CHAPTER XIII

ON INFINITE INTEGRALS

140. Convergence and divergence. The definite integral, and hence for theoretical purposes the indefinite integral, has been defined,

$$\int_a^b f(x)\,dx, \qquad F(x) = \int_a^x f(x)\,dx,$$

when the function $f(x)$ is *limited* in the interval a to b, or a to x; the proofs of various propositions have depended essentially on the fact that *the integrand remained finite over the finite interval of integration* (§§ 16–17, 28–30). Nevertheless problems which call for the determination of the area between a curve and its asymptote, say the area under the witch or cissoid,

$$\int_{-\infty}^{+\infty} \frac{8\,a^3dx}{x^2+4\,a^2} = 4\,a^2 \tan^{-1}\frac{x}{2\,a}\bigg|_{-\infty}^{+\infty} = 4\,\pi a^2, \qquad 2\int_0^{2a} \frac{x^{\frac{3}{2}}dx}{\sqrt{2\,a-x}} = 3\,\pi a^2,$$

have arisen and have been treated as a matter of course.* The integrals of this sort require some special attention.

When the integrand of a definite integral becomes infinite within or at the extremities of the interval of integration, or when one or both of the limits of integration become infinite, the integral is called an infinite integral and is defined, not as the limit of a sum, but as the limit of an integral with a variable limit, that is, as the limit of a function. Thus

$$\int_a^\infty f(x)\,dx = \lim_{x=\infty}\left[F(x) = \int_a^x f(x)dx\right], \qquad \text{infinite upper limit,}$$

$$\int_a^b f(x)\,dx = \lim_{x \doteq b}\left[F(x) = \int_a^x f(x)\,dx\right], \qquad \text{integrand } f(b) = \infty.$$

These definitions may be illustrated by figures which show the connection with the idea of area between a curve and its asymptote. Similar definitions would be given if the lower limit were $-\infty$ or if the integrand became infinite at $x = a$. If the integrand were infinite at some intermediate point of the interval, the interval would be subdivided into two intervals and the definition would be applied to each part.

* Here and below the construction of figures is left to the reader.

Now the behavior of $F(x)$ as x approaches a definite value or becomes infinite may be of three distinct sorts; for $F(x)$ may approach a definite finite quantity, or it may become infinite, or it may oscillate without approaching any finite quantity or becoming definitely infinite. The examples

$$\int_0^\infty \frac{8a^3dx}{x^2+4a^2} = \lim_{x=\infty}\left[\int_0^x \frac{8a^3dx}{x^2+4a^2} = 4a^2\tan^{-1}\frac{x}{2a}\right] = 2\pi a^2, \quad \text{a limit,}$$

$$\int_1^\infty \frac{dx}{x} = \lim_{x=\infty}\left[\int_1^x \frac{dx}{x} = \log x\right], \quad \text{becomes infinite, no limit,}$$

$$\int_0^\infty \cos x dx = \lim_{x=\infty}\left[\int_0^x \cos x dx = \sin x\right], \quad \text{oscillates, no limit,}$$

illustrate the three modes of behavior in the case of an infinite upper limit. In the first case, *where the limit exists, the infinite integral is said to converge;* in the other two cases, where the limit does not exist, the integral is said to *diverge.*

If the indefinite integral can be found as above, the question of the convergence or divergence of an infinite integral may be determined and the value of the integral may be obtained in the case of convergence. If the indefinite integral cannot be found, it is of prime importance to know whether the definite infinite integral converges or diverges; for there is little use trying to compute the value of the integral if it does not converge. As the infinite limits or the points where the integrand becomes infinite are the essentials in the discussion of infinite integrals, the integrals will be written with only one limit, as

$$\int^\infty f(x)\,dx, \qquad \int^b f(x)\,dx, \qquad \int_a f(x)\,dx.$$

To discuss a more complicated combination, one would write

$$\int_0^\infty \frac{e^{-x}dx}{\sqrt{x^3}\log x} = \int_0^\xi + \int_\xi^1 + \int_1^\Xi + \int_\Xi^\infty \frac{e^{-x}dx}{\sqrt{x}\log x},$$

and treat all four of the infinite integrals

$$\int_0 \frac{e^{-x}dx}{\sqrt{x^3}\log x}, \quad \int^1 \frac{e^{-x}dx}{\sqrt{x^3}\log x}, \quad \int_1 \frac{e^{-x}dx}{\sqrt{x^3}\log x}, \quad \int^\infty \frac{e^{-x}dx}{\sqrt{x^3}\log x}.$$

Now by definition a function $E(x)$ is called an E-function in the neighborhood of the value $x = a$ when the function is continuous in the neighborhood of $x = a$ and approaches a limit which is neither zero nor infinite (p. 62). *The behavior of the infinite integrals of a function* *which does not change sign and of the product of that function by an E-function are identical as far as convergence or divergence are concerned.* Consider the proof of this theorem in a special case, namely,

$$\int^{\infty} f(x)\,dx, \qquad \int^{\infty} f(x)\,E(x)\,dx, \tag{1}$$

where $f(x)$ may be assumed to remain positive for large values of x and $E(x)$ approaches a positive limit as x becomes infinite. Then if K be taken sufficiently large, both $f(x)$ and $E(x)$ have become and will remain positive and finite. By the Theorem of the Mean (Ex. 5, p. 29)

$$m\int_K^x f(x)\,dx < \int_K^x f(x)\,E(x)\,dx < M\int_K^x f(x)\,dx, \qquad x > K,$$

where m and M are the minimum and maximum values of $E(x)$ between K and ∞. Now let x become infinite. As the integrands are positive, the integrals must increase with x. Hence (p. 35)

if $\int_K^{\infty} f(x)\,dx$ converges, $\int_K^{\infty} f(x)\,E(x)\,dx < M\int_K^{\infty} f(x)\,dx$ converges,

if $$\int_K^{\infty} f(x)\,E(x)\,dx \text{ converges,}$$

$$\int_K^{\infty} f(x)\,dx < \frac{1}{m}\int_K^{\infty} f(x)\,E(x)\,dx \text{ converges;}$$

and divergence may be treated in the same way. Hence the integrals (1) converge or diverge together. The same treatment could be given for the case the integrand became infinite and for all the variety of hypotheses which could arise under the theorem.

This theorem is one of the most useful and most easily applied for determining the convergence or divergence of an infinite integral with an integrand which does not change sign. Thus consider the case

$$\int^{\infty}\frac{xdx}{(ax+x^2)^{\frac{3}{2}}} = \int^{\infty}\left[\frac{x^2}{ax+x^2}\right]^{\frac{3}{2}}\frac{dx}{x^2}, \qquad E(x) = \left[\frac{x^2}{ax+x^2}\right]^{\frac{3}{2}}, \qquad \int^{\infty}\frac{dx}{x^2} = -\frac{1}{x}\bigg|^{\infty}.$$

Here a simple rearrangement of the integrand throws it into the product of a function $E(x)$, which approaches the limit 1 as x becomes infinite, and a function $1/x^2$, the integration of which is possible. Hence by the theorem the original integral converges. This could have been seen by integrating the original integral; but the integration is not altogether short. Another case, in which the integration is not possible, is

$$\int^1\frac{dx}{\sqrt{1-x^4}} = \int^1\frac{1}{\sqrt{1+x^2}\sqrt{1+x}}\frac{dx}{\sqrt{1-x}},$$

$$E(x) = \frac{1}{\sqrt{1+x^2}\sqrt{1+x}}, \qquad \int^1\frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x}\,\bigg|^1.$$

Here $E(1) = \frac{1}{2}$. The integral is again convergent. A case of divergence would be

$$\int_0 \frac{dx}{(2x - x^2)^{\frac{3}{2}}} = \int_0 \frac{1}{(2-x)^{\frac{3}{2}}} \frac{dx}{x^{\frac{3}{2}}}, \qquad E(x) = \frac{1}{(2-x)^{\frac{3}{2}}}, \qquad \int_0 \frac{dx}{x^{\frac{3}{2}}} = -\frac{2}{\sqrt{x}}\bigg|_0.$$

141. The interpretation of a definite integral as an area will suggest another form of test for convergence or divergence in case the integrand does not change sign. Consider two functions $f(x)$ and $\psi(x)$ both of which are, say, positive for large values of x or in the neighborhood of a value of x for which they become infinite. *If the curve $y = \psi(x)$ remains above $y = f(x)$, the integral of $f(x)$ must converge if the integral of $\psi(x)$ converges, and the integral of $\psi(x)$ must diverge if the integral of $f(x)$ diverges.* This may be proved from the definition. For $f(x) < \psi(x)$ and

$$\int_K^x f(x)\,dx < \int_K^x \psi(x)\,dx \quad \text{or} \quad F(x) < \Psi(x).$$

Now as x approaches b or ∞, the functions $F(x)$ and $\Psi(x)$ both increase. If $\Psi(x)$ approaches a limit, so must $F(x)$; and if $F(x)$ increases without limit, so must $\Psi(x)$.

As the relative behavior of $f(x)$ and $\psi(x)$ is consequential *only near* particular values of x or when x is very great, the conditions may be expressed in terms of limits, namely: *If $\psi(x)$ does not change sign and if the ratio $f(x)/\psi(x)$ approaches a finite limit (or zero), the integral of $f(x)$ will converge if the integral of $\psi(x)$ converges; and if the ratio $f(x)/\psi(x)$ approaches a finite limit (not zero) or becomes infinite, the integral of $f(x)$ will diverge if the integral of $\psi(x)$ diverges.* For in the first case it is possible to take x so near its limit or so large, as the case may be, that the ratio $f(x)/\psi(x)$ shall be less than any assigned number G greater than its limit; then the functions $f(x)$ and $G\psi(x)$ satisfy the conditions established above, namely $f < G\psi$, and the integral of $f(x)$ converges if that of $\psi(x)$ does. In like manner in the second case it is possible to proceed so far that the ratio $f(x)/\psi(x)$ shall have become to remain greater than any assigned number g less than its limit; then $f > g\psi$, and the result above may be applied to show that the integral of $f(x)$ diverges if that of $\psi(x)$ does.

For an infinite upper limit a direct integration shows that

$$\int^\infty \frac{dx}{x^k} = \frac{-1}{k-1}\frac{1}{x^{k-1}}\bigg|^\infty \quad \text{or } \log x\bigg|^\infty, \quad \begin{array}{r}\text{converges if } k > 1,\\ \text{diverges if } k \leqq 1.\end{array} \tag{2}$$

Now if the *test function* $\phi(x)$ be chosen as $1/x^k = x^{-k}$, the ratio $f(x)/\phi(x)$ becomes $x^k f(x)$, and *if the limit of the product $x^k f(x)$ exists* *and may be shown to be finite (or zero) as x becomes infinite for any choice of k greater than* 1, *the integral of* $f(x)$ *to infinity will converge; but if the product approaches a finite limit (not zero) or becomes infinite for any choice of k less than or equal to* 1, *the integral diverges.* This may be stated as: The integral of $f(x)$ to infinity will converge if $f(x)$ is an infinitesimal of order higher than the first relative to $1/x$ as x becomes infinite, but will diverge if $f(x)$ is an infinitesimal of the first or lower order. In like manner

$$\int^b \frac{dx}{(b-x)^k} = \frac{1}{k-1}\frac{1}{(b-x)^{k-1}}\Bigg|^b \text{ or } -\log(b-x)\Bigg|^b, \quad \begin{matrix}\text{converges if } k<1,\\ \text{diverges if } k \geqq 1,\end{matrix} \tag{3}$$

and it may be stated that: The integral of $f(x)$ to b will converge if $f(x)$ is an infinite of order less than the first relative to $(b-x)^{-1}$ as x approaches b, but will diverge if $f(x)$ is an infinite of the first or higher order. The proof is left as an exercise. See also Ex. 3 below.

As an example, let the integral $\int_0^\infty x^n e^{-x}dx$ be tested for convergence or divergence. If $n>0$, the integrand never becomes infinite, and the only integral to examine is that to infinity; but if $n<0$ the integral from 0 has also to be considered. Now the function e^{-x} for large values of x is an infinitesimal of infinite order, that is, the limit of $x^{k+n}e^{-x}$ is zero for any value of k and n. Hence the integrand $x^n e^{-x}$ is an infinitesimal of order higher than the first and the integral to infinity converges under all circumstances. For $x=0$, the function e^{-x} is finite and equal to 1; the order of the infinite $x^n e^{-x}$ will therefore be precisely the order n. Hence the integral from 0 converges when $n > -1$ and diverges when $n \leqq -1$. Hence the function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \qquad \alpha > 0,$$

defined by the integral containing the parameter α, will be defined for all positive values of the parameter, but not for negative values nor for 0.

Thus far tests have been established only for integrals in which the integrand does not change sign. There is a general test, not particularly useful for practical purposes, but highly useful in obtaining theoretical results. It will be treated merely for the case of an infinite limit. Let

$$F(x) = \int_K^x f(x)\,dx, \qquad F(x'') - F(x') = \int_{x'}^{x''} f(x)\,dx, \qquad x',\, x'' > K. \tag{4}$$

Now (Ex. 3, p. 44) the necessary and sufficient condition that $F(x)$ approach a limit as x becomes infinite is that $F(x'') - F(x')$ shall approach the limit 0 when x' and x'', regarded as independent variables, become infinite; by the definition, then, this is the necessary and sufficient condition that the integral of $f(x)$ to infinity shall converge. Furthermore *if* $$\int^{\infty} |f(x)|\,dx \quad \textit{converges, then} \int^{\infty} f(x)\,dx \tag{5}$$

must converge and is said to be *absolutely convergent.* The proof of this important theorem is contained in the above and in

$$\int_{x'}^{x''} f(x)\,dx \leqq \int_{x'}^{x''} |f(x)|\,dx.$$

To see whether an integral is absolutely convergent, the tests established for the convergence of an integral with a positive integrand may be applied to the integral of the absolute value, or some obvious direct *method of comparison* may be employed; for example,

$$\int^{\infty} \frac{\cos x dx}{a^2 + x^2} \leqq \int^{\infty} \frac{1\,dx}{a^2 + x^2} \text{ which converges,}$$

and it therefore appears that the integral on the left converges absolutely. When the convergence is not absolute, the question of convergence may sometimes be settled by *integration by parts.* For suppose that the integral may be written as

$$\int^{x} f(x)\,dx = \int^{x} \phi(x)\psi(x)\,dx = \left[\phi(x)\int\psi(x)\,dx\right]^{x} - \int^{x}\phi'(x)\int\psi(x)\,dx^2$$

by separating the integrand into two factors and integrating by parts. Now if, when x becomes infinite, each of the right-hand terms approaches a limit, then

$$\int^{\infty} f(x)\,dx = \lim_{x=\infty}\left[\phi(x)\int\psi(x)\,dx\right]^{x} - \lim_{x=\infty}\int^{x}\phi'(x)\int\psi(x)\,dxdx,$$

and the integral of $f(x)$ to infinity converges.

As an example consider the convergence of $\int^{\infty}\frac{x\cos x dx}{a^2+x^2}$. Here $\int^{\infty}\frac{x|\cos x|dx}{a^2+x^2}$ does not appear to be convergent; for, apart from the factor $|\cos x|$ which oscillates between 0 and 1, the integrand is an infinitesimal of only the first order and the integral of such an integrand does not converge; the original integral is therefore apparently not absolutely convergent. However, an integration by parts gives

$$\int^{x}\frac{x\cos x dx}{a^2+x^2} = \frac{x\sin x}{a^2+x^2}\bigg|^{x} - \int^{x}\frac{x^2-a^2}{(x^2+a^2)^2}\cos x dx,$$

$$\lim_{x=\infty}\frac{x\sin x}{a^2+x^2} = 0, \qquad \int^{x}\frac{x^2-a^2}{(x^2+a^2)^2}\cos x dx < \int^{x}\frac{dx}{x^2}.$$

Now the integral on the right is seen to be convergent and, in fact, absolutely convergent as x becomes infinite. The original integral therefore must approach a limit and be convergent as x becomes infinite.

EXERCISES

1. Establish the convergence or divergence of these infinite integrals:

$(\alpha)\ \int^{\infty}\frac{dx}{x\sqrt{1+x^2}}$, $(\beta)\ \int^{\infty}\frac{x^2dx}{(a^2+x^2)^2}$, $(\gamma)\ \int^{\infty}\frac{x^2dx}{(a^2+x^2)^{\frac{3}{2}}}$,

$(\delta)\ \int_0 x^{\alpha-1}(1-x)^{\beta-1}dx$ (to have an infinite integral, α must be less than 1),

$(\epsilon)\ \int^1 x^{\alpha-1}(1-x)^{\beta-1}dx$, $(\zeta)\ \int_0^a\frac{dx}{\sqrt{ax-x^2}}$, $(\eta)\ \int_1^{\infty}\frac{dx}{x\sqrt{x^2-1}}$,

$(\theta)\ \int_0^{\infty}\frac{dx}{1-x^4}$, $(\iota)\ \int_0^2\frac{xdx}{(1-x)^{\frac{1}{3}}}$, $(\kappa)\ \int_0^2\frac{x^{\alpha-1}}{1-x}dx$,

$(\lambda)\ \int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, $k<1$, $k=1$, $(\mu)\ \int_0^1\sqrt{\frac{1-k^2x^2}{1-x^2}}dx$, $k<1$.

2. Point out the peculiarities which make these integrals infinite integrals, and test the integrals for convergence or divergence:

$(\alpha)\ \int_0^1\left(\log\frac{1}{x}\right)^n dx$, conv. if $n>-1$, div. if $n\leqq-1$, $(\beta)\ \int_0^1\frac{\log x}{1-x}dx$,

$(\gamma)\ \int_0^1(-\log x)^n dx$, $(\delta)\ \int_0^{\frac{\pi}{2}}\log\sin x dx$, $(\epsilon)\ \int_0^{\pi}x\log\sin x dx$,

$(\zeta)\ \int^{\infty}\log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$, $(\eta)\ \int_0^{\pi}\frac{dx}{(\sin x+\cos x)^k}$, $(\theta)\ \int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx$,

$(\iota)\ \int_0^{\infty}\frac{e^{-x}dx}{\sqrt{x}\log(x+1)}$, $(\kappa)\ \int_0^{\infty}x^{\frac{1}{x}}dx$, $(\lambda)\ \int_0^1\log x\tan\frac{\pi x}{2}dx$,

$(\mu)\ \int_0^{\infty}\frac{x^{\alpha-1}}{1+x}dx$, $(\nu)\ \int_{-\infty}^{+\infty}e^{-x^2}dx$, $(o)\ \int_0^{\infty}\frac{x^{\alpha-1}dx}{(1+x)^2}$,

$(\pi)\ \int_0^{\infty}\frac{\sin^2 x}{x^2}dx$, $(\rho)\ \int_0^1\frac{\log x dx}{\sqrt{1-x^2}}$, $(\sigma)\ \int_0^{\infty}e^{-\left(x-\frac{a}{x}\right)^2}$,

$(\tau)\ \int_0^{\infty}\frac{x^{\alpha-1}\log x}{1+x}dx$, $(\upsilon)\ \int_0^{\infty}\frac{\log(1+a^2x^2)}{1+b^2x^2}dx$, $(\chi)\ \int_0^{\infty}e^{-a^2x^2}\cosh\beta x dx$.

3. Point out the similarities and differences of the method of E-functions and of test functions. Compare also with the work of this section the remark that the determination of the order of an infinitesimal or infinite is a problem in indeterminate forms (p. 63). State also whether it is necessary that $f(x)/\psi(x)$ or $x^kf(x)$ should approach a limit, or whether it is sufficient that the quantity remain finite. Distinguish "of order higher" (p. 356) from "of higher order" (p. 63); see Ex. 8, p. 66.

4. Discuss the convergence of these integrals and prove the convergence is absolute in all cases where possible:

$(\alpha)\ \int^{\infty}\frac{\sin x}{x^k}dx$, $(\beta)\ \int^{\infty}\cos x^2 dx$, $(\gamma)\ \int^{\infty}\frac{\cos\sqrt{x}}{x^k}dx$,

$(\delta)\ \int_0^{\infty}\frac{e^{-\alpha x}\sin\beta x}{x}dx$, $(\epsilon)\ \int_0^{\infty}e^{-a^2x^2}\cos\beta x dx$, $(\zeta)\ \int_0^{\infty}\sqrt{\frac{a^2+x^2}{x^3}}dx$,

(η) $\int_0^\infty \frac{x\sin x}{x^2+k^2}dx$, ($\theta$) $\int_0^\infty e^{-ax}\cos bx dx$, ($\iota$) $\int_0^\infty \frac{\cos x}{\sqrt{x}}dx$,

(κ) $\int_0^\infty x^{\alpha-1}e^{-x\cos\beta}\cos(x\sin\beta)\,dx$, ($\lambda$) $\int_0^\infty \frac{\sin x\cos\alpha x}{x}dx$,

(μ) $\int_0^\infty \cos x^2\cos 2\,\alpha x dx$, ($\nu$) $\int_0^\infty \sin\left(\frac{x^2}{2}+\frac{\alpha^2}{2x^2}\right)dx$, ($o$) $\int_0^\infty \frac{\sin^k x^l}{x^m}dx$.

5. If $f_1(x)$ and $f_2(x)$ are two limited functions integrable (in the sense of §§ 28–30) over the integral $a \leqq x \leqq b$, show that their product $f(x)=f_1(x)f_2(x)$ is integrable over the interval. Note that in any interval δ_i, the relations $m_{1i}m_{2i} \leqq m_i \leqq M_i \leqq M_{1i}M_{2i}$ and $M_{1i}M_{2i}-m_{1i}m_{2i} = M_{1i}M_{2i}-M_{1i}m_{2i}+M_{1i}m_{2i}-m_{1i}m_{2i} = M_{1i}O_{2i}+m_{2i}O_{1i}$ hold. Show further that

$$\int_a^b f_1(x)f_2(x)\,dx = \lim\sum f_1(\xi_i)f_2(\xi_i)\delta_i$$

$$= \lim\sum f_1(\xi_i)\left[\int_{x_i}^{x_{i+1}}f_2(x)\,dx - \int_{x_i}^{x_{i+1}}\{f_2(\xi_i)-f_2(x)\,dx\}\right],$$

or $$\int_a^b f(x)\,dx = \lim\sum f_1(\xi_i)\int_{x_i}^{x_{i+1}}f_2(x)\,dx$$

$$= \lim\sum f_1(\xi_i)\left[\int_{x_i}^{b}f_2(x)\,dx - \int_{x_{i+1}}^{b}f_2(x)\,dx\right],$$

or $$\int_a^b f(x)\,dx = f_1(\xi_1)\int_a^b f_2(x)\,dx + \lim\sum[f_2(\xi_i)-f_2(\xi_{i-1})]\int_{x_i}^b f_2(x)\,dx.$$

6. *The Second Theorem of the Mean.* If $f(x)$ and $\phi(x)$ are two limited functions integrable in the interval $a \leqq x \leqq b$, and if $\phi(x)$ is positive, nondecreasing, and less than K, then

$$\int_a^b \phi(x)f(x)\,dx = K\int_\xi^b f(x)\,dx, \qquad a \leqq \xi \leqq b.$$

And, more generally, if $\phi(x)$ satisfies $-\infty < k \leqq \phi(x) \leqq K < \infty$ and is either nondecreasing or nonincreasing throughout the interval, then

$$\int_a^b \phi(x)f(x)\,dx = k\int_a^\xi f(x)\,dx + K\int_\xi^b f(x)\,dx, \qquad a \leqq \xi \leqq b.$$

In the first case the proof follows from Ex. 5 by noting that the integral of $\phi(x)f(x)$ may be regarded as the limit of the sum

$$\phi(\xi_1)\int_a^b f(x)\,dx + \sum[\phi(\xi_i)-\phi(\xi_{i-1})]\int_{x_i}^b f(x)\,dx + [K-\phi(\xi_n)]\int_{x_n}^b f(x)\,dx,$$

where the restrictions on $\phi(x)$ make the coefficients of the integrals all positive or zero, and where the sum may consequently be written as

$$\mu[\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1)+\cdots+\phi(\xi_n)-\phi(\xi_{n-1})+K-\phi(\xi_n)] = \mu K$$

if μ be a properly chosen mean value of the integrals which multiply these coefficients; as the integrals are of the form $\int_\xi^b f(x)\,dx$ where $\xi = a, x_1, \cdots, x_n$, it follows that μ must be of the same form where $a \leqq \xi \leqq b$. The second form of the theorem follows by considering the function $\phi - k$ or $k - \phi$.

7. If $\phi(x)$ is a function varying always in the same sense and approaching a finite limit as x becomes infinite, the integral $\int^{\infty}\phi(x)f(x)\,dx$ will converge if $\int^{\infty}f(x)\,dx$ converges. Consider

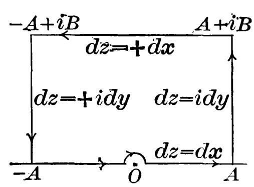
$$\int_{x'}^{x''}\phi(x)f(x)\,dx = \phi(x')\int_{x'}^{\xi}f(x)\,dx + \phi(x'')\int_{\xi}^{x''}f(x)\,dx.$$

8. If $\phi(x)$ is a function varying always in the same sense and approaching 0 as a limit when $x = \infty$, and if the integral $F(x)$ of $f(x)$ remains finite when $x = \infty$, then the integral $\int^{\infty}\phi(x)f(x)\,dx$ is convergent. Consider

$$\int_{x'}^{x''}\phi(x)f(x)\,dx = \phi(x')\,[F(\xi) - F(x')] + \phi(x'')\,[F(x'') - F(\xi)].$$

This test is very useful in practice; for many integrals are of the form $\int^{\infty}\phi(x)\sin x\,dx$ where $\phi(x)$ constantly decreases or increases toward the limit 0 when $x = \infty$; all these integrals converge.

142. The evaluation of infinite integrals. After an infinite integral has been proved to converge, the problem of calculating its value still remains. No general method is to be had, and for each integral some special device has to be discovered which will lead to the desired result. *This may frequently be accomplished by choosing a function $F(z)$ of the complex variable $z = x + iy$ and integrating the function around some closed path in the z-plane.* It is known that if the points where $F(z) = X(x, y) + iY(x, y)$ ceases to have a derivative $F'(z)$, that is, where $X(x, y)$ and $Y(x, y)$ cease to have continuous first partial derivatives satisfying the relations $X'_x = Y'_y$ and $X'_y = -Y'_x$, are cut out of the plane, the integral of $F(z)$ around any closed path which does not include any of the excised points is zero (§ 124). It is sometimes possible to select such a function $F(z)$ and such a path of integration that part of the integral of the complex function reduces to the given infinite integral while the rest of the integral of the complex function may be computed. Thus there arises an equation which determines the value of the infinite integral.



Consider the integral $\int_0^{\infty}\frac{\sin x}{x}\,dx$ which is known to converge. Now

$$\int_0^{\infty}\frac{\sin x}{x}\,dx = \int_0^{\infty}\frac{e^{ix} - e^{-ix}}{2\,ix}\,dx = \int_0^{\infty}\frac{e^{ix}}{2\,ix} - \int_0^{\infty}\frac{e^{-ix}}{2\,ix}\,dx$$

suggests at once that the function e^{iz}/z be examined. This function has a definite derivative at every point except $z = 0$, and the origin is therefore the only point which has to be cut out of the plane. The integral of e^{iz}/z around any path such as that marked in the figure * is therefore zero. Then if a is small and A is large,

$$0 = \int_{\bigcirc} \frac{e^{iz}}{z} dz = \int_a^A \frac{e^{ix}}{x} dx + \int_0^B \frac{e^{iA-y}}{A+iy} i dy + \int_A^{-A} \frac{e^{ix-B}}{x+iB} dx$$

$$+ \int_B^0 \frac{e^{-iA-y}}{-A+iy} i dy + \int_{-A}^{-a} \frac{e^{ix}}{x} dx + \int_{-a}^{+a} \frac{e^{iz}}{z} dz.$$

But $\displaystyle\int_{-A}^{-a} \frac{e^{ix}}{x} dx = -\int_{-a}^{-A} \frac{e^{ix}}{x} dx = -\int_a^A \frac{e^{-ix}dx}{x}$ and $\displaystyle\int_{-a}^{+a} \frac{e^{iz}}{z} dz = \int_{-a}^{+a} \frac{1+\eta}{z} dz$;

the first by the ordinary rules of integration and the second by Maclaurin's Formula. Hence

$$0 = \int_{\bigcirc} \frac{e^{iz}}{z} dz = \int_a^A \frac{e^{ix} - e^{-ix}}{x} + \int_{-a}^{+a} \frac{dz}{z} + \text{four other integrals.}$$

It will now be shown that by taking the rectangle sufficiently large and the semicircle about the origin sufficiently small each of the four integrals may be made as small as desired. The method is to replace each integral by a larger one which may be evaluated.

$$\left| \int_0^B \frac{e^{iA-y}}{A+iy} i dy \right| \leqq \int_0^B \frac{|e^{iA}| e^{-y}}{|A+iy|} |i| dy < \int_0^B \frac{1}{A} e^{-y} dy < \frac{B}{A}.$$

These changes involve the facts that the integral of the absolute value is as great as the absolute value of the integral and that $e^{iA-y} = e^{iA}e^{-y}$, $|e^{iA}| = 1$, $|A+iy| > A$, $e^{-y} < 1$. For the relations $|e^{iA}| = 1$ and $|A+iy| > A$, the interpretation of the quantities as vectors suffices (§§ 71–74); that the integral of the absolute value is as great as the absolute value of the integral follows from the same fact for a sum (p. 154). The absolute value of a fraction is enlarged if that of its numerator is enlarged or that of its denominator diminished. In a similar manner

$$\left| \int_A^{-A} \frac{e^{ix-B}}{x+iB} dx \right| < \int_{-A}^{A} \frac{e^{-B}}{B} dx = 2e^{-B}\frac{A}{B}, \qquad \left| \int_B^0 \frac{e^{-iA-y}}{-A+iy} i dy \right| < \frac{B}{A}.$$

Furthermore $$\left| \int_{-a}^{+a} \frac{\eta}{z} dz \right| \leqq \int_{-a}^{+a} |\eta| \frac{|dz|}{|z|} = \int_0^\pi |\eta| d\phi,$$

$$\int_{-a}^{+a} \frac{dz}{z} = \int_\pi^0 \frac{re^{\phi i} i d\phi}{re^{\phi i}} = -\pi i.$$

Then $$0 = \int_{\bigcirc} \frac{e^{iz}}{z} dz = \int_a^A 2i \frac{\sin x}{x} dx - \pi i + R, \qquad |R| < 2\frac{B}{A} + 2e^{-B}\frac{A}{B} + \pi\epsilon,$$

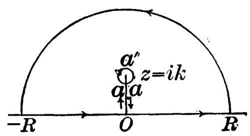
where ϵ is the greatest value of $|\eta|$ on the semicircle. Now let the rectangle be so chosen that $A = Be^{\frac{1}{2}B}$; then $|R| < 4e^{-\frac{1}{2}B} + \pi\epsilon$. By taking B sufficiently large $e^{-\frac{1}{2}B}$ may be made as small as desired; and by taking the semicircle sufficiently

* It is also possible to integrate along a semicircle from A to $-A$, or to come back directly from iB to the origin and separate real from imaginary parts. These variations in method may be left as exercises.

small, ϵ may be made as small as desired. This amounts to saying that, for A sufficiently large and for a sufficiently small, R is negligible. In other words, by taking A large enough and a small enough $\int_a^A \frac{\sin x}{x}$ may be made to differ from $\frac{\pi}{2}$ by as little as desired. As the integral from zero to infinity converges and may be regarded as the limit of the integral from a to A (is so defined, in fact), the integral from zero to infinity must also differ from $\frac{1}{2}\pi$ by as little as desired. But if two constants differ from each other by as little as desired, they must be equal. Hence

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}. \tag{6}$$

As a second example consider what may be had by integrating $e^{iz}/(z^2+k^2)$ over an appropriate path. The denominator will vanish when $z = \pm ik$ and there are two points to exclude in the z-plane. Let the integral be extended over the closed path as indicated. There is no need of integrating back and forth along the double line Oa, because the function takes on the same values and the integrals destroy each other. Along the large semicircle $z = Re^{i\phi}$ and $dz = Rie^{i\phi}d\phi$. Moreover



$$\int_{-R}^{0} \frac{e^{ix}dx}{x^2+k^2} = -\int_0^{-R} \frac{e^{ix}dx}{x^2+k^2} = \int_0^R \frac{e^{-ix}dx}{x^2+k^2} \quad \text{by elementary rules.}$$

Hence $$\int_{-R}^{0} \frac{e^{ix}dx}{x^2+k^2} + \int_0^R \frac{e^{ix}dx}{x^2+k^2} = \int_0^R \frac{e^{ix}+e^{-ix}}{x^2+k^2}dx = 2\int_0^R \frac{\cos x}{x^2+k^2}dx,$$

and $$0 = \int_O \frac{e^{iz}}{z^2+k^2}dz = 2\int_0^R \frac{\cos x}{x^2+k^2}dx + \int_0^\pi \frac{e^{iRe^{i\phi}}Rie^{i\phi}d\phi}{R^2e^{2i\phi}+k^2} + \int_{aa'a} \frac{e^{iz}dz}{z^2+k^2}.$$

Now $$|e^{iRe^{i\phi}}| = |e^{iR(\cos\phi + i\sin\phi)}| = |e^{-R\sin\phi}e^{iR\cos\phi}| = e^{-R\sin\phi}.$$

Moreover $|R^2e^{2i\phi}+k^2|$ cannot possibly exceed R^2-k^2 and can equal it only when $\phi = \frac{1}{2}\pi$. Hence

$$\left|\int_0^\pi \frac{e^{iRe^{i\phi}}Rie^{i\phi}d\phi}{R^2e^{2i\phi}+k^2}\right| \leqq \int_0^\pi \frac{Re^{-R\sin\phi}}{R^2-k^2}d\phi = 2\int_0^{\frac{\pi}{2}} \frac{Re^{-R\sin\phi}}{R^2-k^2}d\phi.$$

Now by Ex. 28, p. 11, $\sin\phi > 2\phi/\pi$. Hence the integral may be further increased.

$$\left|\int_0^\pi \frac{e^{iRe^{i\phi}}Rie^{i\phi}d\phi}{R^2e^{2i\phi}+k^2}\right| < 2\int_0^{\frac{\pi}{2}} \frac{Re^{-R\frac{2\phi}{\pi}}d\phi}{R^2-k^2} = \frac{\pi}{R^2-k^2}(e^{-R}-1).$$

Moreover, $$\int_{aa'a} \frac{e^{iz}dz}{z^2+k^2} = \int_{aa'a} \frac{e^{iz}}{z+ik}\frac{dz}{z-ik} = \int_{aa'a}\left(\frac{e^{-k}}{2ki}+\eta\right)\frac{dz}{z-ik},$$

where η is uniformly infinitesimal with the radius of the small circle. But

$$\int_{aa'a} \frac{dz}{z-ik} = -2\pi i, \quad \text{and} \quad \int_{aa'a} \frac{e^{iz}dz}{z^2+k^2} = -\frac{2\pi e^{-k}}{2k} + \zeta,$$

where $|\zeta| \leqq 2\pi\epsilon$ if ϵ is the largest value of $|\eta|$. Hence finally

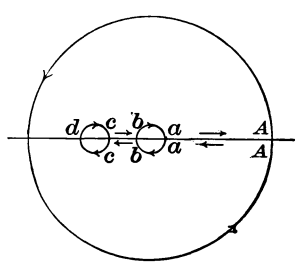
$$0 = 2\int_0^R \frac{\cos x}{x^2+k^2}dx - \frac{\pi}{k}e^{-k} + \zeta + \frac{\pi}{R^2-k^2}(e^{-R}-1).$$

By taking the small circle small enough and the large circle large enough, the last two terms may be made as near zero as desired. Hence

$$\int_0^\infty \frac{\cos x}{x^2+k^2}dx = \frac{\pi e^{-k}}{2k}. \tag{7}$$

It may be noted that, by the work of § 126, $\int_{aa'a} \frac{e^{iz}}{z+ki}\frac{dz}{z-ki} = -2\pi i\frac{e^{-k}}{2ki}$ is exact and not merely approximate, and remains exact for any closed curve about $z = ki$ which does not include $z = -ki$. That it is approximate in the small circle follows immediately from the continuity of $e^{iz}/(z+ki) = e^{-k}/2ki + \eta$ and a direct integration about the circle.

As a third example of the method let $\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx$ be evaluated. This integral will converge if $0 < \alpha < 1$, because the infinity at the origin is then of order less than the first and the integrand is an infinitesimal of order higher than the first for large values of x. The function $z^{\alpha-1}/(1+z)$ becomes infinite at $z = 0$ and $z = -1$, and these points must be excluded. The path marked in the figure is a closed path which does not contain them. Now here the integral back and forth along the line aA cannot be neglected; for the function has a fractional or irrational power $z^{\alpha-1}$ in the numerator and is therefore not single valued. In fact, when z is given, the function $z^{\alpha-1}$ is determined as far as its absolute value is concerned, but its angle may take on any addition of the form $2\pi k(\alpha-1)$ with k integral. Whatever value of the function is assumed at one point of the path, the values at the other points must be such as to piece on continuously when the path is followed. Thus the values along the line aA outward will differ by $2\pi(\alpha-1)$ from those along Aa inward because the turn has been made about the origin and the angle of z has increased by 2π. The double line bc and cb, however, may be disregarded because no turn about the origin is made in describing cdc. Hence, remembering that $e^{\pi i} = -1$,



$$0 = \int_O \frac{z^{\alpha-1}}{1+z}dz = \int_O \frac{r^{\alpha-1}e^{(\alpha-1)\phi i}}{1+re^{\phi i}}d(re^{\phi i}) = \int_a^A \frac{r^{\alpha-1}}{1+r}dr + \int_0^{2\pi} \frac{A^\alpha e^{\alpha\phi i}}{1+Ae^{\phi i}}id\phi$$

$$+ \int_A^a \frac{r^{\alpha-1}e^{2\pi(\alpha-1)i}}{1+re^{2\pi i}}e^{2\pi i}dr + \int_{abba}\frac{z^{\alpha-1}}{1+z}dz + \int_{cdc}\frac{z^{\alpha-1}}{1+z}dz.$$

Now $$\int_a^A \frac{r^{\alpha-1}}{1+r}dr + \int_A^a \frac{r^{\alpha-1}e^{2\pi\alpha i}}{1+r}dr = \int_a^A \frac{r^{\alpha-1}}{1+r}(1-e^{2\pi\alpha i})dr,$$

$$\left|\int_0^{2\pi} \frac{A^\alpha e^{\alpha\phi i}}{1+Ae^{\phi i}}id\phi\right| \leqq \int_0^{2\pi}\frac{A^\alpha}{A-1}\left|e^{\alpha\phi i}\right|d\phi = \frac{2\pi A^\alpha}{A-1},$$

$$\left|\int_{abba}\frac{z^{\alpha-1}}{1+z}dz\right| = \left|\int_{2\pi}^0 \frac{a^\alpha e^{\alpha\phi i}}{1+ae^{\phi i}}id\phi\right| \leqq \int_0^{2\pi}\frac{a^\alpha}{1-a}d\phi = \frac{2\pi a^\alpha}{1-a},$$

$$\int_{cdc} \frac{z^{\alpha-1}}{1+z}dz = \int z^{\alpha-1}\frac{dz}{1+z} = -2\pi i(-1)^{\alpha-1} = -2\pi i e^{\pi(\alpha-1)i} = 2\pi i e^{\pi\alpha i}.$$

Hence $\quad 0 = (1-e^{2\pi\alpha i})\int_a^A \frac{r^{\alpha-1}}{1+r}dr + 2\pi i e^{\pi\alpha i} + \zeta, \qquad |\zeta| < \frac{2\pi A^\alpha}{A-1} + \frac{2\pi a^\alpha}{1-a}.$

If A be taken sufficiently large and a sufficiently small, ζ may be made as small as desired. Then by the same reasoning as before it follows that

$$0 = (1-e^{2\pi\alpha i})\int_0^\infty \frac{r^{\alpha-1}}{1+r}dr + 2\pi i e^{\pi\alpha i}, \quad \text{or} \quad 0 = -\sin\pi\alpha\int_0^\infty \frac{r^{\alpha-1}}{1+r}dr + \pi,$$

and

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin\alpha\pi}. \tag{8}$$

143. One integral of particular importance is $\int_0^\infty e^{-x^2}dx$. The evaluation may be made by a device which is rarely useful. Write

$$\int_0^A e^{-x^2}dx = \left[\int_0^A e^{-x^2}dx\int_0^A e^{-y^2}dy\right]^{\frac{1}{2}} = \left[\int_0^A\int_0^A e^{-x^2-y^2}dxdy\right]^{\frac{1}{2}}.$$

The passage from the product of two integrals to the double integral may be made because neither the limits nor the integrands of either integral depend on the variable in the other. Now transform to polar coördinates and integrate over a quadrant of radius A.

$$\int_0^A\int_0^A e^{-x^2-y^2}dxdy = \int_0^{\frac{\pi}{2}}\int_0^A e^{-r^2}rdrd\theta + R = \frac{1}{4}\pi(1-e^{-A^2}) + R,$$

where R denotes the integral over the area between the quadrant and square, an area less than $\frac{1}{2}A^2$ over which $e^{-r^2} \leqq e^{-A^2}$. Then

$$R < \tfrac{1}{2}A^2e^{-A^2}, \qquad \left|\int_0^A\int_0^A e^{-x^2-y^2}dxdy - \tfrac{1}{4}\pi\right| < \tfrac{1}{2}A^2e^{-A^2}.$$

Now A may be taken so large that the double integral differs from $\frac{1}{4}\pi$ by as little as desired, and hence for sufficiently large values of A the simple integral will differ from $\frac{1}{2}\sqrt{\pi}$ by as little as desired. Hence *

$$\int_0^\infty e^{-x^2}dx = \tfrac{1}{2}\sqrt{\pi}. \tag{9}$$

* It should be noticed that the proof just given does not require the theory of infinite double integrals nor of change of variable; the whole proof consists merely in finding a number $\frac{1}{2}\sqrt{\pi}$ from which the integral may be shown to differ by as little as desired. This was also true of the proofs in § 142; no theory had to be developed and no limiting processes were used. In fact the evaluations that have been performed show of themselves that the infinite integrals converge. For when it has been shown that an integral with a large enough upper limit and a small enough lower limit can be made to differ from a certain constant by as little as desired, it has thereby been proved that that integral from zero to infinity must converge to the value of that constant.

When some infinite integrals have been evaluated, others may be obtained from them by various operations, such as integration by parts and change of variable. It should, however, be borne in mind that the rules for operating with definite integrals were established only for finite integrals and must be *reëstablished* for infinite integrals. From the direct application of the definition it follows that the integral of a function times a constant is the product of the constant by the integral of the function, and that the sum of the integrals of two functions taken between the same limits is the integral of the sum of the functions. But it cannot be inferred conversely that an integral may be resolved into a sum as

$$\int_a^b [f(x) + \phi(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b \phi(x)\,dx$$

when one of the limits is infinite or one of the functions becomes infinite in the interval. For, the fact that the integral on the left converges is no guarantee that either integral upon the right will converge; all that can be stated is that *if one of the integrals on the right converges, the other will,* and the equation will be true. The same remark applies to integration by parts,

$$\int_a^b f(x)\,\phi'(x)\,dx = \Big[f(x)\,\phi(x)\Big]_a^b - \int_a^b f'(x)\,\phi(x)\,dx.$$

If, in the process of taking the limit which is required in the definition of infinite integrals, *two of the three terms in the equation approach limits, the third will approach a limit,* and the equation will be true for the infinite integrals.

The formula for the change of variable is

$$\int_{x=\phi(t)}^{x=\phi(T)} f(x)\,dx = \int_t^T f[\phi(t)]\,\phi'(t)\,dt,$$

where it is assumed that the derivative $\phi'(t)$ is continuous and does not vanish in the interval from t to T (although either of these conditions may be violated at the extremities of the interval). As these two quantities are equal, they will approach equal limits, provided they approach limits at all, when the limit

$$\int_{a=\phi(t_0)}^{b=\phi(t_1)} f(x)\,dx = \int_{t_0}^{t_1} f[\phi(t)]\,\phi'(t)\,dt$$

required in the definition of an infinite integral is taken, where one of the four limits a, b, t_0, t_1 is infinite or one of the integrands becomes infinite at the extremity of the interval. *The formula for the change of variable is therefore applicable to infinite integrals.* It should be noted that the proof applies only to infinite limits and infinite values of the integrand at the extremities of the interval of integration; in case the integrand becomes infinite within the interval, the change of variable should be examined in each subinterval just as the question of convergence was examined.

As an example of the change of variable consider $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ and take $x = \alpha x'$.

$$\int_{x=0}^{x=\infty} \frac{\sin \alpha x'}{x'} dx' = \int_{x'=0}^{+\infty} \frac{\sin \alpha x'}{x'} dx' \text{ or } = \int_{x'=0}^{-\infty} \frac{\sin \alpha x'}{x'} dx' = -\int_{x'=0}^{x'=\infty} \frac{\sin \alpha x'}{x'} dx',$$

according as α is positive or negative. Hence the results

$$\int_0^\infty \frac{\sin \alpha x}{x} dx = +\frac{\pi}{2} \quad \text{if} \quad \alpha > 0 \quad \text{and} \quad -\frac{\pi}{2} \quad \text{if} \quad \alpha < 0. \tag{10}$$

Sometimes changes of variable or integrations by parts will lead back to a given integral in such a way that its value may be found. For instance take

$$I = \int_0^{\frac{\pi}{2}} \log \sin x dx = -\int_{\frac{\pi}{2}}^0 \log \cos y dy = \int_0^{\frac{\pi}{2}} \log \cos y dy, \qquad y = \frac{\pi}{2} - x.$$

Then
$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\, dx = \int_0^{\frac{\pi}{2}} \log \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x dx - \frac{\pi}{2}\log 2 = \int_0^{\frac{\pi}{2}} \log \sin x dx - \frac{\pi}{2} \log 2.$$

Hence
$$I = \int_0^{\frac{\pi}{2}} \log \sin x dx = -\frac{\pi}{2} \log 2. \tag{11}$$

Here the first change was $y = \frac{1}{2}\pi - x$. The new integral and the original one were then added together (the variable indicated under the sign of a definite integral is immaterial, p. 26), and the sum led back to the original integral by virtue of the substitution $y = 2x$ and the fact that the curve $y = \log \sin x$ is symmetrical with respect to $x = \frac{1}{2}\pi$. This gave an equation which could be solved for I.

EXERCISES

1. Integrate $\frac{ze^{iz}}{z^2 + k^2}$, as for the case of (7), to show $\int_0^\infty \frac{x \sin x}{x^2 + k^2} dx = \frac{\pi}{2} e^{-k}$.

2. By direct integration show that $\int_0^\infty e^{-(a-bi)z} dz$ converges to $(a - bi)^{-1}$, when $a > 0$ and the integral is extended along the line $y = 0$. Thus prove the relations

$$\int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}, \qquad \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}, \qquad a > 0.$$

Along what lines issuing from the origin would the given integral converge?

3. Show $\int_0^\infty \frac{x^{\alpha-1}dx}{(1+x)^2} = \frac{(1-\alpha)\pi}{\sin \alpha\pi}$. To integrate about $z = -1$ use the binomial expansion $z^{\alpha-1} = [-1+1+z]^{\alpha-1} = (-1)^{\alpha-1}[1+(1-\alpha)(1+z)+\eta(1+z)]$, η small.

4. Integrate e^{-z^2} around a circular sector with vertex at $z = 0$ and bounded by the real axis and a line inclined to it at an angle of $\frac{1}{4}\pi$. Hence show

$$-e^{\frac{1}{4}\pi i}\int_0^\infty (\cos r^2 - i\sin r^2)\,dr = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

5. Integrate e^{-z^2} around a rectangle $y = 0$, $y = B$, $x = \pm A$, and show

$$\int_0^\infty e^{-x^2}\cos 2\,ax dx = \tfrac{1}{2}\sqrt{\pi}e^{-a^2}, \qquad \int_{-\infty}^\infty e^{-x^2}\sin 2\,ax dx = 0.$$

6. Integrate $z^{\alpha-1}e^{-z}$, $0 < \alpha$, along a sector of angle $q < \frac{1}{2}\pi$ to show

$$\sec \alpha q \int_0^\infty x^{\alpha-1}e^{-x\cos q}\cos(x\sin q)\,dx$$
$$= \csc \alpha q \int_0^\infty x^{\alpha-1}e^{-x\cos q}\sin(x\sin q)\,dx = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

7. Establish the following results by the proper change of variable:

(α) $\int_0^\infty \frac{\cos \alpha x}{x^2+k^2}dx = \frac{\pi e^{-\alpha k}}{2k}$, $\alpha > 0$, $\qquad (\beta)$ $\int_0^\infty \frac{x^{\alpha-1}dx}{\beta + x} = \frac{\pi\beta^{\alpha-1}}{\sin \alpha\pi}$, $\beta > 0$,

(γ) $\int_0^\infty e^{-\alpha^2x^2}dx = \frac{1}{2\alpha}\sqrt{\pi}$, $\qquad (\delta)$ $\int_0^\infty e^{-\alpha x}\frac{1}{\sqrt{x}}dx = \sqrt{\frac{\pi}{\alpha}}$,

(ϵ) $\int_0^\infty e^{-\alpha^2x^2}\cos bx dx = \frac{\sqrt{\pi}e^{-\frac{b^2}{4\alpha^2}}}{2\alpha}$, $\alpha > 0$, $\qquad (\zeta)$ $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$,

(η) $\int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$, $\qquad (\theta)$ $\int_0^1 \frac{\log x dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2$.

8. By integration by parts or other devices show the following:

(α) $\int_0^\pi x\log\sin x dx = -\frac{1}{2}\pi^2\log 2$, $\qquad (\beta)$ $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$,

(γ) $\int_0^\infty \frac{\sin x\cos \alpha x}{x}dx = \frac{\pi}{2}$ if $-1<\alpha<1$, or $\frac{\pi}{4}$ if $\alpha = \pm 1$, or 0 if $|\alpha|>1$,

(δ) $\int_0^\infty x^2e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{4\alpha^3}$, $\qquad (\epsilon)$ $\int_0^\infty x^4e^{-\alpha^2x^2}dx = \frac{3\sqrt{\pi}}{8\alpha^5}$,

(ζ) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ if $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, $\qquad (\eta)$ $\int_0^\pi \frac{x\sin x dx}{1+\cos^2 x} = \frac{\pi^2}{4}$,

(θ) $\int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \pi\log 2$, by virtue of $x = \tan y$.

9. Suppose $\int_a^\infty f(x)\frac{dx}{x}$, where $a>0$, converges. Then if $p>0$, $q>0$,

$$\int_0^\infty \frac{f(px)-f(qx)}{x}dx = \lim_{a \doteq 0}\left[\int_a^\infty \frac{f(px)-f(qx)}{x}dx = \int_{pa}^\infty \frac{f(\xi)}{\xi}d\xi - \int_{qa}^\infty \frac{f(\xi)}{\xi}d\xi\right].$$

Show $$\int_0^\infty \frac{f(px)-f(qx)}{x}dx = \lim_{a \doteq 0}\int_{pa}^{qa} f(x)\frac{dx}{x} = f(0)\log\frac{q}{p}.$$

Hence (α) $\displaystyle\int_0^\infty \frac{\sin px - \sin qx}{x}dx = 0,$ (β) $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{x}dx = \log\frac{q}{p},$

(γ) $\displaystyle\int_0^1 \frac{x^{p-1}-x^{q-1}}{\log x}dx = \log\frac{q}{p},$ (δ) $\displaystyle\int_0^\infty \frac{\cos x - \cos ax}{x}dx = \log a.$

10. If $f(x)$ and $f'(x)$ are continuous, show by integration by parts that

$\displaystyle\lim_{k=\infty}\int_a^b f(x)\sin kx dx = 0.$ Hence prove $\displaystyle\lim_{k=\infty}\int_0^a f(x)\frac{\sin kx}{x}dx = \frac{\pi}{2}f(0).$

$$\left[\text{Write} \quad \int_0^a f(x)\frac{\sin kx}{x}dx = f(0)\int_0^a \frac{\sin kx}{x}dx + \int_0^a \frac{f(x)-f(0)}{x}\sin kx dx.\right]$$

Apply Ex. 6, p. 359, to prove these formulas under general hypotheses.

11. Show that $\displaystyle\lim_{k=\infty}\int_a^b f(x)\frac{\sin kx}{x}dx = 0$ if $b>a>0$. Hence note that

$$\lim_{k=\infty}\lim_{a\doteq 0}\int_a^b f(x)\frac{\sin kx}{x}dx \neq \lim_{a\doteq 0}\lim_{k=\infty}\int_a^b f(x)\frac{\sin kx}{x}dx, \quad \text{unless} \quad f(0)=0.$$

144. Functions defined by infinite integrals. If the integrand of an integral contains a parameter (§ 118), the integral defines a function of the parameter for every value of the parameter for which it converges. The continuity and the differentiability and integrability of the function have to be treated. Consider first the case of an infinite limit

$$\int_a^\infty f(x, \alpha)dx = \int_a^x f(x, \alpha)dx + R(x, \alpha), \qquad R = \int_x^\infty f(x, \alpha)dx.$$

If this integral is to converge for a given value $\alpha = \alpha_0$, it is necessary that the remainder $R(x, \alpha_0)$ can be made as small as desired by taking x large enough, and shall remain so for all larger values of x. In like manner if the integrand becomes infinite for the value $x = b$, the condition that

$$\int_a^b f(x, \alpha)dx = \int_a^x f(x, \alpha)dx + R(x, \alpha), \qquad R = \int_x^b f(x, \alpha)dx$$

converge is that $R(x, \alpha_0)$ can be made as small as desired by taking x near enough to b, and shall remain so for nearer values.

Now for different values of α, the least values of x which will make $|R(x, \alpha)| \leqq \epsilon$, when ϵ is assigned, will probably differ. The infinite integrals are said to *converge uniformly* for a range of values of α such as

$\alpha_0 \leqq \alpha \leqq \alpha_1$ when it is possible to take x so large (or x so near b) that $|R(x, \alpha)| < \epsilon$ holds (and continues to hold for all larger values, or values nearer b) simultaneously for all values of α in the range $\alpha_0 \leqq \alpha \leqq \alpha_1$. The most useful test for uniform convergence is contained in the theorem: *If a positive function $\phi(x)$ can be found such that*

$$\int^{\infty} \phi(x)\,dx \quad \textit{converges and} \quad \phi(x) \geqq f(x, \alpha)$$

for all large values of x and for all values of α in the interval $\alpha_0 \leqq \alpha \leqq \alpha_1$, the integral of $f(x, \alpha)$ to infinity converges uniformly (and absolutely) for the range of values in α. The proof is contained in the relation

$$\left| \int_x^{\infty} f(x, \alpha)\,dx \right| \leqq \int_x^{\infty} \phi(x)\,dx < \epsilon,$$

which holds for all values of α in the range. There is clearly a similar theorem for the case of an infinite integrand. See also Ex. 18 below.

Fundamental theorems are: * Over any interval $\alpha_0 \leqq \alpha \leqq \alpha_1$ where an infinite integral converges uniformly the integral defines a continuous function of α. This function may be integrated over any finite interval where the convergence is uniform by integrating with respect to α under the sign of integration with respect to x. The function may be differentiated at any point α_ξ of the interval $\alpha_0 \leqq \alpha \leqq \alpha_1$ by differentiating with respect to α under the sign of integration with respect to x provided the integral obtained by this differentiation converges uniformly for values of α in the neighborhood of α_ξ. Proofs of these theorems are given immediately below. †

To prove that the function is continuous if the convergence is uniform let

$$\psi(\alpha) = \int_a^{\infty} f(x, \alpha)\,dx = \int_a^{x} f(x, \alpha)\,dx + R(x, \alpha), \qquad \alpha_0 \leqq \alpha \leqq \alpha_1,$$

$$\psi(\alpha + \Delta\alpha) = \int_a^{x} f(x, \alpha + \Delta\alpha)\,dx + R(x, \alpha + \Delta\alpha),$$

$$|\Delta\psi| \leqq \left| \int_a^{x} [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]\,dx \right| + |R(x, \alpha + \Delta\alpha)| + |R(x, \alpha)|.$$

* It is of course assumed that $f(x, \alpha)$ is continuous in (x, α) for all values of x and α under consideration, and in the theorem on differentiation it is further assumed that $f'_\alpha(x, \alpha)$ is continuous.

† It should be noticed, however, that although the conditions which have been imposed are *sufficient* to establish the theorems, they are *not necessary*; that is, it may happen that the function will be continuous and that its derivative and integral may be obtained by operating under the sign although the convergence is not uniform. In this case a special investigation would have to be undertaken; and if no process for justifying the continuity, integration, or differentiation could be devised, it might be necessary in the case of an integral occurring in some application to assume that the formal work led to the right result if the result looked reasonable from the point of view of the problem under discussion, — the chance of getting an erroneous result would be tolerably small.

Now let x be taken so large that $|R| < \epsilon$ for all α's and for all larger values of x — the condition of uniformity. Then the finite integral (§ 118)

$$\int_a^x f(x, \alpha)\,dx \quad \text{is continuous in } \alpha \text{ and hence} \quad \left|\int_a^x [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]\,dx\right.$$

can be made less than ϵ by taking $\Delta\alpha$ small enough. Hence $|\Delta\psi| < 3\epsilon$; that is, by taking $\Delta\alpha$ small enough the quantity $|\Delta\psi|$ may be made less than any assigned number 3ϵ. The continuity is therefore proved.

To prove the integrability under the sign a like use is made of the condition of uniformity and of the earlier proof for a finite integral (§ 120).

$$\int_{\alpha_0}^{\alpha_1} \psi(\alpha)\,d\alpha = \int_{\alpha_0}^{\alpha_1}\int_a^x f(x, \alpha)\,dx d\alpha + \int_{\alpha_0}^{\alpha_1} R dx = \int_a^x \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\,d\alpha dx + \zeta.$$

Now let x become infinite. The quantity ζ can approach no other limit than 0; for by taking x large enough $R < \epsilon$ and $|\zeta| < \epsilon(\alpha_1 - \alpha_0)$ independently of α. Hence as x becomes infinite, the integral converges to the constant expression on the left and

$$\int_{\alpha_0}^{\alpha_1} \psi(\alpha)\,d\alpha = \int_a^\infty \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\,d\alpha dx.$$

Moreover if the integration be to a variable limit for α, then

$$\Psi(\alpha) = \int_{\alpha_0}^{\alpha} \psi(\alpha)\,d\alpha = \int_a^\infty \int_{\alpha_0}^{\alpha} f(x, \alpha)\,d\alpha dx = \int_a^\infty F(x, \alpha)\,dx.$$

Also $$\left|\int_x^\infty F(x, \alpha)\,dx\right| = \left|\int_x^\infty \int_{\alpha_0}^{\alpha} f(x, \alpha)\,d\alpha dx\right| = \left|\int_{\alpha_0}^{\alpha}\int_x^\infty f(x, \alpha)\,dx d\alpha\right| < \epsilon(\alpha - \alpha_0).$$

Hence it appears that the remainder for the new integral is less than $\epsilon(\alpha_1 - \alpha_0)$ for all values of α; the convergence is therefore uniform and a second integration may be performed if desired. Thus *if an infinite integral converges uniformly, it may be integrated as many times as desired under the sign.* It should be noticed that the proof fails to cover the case of integration to an infinite upper limit for α.

For the case of differentiation it is necessary to show that

$$\int_a^\infty f'_\alpha(x, \alpha_\xi)\,dx = \phi'(\alpha_\xi). \qquad \text{Consider} \int_a^\infty f'_\alpha(x, \alpha)\,dx = \omega(\alpha).$$

As the infinite integral is assumed to converge uniformly by the statement of the theorem, it is possible to integrate with respect to α under the sign. Then

$$\int_{\alpha_\xi}^{\alpha} \omega(\alpha)\,d\alpha = \int_a^\infty \int_{\alpha_\xi}^{\alpha} f'_\alpha(x, \alpha)\,d\alpha dx = \int_a^\infty [f(x, \alpha) - f(x, \alpha_\xi)]\,dx = \phi(\alpha) - \phi(\alpha_\xi).$$

The integral on the left may be differentiated with respect to α, and hence $\phi(\alpha)$ must be differentiable. The differentiation gives $\omega(\alpha) = \phi'(\alpha)$ and hence $\omega(\alpha_\xi) = \phi'(\alpha_\xi)$. The theorem is therefore proved. This theorem and the two above could be proved in analogous ways in the case of an infinite integral due to the fact that the integrand $f(x, \alpha)$ became infinite at the ends of (or within) the interval of integration with respect to x; the proofs need not be given here.

145. The method of integrating or differentiating under the sign of integration may be applied to evaluate infinite integrals when the conditions of uniformity are properly satisfied, in precisely the same manner as the method was previously applied to the case of finite integrals where the question of the uniformity of convergence did not arise (§§ 119–120). The examples given below will serve to illustrate how the method works and in particular to show how readily the test for uniformity may be applied in some cases. Some of the examples are purposely chosen identical with some which have previously been treated by other methods.

Consider first an integral which may be found by direct integration, namely,

$$\int_0^\infty e^{-ax}\cos bx dx = \frac{a}{a^2+b^2}. \qquad \text{Compare } \int_0^\infty e^{-ax}dx = \frac{1}{a}.$$

The integrand e^{-ax} is a positive quantity greater than or equal to $e^{-ax}\cos bx$ for all values of b. Hence, by the general test, the first integral regarded as a function of b converges uniformly for all values of b, defines a continuous function, and may be integrated between any limits, say from 0 to b. Then

$$\int_0^b\int_0^\infty e^{-ax}\cos bx dx db = \int_0^\infty\int_0^b e^{-ax}\cos bx db dx$$
$$= \int_0^\infty e^{-ax}\frac{\sin bx}{x}dx = \int_0^b \frac{a db}{a^2+b^2} = \tan^{-1}\frac{b}{a}.$$

Integrate again. $$\int_0^\infty\int_0^b e^{-ax}\frac{\sin bx}{x}db dx = \int_0^\infty e^{-ax}\frac{1-\cos bx}{x^2}dx$$
$$= b\tan^{-1}\frac{b}{a} - \frac{a}{2}\log(a^2+b^2).$$

Compare $$\int_0^\infty e^{-ax}\frac{1-\cos bx}{x^2}dx \quad \text{and} \quad \int_0^\infty \frac{1-\cos bx}{x^2}dx.$$

Now as the second integral has a positive integrand which is never less than the integrand of the first for any positive value of a, the first integral converges uniformly for all positive values of a including 0, is a continuous function of a, and the value of the integral for $a = 0$ may be found by setting a equal to 0 in the integrand. Then

$$\int_0^\infty \frac{1-\cos bx}{x^2}dx = \lim_{a \doteq 0}\left[b\tan^{-1}\frac{b}{a} - \frac{a}{2}\log(a^2+b^2)\right] = |b|\frac{\pi}{2}.$$

The change of the variable to $x' = \frac{1}{2}x$ and an integration by parts give respectively

$$\int_0^\infty \frac{\sin^2 bx}{x^2}dx = \frac{\pi}{2}|b|, \qquad \int_0^\infty \frac{\sin bx}{x}dx = +\frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2}, \quad \text{as} \quad b>0 \quad \text{or} \quad b<0.$$

This last result might be obtained *formally* by taking the limit

$$\lim_{a \doteq 0}\int_0^\infty e^{-ax}\frac{\sin bx}{x}dx = \int_0^\infty \frac{\sin bx}{x}dx = \tan^{-1}\frac{b}{0} = \pm\frac{\pi}{2}$$

after the first integration; but such a process would be unjustifiable without first showing that the integral was a continuous function of a for small positive values of a and for 0. In this case $|x^{-1}e^{-ax}\sin bx| \leqq |x^{-1}\sin x|$, but as the integral of $|x^{-1}\sin bx|$ does not converge, the test for uniformity fails to apply. Hence the limit would not be justified without special investigation. Here the limit does give the right result, but a simple case where the integral of the limit is not the limit of the integral is

$$\lim_{b \doteq 0}\int_0^\infty \frac{\sin bx}{x}dx = \lim_{b \doteq 0}\left(\pm\frac{\pi}{2}\right) = \pm\frac{\pi}{2} \neq \int_0^\infty \lim_{b \doteq 0}\frac{\sin bx}{x}dx \int_0^\infty \frac{0}{x}dx = 0.$$

As a second example consider the evaluation of $\int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2} dx$. Differentiate.

$$\phi'(a) = \frac{d}{da}\int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2} dx = 2\int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2}\left(x-\frac{a}{x}\right)\frac{1}{x}dx$$
$$= 2\int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2}\left(1-\frac{a}{x^2}\right)dx.$$

To justify the differentiation this last integral must be shown to converge uniformly. In the first place note that the integrand does not become infinite at the origin, although one of its factors does. Hence the integral is infinite only by virtue of its infinite limit. Suppose $a \geqq 0$; then for large values of x

$$e^{-\left(x-\frac{a}{x}\right)^2}\left(1-\frac{a}{x^2}\right) \leqq e^{2a}e^{-x^2} \quad \text{and} \quad \int^\infty e^{-x^2}dx \quad \text{converges (§ 143).}$$

Hence the convergence is uniform when $a \geqq 0$, and the differentiation is justified. But, by the change of variable $x' = -a/x$, when $a > 0$,

$$\int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2}\frac{adx}{x^2} = \int_0^\infty e^{-\left(-\frac{a}{x'}+x'\right)^2}dx' = \int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2}dx.$$

Hence the derivative above found is zero; $\phi'(a) = 0$ and

$$\phi(a) = \int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2}dx = \text{const.} = \int_0^\infty e^{-x^2}dx = \tfrac{1}{2}\sqrt{\pi};$$

for the integral converges uniformly when $a \geqq 0$ and its constant value may be obtained by setting $a = 0$. As the convergence is uniform for any range of values of a, the function is everywhere continuous and equal to $\frac{1}{2}\sqrt{\pi}$.

As a third example calculate the integral $\phi(b) = \int_0^\infty e^{-a^2x^2}\cos bx dx$. Now

$$\frac{d\phi}{db} = \int_0^\infty -xe^{-a^2x^2}\sin bx dx = \frac{1}{2a^2}\Big[e^{-a^2x^2}\sin bx\Big]_0^\infty - \frac{b}{2a^2}\int_0^\infty e^{-a^2x^2}\cos bx dx.$$

The second step is obtained by integration by parts. The previous differentiation is justified by the fact that the integral of $xe^{-a^2x^2}$, which is greater than the integrand of the derived integral, converges. The differential equation may be solved.

$$\frac{d\phi}{db} = -\frac{b}{2a^2}\phi, \qquad \phi = Ce^{-\frac{b^2}{4a^2}}, \qquad \phi(0) = \int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}.$$

Hence $$\phi(b) = \phi(0)e^{-\frac{b^2}{4a^2}} = \int_0^\infty e^{-a^2x^2}\cos bx dx = \frac{\sqrt{\pi}e^{-\frac{b^2}{4a^2}}}{2a}.$$

In determining the constant C, the function $\phi(b)$ is assumed continuous, as the integral for $\phi(b)$ obviously converges uniformly for all values of b.

146. The question of the integration under the sign is naturally connected with the question of infinite double integrals. The double integral $\int f(x, y)\,dA$ over an area A is said to be an infinite integral if that area extends out indefinitely in any direction or if the function $f(x, y)$ becomes infinite at any point of the area. The definition of convergence is analogous to that given before in the case of infinite simple integrals. If the area A is infinite, it is replaced by a finite area A' which is allowed to expand so as to cover more and more of the area A. If the function $f(x, y)$ becomes infinite at a point or along a line in the area A, the area A is replaced by an area A' from which the singularities of $f(x, y)$ are excluded, and again the area A' is allowed to expand and approach coincidence with A. If then the double integral extended over A' approaches a definite limit which is independent of how A' approaches A, the double integral is said to converge. As

$$\iint f(x, y)\, dxdy = \iint \left| J\left(\frac{x, y}{u, v}\right)\right| f(\phi, \psi)\, dudv,$$

where $x = \phi(u, v)$, $y = \psi(u, v)$, is the rule for the change of variable and is applicable to A', it is clear that if either side of the equality approaches a limit which is independent of how A' approaches A, the other side must approach the same limit.

The theory of infinite double integrals presents numerous difficulties, the solution of which is beyond the scope of this work. It will be sufficient to point out in a simple case the questions that arise, and then state without proof a theorem which covers the cases which arise in practice. Suppose the region of integration is a complete quadrant so that the limits for x and y are 0 and ∞. The first question is, If the double integral converges, may it be evaluated by successive integration as

$$\int f(x, y)\, dA = \int_{x=0}^{\infty}\int_{y=0}^{\infty} f(x, y)\, dydx = \int_{y=0}^{\infty}\int_{x=0}^{\infty} f(x, y)\, dxdy?$$

And conversely, if one of the iterated integrals converges so that it may be evaluated, does the other one, and does the double integral, converge to the same value? A part of this question also arises in the case of a function defined by an infinite integral. For let

$$\phi(x) = \int_{y=0}^{\infty} f(x, y)\, dy \quad \text{and} \quad \int_{x=0}^{\infty}\phi(x)\, dx = \int_{x=0}^{\infty}\int_{y=0}^{\infty} f(x, y)\, dydx,$$

it being assumed that $\phi(x)$ converges except possibly for certain values of x, and that the integral of $\phi(x)$ from 0 to ∞ converges. The question arises, May the integral of $\phi(x)$ be evaluated by integration under the sign? The proofs given in § 144 for uniformly convergent integrals integrated over a finite region do not apply to this case of an infinite integral. In any particular given integral special methods may possibly be devised to justify for that case the desired transformations. But most cases are covered by a theorem due to de la Vallée-Poussin: *If the* *function $f(x, y)$ does not change sign and is continuous except over a finite number of lines parallel to the axes of x and y, then the three integrals*

$$\int f(x, y)\,dA, \quad \int_{x=0}^{\infty}\int_{y=0}^{\infty} f(x, y)\,dydx, \quad \int_{y=0}^{\infty}\int_{x=0}^{\infty} f(x, y)\,dxdy, \tag{12}$$

*cannot lead to different determinate results; that is, if any two of them lead to definite results, those results are equal.** The chief use of the theorem is to establish the equality of the two iterated integrals when each is known to converge; the application requires no test for uniformity and is very simple.

As an example of the use of the theorem consider the evaluation of

$$I = \int_0^{\infty} e^{-x^2}dx = \int_0^{\infty} \alpha e^{-\alpha^2 x^2}\,dx.$$

Multiply by $e^{-\alpha^2}$ and integrate from 0 to ∞ with respect to α.

$$Ie^{-\alpha^2} = \int_0^{\infty} \alpha e^{-\alpha^2(1+x^2)}dx, \qquad I\int_0^{\infty} e^{-\alpha^2}\,d\alpha = I^2 = \int_0^{\infty}\int_0^{\infty} \alpha e^{-\alpha^2(1+x^2)}\,dxd\alpha.$$

Now the integrand of the iterated integral is positive and the integral, being equal to I^2, has a definite value. If the order of integrations is changed, the integral

$$\int_0^{\infty}\int_0^{\infty} \alpha e^{-\alpha^2(1+x^2)}\,d\alpha dx = \int_0^{\infty} \frac{1}{1+x^2}\frac{dx}{2} = \frac{1}{2}\tan^{-1}\infty = \frac{\pi}{4}$$

is seen also to lead to a definite value. Hence the values I^2 and $\frac{1}{4}\pi$ are equal.

EXERCISES

1. Note that the two integrands are continuous functions of (x, α) in the whole region $0 \leqq \alpha < \infty$, $0 \leqq x < \infty$ and that for each value of α the integrals converge. Establish the forms given to the remainders and from them show that it is not possible to take x so large that for all values of α the relation $|R(x, \alpha)| < \epsilon$ is satisfied, but may be satisfied for all α's such that $0 < \alpha_0 \leqq \alpha$. Hence infer that the convergence is nonuniform about $\alpha = 0$, but uniform elsewhere. Note that the functions defined are not continuous at $\alpha = 0$, but are continuous for all other values.

$$(\alpha) \quad \int_0^{\infty} \alpha e^{-\alpha x}\,dx, \quad R(x, \alpha) = \int_x^{\infty} \alpha e^{-\alpha x}\,dx = e^{-\alpha x} - 1,$$

$$(\beta) \quad \int_0^{\infty} \frac{\sin \alpha x}{x}\,dx, \quad R(x, \alpha) = \int_x^{\infty} \frac{\sin \alpha x}{x}\,dx = \int_{\alpha x}^{\infty} \frac{\sin x}{x}\,dx.$$

2. Repeat in detail the proofs relative to continuity, integration, and differentiation in case the integral is infinite owing to an infinite integrand at $x = b$.

* The theorem may be generalized by allowing $f(x, y)$ to be discontinuous over a finite number of curves each of which is cut in only a finite limited number of points by lines parallel to the axis. Moreover, the function may clearly be allowed to change sign to a certain extent, as in the case where $f > 0$ when $x > a$, and $f < 0$ when $0 < x < a$, etc., where the integral over the whole region may be resolved into the sum of a finite number of integrals. Finally, if the integrals are absolutely convergent and the integrals of $|f(x, y)|$ lead to definite results, so will the integrals of $f(x, y)$.

3. Show that differentiation under the sign is allowable in the following cases, and hence derive the results that are given :

(α) $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}},\ \alpha > 0,\ \int_0^\infty x^{2n}e^{-ax^2}dx = \frac{\sqrt{\pi}}{2}\frac{1\cdot 3\cdots(2n-1)}{2^n\alpha^{n+\frac{1}{2}}},$

(β) $\int_0^\infty xe^{-ax^2}dx = \frac{1}{2\alpha},\ \alpha > 0,\ \int_0^\infty x^{2n+1}e^{-ax^2}dx = \frac{1\cdot 2\cdots n}{2\alpha^{n+1}},$

(γ) $\int_0^\infty \frac{dx}{x^2+k} = \frac{\pi}{2}\frac{1}{\sqrt{k}},\ k > 0,\ \int_0^\infty \frac{dx}{(x^2+k)^{n+1}} = \frac{\pi}{2}\frac{1\cdot 3\cdots(2n-1)}{2^n n!\,k^{n+\frac{1}{2}}},$

(δ) $\int_0^1 x^n dx = \frac{1}{n+1},\ n > -1,\ \int_0^1 x^n(-\log x)^m dx = \frac{m!}{(n+1)^{m+1}},$

(ϵ) $\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin\alpha\pi},\ 0<\alpha<1,\ \int_0^\infty \frac{x^{\alpha-1}\log x}{1+x}dx = \frac{\pi^2\cos\alpha\pi}{\cos^2\alpha\pi - 1}.$

4. Establish the right to integrate and hence evaluate these :

(α) $\int_0^\infty e^{-\alpha x}dx,\ 0<\alpha_0 \leqq \alpha,\ \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx = \log\frac{b}{a},\ b,\ a \geqq \alpha_0,$

(β) $\int_0^1 x^\alpha dx,\ -1<\alpha_0<\alpha,\ \int_0^1 \frac{x^a - x^b}{\log x}dx = \log\frac{a+1}{b+1},\ b,\ a \geqq \alpha_0,$

(γ) $\int_0^\infty e^{-\alpha x}\cos mx dx,\ 0<\alpha_0\leqq\alpha,\ \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos mx dx = \frac{1}{2}\log\frac{b^2+m^2}{a^2+m^2},$

(δ) $\int_0^\infty e^{-\alpha x}\sin mx dx,\ 0<\alpha_0\leqq\alpha,\ \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin mx dx = \tan^{-1}\frac{b}{m} - \tan^{-1}\frac{a}{m},$

(ϵ) $\int_0^\infty e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2\alpha},\ 0<\alpha_0\leqq\alpha,\ \int_0^\infty e^{-\frac{a^2}{x^2}} - e^{-\frac{b^2}{x^2}}dx = (b-a)\sqrt{\pi}.$

5. Evaluate : (α) $\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}dx = \cot^{-1}\frac{\beta}{\alpha},$

(β) $\int_0^\infty e^{-x}\frac{1-\cos\alpha x}{x}dx = \log\sqrt{1+\alpha^2},$ (γ) $\int_0^\infty e^{-x^2}\frac{\sin 2\alpha x}{x}dx,$

(δ) $\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}e^{-2a},\ a\geqq 0,$ (ϵ) $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}dx.$

6. If $0<a<b$, obtain from $\int_0^\infty e^{-rx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{r}}$ and justify the relations:

$$\int_a^b \frac{\sin r}{\sqrt{r}}dr = \frac{2}{\sqrt{\pi}}\int_a^b\int_0^\infty e^{-rx^2}\sin r\,dx\,dr = \frac{2}{\sqrt{\pi}}\int_0^\infty\int_a^b e^{-rx^2}\sin r\,dr\,dx$$

$$= \frac{2}{\sqrt{\pi}}\left[\sin a\int_0^\infty \frac{e^{-ax^2}x^2dx}{1+x^4} - \sin b\int_0^\infty\frac{e^{-bx^2}x^2dx}{1+x^4}\right.$$
$$\left. + \cos a\int_0^\infty\frac{e^{-ax^2}dx}{1+x^4} - \cos b\int_0^\infty\frac{e^{-bx^2}dx}{1+x^4}\right],$$

$$\int_0^r \frac{\sin r}{\sqrt{r}}dr = \sqrt{\frac{\pi}{2}} - \frac{2}{\sqrt{\pi}}\left[\sin r\int_0^\infty\frac{e^{-rx^2}x^2dx}{1+x^4} + \cos r\int_0^\infty\frac{e^{-rx^2}dx}{1+x^4}\right].$$

Similarly, $\int_0^r \frac{\cos r}{\sqrt{r}}dr = \sqrt{\frac{\pi}{2}} - \frac{2}{\pi}\left[\cos r\int_0^\infty \frac{e^{-rx^2}x^2dx}{1+x^4} - \sin r\int_0^\infty \frac{e^{-rx^2}dx}{1+x^4}\right].$

Also $\int_0^\infty \frac{\sin r}{\sqrt{r}}dr = \int_0^\infty \frac{\cos r}{\sqrt{r}}dr = \sqrt{\frac{\pi}{2}}, \quad \int_0^\infty \sin\frac{\pi}{2}r^2dr = \int_0^\infty \cos\frac{\pi}{2}r^2dr = \frac{1}{2}.$

7. Given that $\frac{1}{1+x^2} = 2\int_0^\infty \alpha e^{-\alpha^2(1+x^2)}d\alpha$, show that

$$\int_0^\infty \frac{1+\cos mx}{1+x^2}dx = \frac{\pi}{2}(1+e^{-m}) \quad \text{and} \quad \int_0^\infty \frac{\cos mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}, \quad m>0.$$

8. Express $R(x,\alpha) = \int_x^\infty \frac{x\sin\alpha x}{1+x^2}dx$, by integration by parts and also by substituting x' for αx, in such a form that the uniform convergence for α such that $0 < \alpha_0 \leqq \alpha$ is shown. Hence from Ex. 7 prove

$$\int_0^\infty \frac{x\sin\alpha x}{1+x^2}dx = \frac{\pi}{2}e^{-\alpha}, \qquad \alpha > 0 \qquad \text{(by differentiation).}$$

Show that this integral does not satisfy the test for uniformity given in the text; also that for $\alpha = 0$ the convergence is not uniform and that the integral is also discontinuous.

9. If $f(x, \alpha, \beta)$ is continuous in (x, α, β) for $0 \leqq x < \infty$ and for all points (α, β) of a region in the $\alpha\beta$-plane, and if the integral $\phi(\alpha, \beta) = \int_0^\infty f(x, \alpha, \beta)dx$ converges uniformly for said values of (α, β), show that $\phi(\alpha, \beta)$ is continuous in (α, β). Show further that if $f'_\alpha(x, \alpha, \beta)$ and $f'_\beta(x, \alpha, \beta)$ are continuous and their integrals converge uniformly for said values of (α, β), then

$$\int_0^\infty f'_\alpha(x, \alpha, \beta)\,dx = \phi'_\alpha, \qquad \int_0^\infty f'_\beta(x, \alpha, \beta)\,dx = \phi'_\beta,$$

and ϕ'_α, ϕ'_β are continuous in (α, β). The proof in the text holds almost verbatim.

10. If $f(x, \gamma) = f(x, \alpha + i\beta)$ is a function of x and the complex variable $\gamma = \alpha + i\beta$ which is continuous in (x, α, β), that is, in (x, γ) over a region of the γ-plane, etc., as in Ex. 9, and if $f'_\gamma(x, \gamma)$ satisfies the same conditions, show that

$$\phi(\gamma) = \int_0^\infty f(x, \gamma)\,dx \text{ defines an analytic function of } \gamma \text{ in said region.}$$

11. Show that $\int_0^\infty e^{-\gamma x^2}dx$, $\gamma = \alpha + i\beta$, $\alpha \geqq \alpha_0 > 0$, defines an analytic function of γ over the whole γ-plane to the right of the vertical $\alpha = \alpha_0$. Hence infer

$$\phi(\gamma) = \int_0^\infty e^{-\gamma x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\gamma}} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha+i\beta}}, \qquad \alpha \geqq \alpha_0 > 0.$$

Prove
$$\int_0^\infty e^{-\alpha x^2}\cos\beta x^2dx = \frac{1}{2}\sqrt{\frac{\pi}{2}\,\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2}},$$

$$\int_0^\infty e^{-\alpha x^2}\sin\beta x^2dx = \frac{1}{2}\sqrt{\frac{\pi}{2}\,\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2}}.$$

12. Integrate $\int_x^\infty \frac{1}{x} e^{-\alpha x^2} x \cos \beta x^2 dx$ of Ex. 11 by parts with $x \cos \beta x^2 dx = du$ to show that the convergence is uniform at $\alpha = 0$. Hence find $\int_0^\infty \cos \beta x^2 dx$.

13. From $\int_{-\infty}^{+\infty} \cos x^2 dx = \int_{-\infty}^{+\infty} \cos (x + \alpha)^2 dx = \sqrt{\frac{\pi}{2}} = \int_{-\infty}^{+\infty} \sin (x + \alpha)^2 dx$, with the results $\int_{-\infty}^{+\infty} \cos x^2 \sin 2\, \alpha x dx = \int_{-\infty}^{+\infty} \sin x^2 \sin 2\, \alpha x dx = 0$ due to the fact that $\sin x$ is an odd function, establish the relations

$$\int_0^\infty \cos x^2 \cos 2\, \alpha x dx = \frac{\sqrt{\pi}}{2} \cos \left(\frac{\pi}{4} - \alpha^2\right), \quad \int_0^\infty \sin x^2 \cos 2\, \alpha x dx = \frac{\sqrt{\pi}}{2} \sin \left(\frac{\pi}{4} - \alpha^2\right).$$

14. Calculate: $(\alpha)\ \int_0^\infty e^{-a^2x^2} \cosh bx dx,\qquad (\beta)\ \int_0^\infty x e^{-ax} \cos bx dx,$

and (together) $(\gamma)\ \int_0^\infty \cos \left(\frac{x^2}{2} \pm \frac{a^2}{2x^2}\right) dx,\qquad (\delta)\ \int_0^\infty \sin \left(\frac{x^2}{2} \pm \frac{a^2}{2x^2}\right) dx.$

15. In continuation of Exs. 10–11, p. 368, prove at least formally the relations:

$$\lim_{k=\infty} \int_{-a}^0 f(x) \frac{\sin kx}{x} dx = \frac{\pi}{2} f(0), \qquad \lim_{k=\infty} \frac{1}{\pi} \int_{-a}^a f(x) \frac{\sin kx}{x} dx = f(0),$$

$$\int_0^k \int_{-a}^a f(x) \cos kx dx dk = \int_{-a}^a \int_0^k f(x) \cos kx dk dx = \int_{-a}^a f(x) \frac{\sin kx}{x} dx,$$

$$\frac{1}{\pi} \int_0^\infty \int_{-a}^a f(x) \cos kx dx dk = \lim_{k=\infty} \frac{1}{\pi} \int_{-a}^a f(x) \frac{\sin kx}{x} dx = f(0),$$

$$\frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(x) \cos kx dx dk = f(0), \qquad \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(x) \cos k(x - t) dx dk = f(t).$$

The last form is known as Fourier's Integral; it represents a function $f(t)$ as a double infinite integral containing a parameter. Wherever possible, justify the steps after placing sufficient restrictions on $f(x)$.

16. From $\int_0^\infty e^{-xy} dy = \frac{1}{x}$ prove $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log \frac{b}{a}\cdot$ Prove also

$$\int_0^\infty x^{n-1} e^{-x} dx \int_0^\infty x^{m-1} e^{-x} dx$$

$$= 2 \int_0^\infty r^{2n+2m-2} e^{-r^2} dr^2 \int_0^{\frac{\pi}{2}} \sin^{2n-1}\phi \cos^{2m-1}\phi d\phi.$$

17. Treat the integrals (12) by polar coördinates and show that

$$\int f(x, y)\, dA = \int_0^{\frac{\pi}{2}} \int_0^\infty f(r \cos \phi, r \sin \phi)\, r dr d\phi$$

will converge if $|f| < r^{-2-k}$ as r becomes infinite. If $f(x, y)$ becomes infinite at the origin, but $|f| < r^{-2+k}$, the integral converges as r approaches zero. Generalize these results to triple integrals and polar coördinates in space; the only difference is that 2 becomes 3.

18. As in Exs. 1, 8, 12, uniformity of convergence may often be tested directly, without the test of page 369; treat the integrand $x^{-1} e^{-ax} \sin bx$ of page 371, where that test failed.