, the trammel is the simplest mechanism for drawing ellipses. It was described by Proclus, but it is also attributed to Archimedes.

Trammel as a toy

Al-Tusi (1201–1274) was among the first of several Persian astronomers who found some serious shortcomings in Ptolemy's planetary model based on mechanical principles and modified it. He devised a few instruments for astronomical observations, but his best-known device is the so-called *Tusi couple*. Later, this mechanism was called in kinematics "a planetary motion mechanism" and was used as one of the straight-line mechanisms in order to convert circular motion into straight line motion in machines (see discussion below).

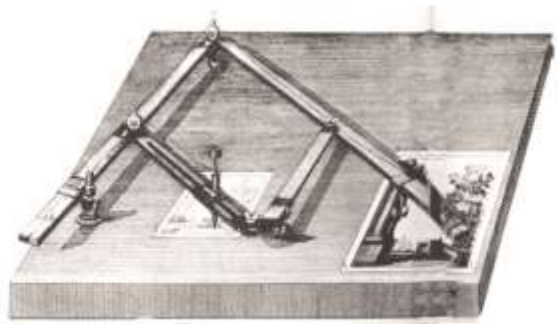
Planetary gear or Tusi couple mechanism
(Kinematic model collection, Cornell University, photo Prof. Francis Moon)

The philosophical approach to the description of motion mathematically was done in the thirteenth century in the so-called Merton School by Thomas Bradwardine (ca. 1290–1349) and others. In the fourteenth century, motion was discussed in writings by Jean Buridan (ca. 1300–ca. 1358) and Nicole Oresme (ca. 1320–1382). Oresme represented motions geometrically by plotting primitive graphs. Motion and mathematics were important objects of interest in research done by Tycho Brahe (1546–1601) and Galileo Galilei (1564–1642).

One of the most significant turning points in the development in technology was learning how to transform continuous circular motion into reciprocal or straight-line motion. Rotary motion was available to humans using nature forces—waterwheels, windmills. But this kind of motion was not enough—for example, to saw logs into boards, rectilinear motion was needed. This transformation was achieved by the use of gears and linkages. Both later became important subjects of mathematical interest. To construct the most efficient shape of gear teeth, geometers were studying cycloids (Nicolas of Cusa in 1451, Galileo in 1599) and epicycloids (Albrect Dürer in 1525). Apollonius and Ptolemy were discussing the motion of planets in geometrical terms, and that is where a mention of epicycloids appears for the first time.

In 1557, Girolamo Cardano first published a mathematical theory of gears. In 1694, Philippe de la Hire published a full mathematical analysis of epicycloids and recommended involute curves for gearing, but in practice it was not used for another 150 years. In 1733, Charles Camus expanded la Hire's work and developed theories of mechanisms. In 1754, Leonard Euler worked out design principles for involute gearing.

Another mechanism based on geometrical proportions and known since ancient times is a pantograph. It can be called the earliest copying machine, making exact duplicates of written documents. Artists adopted pantographs for duplicating drawings and enlarging sketches. One of them was Leonardo da Vinci, who used a pantograph to duplicate his sketch on canvas. Later pantographs were adopted specifically for duplicating paintings—first the pantograph would be used to trace the outlines and then the shapes would be filled with the paint. Sculptors and carvers adopted pantographs for tracing master drawings onto blocks of marble or wood. In the eighteenth century, the pantograph was used to cut out typeset letters for printings and engravings. In the nineteenth century, pantographs were advanced enough to duplicate sculptures. One of the first such duplicated sculptures was Michelangelo's sculpture *David*. Heavy-duty pantographs are still used for engraving and contour milling.

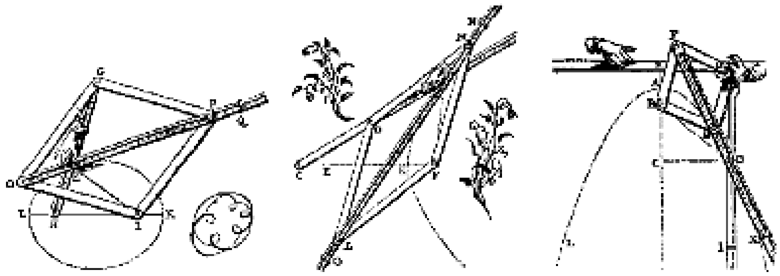


Pantograph (Diderot and d'Alambert *Encyclopedia*, 1755–1780)

Leonardo da Vinci had ideas about several mechanisms that would trace various mathematical curves. Mechanical devices for drawing curves were used also by Albrecht Durer.

By the beginning of the seventeenth century, mathematicians had developed a new "language" for representing various arithmetic concepts and relationships: symbolic algebra. Geometry, however, still was considered as the more trusted form of expressing mathematical thought, partially due to the tradition of authority of Euclid's *Elements*, where even the concepts of number theory were expressed in geometrical form. The scientific revolution prompted experiments in representing geometric concepts and constructions in this new symbolic language. In seventeenth-century Europe, questions of appropriate forms of representation were dominant intellectual activities. They appeared not only in mathematics and physics but maybe even more in religious, political, legal, and philosophical discussions.

Therefore, it is not surprising that Descartes and Leibniz were paying so much attention to symbolic representations of their mathematical ideas; they considered these investigations as part of their extensive philosophical works. Descartes' *Geometry* was originally published as an appendix to his philosophical work *Discourse of the Method*. Political thinkers of the time, like Thomas Hobbes, commented on the latest developments in mathematics and physics. Descartes *Geometry* is considered to be the start of analytical geometry—using algebraic methods for solving geometry problems. But nowhere in his book had he written symbolic equations. He studied curves that were constructed by mechanical devices. After the curves had been drawn, Descartes would introduce coordinates and analyze the motion that resulted in the particular curve and arrive at an "equation" written out as a sentence that would represent this curve. Curves were creating equations and not the other way around—the way we are used to studying curves today. Descartes used equations to create a taxonomy of curves. He could assume his audience would be familiar with Euclid's *Elements* and Apollonius' *Conic Sections*.



Schooten's illustrations for Descartes' *Geometry*

Descartes was not alone: independently Fermat came up with ideas of analytical geometry. Roberval, Wallis, Cavalieri, and Newton all tried to express their geometrical findings about the motion in symbolic language. These efforts culminated in creating calculus by the late seventeenth century

As we will discuss in Chapter 21, there was an interaction between mathematics and mechanics that led to marvelous machine design and continues to the modern mathematics of rigidity and robotics.

For more details on the Motion/Machines Strand, see [**ME**: De- Camp], [**ME**: Dyson], [**ME**: Ferguson 2001], [**ME**: Kirby], [**ME**: Moon], [**ME**: Ramelli], [**ME**: Williams].