CHAPTER XIV

SPECIAL FUNCTIONS DEFINED BY INTEGRALS

147. The Gamma and Beta functions. The two integrals

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx, \qquad \mathrm{B}(m,\, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{1}$$

converge when $n > 0$ and $m > 0$, and hence define functions of the parameters n or n and m for all positive values, zero not included. Other forms may be obtained by changes of variable. Thus

$$\Gamma(n) = 2\int_0^\infty y^{2n-1}e^{-y^2}dy, \qquad \text{by} \quad x = y^2, \tag{2}$$

$$\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy, \qquad \text{by} \quad e^{-x} = y, \tag{3}$$

$$\mathrm{B}(m,\, n) = \int_0^1 y^{n-1}(1-y)^{m-1}dy = \mathrm{B}(n,\, m), \quad \text{by} \quad x = 1 - y, \tag{4}$$

$$\mathrm{B}(m,\, n) = \int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}}, \qquad \text{by} \quad x = \frac{y}{1+y}, \tag{5}$$

$$\mathrm{B}(m,\, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi \cos^{2n-1}\phi\, d\phi, \qquad \text{by} \quad x = \sin^2\phi. \tag{6}$$

If the original form of $\Gamma(n)$ be integrated by parts, then

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \frac{1}{n}x^n e^{-x}\bigg]_0^\infty + \frac{1}{n}\int_0^\infty x^n e^{-x}dx = \frac{1}{n}\Gamma(n+1).$$

The resulting relation $\Gamma(n+1) = n\Gamma(n)$ shows that the values of the Γ-function for $n+1$ may be obtained from those for n, and that consequently the values of the function will all be determined if the values over a unit interval are known. Furthermore

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) \\ &= n(n-1)\cdots(n-k)\Gamma(n-k) \end{aligned} \tag{7}$$

is found by successive reduction, where k is any integer less than n. If in particular n is an integer and $k = n - 1$, then

$$\Gamma(n+1) = n(n-1)\cdots 2\cdot 1\cdot \Gamma(1) = n!\,\Gamma(1) = n!; \tag{8}$$

since when $n = 1$ a direct integration shows that $\Gamma(1) = 1$. Thus *for integral values of n the Γ-function is the factorial;* and for other than integral values it may be regarded as a sort of generalization of the factorial.

Both the Γ- and B-functions are continuous for all values of the parameters greater than, but not including, zero. To prove this it is sufficient to show that the convergence is uniform. Let n be any value in the interval $0 < n_0 \leqq n \leqq N$; then

$$\int_0 x^{n-1}e^{-x}dx \leqq \int_0 x^{n_0-1}e^{-x}dx, \qquad \int^{\infty} x^{n-1}e^{-x}dx \leqq \int^{\infty} x^{N-1}e^{-x}dx.$$

The two integrals converge and the general test for uniformity (§ 144) therefore applies; the application at the lower limit is not necessary except when $n < 1$. Similar tests apply to $\mathrm{B}(m, n)$. Integration with respect to the parameter may therefore be carried under the sign. The derivatives

$$\frac{d^k\Gamma(n)}{dn^k} = \int_0^{\infty} x^{n-1}e^{-x}(\log x)^k dx \tag{9}$$

may also be had by differentiating under the sign; for these derived integrals may likewise be shown to converge uniformly.

By multiplying two Γ-functions expressed as in (2), treating the product as an iterated or double integral extended over a whole quadrant, and evaluating by transformation to polar coördinates (all of which is justifiable by § 146, since the integrands are positive and the processes lead to a determinate result), the B-function may be expressed in terms of the Γ-function.

$$\Gamma(n)\Gamma(m) = 4\int_0^{\infty} x^{2n-1}e^{-x^2}dx \int_0^{\infty} y^{2m-1}e^{-y^2}dy = 4\int_0^{\infty} x^{2n-1}y^{2m-1}e^{-x^2-y^2}dxdy$$

$$= 4\int_0^{\infty} r^{2n+2m-1}e^{-r^2}dr \int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi \cos^{2n-1}\phi d\phi = \Gamma(n+m)\,\mathrm{B}(m, n).$$

Hence
$$\mathrm{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \mathrm{B}(n, m). \tag{10}$$

The result is symmetric in m and n, as must be the case inasmuch as the B-function has been seen by (4) to be symmetric.

That $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ follows from (9) of § 143 after setting $n = \frac{1}{2}$ in (2); it may also be deduced from a relation of importance which is obtained from (10) and (5), and from (8) of § 142, namely, if $n < 1$,

$$\frac{\Gamma(n)\Gamma(1-n)}{\Gamma(1) = 1} = \mathrm{B}(n, 1-n) = \int_0^{\infty} \frac{y^{n-1}}{1+y}dy = \frac{\pi}{\sin n\pi},$$

or
$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}. \tag{11}$$

As it was seen that all values of $\Gamma(n)$ could be had from those in a unit interval, say from 0 to 1, the relation (11) shows that the interval may be further reduced to $\frac{1}{2} \leqq n \leqq 1$ and that the values for the interval $0 < 1 - n < \frac{1}{2}$ may then be found.

148. By suitable changes of variable a great many integrals may be reduced to B- and Γ-integrals and thus expressed in terms of Γ-functions. Many of these types are given in the exercises below; a few of the most important ones will be taken up here. By $y = ax$,

$$\int_0^a x^{m-1}(a-x)^{n-1}dx = a^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}dy = a^{m+n-1}\mathrm{B}(m, n)$$

or
$$\int_0^a x^{m-1}(a-x)^{n-1} = a^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad a > 0. \tag{12}$$

Next let it be required to evaluate the triple integral

$$I = \iiint x^{l-1}y^{m-1}z^{n-1}dxdydz, \qquad x + y + z \leqq 1,$$

over the volume bounded by the coördinate planes and $x + y + z = 1$, that is, over all positive values of x, y, z such that $x + y + z \leqq 1$. Then

$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}y^{m-1}z^{n-1}dzdydx$$

$$= \frac{1}{n}\int_0^1\int_0^{1-x} x^{l-1}y^{m-1}(1-x-y)^n dydx.$$

By (12) $\displaystyle\int_0^{1-x} y^{m-1}(1-x-y)^n dy = \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n)}(1-x)^{m+n}.$

Then
$$I = \frac{\Gamma(m)\Gamma(n+1)}{n\Gamma(m+n)}\int_0^1 x^{l-1}(1-x)^{m+n}dx$$

$$= \frac{\Gamma(m)\Gamma(n+1)}{n\Gamma(m+n)}\,\frac{\Gamma(l)\Gamma(m+n)}{\Gamma(l+m+n+1)}.$$

This result may be simplified by (7) and by cancellation. Then

$$I = \iiint x^{l-1}y^{m-1}z^{n-1}dxdydz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}. \tag{13}$$

There are simple modifications and generalizations of these results which are sometimes useful. For instance if it were desired to evaluate I over the range of positive values such that $x/a + y/b + z/c \leqq h$, the change $x = ah\xi$, $y = bh\eta$, $z = ch\zeta$ gives

$$I = a^l b^m c^n h^{l+m+n}\iiint \xi^{l-1}\eta^{m-1}\zeta^{n-1}d\xi d\eta d\zeta, \qquad \xi + \eta + \zeta \leqq 1,$$

$$I = \iiint x^{l-1}y^{m-1}z^{n-1}dxdydz = a^l b^m c^n\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}h^{l+m+n}, \qquad \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leqq h.$$

The value of this integral extended over the lamina between two parallel planes determined by the values h and $h + dh$ for the constant h would be

$$dI = a^l b^m c^n \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} h^{l+m+n-1} dh.$$

Hence if the integrand contained a function $f(h)$, the reduction would be

$$\iiint x^{l-1} y^{m-1} z^{n-1} f\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) dxdydz$$
$$= a^l b^m c^n \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} \int_0^H f(h) h^{l+m+n-1} dh$$

if the integration be extended over all values $x/a + y/b + z/c \leqq H$.

Another modification is to the case of the integral extended over a volume

$$J = \iiint x^{l-1} y^{m-1} z^{n-1} dxdydz, \qquad \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leqq h,$$

which is the octant of the surface $(x/a)^p + (y/b)^q + (z/c)^r = h$. The reduction to

$$J = \frac{a^l b^m c^n h^{\frac{l}{p}+\frac{m}{q}+\frac{n}{r}}}{pqr} \iiint \xi^{\frac{l}{p}-1} \eta^{\frac{m}{q}-1} \zeta^{\frac{n}{r}-1} d\xi d\eta d\zeta, \qquad \xi + \eta + \zeta \leqq 1,$$

is made by $\xi h = \left(\frac{x}{a}\right)^p$, $\eta h = \left(\frac{y}{b}\right)^q$, $\zeta h = \left(\frac{z}{c}\right)^r$, $dx = \frac{a}{p} h^{\frac{1}{p}} \xi^{\frac{1}{p}-1}, \cdots$.

$$J = \iiint x^{l-1} y^{m-1} z^{n-1} dxdydz = \frac{a^l b^m c^n}{pqr} \frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)} h^{\frac{l}{p}+\frac{m}{q}+\frac{n}{r}}.$$

This integral is of importance because the bounding surface here occurring is of a type tolerably familiar and frequently arising; it includes the ellipsoid, the surface $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$, the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$. By taking $l = m = n = 1$ the volumes of the octants are expressed in terms of the Γ-function; by taking first $l = 3$, $m = n = 1$, and then $m = 3$, $l = n = 1$, and adding the results, the moments of inertia about the z-axis are found.

Although the case of a triple integral has been treated, the results for a double integral or a quadruple integral or integral of higher multiplicity are made obvious. For example,

$$\iint x^{l-1} y^{m-1} dxdy = a^l b^m h^{l+m} \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}, \qquad \frac{x}{a} + \frac{y}{b} \leqq h,$$

$$\iint x^{l-1} y^{m-1} dxdy = \frac{a^l b^m}{pq} \frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+1\right)} h^{\frac{l}{p}+\frac{m}{q}}, \qquad \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q \leqq h,$$

$$\iint x^{l-1} y^{m-1} f\left[\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q\right] dxdy = \frac{a^l b^m}{pq} \frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}\right)} \int_0^H f(h) h^{\frac{l}{p}+\frac{m}{q}-1} dh,$$
$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q \leqq H,$$

$$\iiiint x^{k-1}y^{l-1}z^{m-1}t^{n-1}dxdydzdt = \frac{a^kb^lc^md^n}{pqrs}\frac{\Gamma\left(\frac{k}{p}\right)\Gamma\left(\frac{l}{q}\right)\Gamma\left(\frac{m}{r}\right)\Gamma\left(\frac{n}{s}\right)}{\Gamma\left(\frac{k}{p}+\frac{l}{q}+\frac{m}{r}+\frac{n}{s}+1\right)},$$

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r+\left(\frac{t}{d}\right)^s \leqq 1.$$

149. If the product (11) be formed for each of $\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}$, and the results be multiplied and reduced by Ex. 19 below, then

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}. \tag{14}$$

The logarithms may be taken and the result be divided by n.

$$\sum_{k=1}^{n}\log\Gamma\left(\frac{k}{n}\right)\cdot\frac{1}{n}=\left(\frac{1}{2}-\frac{1}{2n}\right)\log 2\pi-\frac{1}{2}\frac{\log n}{n}.$$

Now if n be allowed to become infinite, the sum on the left is that formed in computing an integral if $dx = 1/n$. Hence

$$\lim_{n=\infty}\sum\log\Gamma(x_i)\Delta x=\int_0^1\log\Gamma(x)\,dx=\log\sqrt{2\pi}. \tag{15}$$

Then
$$\int_0^1\log\Gamma(a+x)\,dx=a(\log a-1)+\log\sqrt{2\pi} \tag{15'}$$

may be evaluated by differentiating under the sign (Ex. 12 (θ), p. 288).

By the use of differentiation and integration under the sign, the expressions for the first and second logarithmic derivatives of $\Gamma(n)$ and for $\log\Gamma(n)$ itself may be found as definite integrals. By (9) and the expression of Ex. 4 (α), p. 375, for $\log x$,

$$\Gamma'(n)=\int_0^\infty x^{n-1}e^{-x}\log x\,dx=\int_0^\infty x^{n-1}e^{-x}\int_0^\infty\frac{e^{-\alpha}-e^{-\alpha x}}{\alpha}\,d\alpha dx.$$

If the iterated integral be regarded as a double integral, the order of the integrations may be inverted; for the integrand maintains a positive sign in the region $1 < x < \infty$, $0 < \alpha < \infty$, and a negative sign in the region $0 < x < 1$, $0 < \alpha < \infty$, and the integral from 0 to ∞ in x may be considered as the sum of the integrals from 0 to 1 and from 1 to ∞, — to each of which the inversion is applicable (§ 146) because the integrand does not change sign and the results (to be obtained) are definite. Then by Ex. 1 (α),

$$\Gamma'(n)=\int_0^\infty\int_0^\infty x^{n-1}e^{-x}\frac{e^{-\alpha}-e^{-\alpha x}}{\alpha}\,dxd\alpha=\Gamma(n)\int_0^\infty\left(e^{-\alpha}-\frac{1}{(1+\alpha)^n}\right)\frac{d\alpha}{\alpha}$$

or $$\frac{\Gamma'(n)}{\Gamma(n)} = \frac{d}{dn}\log\Gamma(n) = \int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)^n}\right)\frac{d\alpha}{\alpha}. \tag{16}$$

This value may be simplified by subtracting from it the particular value $-\gamma = \Gamma'(1)/\Gamma(1) = \Gamma'(1)$ found for $n = 1$. Then

$$\frac{\Gamma'(n)}{\Gamma(n)} - \frac{\Gamma'(1)}{\Gamma(1)} = \frac{\Gamma'(n)}{\Gamma(n)} + \gamma = \int_0^\infty \left(\frac{1}{1+\alpha} - \frac{1}{(1+\alpha)^n}\right)\frac{d\alpha}{\alpha}.$$

The change of $1+\alpha$ to $1/\alpha$ or to e^α gives

$$\frac{\Gamma'(n)}{\Gamma(n)} + \gamma = \int_0^1 \frac{1-\alpha^{n-1}}{1-\alpha}\,d\alpha = \int_0^\infty \frac{e^{-\alpha} - e^{-\alpha n}}{1-e^{-\alpha}}\,d\alpha. \tag{17}$$

Differentiate: $$\frac{d^2}{dn^2}\log\Gamma(n) = \int_0^\infty \frac{\alpha e^{-\alpha n}}{1-e^{-\alpha}}\,d\alpha. \tag{18}$$

To find $\log\Gamma(n)$ integrate (16) from $n = 1$ to $n = n$. Then

$$\log\Gamma(n) = \int_0^\infty \left[(n-1)e^{-\alpha} - \frac{(1+\alpha)^{-1} - (1+\alpha)^{-n}}{\log(1+\alpha)}\right]\frac{d\alpha}{\alpha}, \tag{19}$$

since $\Gamma(1) = 1$ and $\log\Gamma(1) = 0$. As $\Gamma(2) = 1$,

$$\log\Gamma(2) = 0 = \int_0^\infty \left[\frac{e^{-\alpha}}{\alpha} - \frac{(1+\alpha)^{-2}}{\log(1+\alpha)}\right]d\alpha,$$

and $$\log\Gamma(n) = \int_0^\infty \left[\frac{n-1}{(1+\alpha)^2} - \frac{(1+\alpha)^{-1} - (1+\alpha)^{-n}}{\alpha}\right]\frac{d\alpha}{\log(1+\alpha)}$$

by subtracting from (19) the quantity $(n-1)\log\Gamma(2) = 0$. Finally

$$\log\Gamma(n) = \int_{-\infty}^0 \left[\frac{e^{\alpha n} - e^\alpha}{e^\alpha - 1} - (n-1)e^\alpha\right]\frac{d\alpha}{\alpha} \tag{19'}$$

if $1+\alpha$ be changed to $e^{-\alpha}$. The details of the reductions and the justification of the differentiation and integration will be left as exercises.

An approximate expression or, better, an *asymptotic expression*, that is, an expression with *small percentage error*, may be found for $\Gamma(n+1)$ when n is *large*. Choose the form (2) and note that the integrand $y^{2n+1}e^{-y^2}$ rises from 0 to a maximum at the point $y^2 = n + \frac{1}{2}$ and falls away again to 0. Make the change of variable $y = \sqrt{\alpha} + w$, where $\alpha = n + \frac{1}{2}$, so as to bring the origin under the maximum. Then

$$\Gamma(n+1) = 2\int_{-\sqrt{\alpha}}^\infty (\sqrt{\alpha} + w)^{2\alpha} e^{-\alpha - 2\sqrt{\alpha}w - w^2}\,dw,$$

or $$\Gamma(n+1) = 2\alpha^\alpha e^{-\alpha}\int_{-\sqrt{\alpha}}^\infty e^{2\alpha\log\left(1+\frac{w}{\sqrt{\alpha}}\right) - 2\sqrt{\alpha}w - w^2}\,dw.$$

Now $$2\alpha\log\left(1+\frac{w}{\sqrt{\alpha}}\right) - 2\sqrt{\alpha}w \leqq 0, \qquad -\sqrt{\alpha} < w < \infty.$$

The integrand is therefore always less than e^{-w^2}, except when $w \doteq 0$ and the integrand becomes 1. Moreover, as w increases, the integrand falls off very rapidly, and the chief part of the value of the integral may be obtained by integrating between rather narrow limits for w, say from -3 to $+3$. As α is large by hypothesis, the value of $\log\left(1 + w/\sqrt{\alpha}\right)$ may be obtained for small values of w from Maclaurin's Formula. Then

$$\Gamma(n+1) = 2\,\alpha^{\alpha}e^{-\alpha}\int_{-c}^{c} e^{-2w^2(1-\epsilon)}dw$$

is an approximate form for $\Gamma(n+1)$, where the quantity ϵ is about $\frac{2}{3}\,w/\sqrt{\alpha}$ and where the limits $\pm c$ of the integral are small relative to $\sqrt{\alpha}$. But as the integrand falls off so rapidly, there will be little error made in extending the limits to ∞ after dropping ϵ. Hence approximately

$$\Gamma(n+1) = 2\,\alpha^{\alpha}e^{-\alpha}\int_{-\infty}^{\infty} e^{-2w^2}\,dw = \sqrt{2\pi}\alpha^{\alpha}e^{-\alpha},$$

or $$\Gamma(n+1) = \sqrt{2\pi}\,(n+\tfrac{1}{2})^{n+\frac{1}{2}}\,e^{-(n+\frac{1}{2})}(1+\eta), \tag{20}$$

where η is a small quantity approaching 0 as n becomes infinite.

EXERCISES

1. Establish the following formulas by changes of variable.

(α) $\Gamma(n) = \alpha^n\int_0^{\infty} x^{n-1}e^{-\alpha x}dx,\quad \alpha > 0,$ (β) $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{n}{2}+\frac{1}{2},\,\frac{1}{2}\right),$

(γ) $\mathrm{B}(n, n) = 2^{1-2n}\,\mathrm{B}(n, \tfrac{1}{2})$ by (6), (δ) $\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \frac{1}{2}\,\mathrm{B}(\frac{1}{2}m, n),$

(ϵ) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx = \frac{\mathrm{B}(m,n)}{a^n(1+a)^m} = \frac{1}{a^n(1+a)^m}\,\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, take $\frac{x}{x+a} = \frac{y}{1+a}$,

(ζ) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}dx}{[ax+b(1-x)]^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{a^m b^n\Gamma(m+n)}$, take $x = \frac{by}{a(1-y)+by}$,

(η) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}dx}{(b+cx)^{m+n}} = \frac{\mathrm{B}(m,n)}{b^n(b+c)^m},$ (θ) $\int_0^1 \frac{x^{2n}dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n+1)},$

(ι) $\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\,\mathrm{B}\left(p+1,\,\frac{m+1}{n}\right),$ (κ) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\,\frac{\Gamma(n^{-1})}{\Gamma(n^{-1}+\frac{1}{2})}.$

2. From $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ make a table of the values for every integer and half integer from 0 to 5 and plot the curve $y = \Gamma(x)$ from them.

3. By the aid of (10) and Ex. 1 (γ) prove the relations

$$\sqrt{\pi}\Gamma(2a) = 2^{2a-1}\Gamma(a)\,\Gamma(a+\tfrac{1}{2}), \qquad \sqrt{\pi}\Gamma(n) = 2^{n-1}\Gamma(\tfrac{1}{2}n)\,\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}).$$

4. Given that $\Gamma(1.75) = 0.9191$, add to the table of Ex. 2 the values of $\Gamma(n)$ for every quarter from 0 to 3 and add the points to the plot.

5. With the aid of the Γ-function prove these relations (see Ex. 1):

$$(\alpha)\ \int_0^{\frac{\pi}{2}}\sin^n x dx = \int_0^{\frac{\pi}{2}}\cos^n x dx = \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\frac{\pi}{2} \quad\text{or}\quad \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n},$$

as n is even or odd.

$$(\beta)\ \int_0^1 \frac{x^{2n}dx}{\sqrt{1-x^2}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{\pi}{2}, \qquad (\gamma)\ \int_0^1 \frac{x^{2n+1}dx}{\sqrt{1-x^2}} = \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)},$$

$$(\delta)\ \int_0^a x^2\sqrt{a^2-x^2}dx = \frac{\pi a^4}{16}, \qquad (\epsilon)\ \int_0^a x^2(a^2-x^2)^{\frac{3}{2}}dx = \frac{3\pi a^6}{96},$$

(ζ) Find $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ to four decimals, $\quad(\eta)$ Find $\int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}}$.

6. Find the areas of the quadrants of these curves:

(α) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $\quad(\beta)$ $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $\quad(\gamma)$ $x^2 + y^{\frac{2}{3}} = 1$,

(δ) $x^2/a^2 + y^2/b^2 = 1$, $\quad(\epsilon)$ the evolute $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.

7. Find centers of gravity and moments of inertia about the axes in Ex. 6.

8. Find volumes, centers of gravity, and moments of inertia of the octants of

(α) $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$, $\quad(\beta)$ $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$, $\quad(\gamma)$ $x^2 + y^2 + z^{\frac{2}{3}} = 1$.

9. (α) The sum of four proper fractions does not exceed unity; find the average value of their product. (β) The same if the sum of the squares does not exceed unity. (γ) What are the results in the case of k proper fractions?

10. Average $e^{-ax^2-by^2}$ under the supposition $ax^2 + by^2 \leqq H$.

11. Evaluate the definite integral (15′) by differentiation under the sign.

12. From (18) and $1 < \frac{\alpha}{1-e^{-\alpha}} < 1 + \alpha$ show that the magnitude of $D^2 \log\Gamma(n)$ is about $1/n$ for large values of n.

13. From Ex. 12, and Ex. 23, p. 76, show that the error in taking

$$\log\Gamma\left(n+\frac{1}{2}\right) \quad\text{for}\quad \int_n^{n+1}\log\Gamma(x)dx \quad\text{is about}\quad \frac{1}{24n+12}\log\Gamma\left(n+\frac{1}{2}\right).$$

14. Show that $\int_n^{n+1}\log\Gamma(x)dx = \int_0^1\log\Gamma(n+x)dx$ and hence compare (15′), (20), and Ex. 13 to show that the small quantity η is about $(24n+12)^{-1}$.

15. Use a four-place table to find the logarithms of 5! and 10!. Find the logarithms of the approximate values by (20), and determine the percentage errors.

16. Assume $n = 11$ in (17) and evaluate the first integral. Take the logarithmic derivative of (20) to find an approximate expression for $\Gamma'(n)/\Gamma(n)$, and in particular compute the value for $n = 11$. Combine the results to find $\gamma = 0.578$. By more accurate methods it may be shown that Euler's Constant $\gamma = 0.577,215,665\cdots$.

17. Integrate (19′) from n to $n + 1$ to find a definite integral for (15′). Subtract the integrals and add $\frac{1}{2}\log n = \int_{-\infty}^0 \frac{e^{\alpha n}-e^{\alpha}}{2}\frac{d\alpha}{\alpha}$. Hence find

$$\log\Gamma(n) - n(\log n - 1) - \log\sqrt{2\pi} + \frac{1}{2}\log n = \int_{-\infty}^0\left[\frac{1}{e^{\alpha}-1} - \frac{1}{\alpha} + \frac{1}{2}\right]e^{\alpha n}\frac{d\alpha}{\alpha}.$$

18. Obtain *Stirling's approximation*, $\Gamma(n+1) = \sqrt{2\pi n}\, n^n e^{-n}$, either by comparing it with the one already found or by applying the method of the text, with the substitution $x = n + \sqrt{2n}\,y$, to the original form (1) of $\Gamma(n+1)$.

19. The relation $\prod_{k=1}^{k=n-1} \sin\frac{k\pi}{n} = \sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$ may be obtained from the roots of unity (§ 72); for $x^n - 1 = (x-1)\prod\left(x - e^{-\frac{2k\pi i}{n}}\right)$,

$$n = \lim_{x \doteq 1}\frac{x^n - 1}{x - 1} = \prod_{k=1}^{k=n-1}\left(1 - e^{-\frac{2k\pi i}{n}}\right), \qquad \prod_{k=1}^{k=n-1}\frac{e^{\frac{k\pi i}{n}}}{2i} = \frac{e^{(n-1)\frac{\pi i}{2}}}{(2i)^{n-1}} = \frac{1}{2^{n-1}}.$$

150. The error function. Suppose that measurements to determine the magnitude of a certain object be made, and let $m_1, m_2, \cdots, m_n$ be a set of n determinations each made independently of the other and each worthy of the same weight. Then the quantities

$$q_1 = m_1 - m, \qquad q_2 = m_2 - m, \qquad \cdots, \qquad q_n = m_n - m,$$

which are the differences between the observed values and the assumed value m, are the errors committed; their sum is

$$q_1 + q_2 + \cdots + q_n = (m_1 + m_2 + \cdots + m_n) - mn.$$

It will be taken as a fundamental axiom that on the average the errors in excess, the positive errors, and the errors in defect, the negative errors, are evenly balanced so that their sum is zero. In other words it will be assumed that the mean value

$$nm = m_1 + m_2 + \cdots + m_n \quad \text{or} \quad m = \frac{1}{n}(m_1 + m_2 + \cdots + m_n) \tag{21}$$

is the most probable value for m as determined from $m_1, m_2, \cdots, m_n$. Note that the average value m is that which makes the sum of the squares of the errors a minimum; hence the term "least squares."

Before any observations have been taken, the chance that any particular error q should be made is 0, and the chance that an error lie within infinitesimal limits, say between q and $q + dq$, is infinitesimal; let the chance be assumed to be a function of the size of the error, and write $\phi(q)dq$ as the chance that an error lie between q and $q + dq$. It may be seen that $\phi(q)$ may be expected to decrease as q increases; for, under the reasonable hypothesis that an observer is not so likely to be far wrong as to be somewhere near right, the chance of making an error between 8.0 and 8.1 would be less than that of making an error between 1.0 and 1.1. The function $\phi(q)$ is called the error function. It will be said that the chance of making an error q_i is $\phi(q_i)$; to put it more precisely, this means simply that $\phi(q_i)dq$ is the chance of making an error which lies between q_i and $q_i + dq$.

It is a fundamental principle of the theory of chance that the chance that several independent events take place is the product of the chances for each separate event. The probability, then, that the errors $q_1, q_2, \cdots, q_n$ be made is the product

$$\phi(q_1)\,\phi(q_2)\cdots\phi(q_n) = \phi(m_1 - m)\,\phi(m_2 - m)\cdots\phi(m_n - m). \quad (22)$$

The fundamental axiom (21) is that this probability is a maximum when m is the arithmetic mean of the measurements $m_1, m_2, \cdots, m_n$; for the errors, measured from the mean value, are on the whole less than if measured from some other value.* If the probability is a maximum, so is its logarithm; and the derivative of the logarithm of (22) with respect to m is

$$\frac{\phi'(m_1 - m)}{\phi(m_1 - m)} + \frac{\phi'(m_2 - m)}{\phi(m_2 - m)} + \cdots + \frac{\phi'(m_n - m)}{\phi(m_n - m)} = 0$$

when $q_1 + q_2 + \cdots + q_n = (m_1 - m) + (m_2 - m) + \cdots + (m_n - m) = 0.$ It remains to determine ϕ from these relations.

For brevity let $F(q)$ be the function $F = \phi'/\phi$ which is the ratio of $\phi'(q)$ to $\phi(q)$. Then the conditions become

$$F(q_1) + F(q_2) + \cdots + F(q_n) = 0 \quad \text{when} \quad q_1 + q_2 + \cdots + q_n = 0.$$

In particular if there are only two observations, then

$$F(q_1) + F(q_2) = 0 \quad \text{and} \quad q_1 + q_2 = 0 \quad \text{or} \quad q_2 = -q_1.$$

Then $\qquad F(q_1) + F(-q_1) = 0 \quad \text{or} \quad F(-q) = -F(q).$

Next if there are three observations, the results are

$$F(q_1) + F(q_2) + F(q_3) = 0 \quad \text{and} \quad q_1 + q_2 + q_3 = 0.$$

Hence $\qquad F(q_1) + F(q_2) = -F(q_3) = F(-q_3) = F(q_1 + q_2).$

Now from $\qquad F(x) + F(y) = F(x + y)$

the function F may be determined (Ex. 9, p. 45) as $F(x) = Cx$. Then

$$F(q) = \frac{\phi'(q)}{\phi(q)} = Cq, \qquad \log\phi(q) = \frac{1}{2}Cq^2 + K,$$

and $\qquad \phi(q) = e^{\frac{1}{2}Cq^2 + K} = Ge^{\frac{1}{2}Cq^2}.$

This determination of ϕ contains two arbitrary constants which may be further determined. In the first place, note that C is negative, for $\phi(q)$ decreases as q increases. Let $\frac{1}{2}C = -k^2$. In the second place, the

* The derivation of the expression for ϕ is physical rather than logical in its argument. The real justification or proof of the validity of the expression obtained is *à posteriori* and depends on the experience that in practice errors do follow the law (24).

error q must lie within the interval $-\infty < q < +\infty$ which comprises all possible values. Hence

$$\int_{-\infty}^{+\infty} \phi(q)\,dq = 1, \qquad G\int_{-\infty}^{+\infty} e^{-k^2q^2}dq = 1. \tag{23}$$

For the chance that an error lie between q and $q + dq$ is ϕdq, and if an interval $a \leqq q \leqq b$ be given, the chance of an error in it is

$$\sum_a^b \phi(q)\,dq \quad \text{or, better,} \quad \lim \sum_a^b \phi(q)\,dq = \int_a^b \phi(q)\,dq,$$

and finally the chance that $-\infty < q < +\infty$ represents a certainty and is denoted by 1. The integral (23) may be evaluated (§ 143). Then $G\sqrt{\pi}/k = 1$ and $G = k/\sqrt{\pi}$. Hence *

$$\phi(q) = \frac{k}{\sqrt{\pi}} e^{-k^2q^2}. \tag{24}$$

The remaining constant k is essential; it measures the accuracy of the observer. If k is large, the function $\phi(q)$ falls very rapidly from the large value $k/\sqrt{\pi}$ for $q = 0$ to very small values, and it appears that the observer is far more likely to make a small error than a large one; but if k is small, the function ϕ falls very slowly from its value $k/\sqrt{\pi}$ for $q = 0$ and denotes that the observer is almost as likely to make reasonably large errors as small ones.

151. If only the numerical value be considered, the probability that the error lie numerically between q and $q + dq$ is

$$\frac{2k}{\sqrt{\pi}} e^{-k^2q^2}dq, \quad \text{and} \quad \frac{2k}{\sqrt{\pi}}\int_0^{\xi} e^{-k^2q^2}dq$$

is the chance that an error be numerically less than ξ. Now

$$\psi(\xi) = \frac{2k}{\sqrt{\pi}}\int_0^{\xi} e^{-k^2q^2}dq = \frac{2}{\sqrt{\pi}}\int_0^{k\xi} e^{-x^2}dx \tag{25}$$

is a function defined by an integral with a variable upper limit, and the problem of computing the value of the function for any given value of ξ reduces to the problem of computing the integral. The integrand may be expanded by Maclaurin's Formula

$$\begin{aligned} e^{-x^2} &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}e^{-\theta x^2}}{5!}, \qquad 0 < \theta < 1, \\ \int_0^x e^{-x^2}dx &= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - R, \qquad R < \frac{x^{11}}{1320}. \end{aligned} \tag{26}$$

* The reader may now verify the fact that, with ϕ as in (24), the product (22) is a maximum if the sum of the squares of the errors is a minimum as demanded by (21).

For small values of x this series is satisfactory; for $x \leqq \frac{1}{2}$ it will be accurate to five decimals.

The *probable error* is the technical term used to denote that error ξ which makes $\psi(\xi) = \frac{1}{2}$; that is, the error such that the chance of a smaller error is $\frac{1}{2}$ and the chance of a larger error is also $\frac{1}{2}$. This is found by solving for x the equation

$$\frac{\sqrt{\pi}}{2} \cdot \frac{1}{2} = 0.44311 = \int_0^x e^{-x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216}.$$

The first term alone indicates that the root is near $x = .45$, and a trial with the first three terms in the series indicates the root as between $x = .47$ and $x = .48$. With such a close approximation it is easy to fix the root to four places as

$$x = k\xi = 0.4769 \quad \text{or} \quad \xi = 0.4769\, k^{-1}. \tag{27}$$

That the probable error should depend on k is obvious.

For large values of $x = k\xi$ the method of expansion by Maclaurin's Formula is a very poor one for calculating $\psi(\xi)$; too many terms are required. It is therefore important to obtain an *expansion according to descending powers of* x. Now

$$\int_0^x e^{-x^2} dx = \int_0^\infty e^{-x^2} dx - \int_x^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} - \int_x^\infty e^{-x^2} dx$$

and $$\int_x^\infty e^{-x^2} dx = \int_x^\infty \frac{1}{x} x e^{-x^2} dx = \left[-\frac{e^{-x^2}}{2x} \right]_x^\infty - \frac{1}{2} \int_x^\infty \frac{e^{-x^2} dx}{x^2}.$$

The limits may be substituted in the first term and the method of integration by parts may be applied again. Thus

$$\begin{aligned} \int_x^\infty e^{-x^2} dx &= \frac{e^{-x^2}}{2x}\left(1 - \frac{1}{2x^2}\right) + \frac{1 \cdot 3}{2^2} \int_x^\infty \frac{e^{-x^2} dx}{x^4} \\ &= \frac{e^{-x^2}}{2x}\left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{2^2 x^4}\right) - \frac{1 \cdot 3 \cdot 5}{2^3} \int_x^\infty \frac{e^{-x^2} dx}{x^6}, \end{aligned}$$

and so on indefinitely. It should be noticed, however, that the term

$$T = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n x^{2n}} \frac{e^{-x^2}}{2x} \text{ diverges as } n = \infty.$$

In fact although the denominator is multiplied by $2x^2$ at each step, the numerator is multiplied by $2n - 1$, and hence after the integrations by parts have been applied so many times that $n > x^2$ the terms in the parenthesis begin to increase. It is worse than useless to carry the integrations further. The integral which remains is (Ex. 5, p. 29)

$$\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}\int_x^\infty \frac{e^{-x^2}dx}{x^{2n+2}} < \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}x^{2n+1}}e^{-x^2} < T.$$

Thus the integral is less than the last term of the parenthesis, and it is possible to write the *asymptotic series*

$$\int_0^x e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} - \frac{e^{-x^2}}{2x}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{2^2x^4} - \frac{1\cdot 3\cdot 5}{2^3x^6} + \cdots\right) \tag{28}$$

with the assurance that *the value obtained by using the series will differ from the true value by less than the last term which is used in the series.* This kind of series is of frequent occurrence.

In addition to the probable error, the *average numerical error* and the *mean square error,* that is, the average of the square of the error, are important. In finding the averages the probability $\phi(q)\,dq$ may be taken as the weight; in fact the probability is in a certain sense the simplest weight because the sum of the weights, that is, the sum of the probabilities, is 1 if an average over the whole range of possible values is desired. For the average numerical error and mean square error

$$\begin{aligned}\overline{|q|} &= \frac{2k}{\sqrt{\pi}}\int_0^\infty qe^{-k^2q^2}dq = \frac{1}{k\sqrt{\pi}} = \frac{0.5643}{k},\\ \overline{q^2} &= \frac{2k}{\sqrt{\pi}}\int_0^\infty q^2e^{-k^2q^2}dq = \frac{1}{2k^2}, \qquad \sqrt{\overline{q^2}} = \frac{0.7071}{k}.\end{aligned} \tag{29}$$

It is seen that the average error is greater than the probable error, and that the square root of the mean square error is still larger. In the case of a given set of n observations the averages may actually be computed as

$$\begin{aligned}\overline{|q|} &= \frac{|q_1| + |q_2| + \cdots + |q_n|}{n} = \frac{1}{k\sqrt{\pi}}, \qquad & k &= \frac{1}{\overline{|q|}\sqrt{\pi}},\\ \overline{q^2} &= \frac{q_1^2 + q_2^2 + \cdots + q_n^2}{n} = \frac{1}{2k^2}, \qquad & k &= \frac{1}{\sqrt{\overline{q^2}}\sqrt{2}}.\end{aligned}$$

Moreover, $$\pi\,\overline{|q|}^2 = 2\,\overline{q^2}.$$

It cannot be expected that the two values of k thus found will be precisely equal or that the last relation will be exactly fulfilled; but so well does the theory of errors represent what actually arises in practice that unless the two values of k are nearly equal and the relation nearly satisfied there are fair reasons for suspecting that the observations are not bona fide.

152. Consider the question of the application of these theories to the errors made in rifle practice on a target. Here there are two errors, one due to the fact that the shots may fall to the right or left of the central vertical, the other to their falling above or below the central horizontal. In other words, each of the coördinates (x, y) of the position of a shot will be regarded as subject to the law of errors independently of the other. Then

$$\frac{k}{\sqrt{\pi}}e^{-k^2x^2}dx, \qquad \frac{k'}{\sqrt{\pi}}e^{-k'^2y^2}dy, \qquad \frac{kk'}{\pi}e^{-k^2x^2-k'^2y^2}dxdy$$

will be the probabilities that a shot fall in the vertical strip between x and $x + dx$, in the horizontal strip between y and $y + dy$, or in the small rectangle common to the two strips. Moreover it will be assumed that the accuracy is the same with respect to horizontal and vertical deviations, so that $k = k'$.

These assumptions may appear too special to be reasonable. In particular it might seem as though the accuracies in the two directions would be very different, owing to the possibility that the marksman's aim should tremble more to the right and left than up and down, or vice versa, so that $k \neq k'$. In this case the shots would not tend to lie at equal distances in all directions from the center of the target, but would dispose themselves in an elliptical fashion. Moreover as the shooting is done from the right shoulder it might seem as though there would be some inclined line through the center of the target along which the accuracy would be least, and a line perpendicular to it along which the accuracy would be greatest, so that the disposition of the shots would not only be elliptical but inclined. To cover this general assumption the probability would be taken as

$$Ge^{-k^2x^2-2\lambda xy-k'^2y^2}dxdy, \quad \text{with} \quad G\int_{-\infty}^{+\infty}\int e^{-k^2x^2-2\lambda xy-k'^2y^2}dxdy = 1$$

as the condition that the shots lie somewhere. See the exercises below.

With the special assumptions, it is best to transform to polar coördinates. The important quantities to determine are the average distance of the shots from the center, the mean square distance, the probable distance, and the most probable distance. It is necessary to distinguish carefully between the probable distance, which is by definition the distance such that half the shots fall nearer the center and half fall farther away, and the most probable distance, which by definition is that distance which occurs most frequently, that is, the distance of the ring between r and $r + dr$ in which most shots fall.

The probability that the shot lies in the element $rdrd\phi$ is

$$\frac{k^2}{\pi}e^{-k^2r^2}rdrd\phi, \quad \text{and} \quad 2k^2e^{-k^2r^2}rdr,$$

obtained by integrating with respect to ϕ, is the probability that the shot lies in the ring from r to $r + dr$. The *most probable* distance r_p is that which makes this a maximum, that is,

$$\frac{d}{dr}(e^{-k^2r^2}r) = 0 \quad \text{or} \quad r_p = \frac{1}{\sqrt{2}\,k} = \frac{0.7071}{k}. \tag{30}$$

The *mean* distance and the *mean square* distance are respectively

$$\begin{aligned} \bar{r} &= \int_0^\infty 2\,k^2 e^{-k^2r^2} r^2 dr = \frac{\sqrt{\pi}}{2\,k}, & \bar{r} &= \frac{0.8862}{k}, \\ \overline{r^2} &= \int_0^\infty 2\,k^2 e^{-k^2r^2} r^3 dr = \frac{1}{k^2}, & \sqrt{\overline{r^2}} &= \frac{1.0000}{k}. \end{aligned} \tag{30'}$$

The *probable* distance r_ξ is found by solving the equation

$$\frac{1}{2} = \int_0^{r_\xi} 2\,k^2 e^{-k^2r^2} r dr = 1 - e^{-k^2r_\xi^2}, \qquad r_\xi = \frac{\sqrt{\log 2}}{k} = \frac{0.8326}{k}. \tag{30''}$$

Hence $\qquad r_p < r_\xi < \bar{r} < \sqrt{\overline{r^2}}.$

The chief importance of these considerations lies in the fact that, owing to Maxwell's assumption, analogous considerations may be applied to the velocities of the molecules of a gas. Let u, v, w be the component velocities of a molecule in three perpendicular directions so that $V = (u^2 + v^2 + w^2)^{\frac{1}{2}}$ is the actual velocity. The assumption is made that the individual components u, v, w obey the law of errors. The probability that the components lie between the respective limits u and $u + du$, v and $v + dv$, w and $w + dw$ is

$$\frac{k^3}{\pi\sqrt{\pi}} e^{-k^2u^2 - k^2v^2 - k^2w^2} du dv dw, \quad \text{and} \quad \frac{k^3}{\pi\sqrt{\pi}} e^{-k^2V^2} V^2 \sin\theta dV d\theta d\phi$$

is the corresponding expression in polar coördinates. There will then be a most probable, a probable, a mean, and a mean square velocity. Of these, the last corresponds to the mean kinetic energy and is subject to measurement.

EXERCISES

1. If $k = 0.04475$, find to three places the probability of an error $\xi < 12$.

2. Compute $\int_0^x e^{-x^2}dx$ to three places for (α) $x = 0.2$, (β) $x = 0.8$.

3. State how many terms of (28) should be taken to obtain the best value for the integral to $x = 2$ and obtain that value.

4. How accurately will (28) determine $\int_0^4 e^{-x^2}dx - \frac{1}{2}\sqrt{\pi}$? Compute.

5. Obtain these asymptotic expansions and extend them to find the general law. Show that the error introduced by omitting the integral is less than the last term retained in the series. Show further that the general term diverges when n becomes infinite.

$$(\alpha)\ \int_0^x \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{\sin x^2}{2x} - \frac{\cos x^2}{2^2x^3} + \frac{1\cdot 3}{2^2}\int_x^\infty \cos x^2 \frac{dx}{x^4},$$

$$(\beta)\ \int_0^x \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} - \frac{\cos x^2}{2x} - \frac{\sin x^2}{2^2x^3} + \frac{1\cdot 3}{2^2}\int_x^\infty \sin x^2 \frac{dx}{x^4},$$

$$(\gamma)\ \int_0^x \frac{\sin x}{x} dx,\ x \text{ large}, \qquad (\delta)\ \int_0^x \left(\frac{\sin x}{x}\right)^2 dx,\ x \text{ large}.$$

6. (α) Find the value of the average of any odd power $2n+1$ of the error; (β) also for the average of any even power; (γ) also for any power.

7. The observations 195, 225*, 190, 210, 205, 180*, 170*, 190, 200, 210, 210, 220*, 175*, 192 were obtained for deflections of a galvanometer. Compute k from the mean error and mean square error and compare the results. Suppose the observations marked *, which show great deviations, were discarded; compute k by the two methods and note whether the agreement is so good.

8. Find the average value of the product qq' of two errors selected at random and the average of the product $|q|\cdot|q'|$ of numerical values.

9. Show that the various velocities for a gas are $V_p = \frac{1}{k}$, $V_\xi = \frac{1.0875}{k}$, $\overline{V} = \frac{2}{\sqrt{\pi}k} = \frac{1.1284}{k}$, $\sqrt{\overline{V^2}} = \frac{\sqrt{3}}{\sqrt{2}\,k} = \frac{1.2247}{k}$.

10. For oxygen (at 0° C. and 76 cm. Hg.) the square root of the mean square velocity is 462.2 meters per second. Find k and show that only about 13 or 14 molecules to the thousand are moving as slow as 100 m./sec. What speed is most probable?

11. Under the general assumption of ellipticity and inclination in the distribution of the shots show that the area of the ellipse $k^2x^2 + 2\lambda xy + k'^2y^2 = H$ is $\pi H(k^2k'^2 - \lambda^2)^{-\frac{1}{2}}$, and the probability may be written $Ge^{-H}\pi(k^2k'^2 - \lambda^2)^{-\frac{1}{2}}dH$.

12. From Ex. 11 establish the relations $(\alpha)\ G = \frac{1}{\pi}\sqrt{k^2k'^2 - \lambda^2}$,

$$(\beta)\ \overline{x^2} = \frac{k'^2}{2(k^2k'^2 - \lambda^2)}, \qquad (\gamma)\ \overline{y^2} = \frac{k^2}{2(k^2k'^2 - \lambda^2)}, \qquad (\delta)\ \overline{xy} = \frac{-\lambda}{2(k^2k'^2 - \lambda^2)}.$$

13. Find H_p, $H_\xi = 0.693$, $\overline{H}$, $\overline{H^2}$ in the above problem.

14. Take 20 measurements of some object. Determine k by the two methods and compare the results. Test other points of the theory.

153. Bessel functions. The use of a definite integral to define functions which satisfy a given differential equation may be illustrated by the treatment of $xy'' + (2n+1)y' + xy = 0$, which at the same time will afford a new investigation of some functions which have previously been briefly discussed (§§ 107–108). To obtain a solution of this equation, or of any equation, in the form of a definite integral, some special type of integrand is assumed in part and the remainder of the integrand and the limits for the integral are then determined so that the equation is satisfied. In this case try the form

$$y(x)=\int e^{ixt}Tdt,\qquad y'=\int ite^{ixt}Tdt,\qquad y''=\int -t^2e^{ixt}Tdt,$$

where T is a function of t, and the derivatives are found by differentiating under the sign. Integrate y and y'' by parts and substitute in the equation. Then

$$(1-t^2)\frac{T}{x}e^{ixt}\Big]-\int e^{ixt}[T'(1-t^2)-(2n-1)tT]dt=0,$$

where the bracket after the first term means that the difference of the values for the upper and lower limit of the integral are to be taken; these limits and the form of T remain to be determined so that the expression shall really be zero.

The integral may be made to vanish by so choosing T that the bracket vanishes; this calls for the integration of a simple differential equation. The result then is

$$T=(1-t^2)^{n-\frac{1}{2}},\qquad (1-t^2)^{n+\frac{1}{2}}\frac{T}{x}e^{ixt}\Big]=0.$$

The integral vanishes, and the integrated term will vanish provided $t=\pm 1$ or $e^{ixt}=0$. If x be assumed to be real and positive, the exponential will approach 0 when $t=1+iK$ and K becomes infinite. Hence

$$y(x)=\int_{-1}^{+1}e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt\quad\text{and}\quad z(x)=\int_{+1}^{1+i\infty}e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt\qquad(31)$$

are solutions of the differential equation. In the first the integral is an infinite integral when $n<+\frac{1}{2}$ and fails to converge when $n\leqq-\frac{1}{2}$. The solution is therefore defined only when $n>-\frac{1}{2}$. The second integral is always an infinite integral because one limit is infinite. The examination of the integrals for uniformity is found below.

Consider $\int_{-1}^{+1}e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt$ with $n<\frac{1}{2}$ so that the integral is infinite.

$$\int_{-1}^{+1}e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt=\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}\cos xtdt+i\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}\sin xtdt.$$

From considerations of symmetry the second integral vanishes. Then

$$\left|\int_{-1}^{+1}e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt\right|=\left|\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}\cos xtdt\right|\leqq\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}dt.$$

This last integral with a positive integrand converges when $n>-\frac{1}{2}$, and hence the given integral converges uniformly for all values of x and defines a continuous function. The successive differentiations under the sign give the results

$$-\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}t\sin xtdt, \qquad -\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}t^2\cos xtdt.$$

These integrals also converge uniformly, and hence the differentiations were justifiable. The second integral (31) may be written with $t = 1 + iu$, as

$$\left| i\int_{u=0}^{\infty} e^{ix(1+iu)}(1-\overline{1+iu}^2)^{n-\frac{1}{2}}du\right| \leqq \int_0^{\infty} e^{-ux}(4u^2+u^4)^{\frac{1}{2}n-\frac{1}{4}}du.$$

This integral converges for all values of $x > 0$ and $n > -\frac{1}{2}$. Hence the given integral converges uniformly for all values of $x \geqq x_0 > 0$, and defines a continuous function; when $x = 0$ it is readily seen that the integral diverges and could not define a continuous function. It is easy to justify the differentiations as before.

The first form of the solution may be expanded in series.

$$y(x) = \int_{-1}^{+1} e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt = \int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}\cos xtdt$$

$$= 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xtdt \tag{32}$$

$$= 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\left(1-\frac{x^2t^2}{2!}+\frac{x^4t^4}{4!}-\frac{x^6t^6}{6!}+\theta\frac{x^8t^8}{8!}\right)dt, \quad 0<|\theta|<1.$$

The expansion may be carried to as many terms as desired. Each of the terms separately may be integrated by B- or Γ-functions.

$$2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\frac{x^{2k}t^{2k}}{2k!} = 2\frac{x^{2k}}{\Gamma(2k+1)}\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\cos^{2k}\phi d\phi$$

$$= \frac{x^{2k}\Gamma(n-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+1)\Gamma(n+k+1)} = \frac{x^{2k}\Gamma(n+\frac{1}{2})\sqrt{\pi}}{2^{2k}\Gamma(k+1)\Gamma(n+k+1)},$$

and
$$J_n(x) = \frac{x^n y(x)}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} = \sum_{k=0}\frac{(-1)^k x^{n+2k}}{2^{n+2k}\Gamma(k+1)\Gamma(n+k+1)} \tag{33}$$

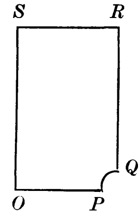
is then taken as the definition of the special function $J_n(x)$, where the expansion may be carried as far as desired, with the coefficient θ for the last term. If n is an integer, the Γ-functions may be written as factorials.

154. The second solution of the differential equation, namely

$$z(x) = y_1(x) + iy_2(x) = \int_1^{1+i\infty} -2e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt, \tag{31'}$$

where the coefficient -2 has been inserted for convenience, is for some purposes more useful than the first. It is complex, and, as the equation is real and x is taken as real, it affords two solutions, namely its real part and its pure imaginary part, each of which must satisfy the equation. As $y(x)$ converges for $x = 0$ and $z(x)$ diverges for $x = 0$, so that $y_1(x)$ or

$y_2(x)$ diverges, it follows that $y(x)$ and $y_1(x)$ or $y(x)$ and $y_2(x)$ must be independent; and as the equation can have but two independent solutions, one of the pairs of solutions must constitute a complete solution. It will now be shown that $y_1(x) = y(x)$ and that $Ay(x) + By_2(x)$ is therefore the complete solution of $xy'' + (2n+1)y' + xy = 0$.



Consider the line integral around the contour $0, 1-\epsilon, 1+\epsilon i, 1+\infty i, \infty i, 0$, or $OPQRS$. As the integrand has a continuous derivative at every point on or within the contour, the integral is zero (§ 124). The integrals along the little quadrant PQ and the unit line RS at infinity may be made as small as desired by taking the quadrant small enough and the line far enough away. The integral along SO is pure imaginary, namely, with $t = iu$,

$$\int_{SO} -2\,e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt = 2\,i\int_O^S e^{-xu}(1+u^2)^{n-\frac{1}{2}}du.$$

The integral along OP is complex, namely

$$\int_{OP} -2\,e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt$$
$$= -2\int_O^P (1-t^2)^{n-\frac{1}{2}}\cos xt dt - 2\,i\int_O^P (1-t^2)^{n-\frac{1}{2}}\sin xt dt.$$

Hence $$0 = -2\int_O^P (1-t^2)^{n-\frac{1}{2}}\cos xt dt - 2\,i\int_O^P (1-t^2)^{n-\frac{1}{2}}\sin xt dt + \zeta_1$$
$$+\int_Q^R -2\,e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt + \zeta_2 + 2\,i\int_O^S e^{-xu}(1+u^2)^{n-\frac{1}{2}}du,$$

where ζ_1 and ζ_2 are small. Equate real and imaginary parts to zero separately after taking the limit.

$$2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt dt = y(x) = \Re\int_1^{1+i\infty} -2\,e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt = y_1(x),$$

$$2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\sin xt dt - 2\int_0^\infty e^{-xu}(1+u^2)^{n-\frac{1}{2}}du$$
$$= \Im\int_1^{1+i\infty} -2\,e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt = y_2(x).$$

The signs $\Re$ and $\Im$ are used to denote respectively real and imaginary parts. The identity of $y(x)$ and $y_1(x)$ is established and the new solution $y_2(x)$ is found as a difference of two integrals.

It is now possible to obtain the important expansion of the solutions $y(x)$ and $y_2(x)$ in *descending* powers of x. For

$$\int_1^{1+i\infty} -2\, e^{ixt}(1-t^2)^{n-\frac{1}{2}}dt = \int_0^{\infty} -2\, ie^{ix-ux}(u^2-2\,iu)^{n-\frac{1}{2}}du, \quad t=1+iu.$$

Since $x \neq 0$, the transformation $ux = v$ is permissible and gives

$$2^{n+\frac{1}{2}}(-i)^{n+\frac{1}{2}}e^{ix}x^{-n-\frac{1}{2}}\int_0^{\infty} e^{-v}v^{n-\frac{1}{2}}\left(1+\frac{vi}{2\,x}\right)^{n-\frac{1}{2}}dv$$

$$= 2^{n+\frac{1}{2}}x^{-n-\frac{1}{2}}e^{i\left[x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right]}\int_0^{\infty} e^{-v}v^{n-\frac{1}{2}} \times$$

$$\left(1+\frac{n-\frac{1}{2}}{2\,x}vi-\frac{(n-\frac{1}{2})(n-\frac{3}{2})}{2\,!\,(2\,x)^2}v^2-\cdots\right)dv.$$

The expansion by the binomial theorem may be carried as far as desired; but as the integration is subsequently to be performed, the values of v must be allowed a range from 0 to ∞ and the use of Taylor's Formula with a remainder is required — the series would not converge. The result of the integration is

$$z(x) = 2^{n+\frac{1}{2}}x^{-n-\frac{1}{2}}\Gamma\left(n+\tfrac{1}{2}\right)e^{i\left[x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right]}[P(x)+iQ(x)], \tag{34}$$

where $\quad Q(x) = \dfrac{n^2-\frac{1}{4}}{2\,x} - \dfrac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})(n^2-\frac{25}{4})}{3\,!\,(2\,x)^3} + \cdots,$

$$P(x) = 1 - \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2\,!\,(2\,x)^2} + \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})(n^2-\frac{25}{4})(n^2-\frac{49}{4})}{4\,!\,(2\,x)^4} - \cdots.$$

Take real and imaginary parts and divide by $2^n x^{-n}\sqrt{\pi}\,\Gamma(n+\frac{1}{2})$. Then

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left[P(x)\cos\left(x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right) - Q(x)\sin\left(x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right)\right],$$

$$K_n(x) = \sqrt{\frac{2}{\pi x}}\left[Q(x)\cos\left(x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right) + P(x)\sin\left(x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right)\right]$$

are two independent Bessel functions which satisfy the equation (35) of § 107. If $n+\frac{1}{2}$ is an integer, P and Q terminate and the solutions are expressed in terms of elementary functions (§ 108); but if $n+\frac{1}{2}$ is not an integer, P and Q are merely asymptotic expressions which do not terminate of themselves, but must be cut short with a remainder term because of their tendency to diverge after a certain point; for tolerably large values of x and small values of n the values of $J_n(x)$ and $K_n(x)$ may, however, be computed with great accuracy by using the first few terms of P and Q.

The integration to find P and Q offers no particular difficulty.

$$\int_0^\infty e^{-v}v^{n-\frac{1}{2}+k}dv = \Gamma(n+\tfrac{1}{2}+k) = (n+k-\tfrac{1}{2})(n+k-\tfrac{3}{2})\cdots(n+\tfrac{1}{2})\Gamma(n+\tfrac{1}{2}).$$

The factors previous to $\Gamma(n+\frac{1}{2})$ combine with $n-\frac{1}{2}$, $n-\frac{3}{2}, \cdots, n-k+\frac{1}{2}$, which occur in the kth term of the binomial expansion and give the numerators of the terms in P and Q. The remainder term must, however, be discussed. The integral form (p. 57) will be used.

$$R_k = \int_0^v \frac{t^{k-1}}{(k-1)!} f^{(k)}(v-t)\,dt,$$

$$f^{(k)} = \left(n-\frac{1}{2}\right)\cdots\left(n-k+\frac{1}{2}\right)\left(\frac{i}{2x}\right)^k\left(1+\frac{vi}{2x}\right)^{n-k-\frac{1}{2}}.$$

Let it be supposed that the expansion has been carried so far that $n-k-\frac{1}{2}<0$. Then $(1+vi/2x)^{n-k-\frac{1}{2}}$ is numerically greatest when $v=0$ and is then equal to 1. Hence

$$|R_k| < \int_0^v \frac{t^{k-1}}{(k-1)!}\frac{|(n-\frac{1}{2})\cdots(n-k+\frac{1}{2})|}{(2x)^k}dt = \frac{v^k}{k!}\frac{|(n-\frac{1}{2})\cdots(n-k+\frac{1}{2})|}{(2x)^k},$$

and
$$\left|\int_0^\infty e^{-v}v^{n-\frac{1}{2}}R_k dv\right| < \frac{\left|\left(n^2-\frac{1}{4}\right)\cdots\left(n^2-\frac{(2k-1)^2}{4}\right)\right|}{k!\,(2x)^k}\Gamma\left(n+\frac{1}{2}\right).$$

It therefore appears that when $k > n-\frac{1}{2}$ the error made in neglecting the remainder is less than the last term kept, and for the maximum accuracy the series for $P+iQ$ should be broken off between the least term and the term just following.

EXERCISES

1. Solve $xy'' + (2n+1)y' - xy = 0$ by trying Te^{xt} as integrand.

$$A\int_{-1}^{+1}(1-t^2)^{n-\frac{1}{2}}e^{xt} + B\int_{-\infty}^{-1}(t^2-1)^{n-\frac{1}{2}}e^{xt}dt, \qquad x>0, \qquad n>-\tfrac{1}{2}.$$

2. Expand the first solution in Ex. 1 into series; compare with $y(ix)$ above.

3. Try $T(1-tx)^m$ on $x(1-x)y'' + [\gamma-(\alpha+\beta+1)x]y' - \alpha\beta\gamma = 0$.

One solution is $\displaystyle\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}dt, \qquad \beta>0, \qquad \gamma>\beta, \qquad |x|<1.$

4. Expand the solution in Ex. 3 into the series, called hypergeometric,

$$B(\beta,\gamma-\beta)\Big[1+\frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\gamma(\gamma+1)(\gamma+2)}x^3+\cdots\Big].$$

5. Establish these results for Bessel's J-functions:

(α) $\displaystyle J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\Gamma(n+\frac{1}{2})}\int_0^\pi \sin^{2n}\phi\cos(x\sin\phi)\,d\phi, \qquad n>-\tfrac{1}{2},$

(β) $\displaystyle J_n(x) = \frac{1}{\pi}\frac{x^n}{3\cdot 5\cdots(2n-1)}\int_0^\pi \sin^{2n}\phi\cos(x\sin\phi)\,d\phi, \qquad n=0,1,2,3\cdots.$

6. Show $\frac{1}{\pi}\int_0^{\pi}\cos(n\phi - x\sin\phi)\,d\phi$ satisfies

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right)y = \frac{\sin n\pi}{\pi}\left(\frac{1}{x} - \frac{n^2}{x}\right).$$

7. Find the equation of the second order satisfied by $\int_0^1 (1-t^2)^{n-\frac{1}{2}}\sin xt\,dt$.

8. Show $J_0(2x) = 1 - x^2 + \frac{x^4}{(2\,!)^2} - \frac{x^6}{(3\,!)^2} + \frac{x^8}{(4\,!)^2} - \frac{x^{10}}{(5\,!)^2} + \cdots$.

9. Compute $J_0(1) = 0.7652$; $J_0(2) = 0.2239$; $J_0(2.405) = 0.0000$.

10. Prove, from the integrals, $J_0'(x) = -J_1(x)$ and $[x^{-n}J_n]' = -x^{-n}J_{n+1}$.

11. Show that four terms in the asymptotic expansion of $P + iQ$ when $n = 0$ give the best result when $x = 2$ and that the error may be about 0.002.

12. From the asymptotic expansions compute $J_0(3)$ as accurately as may be.

13. Show that for large values of x the solutions of $J_n(x) = 0$ are nearly of the form $k\pi - \frac{1}{4}\pi$ and the solutions of $K_n(x) = 0$ of the form $k\pi + \frac{1}{4}\pi$.

14. Sketch the graphs of $y = J_0(x)$ and $y = J_1(x)$ by using the series of ascending powers for small values and the asymptotic expressions for large values of x.

15. From $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi$ show $\int_0^{\infty} e^{-ax}J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$.

16. Show $\int_0^{\infty} e^{-ax}J_0(x)\,dx$ converges uniformly when $a \geqq 0$.

17. Evaluate the following integrals: (α) $\int_0^{\infty} J_0(bx)\,dx = b^{-1}$,

(β) $\int_0^{\infty}\sin ax J_0(bx)\frac{dx}{x} = \frac{\pi}{2}$ or $\sin^{-1}\frac{a}{b}$ as $a > b > 0$ or $b > a > 0$,

(γ) $\int_0^{\infty}\sin ax J_0(bx)\,dx = \frac{1}{\sqrt{a^2-b^2}}$ or 0 as $a^2 > b^2$ or $b^2 > a^2$,

(δ) $\int_0^{\infty}\cos ax\, J_0(bx)\,dx = \frac{1}{\sqrt{b^2-a^2}}$ or 0 as $b^2 > a^2$ or $a^2 > b^2$.

18. If $u = \sqrt{x}J_n(ax)$, show $\frac{d^2u}{dx^2} + \left(a^2 - \frac{n^2-\frac{1}{4}}{x^2}\right)u = 0$. If $v = \sqrt{x}J_n(bx)$,

$$\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 = (b^2 - a^2)\int_0^1 xJ_n(ax)J_n(bx)\,dx.$$

19. With the aid of Ex. 18 establish the relations:

(α) $bJ_n(a)J_{n+1}(b) - aJ_n(b)J_{n+1}(a) = (b^2 - a^2)\int_0^1 xJ_n(ax)J_n(bx)\,dx$,

(β) $aJ_1(a) = a^2\int_0^1 xJ_0(ax)\,dx = \int_0^a xJ_0(x)\,dx$,

(γ) $J_n(a)J_{n+1}(a) + a[J_n(a)J_{n+1}'(a) - J_n'(a)J_{n+1}(a)] = 2a\int_0^1 x[J_n(ax)]^2 dx$.

20. Show $J_0(x) = \frac{2}{\pi}\int_1^{\infty}\frac{\sin xt\,dt}{\sqrt{t^2-1}}$, $K_0(x) = \frac{2}{\pi}\int_1^{\infty}\frac{\cos xt\,dt}{\sqrt{t^2-1}}$.