Part I

PRELIMINARY CONCEPTS

Chapter 1

Frechet manifolds

This Chapter is devoted to the Frechet manifolds, having Frechet vector spaces as their local models. We start our exposition by recalling basic facts on Frechet spaces in Sec. 1.1. In Sec. 1.2 we introduce Frechet manifolds and define various geometric structures on them, including vector bundles and connections, differential forms, symplectic and complex structures.

1.1 Frechet vector spaces

1.1.1 Basic definitions

In contrast with Banach spaces, whose topology is defined by a norm, the topology of a Frechet vector space is determined by a system of seminorms. Recall that

Definition 1. A *seminorm* on a vector space F is a real-valued function $p: F \to \mathbb{R}$, which satisfies the following conditions:

1. $p(f) \geq 0$ for any $f \in F$;
2. $p(f+g) \leq p(f) + p(g)$ for any $f, g \in F$;
3. $p(cf) = |c|p(f)$ for any $f \in F$ and any element c of the basic number field k (we restrict to $k = \mathbb{R}$ and $k = \mathbb{C}$ in the sequel).

As one can see from this definition, the only difference between seminorms and norms is that a seminorm p is not required to satisfy the property: $p(f) = 0 \iff f = 0$.

A system of seminorms $\{p_n\}_{n\in\mathbb{N}}$ determines on the vector space F a unique *topology*, for which

$$f_j \to f \iff p_n(f_j - f) \to 0 \quad \text{for any } n \in \mathbb{N} .$$

This topology is Hausdorff, if the following condition is fulfilled:

$$f = 0 \iff p_n(f) = 0 \quad \text{for all } n \in \mathbb{N} .$$

A sequence $\{f_j\}$ of elements of F is called a *Cauchy sequence* with respect to this topology if $p_n(f_j - f_k) \to 0$ for $j, k \to \infty$ for any $n \in \mathbb{N}$. The space F is *complete*, if any Cauchy sequence in F has a limit in F.

Definition 2. A Hausdorff topological vector space F with the topology, defined by a countable system of seminorms, is called a *Frechet space* iff it is complete.

Example 1. Any *Banach space* is a Frechet space with a system of seminorms, represented by a single norm.

Example 2. The vector space $C^\infty[a,b]$, consisting of C^∞-smooth real-valued functions f on an interval $[a,b]$, is a Frechet space with a system of seminorms

$$p_n(f) = \sum_{j=0}^{n} \sup_{[a,b]} \left| f^{(j)}(x) \right| .$$

Example 3. The vector space $C^\infty(X)$, consisting of C^∞-smooth real-valued functions f on a compact manifold X, is a Frechet space with a system of seminorms

$$p_n(f) = \sum_{|j|=0}^{n} \sup_{X} \left| d^j f(x) \right| .$$

Example 4. Let $V \to X$ be a vector bundle over a compact Riemannian manifold X, provided with a Riemannian metric and connection. Then the vector space $C^\infty(X, V)$, consisting of C^∞-smooth sections f of $V \to X$, is a Frechet space with a system of seminorms

$$p_n(f) = \sum_{|j|=0}^{n} \sup_{X} \left| D^j f(x) \right| ,$$

where $D^j f$ is the jth covariant derivative of a section f, and the "length" $|h|$ of a section h is computed, using the metrics on X and V.

A closed subspace of a Frechet space is also a Frechet space and the same is true for the quotient of a Frechet space by its closed subspace.

Example 5. The vector space $C^\infty_{2\pi}$, consisting of C^∞-smooth real-valued 2π-periodic functions on the real line $\mathbb{R}$, may be identified with the closed subspace in the Frechet space $C^\infty[0, 2\pi]$, consisting of functions $f \in C^\infty[0, 2\pi]$ such that all their derivatives $f^{(j)}$ match together at the end points: $f^{(j)}(0) = f^{(j)}(2\pi)$. It implies that $C^\infty_{2\pi}$ is also a Frechet space.

Many well-known properties of Banach spaces, such as the Hahn-Banach theorem and the closed graph theorem, are fulfilled in Frechet spaces as well.

However, there is a number of properties of Banach spaces, which do not transfer to the Frechet case. For example, the theorem of existence and uniqueness of solutions of ordinary differential equations for Banach spaces do not extend to general Frechet spaces. Another example: the dual of a Frechet space, which is <u>not</u> a Banach space, <u>cannot</u> be a Frechet space. In particular, the dual of the Frechet space $C^\infty(X)$ of C^∞-smooth real-valued functions on a compact manifold X, which is the space $\mathcal{D}'(X)$ of distributions on X, is not a Frechet space. Note also that the space $L(F, G)$ of linear operators, acting from a Frechet space F to another Frechet space G, is not, generally speaking, a Frechet space.

1.1.2 Derivative

Definition 3. Let F and G be Frechet spaces and $A : F \to G$ be a continuous map. The *derivative* of A *at a point* $f \in F$ *in a direction* $h \in F$ is the limit

$$D_f A(h) = \lim_{t\to 0} \frac{A(f+th) - A(f)}{t} \in G \ .$$

The map A is *differentiable at* f *in the direction* h, if this limit exists. The map A is *continuously differentiable* (or belongs to the class $C^1(U)$) on an open subset $U \subset F$, if this limit exists for any $f \in U$ and all $h \in F$ and the map

$$DA : U \times F \longrightarrow G$$

is continuous.

Example 6. Let $f : [a,b] \to F$ be a path in a Frechet space F, i.e. a continuous map from an interval $[a,b]$ to F. Denote by $\mathbf{1}$ the unit vector in $\mathbb{R}$, then the derivative $f'(t)$ (if it exists) coincides with $D_{f(t)}(\mathbf{1})$.

Example 7. A continuous linear map $L : F \to G$ of Frechet spaces belongs to the class C^1 and $D_f L(h) = Lh$ since

$$D_f L(h) = \lim_{t\to 0} \frac{L(f+th) - Lf}{t} = \lim_{t\to 0} \frac{tLh}{t} = Lh \ .$$

Example 8. Let U be a relatively open subset of a band $[a,b] \times \mathbb{R} \subset \mathbb{R}^2_{(x,y)}$ and $F = F(x,y)$ be a smooth function on U. Denote by $\mathcal{U}$ an open subset in $C^\infty[a,b]$, consisting of functions $y = f(x)$, having their graphs inside U. Consider a map $A : \mathcal{U} \longrightarrow C^\infty[a,b]$, given by the formula

$$A(f)(x) = F(x, f(x)) \ .$$

Then A belongs to the class C^1 and

$$D_f A(h)(x) = d_y F(x, f(x)) h(x) \ .$$

Example 9. More generally, let X be a compact manifold and $V \to X$, $W \to X$ be two vector bundles over X. Given an open subset U in V, denote by $\mathcal{U}$ the open subset in $C^\infty(X,V)$, consisting of sections f of $V \to X$, having their image in U: $f(X) \subset U$. Let $F : U \to W$ be an arbitrary smooth *bundle map*, sending any fibre V_p, $p \in X$, into the fibre W_p over the same point p.

Define a *fibrewise operator* $A : \mathcal{U} \longrightarrow C^\infty(X,W)$, acting by the formula

$$A(f) = F \circ f \ .$$

Denote by x a local coordinate on X in a neighborhood of a given point p and by y and z coordinates in the fibres V_p and W_p respectively. Then the map F is given locally by a function $z = F(x,y)$. A section f has a local representation $y = f(x)$, and the bundle operator A is given locally by the formula $A(f)(x) = F(x, f(x))$.

The derivative of A in the chosen local coordinates has the form

$$D_f A(h)(x) = d_y F(x, f(x)) h(x) \ ,$$

where $d_y F$ is the matrix of partial derivatives in y, applied to a vector-valued function h, representing locally a section $h \in C^\infty(X,V)$.

If A is a C^1-map $F \to G$, then

$$D_f A(h_1 + h_2) = D_f A(h_1) + D_f A(h_2) .$$

In other words, a continuously differentiable map A is necessarily linear in h. This important property shows that the derivative "behaves" like a differential with respect to the variable h.

Moreover, a map $A : U \subset F \to G$ is continuously differentiable on a convex open subset $U \subset F$ if and only if there exists a continuous map

$$L : U \times U \times F \longrightarrow G, \quad L = L(f_1, f_2)h ,$$

which is linear in h and for any $f_1, f_2 \in U$ satisfies the relation

$$A(f_1) - A(f_2) = L(f_1, f_2)(f_1 - f_2) .$$

In this case $D_f A(h) = L(f, f)h$.

If two maps $A : F \to G$ and $B : G \to H$ are continuously differentiable, then their composition $B \circ A : F \to H$ is also continuously differentiable and the chain rule for the derivatives is fulfilled

$$D_f[B \circ A](h) = D_{A(f)} B(D_f A(h)) .$$

In particular, if $f(t)$ is a C^1-path in F and $A : F \to G$ is a C^1-map, then $A(f(t))$ is a C^1-path in G and

$$A(f(t))' = D_{f(t)} A(f'(t)) .$$

Suppose now that the basic number field $k = \mathbb{C}$ and $A : U \subset F \to G$ is a map between complex Frechet spaces. We shall call this map *holomorphic* if it belongs to the class $C^1(U)$ and its derivative $DA : F \times F \to G$ is complex linear in $h \in F$.

By iterating the definition of the derivative, one can define higher order derivatives of maps between Frechet spaces. In particular, the *second derivative* of a map $A : F \to G$ is defined by the formula

$$D_f^2 A(h, k) = \lim_{t \to 0} \frac{D_{f+tk} A(h) - D_f A(h)}{t} .$$

A map $A : U \to G$ belongs to the class $C^2(U)$ on an open subset $U \subset F$ if DA belongs to $C^1(U)$, which is equivalent to the existence and continuity of the second derivative as a map $D^2 A : U \times F \times F \to G$.

Similarly to the first derivative, the second derivative $D_f^2 A(h, k)$ is linear separately in h and k if A is of class C^2. Moreover, in this case it can be given by the limit of the second finite difference

$$D_f^2 A(h, k) = \lim_{t,s \to 0} \frac{A(f + th + sk) - A(f + th) - A(f + sk) + A(f)}{ts}$$

and is symmetric in h, k.

By induction, one can define the *nth order derivative* $D_f^n A(h_1, \dots, h_n)$ as the partial derivative of the $(n-1)$th derivative $D_f^{n-1} A(h_1, \dots, h_{n-1})$ with respect to f in the direction of h_n, more precisely:

$$D_f^n A(h_1, \dots, h_n) = \lim_{t \to 0} \frac{D_{f+th_n}^{n-1} A(h_1, \dots, h_{n-1}) - D_f^{n-1} A(h_1, \dots, h_{n-1})}{t} .$$

Again, a map $A : U \to G$ belongs to the class $C^n(U)$ on an open subset $U \subset F$ if $D^n_f A(h_1, \ldots, h_n)$ exists and is continuous as a map $D^n A : U \times F \cdots \times F \to G$. In this case $D^n_f A(h_1, \ldots, h_n)$ is symmetric and linear in $h_1, \ldots, h_n$. We say that a map $A : U \to G$ belongs to the class $C^\infty(U)$ on an open subset $U \subset F$ if it belongs to all classes $C^n(U)$ for $n \in \mathbb{N}$.

1.2 Frechet manifolds

1.2.1 Basic definitions

Definition 4. A *Frechet manifold* is a Hausdorff topological space $\mathcal{X}$, provided with an *atlas*, i.e. a covering of $\mathcal{X}$ by open subsets (*coordinate neighborhoods*) $\{U_\alpha\}$, and a collection of *charts*, i.e. homeomorphisms (*coordinate maps*)

$$\varphi_\alpha : U_\alpha \xrightarrow{\approx} u_\alpha \subset F_\alpha$$

onto open subsets u_α in *model* Frechet spaces F_α. The *transition functions*

$$\varphi_{\beta\alpha} : \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta} \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth (i.e. of class C^∞) maps of Frechet spaces.

If all Frechet spaces F_α in this definition coincide with some Banach spaces E_α, we call such an $\mathcal{X}$ a *Banach manifold*. Respectively, when all F_α coincide with a separable Hilbert space $H = l_2$, we call it a *Hilbert manifold*.

There is one more specification of the above definition in the case when the basic field $k = \mathbb{C}$.

Definition 5. A *complex Frechet manifold* is a Frechet manifold $\mathcal{X}$, for which all model Frechet spaces F_α are complex, and the transition functions $\varphi_{\beta\alpha}$ are holomorphic.

We add the definition of a (closed) Frechet submanifold for the future use.

Definition 6. A closed subset $\mathcal{Y}$ in a Frechet manifold $\mathcal{X}$ is called a *submanifold* of $\mathcal{X}$ if for any point of $\mathcal{Y}$ there exists a coordinate neighborhood U of $\mathcal{X}$ with a coordinate chart, mapping U onto a neighborhood u in the product of Frechet spaces $F \times G$, which identifies $U \cap \mathcal{Y}$ with the subset $u \cap F \times \{0\}$.

Example 10. Let X be a (finite-dimensional) smooth manifold. Then the set of all smooth submanifolds in X, denoted by $\mathcal{S}(X)$, is a Frechet manifold. Indeed, consider a submanifold $S \in \mathcal{S}(X)$, having the normal bundle $NS = (TX|S)/TS$.

Then there exists a local exponential diffeomorphism

$$\exp : v \longrightarrow V \ ,$$

mapping a neighborhood v of the zero section in NS onto a tubular neighborhood V of S in X. This diffeomorphism generates a local coordinate chart φ with

$$\varphi^{-1} : \mathfrak{v} \longrightarrow \mathfrak{V} \ ,$$

mapping the neighborhood $\mathfrak{v}$ of zero in the Frechet space $C^\infty(S, NS)$, consisting of sections of NS with their image in v, onto the neighborhood $\mathfrak{V}$ of the submanifold S in $\mathcal{S}(X)$, consisting of submanifolds in X, lying in V.

Example 11. Let X be a compact smooth manifold and $\pi : E \to X$ is a *smooth bundle*, i.e. E is a smooth manifold, π is a smooth map, whose tangent π_* is everywhere surjective. Then the space of smooth sections of the bundle E, denoted by $C^\infty(X, E)$, is a Frechet manifold.

In order to construct coordinate charts on $C^\infty(X, E)$, we define for a given section f a *vertical vector bundle* $T^v_f E \to X$, *associated with* f, with the fibre at $p \in X$, equal to the kernel of π_*, restricted to $T_{f(p)}E$. Choose a neighborhood u of the zero section of $T^v_f E \to X$ together with a fibrewise diffeomorphism of u onto a tubular neighborhood U of the image $f(X)$ in E. This diffeomorphism generates a local coordinate chart φ with

$$\varphi^{-1} : \mathfrak{u} \longrightarrow \mathfrak{U} ,$$

mapping the neighborhood $\mathfrak{u}$ of the zero section in the Frechet space $C^\infty(X, T^v_f E)$, consisting of sections of $T^v_f E \to X$ with their image in u, onto the neighborhood $\mathfrak{U}$ of f in $C^\infty(X, E)$, consisting of sections of $E \to X$ with their image in U. The transition functions are given by fibrewise operators, as in Ex. 9 from Sec. 1.1.

Example 12. The manifold $C^\infty(X, Y)$ of smooth maps from a smooth compact manifold X into a smooth manifold Y is a particular case of the above construction, when the bundle $E = X \times Y \to X$ is trivial. The group $\mathrm{Diff}(X)$ of diffeomorphisms of X onto itself is an open subspace in $C^\infty(X, Y)$ and so inherits its structure of a Frechet manifold.

Example 13. The latter example is especially interesting for us when X is a circle, which we identify with $S^1 = \{|z| = 1 : z \in \mathbb{C}\}$. In this case the manifold $C^\infty(S^1, Y)$ is called the *space of (free) loops* in the manifold Y.

Consider the simplest example of that sort when Y is also a circle S^1. The manifold $C^\infty(S^1, S^1)$ consists of a countable number of connected components, denoted by $C^\infty_k(S^1, S^1)$ with $k \in \mathbb{Z}$, which are numerated by the index (rotation number) of a map $S^1 \to S^1$. By pulling up to the universal coverings, we can associate with a map $f : S^1 \to S^1$ the map $\tilde{f} : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$, defined up to an additive constant of the form $2\pi n$, $n \in \mathbb{Z}$. In particular, the maps $f \in C^\infty_0(S^1, S^1)$ of index 0 have the pullbacks $\tilde{f}$, which are smooth 2π-periodic functions, i.e. belong to the Frechet space $C^\infty_{2\pi}$ (cf. Ex. 5 in Sec. 1.1). So we have a global coordinate chart for the whole component $C^\infty_0(S^1, S^1)$:

$$\varphi : C^\infty_0(S^1, S^1) \xrightarrow{\approx} C^\infty_{2\pi}/2\pi\mathbb{Z} , \quad f \longmapsto [\tilde{f}] .$$

In the same way, the maps $f \in C^\infty_k(S^1, S^1)$ of index k have the pullbacks $\tilde{f}$, which satisfy the relation: $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi k$. Translating such a function by kx, i.e. replacing $\tilde{f}(x)$ by $\tilde{f}_1(x) := \tilde{f}(x) - kx$, we obtain a 2π-periodic function $\tilde{f}_1$. Hence, we have again a global coordinate chart on $C^\infty_k(S^1, S^1)$:

$$\varphi : C^\infty_k(S^1, S^1) \xrightarrow{\approx} C^\infty_{2\pi}/2\pi\mathbb{Z} .$$

For the whole manifold $C^\infty(S^1, S^1)$ we get a diffeomorphism

$$C^\infty(S^1, S^1) \xrightarrow{\approx} \mathbb{Z} \times C^\infty_{2\pi}/2\pi\mathbb{Z} .$$

Example 14. Consider an open submanifold $\mathrm{Diff}(S^1)$ in $C^\infty(S^1, S^1)$, consisting of all diffeomorphisms of the circle S^1. It has two connected components: the identity component $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms of S^1, preserving its orientation (this component belongs to the subspace $C_1^\infty(S^1, S^1)$), and $\mathrm{Diff}_-(S^1)$, consisting of diffeomorphisms of S^1, reversing its orientation (this component belongs to the subspace $C_{-1}^\infty(S^1, S^1)$).

The maps $f \in \mathrm{Diff}_+(S^1)$ pull back to functions $\tilde{f}$, satisfying the relation

$$\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi \ .$$

They have 2π-periodic derivatives $\tilde{f}'(x)$, which are everywhere positive, since diffeomorphisms f preserve the orientation. We also have:

$$\frac{1}{2\pi}\int_0^{2\pi} \tilde{f}'(x)\, dx = \frac{\tilde{f}(2\pi) - \tilde{f}(0)}{2\pi} = 1 \ ,$$

i.e. the average of $\tilde{f}'(x)$ over the period is equal to 1. Denote by C the subset of $C_{2\pi}^\infty$, consisting of smooth 2π-periodic strictly positive functions on the real line with the average, equal to 1. It is an open convex subset in an affine subspace of codimension 1 in $C_{2\pi}^\infty$, hence a Frechet submanifold. The above argument implies that our manifold $\mathrm{Diff}_+(S^1)$ is diffeomorphic to $S^1 \times C$. Indeed, the function $\tilde{f}$ is defined by $\tilde{f}'$ up to an additive constant $\tilde{f}(0) \in \mathbb{R}$, but the function $\tilde{f}$ itself is defined by $f : S^1 \to S^1$ up to an additive constant $2\pi n \in 2\pi\mathbb{Z}$. Hence, $\tilde{f}'$ determines f up to an element of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since C is contractible, we see that $\mathrm{Diff}_+(S^1)$ *is homotopy equivalent to* S^1.

1.2.2 Frechet vector bundles

Let $\mathcal{X}$, $\mathcal{V}$ be two Frechet manifolds and $\pi : \mathcal{V} \to \mathcal{X}$ be a smooth surjection such that each fibre $\pi^{-1}(x)$, $x \in \mathcal{X}$, of π has the structure of a Frechet vector space.

Definition 7. A Frechet manifold $\mathcal{V}$ is called a *Frechet vector bundle* over $\mathcal{X}$ if the following conditions are satisfied. There exists an atlas $\{U_\alpha\}$ of coordinate neighborhoods in $\mathcal{X}$ such that for any α the preimage $V_\alpha = \pi^{-1}(U_\alpha)$ of the coordinate neighborhood $\{U_\alpha\}$ belongs to a coordinate neighborhood in $\mathcal{V}$. The corresponding coordinate charts have the form

$$\varphi_\alpha : U_\alpha \longrightarrow u_\alpha = \varphi(U_\alpha) \subset F_\alpha \ , \tag{1.1}$$

$$\psi_\alpha : V_\alpha \longrightarrow v_\alpha = \psi_\alpha(V_\alpha) = u_\alpha \times G_\alpha \tag{1.2}$$

and are compatible in the sense that the following diagram is commutative

$$\begin{array}{ccc} V_\alpha = \pi^{-1}(U_\alpha) & \xrightarrow[\psi_\alpha]{} & v_\alpha = u_\alpha \times G_\alpha \subset F_\alpha \times G_\alpha \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{projection}} \\ U_\alpha & \xrightarrow{\varphi_\alpha} & u_\alpha \subset F_\alpha \end{array}$$

The structure of a vector space on π-fibres, induced from the right vertical arrow, coincides with the original one and the transition functions

$$\psi_{\beta\alpha} := \psi_\beta \circ \psi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times G_\alpha \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \times G_\beta$$

are linear in the second variable.

This definition applies with evident modifications to Banach and Hilbert vector bundles. If all Frechet spaces in the above definition, as well as π-fibres, are complex and the transition functions are holomorphic, we obtain the definition of a holomorphic Frechet vector bundle.

Example 15. The *tangent bundle* $T\mathcal{X}$ of a Frechet manifold $\mathcal{X}$ is a Frechet vector bundle. The fibre of $T\mathcal{X}$ at $x \in \mathcal{X}$ is formed by vectors $x'(t)|_{t=0}$, where $x(t)$ is a smooth path in $\mathcal{X}$, emanating from x. The coordinate transition function for $T\mathcal{X}$ are given by the derivatives of coordinate transition functions for $\mathcal{X}$.

Example 16. If, in particular, $\mathcal{X} = C^\infty(X, Y)$, then a path $f : [0, 1] \to C^\infty(X, Y)$ is given by a map $f : [0, 1] \times X \to Y$, i.e. by a 1-parameter family of maps $f_t : X \to Y$, $t \in [0, 1]$. For any $x \in X$ the image $f_t(x)$ for $0 \le t \le 1$ constitutes a path in Y, whose tangent vector at $f_t(x)$ coincides with the derivative $f_t'(x) \in T_{f_t(x)}Y = f_t^*(TY)_x$. Hence, f_t' is a section of the inverse image $f_t^*TY \to X$ of the tangent bundle TY under the map f_t and

$$T_f C^\infty(X, Y) = C^\infty(X, f^*TY) .$$

Example 17. Let X be a (finite-dimensional) smooth manifold and $\mathcal{S}(X)$ be the Frechet manifold of its smooth compact submanifolds (cf. Ex. 10). Then its tangent bundle $T\mathcal{S}(X)$ has the fibre at $S \in \mathcal{S}(X)$, equal to the Frechet space of sections $C^\infty(S, NS)$ of the normal bundle NS.

We shall need later another Frechet vector bundle, related to the Frechet manifold $\mathcal{S}(X)$. Namely, denote by $C^\infty(S)$ the Frechet space of smooth functions on S. Then the union of the spaces $C^\infty(S)$ over all $S \in \mathcal{S}(X)$ is a Frechet vector bundle $C^\infty\mathcal{S}(X) \to \mathcal{S}(X)$. Indeed, a coordinate chart φ on $\mathcal{S}(X)$ in a neighborhood of the submanifold $S \in \mathcal{S}(X)$ maps this neighborhood into the Frechet space $C^\infty(S, NS)$ of smooth sections of the normal bundle NS. Using this map, we can identify diffeomorphically submanifolds S', close to S, with the submanifold S, which corresponds to the zero section of NS. Accordingly, smooth functions on S' will be identified with smooth functions on S, which defines a coordinate chart ψ on $C^\infty\mathcal{S}(X)$ in a neighborhood of S with values in $C^\infty(S, NS) \times C^\infty(S)$, compatible with the coordinate chart φ on $\mathcal{S}(X)$.

Definition 8. A map $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ between Frechet manifolds is called *smooth* if for any point $x \in \mathcal{X}$ we can find coordinate charts φ in a neighborhood of this point and ψ in a neighborhood of its image $y = \mathcal{A}(x)$ such that the composition $\psi \circ \mathcal{A} \circ \varphi^{-1}$, called otherwise a *local representative* of $\mathcal{A}$, is a smooth map of Frechet spaces.

We say that a smooth map $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is an *immersion* (resp. *submersion*) if for any point $x \in \mathcal{X}$ we can find coordinate charts near x and its image $y = \mathcal{A}(x)$ so that the local representative of $\mathcal{A}$ is an immersion (resp. submersion) of Frechet spaces, i.e. it is an inclusion of a summand (resp. projection onto a summand) in a direct sum of Frechet spaces.

Example 18. A smooth map $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ between Frechet manifolds generates a *tangent map* $T(\mathcal{A}) : T\mathcal{X} \to T\mathcal{Y}$ of their tangent bundles. This map sends any fibre $T_x\mathcal{X}$ at $x \in \mathcal{X}$ to the fibre $T_y\mathcal{Y}$ at the image point $y = \mathcal{A}(x) \in \mathcal{Y}$. In a coordinate chart it is given by the derivative of the corresponding local representative. The linear map $D\mathcal{A} : T_x\mathcal{X} \to T_y\mathcal{Y}$, induced by $T(\mathcal{A})$ on the tangent space $T_x\mathcal{X}$, is the *derivative* of $\mathcal{A}$ at x, which agrees with the definition, given in Subsec. 1.1.2, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Frechet spaces.

Definition 9. A smooth map $\pi : \mathcal{E} \to \mathcal{X}$ between Frechet manifolds is called a *Frechet fibre bundle*, if it is a submersion and for any point $x \in \mathcal{X}$ we can find an open neighborhood U of this point such that there exists a Frechet manifold $\mathcal{F}$ and a diffeomorphism $\psi : \pi^{-1}(U) \to \mathcal{F}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\psi]{} & U \times \mathcal{F} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\text{projection}} \\ U & \xrightarrow{\text{id}} & U \end{array}$$

As in the finite-dimensional situation, a smooth map $\mathcal{A} : \mathcal{E}_1 \to \mathcal{E}_2$ of a fibre bundle $\pi_1 : \mathcal{E}_1 \to \mathcal{X}$ to a fibre bundle $\pi_2 : \mathcal{E}_2 \to \mathcal{X}$ is called a *fibre bundle map* if it sends fibres to fibres, i.e. for any $x \in \mathcal{X}$ it sends the fibre $\pi_1^{-1}(x)$ to the fibre $\pi_2^{-1}(x)$.

Example 19. Let $\pi_1 : \mathcal{E}_1 \to \mathcal{X}$ and $\pi_2 : \mathcal{E}_2 \to \mathcal{X}$ be two fibre bundles of Frechet manifolds. Then we can form a new fibre bundle over $\mathcal{X}$, called the *fibre product* of these two bundles, which a closed submanifold in $\mathcal{E}_1 \times \mathcal{E}_2$. Namely, we set

$$\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2 = \{(e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 : \ \pi_1(e_1) = \pi_2(e_2)\} \ .$$

It is a closed subset in $\mathcal{E}_1 \times \mathcal{E}_2$, since $\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2$ coincides with the preimage of the diagonal Δ in $\mathcal{E}_1 \times \mathcal{E}_2$ under the product map $\pi_1 \times \pi_2 : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{X} \times \mathcal{X}$. To prove that it is a fibre bundle over $\mathcal{X}$ and a submanifold in $\mathcal{E}_1 \times \mathcal{E}_2$, take an arbitrary point $x \in \mathcal{X}$ and choose an open neighborhood U so that $\pi_1 : \pi_1^{-1}(U) \to U$ and $\pi_2 : \pi_2^{-1}(U) \to U$ are compatible with the projections $U \times \mathcal{F}_1 \to U$ and $U \times \mathcal{F}_2 \to U$ respectively in the sense of Def. 9. This generates a diffeomorphism $\pi_1^{-1}(U) \times \pi_2^{-1}(U) \subset \mathcal{E}_1 \times \mathcal{E}_2$ into $U \times \mathcal{F}_1 \times U \times \mathcal{F}_2$. Restricting this diffeomorphism to the diagonal Δ in $U \times U$, we obtain for $\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2$ a local diffeomorphism ψ, required in the Def. 9. The same argument shows that $\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2$ is a closed submanifold in $\mathcal{E}_1 \times \mathcal{E}_2$.

1.2.3 Connections

Let $\pi : \mathcal{V} \to \mathcal{X}$ be a Frechet vector bundle over a Frechet manifold $\mathcal{X}$. Given a point $v \in \mathcal{V}$ denote by $V_v = \operatorname{Ker} D\pi$ the subspace in $T_v\mathcal{V}$, formed by vectors, annihilated by the derivative $D\pi : T_v\mathcal{V} \to T_{\pi(v)}\mathcal{X}$. By mimicking the finite-dimensional definition, we want to define a connection $\mathcal{H}$ on $\pi : \mathcal{V} \to \mathcal{X}$ as a rule, assigning to any point $v \in \mathcal{V}$ a subspace H_v in $T_v\mathcal{V}$, complementary to V_v.

The tangent bundle $T\mathcal{V}$ can be considered as a Frechet vector bundle $\pi_{\mathcal{V}} : T\mathcal{V} \to \mathcal{V}$ over $\mathcal{V}$ and also as a Frechet vector bundle $T\pi : T\mathcal{V} \to T\mathcal{X}$ over $T\mathcal{X}$. So we have a natural projection

$$(\pi_{\mathcal{V}}, T\pi) : T\mathcal{V} \longrightarrow \mathcal{V} \oplus T\mathcal{X} \ , \qquad \delta v \longmapsto (\pi_{\mathcal{V}}(\delta v), T\pi(\delta v))$$

for $\delta v \in T\mathcal{V}$. Note that the composite map $\pi \circ \pi_{\mathcal{V}} : T\mathcal{V} \to \mathcal{X}$ provides $T\mathcal{V}$ with a structure of a fibre bundle over $\mathcal{X}$.

Definition 10. A *connection* on a Frechet vector bundle $\pi : \mathcal{V} \to \mathcal{X}$ is a smooth fibre bundle map

$$\mathcal{H} : \mathcal{V} \oplus T\mathcal{X} \longrightarrow T\mathcal{V}$$

of fibre bundles over $\mathcal{X}$ such that

$$(\pi_{\mathcal{V}}, T\pi) \circ \mathcal{H} = \text{id} \quad \text{on} \quad \mathcal{V} \oplus T\mathcal{X}$$

and is bilinear. The latter means that for any $x \in \mathcal{X}$ the restriction of $\mathcal{H}$ to the fibre over x is a map $\mathcal{H}_x : \mathcal{V}_x \oplus T_x\mathcal{X} \to T_x\mathcal{V}$, which is linear in both arguments.

To understand what this definition means in local terms, consider a coordinate neighborhood U in $\mathcal{X}$, over which we have the following identifications

$$TU \longleftrightarrow U \times F \ , \quad \pi^{-1}(U) \longleftrightarrow U \times G \ , \ T(U \times G) \longleftrightarrow (U \times G) \times (F \times G) \ .$$

In these terms our connection $\mathcal{H}$ has the following representation

$$\mathcal{H}(x, v, \xi) = (x, v, H_1(x, v, \xi), H_2(x, v, \xi))$$

where $x \in U$, $v \in G$, $\xi \in F$. Since $(\pi_{\mathcal{V}}, T\pi) \circ \mathcal{H} = \text{id}$ on $\mathcal{V} \oplus T\mathcal{X}$, we have $H_1(x, v, \xi) = \xi$ and the bilinearity condition implies that $H_2(x, v, \xi)$ is bilinear in (v, ξ). We shall denote this map, called the *Christoffel symbol* of the connection $\mathcal{H}$, by

$$\Gamma : U \times G \times F \longrightarrow G \ , \quad \Gamma_x(v, \xi) := H_2(x, v, \xi) \ .$$

Denote, as above, by V the subbundle in $T\mathcal{V}$, given by the kernel $\text{Ker}\, T\pi$ of the tangent map $T\pi : T\mathcal{V} \to T\mathcal{X}$. We call V the *vertical subbundle* of $T\mathcal{V}$. The complementary subbundle H in $T\mathcal{V}$, given by the image $\text{Im}\, \mathcal{H}$ of the map $\mathcal{H} : \mathcal{V} \oplus T\mathcal{X} \to T\mathcal{V}$, is called the *horizontal subbundle* of $T\mathcal{V}$. Note that, while the vertical subbundle V is canonically defined by $\pi : \mathcal{V} \to \mathcal{X}$, the horizontal subbundle H is determined by the connection $\mathcal{H}$.

There is another way to view the connection, based on the notion of *covariant derivative*. The covariant derivative is defined in terms of connection $\mathcal{H}$ as follows. Consider a path $v(t)$ in $\mathcal{V}$, represented in local coordinates as $v(t) = (x(t), g(t))$ with $x(t) \in U$, $g(t) \in G$. Then its covariant derivative $\nabla v(t)$ is equal to

$$\nabla v(t) = (\xi(t), \Xi(t)) \ ,$$

where

$$\xi(t) = x'(t) \ , \quad \Xi(t) = g'(t) - \Gamma_{x(t)}(g(t), \xi(t)) \ .$$

The path $v(t)$ in $\mathcal{V}$, covering the path $x(t)$ in $\mathcal{X}$, is *horizontal* iff $\nabla v(t) = 0$.

For Banach manifolds we can always find for a given path $x(t)$ in $\mathcal{X}$ with the initial value $x(0)$ a uniquely determined horizontal lift $v(t)$ in $\mathcal{V}$, covering $x(t)$. On the contrary, for Frechet manifolds the horizontal lift may not exist and, even if it exists, it may be not unique. This is due to the absence of the existence and uniqueness theorem for the ordinary differential equations in Frechet spaces.

By definition, a *connection on a Frechet manifold* $\mathcal{X}$ is a connection on its tangent bundle $T\mathcal{X}$. If $x(t)$ is a path in $\mathcal{X}$, then its derivative $v(t) := x'(t)$ is a path in $T\mathcal{X}$. Its covariant derivative $\nabla v(t)$ is called otherwise the *acceleration* of $x(t)$. A path $x(t)$ is a *geodesic* of $\mathcal{X}$ iff its acceleration is zero. We say that a connection $\mathcal{H}$ on $T\mathcal{X}$ is *symmetric* if its local representatives $\Gamma_x(\xi, \eta)$ are symmetric in $(\xi, \eta) \in T_x\mathcal{X} \times T_x\mathcal{X}$.

Definition 11. The *curvature* $\mathcal{R}$ of a connection $\mathcal{H}$ on a Frechet vector bundle $\pi : \mathcal{V} \to \mathcal{X}$ is a trilinear map

$$\mathcal{R} : \mathcal{V} \times T\mathcal{X} \times T\mathcal{X} \longrightarrow \mathcal{V} ,$$

given in terms of local representatives by the formula

$$\mathcal{R}_x(v, \xi, \eta) := D\Gamma_x(v, \xi, \eta) - D\Gamma_x(v, \eta, \xi) - \Gamma_x(\Gamma_x(v, \xi), \eta) + \Gamma_x(\Gamma_x(v, \eta), \xi) ,$$

where $\Gamma_x(v, \xi)$ is a local representative of the connection $\mathcal{H}$. This definition does not depend upon the choice of a local chart.

Example 20. Consider the Frechet manifold $C^\infty(X, Y)$ of smooth maps from a compact manifold X into a manifold Y. Suppose that Y has a connection, represented locally by the Christoffel symbol $\Gamma_y(\xi, \eta)$. Then we can define a connection on $C^\infty(X, Y)$ locally by the Christoffel symbol

$$\left(\Gamma_f(\tilde{\xi}, \tilde{\eta})\right)(x) = \Gamma_{f(x)}(\tilde{\xi}(x), \tilde{\eta}(x)) \quad \text{for } x \in X ,$$

where $f \in C^\infty(X, Y)$, $\tilde{\xi}, \tilde{\eta} \in T_f C^\infty(X, Y) = C^\infty(X, f^*TY)$ (cf. Ex.16 in Subsec. 1.2.2). Note that $\tilde{\xi}(x), \tilde{\eta}(x) \in T_{f(x)}Y$.

A path $f(t)$ in $C^\infty(X, Y)$, evaluated at $x \in X$, yields a path $f_t(x)$ in Y. The path $f(t)$ is a geodesic in $C^\infty(X, Y)$ if and only if the path $f_t(x)$ is a geodesic in Y for any $x \in X$. The curvature $\mathcal{R}$ of the introduced connection on $C^\infty(X, Y)$ is given in terms of the curvature R of the connection on Y by the formula

$$\mathcal{R}_f(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})(x) = R_{f(x)}(\tilde{\xi}(x), \tilde{\eta}(x), \tilde{\zeta}(x)) ,$$

i.e. is computed from R pointwise.

Example 21. Consider the Frechet manifold $\mathcal{S}(X)$ of smooth compact submanifolds S in a Riemannian manifold X (cf. Ex. 10 in Subsec. 1.2.1 and Ex. 17 in Subsec. 1.2.2). For any $S \in \mathcal{S}(X)$ and $f \in C^\infty(S)$ we can define vector bundles Tf and Nf over S by setting

$$Tf := \text{graph of } Df = \{(v, D_v f) : v \in TS\} \subset TX \times \mathbb{R}$$

and $Nf = TX \times \mathbb{R}/Tf$.

Then we have the following natural isomorphisms

$$T_S\mathcal{S}(X) = C^\infty(S, NS) , \quad T_{(S,f)}C^\infty\mathcal{S}(X) = C^\infty(S, Nf) .$$

The vector bundle Nf may be included into the following exact sequence of vector bundle maps over S

$$0 \longrightarrow \mathbb{R} \longrightarrow Nf \longrightarrow NS \longrightarrow 0 ,$$

which induces an exact sequence of maps of Frechet vector spaces

$$0 \longrightarrow C^\infty(S) \longrightarrow C^\infty(S, Nf) \longrightarrow C^\infty(S, NS) \longrightarrow 0 .$$

By above isomorphisms, it coincides with the exact sequence

$$0 \longrightarrow C^\infty(S) \longrightarrow T_{(S,f)}C^\infty\mathcal{S}(X) \longrightarrow T_S\mathcal{S}(X) \longrightarrow 0 .$$

The third arrow in this sequence is the tangent map of the vector bundle projection $C^\infty\mathcal{S}(X) \to \mathcal{S}(X)$, while the second arrow realizes $C^\infty(S)$ as the vertical subspace of this bundle at $f \in C^\infty(S)$.

To define a complementary subspace, we need a connection on $C^\infty\mathcal{S}(X)$, which is generated by the Riemannian connection on X. This connection $\mathcal{H}$ may be described as follows. For $S \in \mathcal{S}(X)$ we can identify its normal bundle NS with the subbundle of $TX|S$, consisting of vectors, orthogonal to TS with respect to the Riemannian metric of X. Then $NS \times \mathbb{R}$ would be a complementary subbundle to Tf in $TX \times \mathbb{R}$, so we can identify $Nf = TX \times \mathbb{R}/Tf$ with $NS \times \mathbb{R}$. We set $Hf = NS \times \{0\}$ to be the horizontal subbundle, complementary to the vertical subspace $\{0\} \times \mathbb{R}$. Then $C^\infty(S, Hf)$, which is complementary to the vertical subspace $C^\infty(S)$, will be the horizontal subspace of our connection $\mathcal{H}$. Note that it projects isomorphically onto the space $C^\infty(S, NS) = T_S\mathcal{S}(X)$, since $Hf = NS \times \{0\} \sim NS$.

Let us compute the curvature of this connection. Using the Riemannian connection ∇ on X, we can define covariant derivatives ∇f of $f \in C^\infty(S)$ and $\nabla\xi$ of $\xi \in C^\infty(S, NS)$ and compute their inner product $\nabla f \cdot \nabla\xi$ in $TX|S$. The curvature $\mathcal{R}$ of the connection $\mathcal{H}$ is a trilinear map

$$\mathcal{R} : C^\infty\mathcal{S}(X) \times TC^\infty\mathcal{S}(X) \times TC^\infty\mathcal{S}(X) \longrightarrow C^\infty\mathcal{S}(X) ,$$

which can be interpreted at a point $S \in \mathcal{S}(X)$ as a linear map

$$\mathcal{R}_S : C^\infty(S) \times C^\infty(S, NS) \times C^\infty(S, NS) \longrightarrow C^\infty(S) .$$

This map is given explicitly by the formula

$$\mathcal{R}_S(f, \xi, \eta) = \nabla f \cdot \nabla\xi \cdot \eta - \nabla f \cdot \nabla\eta \cdot \xi .$$

1.2.4 Differential forms

Definition 12. A *differential form* of degree r (or simply an r-*form*) on a Frechet manifold $\mathcal{X}$ is a smooth map

$$\omega : \underbrace{T\mathcal{X} \times \cdots \times T\mathcal{X}}_{r} \longrightarrow \mathbb{C}$$

of the rth direct power $T\mathcal{X} \times \cdots \times T\mathcal{X}$ of the tangent bundle $T\mathcal{X}$ such that for any $x \in \mathcal{X}$ its restriction

$$\omega_x : T_x\mathcal{X} \times \cdots \times T_x\mathcal{X} \longrightarrow \mathbb{C}$$

to $T_x\mathcal{X} \times \cdots \times T_x\mathcal{X}$ is an r-multilinear alternating map. In other words, ω_x is an r-multilinear alternating form on $T_x\mathcal{X}$. We denote the space of r-forms on $\mathcal{X}$ by $\Omega^r(\mathcal{X})$. We shall consider smooth functions on $\mathcal{X}$ as forms of degree 0.

In a coordinate neighborhood U of $\mathcal{X}$ we can identify an r-form ω on U with a smooth map from an open subset of a Frechet space F into the vector space $\Omega^r(F)$ of r-multilinear alternating r-forms on F. If $\xi_1, \ldots, \xi_r$ are smooth vector fields on $U \subset F$, we denote by $\omega(\xi_1, \ldots, \xi_r)$ the map from U to $\mathbb{C}$, whose value at $x \in U$ is equal to $\omega_x(\xi_1(x), \ldots, \xi_r(x))$, i.e. the value of the r-form $\omega_x \in \Omega^r(F)$ on vectors $\xi_1(x), \ldots, \xi_r(x)$ in F.

Differential forms on Frechet manifolds share many properties with differential forms on finite-dimensional manifolds. In particular, one can define their exterior derivative and wedge product similar to the finite-dimensional case.

Definition 13. The *exterior derivative* $d\omega$ of an r-form ω on $\mathcal{X}$ is an $(r+1)$-form on $\mathcal{X}$, which can be defined locally as follows. For any smooth vector fields $\xi_0, \xi_1, \dots, \xi_r$ in a coordinate neighborhood $U \subset F$, the value of $d\omega$ on $\xi_0, \xi_1, \dots, \xi_r$ is equal to

$$\begin{aligned} d\omega(\xi_0, \xi_1, \dots, \xi_r) = &\sum_{i=0}^{r} (-1)^i \xi_i \Big(\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_r)\Big) + \\ &+ \sum_{\substack{i,j=0 \\ i<j}}^{r} (-1)^{i+j} \omega \Big([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_r\Big) \ . \end{aligned} \tag{1.3}$$

This definition does not depend on the choice of the local data in the sense that there is a unique $(r+1)$-form on $\mathcal{X}$, which respects the given local representations (cf. [47], Ch.V, Prop. 3.2).

Example 22. If f is a 0-form on $\mathcal{X}$, i.e. a smooth map $f : \mathcal{X} \to \mathbb{C}$, then df_x for any $x \in \mathcal{X}$ coincides with the tangent map

$$T_x f : T_x \mathcal{X} \longrightarrow T_{f(x)} \mathbb{C} \ .$$

Moreover, for any vector field ξ on $\mathcal{X}$ we have

$$df(\xi) = \xi f \ .$$

If ω is a 1-form on $\mathcal{X}$, then locally

$$d\omega(\xi, \eta) = \xi\left(\omega(\eta)\right) - \eta\left(\omega(\xi)\right) - \omega\left([\xi, \eta]\right) \ .$$

For a 2-form ω we have locally

$$\begin{aligned} d\omega(\xi, \eta, \zeta) = &\ \xi\left(\omega(\eta, \zeta)\right) + \eta\left(\omega(\zeta, \xi)\right) + \zeta\left(\omega(\xi, \eta)\right) - \\ &- \omega\left([\xi, \eta], \zeta\right) - \omega\left([\eta, \zeta], \xi\right) - \omega\left([\zeta, \xi], \eta\right) \ . \end{aligned} \tag{1.4}$$

Definition 14. The *wedge product* of an r-form ω and an s-form ψ on $\mathcal{X}$ is an $(r+s)$-form $\omega \wedge \psi$ on $\mathcal{X}$, which can be defined locally as follows. For any smooth vector fields $\xi_1, \dots \xi_{r+s}$ in a coordinate neighborhood $U \subset F$, the value of $\omega \wedge \psi$ on $\xi_1, \dots \xi_{r+s}$ is equal to

$$(\omega \wedge \psi)(\xi_1, \dots \xi_{r+s}) = \sum_{i=1}^{r+s} (-1)^{\epsilon(\sigma)} \omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}) \psi(\xi_{\sigma(r+1)}, \dots, \xi_{\sigma(r+s)}) \ ,$$

where the sum is taken over all permutations σ of the numbers $(1, \dots, r+s)$ and $\epsilon(\sigma)$ is the parity of σ.

Again, this definition does not depend on the choice of the local data in the sense that there is a unique $(r+s)$-form $\omega \wedge \psi$ on $\mathcal{X}$, which respects the given local representations.

In particular, the wedge product of a function f and a form ω is equal to $f \wedge \omega = f\omega$. One can easily check that the wedge product of two forms ω and ψ on $\mathcal{X}$ is related to the exterior derivative by the usual formula

$$d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^{\deg \omega} \omega \wedge d\psi$$

and the square of d is equal to zero: $dd\omega = 0$.

1.2.5 Symplectic and complex structures

Definition 15. A *symplectic structure* on a Frechet manifold $\mathcal{X}$ is a 2-form ω on $\mathcal{X}$, having the following properties:

1. ω is closed, i.e. $d\omega = 0$;

2. ω is non-degenerate at any point $x \in \mathcal{X}$, i.e. for any $\xi \in T_x\mathcal{X}$, $\xi \neq 0$, there exists an $\eta \in T_x\mathcal{X}$ such that $\omega_x(\xi, \eta) \neq 0$.

A Frechet manifold $\mathcal{X}$, provided with a symplectic structure ω, is called *symplectic*.

Remark 1. Note that we have used here the weakest form of the non-degeneracy condition. For Banach manifolds, modelled locally on a Banach space E, a conventional non-degeneracy condition on ω requires that for any $x \in \mathcal{X}$ the linear operator A_x from $T_x\mathcal{X} \sim E$ to the dual space $T_x^*\mathcal{X} \sim E'$, defined by $\omega_x(\cdot, \eta) = A_x(\cdot)(\eta)$, is invertible for any non-zero $\eta \in T_x^*\mathcal{X}$.

Most of Frechet manifolds, considered in this book, are symplectic in the sense of the Def.15. Moreover, they usually have, along with their symplectic structure, a compatible almost complex structure.

Definition 16. An *almost complex structure* on a Frechet manifold $\mathcal{X}$ is a smooth vector bundle automorphism J of the tangent bundle $T\mathcal{X}$, such that for any $x \in \mathcal{X}$ the restriction J_x of J to $T_x\mathcal{X}$ satisfies the condition

$$J_x^2 = -\mathrm{id} \ .$$

A Frechet manifold $\mathcal{X}$, provided with an almost complex structure, is called *almost complex*.

If J is an almost complex structure on a Frechet manifold $\mathcal{X}$, then the isomorphism J can be extended complex linearly to the *complexified* tangent bundle $T^{\mathbb{C}}\mathcal{X} = T\mathcal{X} \otimes \mathbb{C}$, so that $T^{\mathbb{C}}\mathcal{X}$ decomposes into the direct sum of subbundles

$$T^{\mathbb{C}}\mathcal{X} = T^{1,0}\mathcal{X} \oplus T^{0,1}\mathcal{X} \ ,$$

where for any $x \in \mathcal{X}$ the restriction of J_x to $T_x^{1,0}\mathcal{X}$ is given by the multiplication by i, and the restriction of J_x to $T_x^{0,1}\mathcal{X}$ is given by the multiplication by $-i$. Sections of the bundles $T_x^{1,0}\mathcal{X}$ and $T_x^{0,1}\mathcal{X}$ are called otherwise the *vector fields of type* $(1,0)$ and $(0,1)$ respectively.

We call an almost complex structure J on a Frechet manifold $\mathcal{X}$ *integrable* or *formally integrable* complex structure, if the bracket of any two vector fields on $\mathcal{X}$ of type $(1,0)$ is again a vector field of type $(1,0)$.

Remark 2. An almost complex structure J provides a complex structure on every tangent space $T_x\mathcal{X}$, determined by the action of J_x. In particular, any complex Frechet manifold $\mathcal{X}$ has a natural almost complex structure, given by the multiplication by i on $T_x\mathcal{X}$. Such an almost complex structure is automatically integrable. For finite-dimensional manifolds the *Newlander-Nirenberg theorem* asserts that the converse is also true, namely, any almost complex manifold with an integrable almost complex structure is, in fact, *complex*. It means that one can introduce an atlas of local complex charts on this manifold in such a way that the original almost complex structure in these coordinates will be given by the multiplication by i on tangent spaces. For Frechet manifolds this theorem is, in general, not true (cf. [51]), so in order to show that a given Frechet manifold is complex, it's necessary to construct, following Def.5 from Subsec. 1.2.1, an atlas of local complex charts.

The most important class of Frechet manifolds, considered in this book, is that of Kähler Frechet manifolds, i.e. Frechet manifolds, which are both symplectic and complex, and these two structures are compatible in the sense of the following definition.

Definition 17. A complex symplectic Frechet manifold $\mathcal{X}$ is called a *Kähler Frechet manifold*, if its complex structure J and symplectic structure ω are *compatible* in the following sense:

1. $\omega_x(J_x\xi, J_x\eta) = \omega_x(\xi, \eta)$ for any $\xi, \eta \in T_x\mathcal{X}$, $x \in \mathcal{X}$;
2. symmetric form g on $T\mathcal{X} \times T\mathcal{X}$, defined by
$$g_x(\xi, \eta) := \omega_x(\xi, J_x\eta) \ ,$$
is positively definite for any $x \in \mathcal{X}$.

Such a form g is called the *Kähler metric* on $\mathcal{X}$.

Bibliographic comments

A key reference to Ch.1 is the Hamilton's paper [32] on the Nash–Moser theorem. Its first part is an excellent introduction to the theory of Frechet manifolds. In our exposition (except for Subsecs.1.2.4,1.2.5) we follow closely that paper. The definition Def.10 of the connection on a Frechet vector bundle is borrowed from [47]. The latter book can be recommended for the readers, interested in the theory of infinite-dimensional manifolds with a special emphasis on the Banach case.

Chapter 2

Frechet Lie groups

Definition 18. A *Frechet Lie group* is a Frechet manifold $\mathcal{G}$, provided with the group structure, such that the multiplication

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \ , \quad (g, h) \longmapsto g \cdot h \ ,$$

and "taking-the-inverse"

$$\mathcal{G} \longrightarrow \mathcal{G} \ , \quad g \longmapsto g^{-1} \ ,$$

are smooth maps of Frechet manifolds. The *Frechet Lie algebra* of a Frechet Lie group $\mathcal{G}$ is the tangent space $\mathfrak{G} = T_{\mathbf{1}}\mathcal{G}$ at the unit $\mathbf{1}$ of the group $\mathcal{G}$.

For $g \in \mathcal{G}$ denote by

$$L_g : \mathcal{G} \to \mathcal{G}, \quad L_g(h) = g \cdot h$$
$$R_g : \mathcal{G} \to \mathcal{G}, \quad R_g(h) = h \cdot g$$

respectively the *left* and *right translations* on the group $\mathcal{G}$.

Any element ξ of the Lie algebra $\mathfrak{G}$ generates by left translations a vector field X_ξ on $\mathcal{G}$, invariant under these translations. The correspondence $\xi \longleftrightarrow X_\xi$ allows us to consider elements of the Lie algebra $\mathfrak{G}$ as left-invariant vector fields on the Lie group $\mathcal{G}$. The left-invariant vector fields on $\mathcal{G}$ form a Lie algebra with respect to the bracket of vector fields, which induces a Lie algebra bracket on $T_{\mathbf{1}}\mathcal{G} = \mathfrak{G}$ by the identification $\xi \longleftrightarrow X_\xi$ (this justifies the use of the term "Lie algebra" with respect to $T_{\mathbf{1}}\mathcal{G}$). We note that there exists a unique connection $\mathcal{H}$ on $\mathcal{G}$, called the *Cartan–Maurer connection*, such that the left-invariant vector fields are horizontal with respect to $\mathcal{H}$, its curvature being equal to zero. Of course, the choice of the left-invariant vector fields and left translations in this argument was absolutely ambiguous (though traditional), with the same success we could employ here the right-invariant vector fields and right translations.

If in the definition of a Frechet Lie group the group $\mathcal{G}$ is a Banach (resp. Hilbert) manifold, we say that $\mathcal{G}$ is a *Banach* (resp. *Hilbert) Lie group*.

Suppose that for any element ξ of the Lie algebra $\mathfrak{G}$ there exists a unique 1-parameter subgroup $\gamma_\xi : \mathbb{R} \to \mathcal{G}$ of the group $\mathcal{G}$ such that $\gamma'_\xi(0) = \xi$. Then, as in the finite-dimensional case, we can define the *exponential map*

$$\exp : \mathfrak{G} \longrightarrow \mathcal{G}$$

by setting $\exp \xi := \gamma_\xi(\mathbf{1})$. In particular, for Banach Lie groups $\mathcal{G}$ the above condition is always satisfied. Indeed, any element $\xi \in \mathfrak{G}$ is identified with the left-invariant vector field X_ξ, which can be integrated to a 1-parameter group of transformations $\varphi_\xi^t : \mathcal{G} \to \mathcal{G}$. In this case $\gamma_\xi(t) := \varphi_\xi^t(\mathbf{1})$.

We supplement the definition of Frechet fibre bundles, given in Subsec. 1.2.2 (cf. Def.9), with the definition of a principal Frechet bundle. We say that a Frechet Lie group $\mathcal{G}$ *acts on a Frechet manifold* $\mathcal{X}$, if there is a smooth map

$$\mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X} \ , \quad (g, x) \longmapsto g \cdot x \ ,$$

such that $\mathbf{1} \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

Definition 19. Let $\mathcal{G}$ be a Frechet Lie group, acting on a Frechet manifold $\mathcal{E}$. This manifold is called a *principal Frechet $\mathcal{G}$-bundle*, if there is a smooth submersion $\pi : \mathcal{E} \to \mathcal{X}$ onto another Frechet manifold $\mathcal{X}$, such that for any $x \in \mathcal{X}$ there exists an open neighborhood U of x and a diffeomorphism of its preimage $\pi^{-1}(U)$ in $\mathcal{E}$ onto $U \times \mathcal{G}$, satisfying the following conditions:

1. the action of $\mathcal{G}$ on $\mathcal{E}$ corresponds to the natural action of $\mathcal{G}$ on the second factor of $U \times \mathcal{G}$;
2. the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times \mathcal{G} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \text{projection}} \\ U & \xrightarrow{\text{id}} & U \end{array}$$

is commutative.

We consider next the two most important examples of Frechet Lie groups, playing a special role in this book.

2.1 Group of currents $C^\infty(X, G)$

2.1.1 Basic properties

Let X be a smooth compact manifold and G is a Lie group. The space $C^\infty(X, G)$ of all smooth maps from X into G is a Frechet manifold, as we have pointed out in Subsec.1.2.1 (cf. Ex.12). Let us recall the definition of the structure of a Frechet manifold on $C^\infty(X, G)$ for this particular case.

The exponential map $\exp : \mathfrak{g} \to G$ determines a local diffeomorphism

$$\exp : \mathfrak{u} \longrightarrow U \ ,$$

mapping an open neighborhood $\mathfrak{u}$ of zero in the Lie algebra $\mathfrak{g}$ onto an open neighborhood U of the unit $e \in G$. Using this diffeomorphism, we can construct a local chart in a neighborhood $\mathcal{U} = C^\infty(X, U)$ of the identity $\mathbf{1} := X \to e \in G$ in $C^\infty(X, U)$. It is given by the homeomorphism

$$\chi : \quad \mathfrak{U} := C^\infty(X, \mathfrak{u}) \longrightarrow C^\infty(X, U) = \mathcal{U} \ ,$$

given by the pointwise application of the exponential map $\exp : \mathfrak{u} \to U$. The inverse map $\varphi_{\mathbf{1}} := \chi^{-1} : \mathcal{U} \to \mathfrak{U}$ yields a homeomorphism of the neighborhood $\mathcal{U}$ of the identity $\mathbf{1} \in C^\infty(X, U)$ onto the open subset $\mathfrak{U}$ in the Frechet space $C^\infty(X, \mathfrak{g})$.

The manifold $C^\infty(X, G)$ is a group with respect to the pointwise multiplication. Using this group structure, we can construct local charts at any point of $C^\infty(X, G)$. To define a local chart at an arbitrary point $\gamma \in C^\infty(X, G)$, denote by $\mathcal{U}_\gamma$ a neighborhood of γ of the form $\mathcal{U}_\gamma := \gamma \cdot \mathcal{U}$ and define a local chart φ_γ in the neighborhood $\mathcal{U}_\gamma$ as the composition map

$$\varphi_\gamma := \varphi_{\mathbf{1}} \circ \gamma^{-1} : \mathcal{U}_\gamma \to \mathfrak{U} ,$$

where the map $\gamma^{-1} : \mathcal{U}_\gamma \to \mathcal{U}$ is given by the multiplication by γ^{-1} from the left. The neighborhoods $\{\mathcal{U}_\gamma\}$ and the maps $\{\varphi_\gamma\}$ with $\gamma \in C^\infty(X, G)$ form an open atlas and a system of local charts on $C^\infty(X, G)$, which defines the structure of a smooth Frechet manifold on $C^\infty(X, G)$, modelled on the Frechet space $C^\infty(X, \mathfrak{g})$.

The pointwise multiplication and taking-the-inverse maps in the group $C^\infty(X, G)$ are smooth with respect to the introduced structure of a Frechet manifold, hence $C^\infty(X, G)$ is a Frechet Lie group, called the *group of currents*.

The Lie algebra of $C^\infty(X, G)$ coincides with the Frechet space $C^\infty(X, \mathfrak{g})$, the Lie bracket in $C^\infty(X, \mathfrak{g})$ being given by the pointwise application of the Lie bracket in $\mathfrak{g}$. The exponential map

$$\exp : C^\infty(X, \mathfrak{g}) \longrightarrow C^\infty(X, G) ,$$

given by the pointwise application of the exponential map $\exp : \mathfrak{g} \to G$, is a local homeomorphism in a neighborhood of zero.

Consider now the most important example of the group $C^\infty(X, G)$, corresponding to the case when $X = S^1$. In this case the group $C^\infty(S^1, G)$ is called the *loop group* of the Lie group G, and is denoted by

$$LG := C^\infty(S^1, G) .$$

The Lie algebra of LG coincides with the *loop algebra*

$$L\mathfrak{g} := C^\infty(S^1, \mathfrak{g}) .$$

Since all operations in the loop group LG are defined pointwise, one can expect that the properties of LG will be close to the properties of the group G itself. And this is true in most of the cases, but there are still some differences, demonstrated by the examples below.

Consider first the homotopy structure of LG. Let us introduce the *based loop space*

$$\Omega G := LG/G$$

of G, where G in the denominator is identified with the group of constant maps $S^1 \to g_0 \in G$. We can realize ΩG as the closed submanifold of LG, consisting of the maps $\gamma \in LG$, which send the identity $\mathbf{1} \in LG$ to the unit $e \in G$: $\gamma(\mathbf{1}) = e$. Then the loop group LG will be identified with the direct product $\Omega G \times G$. It is well known (cf.,e.g., [36]) that the homotopy groups of ΩG coincide with the homotopy groups of G, shifted by one:

$$\pi_i(\Omega G) \cong \pi_{i+1}(G) \ .$$

It follows that

$$\pi_i(LG) \cong \pi_i(\Omega G) \oplus \pi_i(G) \cong \pi_{i+1}(G) \oplus \pi_i(G) \ .$$

In particular, $\pi_0(LG)$ is equal to $\pi_1(G) \oplus \pi_0(G)$, i.e. the group LG is connected if and only if G is connected and simply connected. The fundamental group of LG coincides with $\pi_2(G) \oplus \pi_1(G) \cong \pi_1(G)$, since $\pi_2(G) = 0$ for any connected compact Lie group G. Hence, *LG is connected and simply connected if the Lie group G itself is connected and simply connected.*

2.1.2 Exponential map of the loop algebra

As we have pointed out, the exponential map

$$\exp : L\mathfrak{g} \longrightarrow LG$$

of the loop algebra $L\mathfrak{g}$ is given by the pointwise application of the exponential map $\exp : \mathfrak{g} \to G$.

If G is a compact Lie group, then it has the following well-known property. Denote by G° the identity connected component of G. Then the *exponential map* $\exp : \mathfrak{g} \to G^\circ$ *is surjective.* This property is a corollary of the fact that every element of G° belongs to some 1-parameter subgroup of G. However, for the loop group LG it is not true, in general.

Consider, for example, the simply connected group $G = \mathrm{SU}(2)$. Then the element

$$LG \ni \gamma : z \longrightarrow \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} , \quad z \in S^1 ,$$

is not an exponential of any element in the loop algebra $L\mathfrak{g}$.

Indeed, if we suppose that $\gamma = \exp \xi$ for some $\xi \in C^\infty(S^1, \mathfrak{g})$, then the matrix $\gamma(z)$, being a function of $\xi(z)$, should commute with $\xi(z)$ for any $z \in S^1$. It's easy to see that this condition implies that the matrix $\xi(z)$ should be diagonal for any $z \in S^1$, i.e.

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} e^{if(z)} & 0 \\ 0 & e^{-if(z)} \end{pmatrix}$$

for some smooth real-valued function f on S^1. In particular, $z = e^{if(z)}$, which is impossible, since the logarithm $\ln(z)$ does not admit a continuous branch on the circle.

However, one can prove the following property of the loop group LG, which may be considered as a substitution of the surjectivity of $\exp : \mathfrak{g} \to G^\circ$.

Proposition 1. *Let G be a connected compact Lie group. Then the exponential map*

$$\exp : L\mathfrak{g} \longrightarrow (LG)^\circ$$

has a dense image in the connected component of the identity $(LG)^\circ$ of the group LG.

Proof. To prove this assertion, we note first that a connected compact Lie group G is the direct product of a torus and a connected semisimple compact Lie group. Our assertion for the torus is easily checked directly, so it is sufficient to consider the case of a semisimple connected compact Lie group G. In this case the group G can be realized as the connected component of the identity of the automorphism group $\operatorname{Aut}\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ (since an arbitrary semisimple connected compact Lie group G is a finite covering over $(\operatorname{Aut}\mathfrak{g})^\circ$). If this is the case, then the critical points of the exponential map $\exp : \mathfrak{g} \to G$ lie on a closed hypersurface Γ in $\mathfrak{g}$, dividing $\mathfrak{g}$ into an interior convex domain D, containing 0, and its complement. The image of Γ under the exponential map, denoted by $\exp\Gamma$, is contained in a submanifold of G of codimension ≥ 3.

Consider now an arbitrary loop $\gamma(z) \in (LG)^\circ$, passing through $e \in G$: $\gamma(1) = e$. We assert that it can be approximated by smooth loops in $(LG)^\circ$, which are the *exponentials* in LG (we call a loop $\delta(z)$ in LG an exponential, if it can be represented in the form $\delta = \exp\xi$ for some $\xi \in L\mathfrak{g}$).

By smoothly deforming, if necessary, the loop γ, we can approximate it by a smooth loop $\tilde{\gamma} \in (LG)^\circ$, starting at e, such that $\tilde{\gamma}(e^{it})$ does not intersect $\exp\Gamma$ for $0 < t < 2\pi$. Since the exponential $\exp : \mathfrak{g} \to G$ is locally diffeomorphic along $\tilde{\gamma}(e^{it})$ for $t < 2\pi$, we can, beginning from e, choose a continuous logarithm branch of the loop $\tilde{\gamma}(e^{it})$ for $t < 2\pi$. As a result, we obtain a smooth (but, generally speaking, not closed) path $\xi(e^{it})$, $0 \leq t < 2\pi$, in $\mathfrak{g}$ such that $\exp\xi = \tilde{\gamma}$.

The limit ξ_0 of the path $\xi(e^{it})$ for $t \to 2\pi - 0$ belongs to $\bar{D}$. If $\exp\Gamma$ does not contain e, then ξ_0 cannot belong to $\Gamma = \partial D$, because $\exp\xi_0 = e$. Hence, $\xi_0 \in D$, which forces it to be equal to zero (since, otherwise, exp will be equal to e on the whole orbit of ξ_0 in $D \setminus 0$ under the adjoint action Ad, being a smooth submanifold in $\mathfrak{g}$ of a positive dimension). So $\xi(e^{it})$, $0 \leq t \leq 2\pi$, is a smooth loop in $\mathfrak{g}$ such that $\exp\xi = \tilde{\gamma}$, i.e. we have found a logarithm of $\tilde{\gamma}$ in $\mathfrak{g}$.

If $\exp\Gamma$ contains e, then, in contrast with the considered case, it may happen that the limit $\lim_{t\to 2\pi-0}\xi(e^{it}) = \xi_0$ belongs to Γ. But in such a situation the loop $\tilde{\gamma}$ will not be contractible, i.e. $\tilde{\gamma} \notin (LG)^\circ$, contrary to our assumption. To prove it, note that in this case our path $\xi(e^{it})$ is homotopic to a linear path $\xi_0(e^{it}) := t\frac{\xi_0}{2\pi}$, $0 \leq t \leq 2\pi$, with the same endpoints 0 and ξ_0, as $\xi(e^{it})$. Accordingly, the loop $\tilde{\gamma}$ is homotopic to the loop $\gamma_0(e^{it})$ in G, given by

$$\gamma_0 : S^1 \ni e^{it} \longmapsto \exp\left(t\frac{\xi_0}{2\pi}\right) , \quad 0 \leq t \leq 2\pi .$$

But it is easy to see that γ_0 is not contractible in G. So the loop $\tilde{\gamma}$ is also not contractible in G. □

2.1.3 Complexification

The loop group LG, similar to compact Lie groups, admits the complexification.

Recall that the complexification of a Lie algebra $\mathfrak{g}$ coincides with the complex Lie algebra

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} + i\mathfrak{g} .$$

Definition 20. We call by the *complexification* of a connected Lie group G a connected complex Lie group $G^{\mathbb{C}}$, having the following properties:

1. the Lie algebra of $G^{\mathbb{C}}$ coincides with the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$;

2. $G^{\mathbb{C}}$ contains G as a subgroup, i.e. there exists a monomorphism $i : G \to G^{\mathbb{C}}$.

In particular, a group G, which admits the complexification, should have non-trivial homomorphisms into complex Lie groups (the monomorphism i is one of them).

The complexification $G^{\mathbb{C}}$, introduced above, exists and is uniquely defined for any compact connected Lie group G. For example, the complexification of the group $G = S^1$ coincides with the multiplicative group $G^{\mathbb{C}} = \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ of complex numbers, and the complexification of $G = \mathrm{SU}(n)$ coincides with $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. For the non-compact group $\mathrm{SL}(n, \mathbb{R})$ its complexification also coincides with $\mathrm{SL}(n, \mathbb{C})$.

We give an example of a Lie group, which admits no complexification in the above sense. As we have pointed out, the complexification of the group $\mathrm{SL}(2, \mathbb{R})$ coincides with the group $\mathrm{SL}(2, \mathbb{C})$. The group $\mathrm{SL}(2, \mathbb{C})$ is simply connected, while the fundamental group of $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to $\mathbb{Z}$. Let G be a universal covering group of $\mathrm{SL}(2, \mathbb{R})$. Then we have a homomorphism $\pi : G \to \mathrm{SL}(2, \mathbb{R})$, whose kernel is equal to $\mathbb{Z}$. Suppose that G has the complexification $G^{\mathbb{C}}$. Then it should be a covering group of $\mathrm{SL}(2, \mathbb{C})$. Indeed, the composition of π with the natural embedding $i : \mathrm{SL}(2, \mathbb{R}) \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ yields a non-trivial homomorphism of G into the complex group $\mathrm{SL}(2, \mathbb{C})$ with the kernel, equal to $\mathbb{Z}$. This homomorphism extends to a covering homomorphism $G^{\mathbb{C}} \to \mathrm{SL}(2, \mathbb{C})$ with the same kernel. But such a covering cannot exist, since $\mathrm{SL}(2, \mathbb{C})$ is simply connected. The property of the group G, used in this argument, can be reformulated as follows: any homomorphism of G into a connected complex Lie group factors through $\mathrm{SL}(2, \mathbb{R})$ or (still another formulation) the kernel of such a homomorphism should contain $\mathbb{Z}$.

In the case of the loop group $LG = C^\infty(S^1, G)$ of a compact connected Lie group G its complexification coincides with the loop group $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$ of the complexified group $G^{\mathbb{C}}$. The group $LG^{\mathbb{C}}$ is a complex Frechet Lie group, modelled on the Frechet Lie algebra $C^\infty(S^1, \mathfrak{g}^{\mathbb{C}})$.

2.2 Group of diffeomorphisms Diff(X)

Let X be a smooth compact manifold and Diff(X) is the group of diffeomorphisms of X. The group Diff(X) is a Frechet manifold, being an open subset in the Frechet manifold $C^\infty(X, X)$. It is a Frechet Lie group with respect to this Frechet manifold structure.

The group Diff(X) is closely related to the group of currents $C^\infty(X, G)$, considered in the previous Sec.2.1. Namely, Diff(X) acts smoothly on the manifold $C^\infty(X, G)$ by the "reparametrization" of maps from $C^\infty(X, G)$.

The Lie algebra of the group Diff(X) coincides with the Frechet Lie algebra

$$C^\infty(X, TX) =: \mathrm{Vect}(X)$$

of smooth tangent vector fields on X.

The exponential map

$$\exp : \mathrm{Vect}(X) \longrightarrow \mathrm{Diff}(X)$$

can be defined, as in the beginning of this Chapter. Namely, any vector field $\xi \in \mathrm{Vect}(X)$ generates a 1-parameter subgroup of diffeomorphisms φ_t^ξ of X, defined as follows. The image $y(t) := \varphi_t^\xi(x)$ of an arbitrary point $x \in X$ under the action of φ_t^ξ coincides with the value at t of the integral path of the ordinary differential equation $y' = \xi(y)$ with the initial condition: $y = x$ for $t = 0$. We set $\exp \xi := \varphi_1^\xi$.

Restrict now to the case of $X = S^1$, which is the most important for us. As we have already remarked in Subsec.1.2.1 (Ex. 14), the group $\mathrm{Diff}(S^1)$ consists of two connected components, and the connected component of the identity $\mathrm{Diff}_+(S^1)$ is formed by the maps from $\mathrm{Diff}(S^1)$, preserving the orientation of S^1.

The Lie algebra of the group $\mathrm{Diff}(S^1)$ coincides with the algebra $\mathrm{Vect}(S^1)$ of smooth tangent vector fields on the circle S^1. Elements $v \in \mathrm{Vect}(S^1)$ can be written in the form $v = v(\theta)\frac{d}{d\theta}$, where $v(\theta)$ is a smooth 2π-periodic function of θ. The bracket of two vector fields $v_1, v_2 \in \mathrm{Vect}(S^1)$ is given by the standard formula

$$\left[v_1(\theta)\frac{d}{d\theta}, v_2(\theta)\frac{d}{d\theta}\right] = \{v_1(\theta)v_2'(\theta) - v_1'(\theta)v_2(\theta)\}\,\frac{d}{d\theta}\ .$$

Denote by $\mathrm{Vect}^{\mathbb{C}}(S^1)$ the complexification of the Lie algebra $\mathrm{Vect}(S^1)$, identified with the complex Frechet vector space $T_{\mathrm{id}}^{\mathbb{C}}\mathrm{Diff}(S^1)$:

$$\mathrm{Vect}^{\mathbb{C}}(S^1) := \mathrm{Vect}(S^1) \otimes \mathbb{C}\ .$$

It is convenient to represent the coefficients $v(\theta)$ of vector fields $v = v(\theta)\frac{d}{d\theta}$ from $\mathrm{Vect}^{\mathbb{C}}(S^1)$ by their Fourier series

$$v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta}\ , \quad v_n \in \mathbb{C}\ .$$

In these terms the real subalgebra $\mathrm{Vect}(S^1)$ of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ is specified by the relations: $v_{-n} = \bar{v}_n$, $n \in \mathbb{Z}$.

The complexified Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$ has a natural vector space basis, given by the vector fields

$$e_n = ie^{in\theta}\frac{d}{d\theta}\ , \quad n = 0, \pm 1, \pm 2, \dots\ ,$$

satisfying the commutation relations:

$$[e_n, e_m] = (n-m)e_{n+m}\ , \quad m, n \in \mathbb{Z}.$$

2.2.1 Finite-dimensional subalgebras in $\mathrm{Vect}_\omega(S^1)$

Consider the subalgebra $\mathrm{Vect}_\omega(S^1)$ of $\mathrm{Vect}(S^1)$, consisting of vector fields $v(\theta)\frac{d}{d\theta}$ with real analytic coefficients $v(\theta)$. Such $v(\theta)$ are represented by Fourier series of the form

$$v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta}\ , \quad v_{-n} = \bar{v}_n\ ,$$

converging in a neighborhood of S^1 in $\mathbb{C}$.

The Lie algebra $\mathrm{Vect}_\omega(S^1)$ has the following interesting property.

Proposition 2. *There are no finite-dimensional Lie subalgebras in Vect$_\omega(S^1)$ of dimension> 3. Moreover, for any dimension $d = 1, 2, 3$ there exists only one (up to an isomorphism) Lie subalgebra of dimension d in Vect$_\omega(S^1)$.*

Proof. To prove this assertion, we note first that the bracket of two (not identically zero) vector fields $v_1, v_2 \in \mathrm{Vect}_\omega(S^1)$ is identically zero if and only if these fields are linearly dependent, i.e. $\lambda_1 v_1 + \lambda_2 v_2 \equiv 0$ for some constants λ_1, λ_2. So any non-trivial commutative subalgebra in $\mathrm{Vect}_\omega(S^1)$ should be one-dimensional. In particular, the rank of any non-trivial subalgebra in $\mathrm{Vect}_\omega(S^1)$ (i.e. the dimension of its Cartan subalgebra) is equal to 1.

We show that any subalgebra $\mathfrak{g}$ of the Lie algebra $\mathrm{Vect}_\omega(S^1)$ of dimension≥ 3 is *semisimple*, i.e. it contains no non-zero commutative ideals. Suppose, on the contrary, that $\mathfrak{g}$ contains such an ideal, which should be, as we have just noted, one-dimensional. Choose a basis $\{e_1, e_2, e_3, \dots\}$ in $\mathfrak{g}$ so that our ideal is generated by e_1 (by assumption, this basis has, at least, three elements). Then

$$[e_1, e_2] = \lambda e_1 \quad \text{and} \quad [e_1, e_3] = \mu e_1 \ ,$$

where $\lambda, \mu \neq 0$, since e_1, e_2, e_3 are linearly independent. Hence, $[e_1, \mu e_2 - \lambda e_3] = 0$, which implies the linear dependence of e_1, e_2, e_3 in contradiction with our assumption.

Note that the dimension constraint on the Lie algebra $\mathfrak{g}$ in this assertion is essential, since we shall see below that the unique two-dimensional subalgebra, contained in $\mathrm{Vect}_\omega(S^1)$, is not semisimple.

We show next that *any finite-dimensional subalgebra $\mathfrak{g}$ in the algebra Vect$_\omega(S^1)$ of dimension≥ 3 is simple*, i.e. it contains no non-trivial ideals. Indeed, any semisimple algebra $\mathfrak{g}$ is decomposed into the direct sum of simple ideals. If $\mathfrak{g}$ is not simple, then it contains an ideal I of dimension less than $\frac{1}{2}\dim\mathfrak{g}$. We choose a basis in $\mathfrak{g}$ of the form $\{e_1, \dots, e_m, f_1, \dots, f_k\}$, so that the vectors $e_1, \dots, e_m$ form a basis of the ideal I. It's clear that $m \geq 2$ (otherwise, the ideal I would be commutative). The brackets

$$[e_1, f_1] \in I \ , \dots , \ [e_1, f_k] \in I \ , [e_1, e_2] \in I$$

are non-zero (otherwise, the corresponding vectors would be linearly dependent) and so form a collection of $k + 1 > m$ non-zero vectors in the m-dimensional subalgebra I. Hence, they are linearly dependent, which implies, as before, that the vectors $e_1, \dots, e_m, f_1, \dots, f_k$ are linearly dependent, contrary to our assumption.

From the list of simple Lie algebras, one can see that only two simple Lie algebras of dimension 3 can have the properties, described above. Namely, it is the non-compact Lie algebra $\mathrm{sl}_2(\mathbb{R})$ and the compact Lie algebra su(2). By comparing the Lie brackets in the Lie algebras su(2) and $\mathrm{Vect}_\omega(S^1)$, one shows that the second possibility is not realized. A standard embedding of $\mathrm{sl}_2(\mathbb{R})$ into $\mathrm{Vect}_\omega(S^1)$ realizes $\mathrm{sl}_2(\mathbb{R})$ as the Lie subalgebra in $\mathrm{Vect}_\omega(S^1)$, generated by three vector fields $d/d\theta$, $\cos\theta\, d/d\theta$, $\sin\theta\, d/d\theta$. This subalgebra coincides with the Lie algebra of the Möbius group $\mathrm{PSL}_2(\mathbb{R})$ of all fractional linear automorphisms of the unit disc.

Any two-dimensional subalgebra in $\mathrm{Vect}_\omega(S^1)$ is necessarily non-commutative since, as we have seen before, the vanishing of the bracket of two vector fields in $\mathrm{Vect}_\omega(S^1)$ implies their linear dependence. Since all two-dimensional non-commutative Lie algebras are isomorphic, there exists only one (up to an isomorphism) two-dimensional Lie subalgebra in $\mathrm{Vect}_\omega(S^1)$. One of its realizations inside $\mathrm{Vect}_\omega(S^1)$

is given by the subalgebra, generated by two vector fields $v_1 = \cos\theta\, d/d\theta$, $v_2 = d/d\theta + \sin\theta\, d/d\theta$. $\square$

2.2.2 Exponential map of Vect(S^1)

We analyze now the exponential map

$$\exp : \mathrm{Vect}(S^1) \longrightarrow \mathrm{Diff}_+(S^1)$$

in more detail. Recall that this map associates with a tangent vector field $v = v(\theta)\frac{d}{d\theta}$ on the circle S^1 the diffeomorphism $\exp v := \varphi_1^v$, where φ_t^v is the 1-parameter subgroup of diffeomorphisms in $\mathrm{Diff}_+(S^1)$ with the tangent vector v at the identity $\mathrm{id} \in \mathrm{Diff}_+(S^1)$. In other words, $y_\theta(t) := \varphi_t^v(\theta)$ is a solution of the equation $\frac{dy_\theta}{dt} = v(y_\theta)$ with the initial condition $y_\theta(0) = \theta$.

For finite-dimensional Lie groups one proves easily, using the inverse function theorem, that the map exp (whose derivative at zero is equal to the identity) is locally invertible. However, as we have already pointed out several times before, the inverse function theorem is, in general, not true for Frechet manifolds. By this reason we should not be surprised by the following proposition, proved in [32, 65].

Proposition 3. *The exponential map*

$$\exp : \mathit{Vect}(S^1) \to \mathit{Diff}_+(S^1)$$

is neither locally injective, nor locally surjective in any neighborhood of zero.

Proof. We prove first that the exponential is not injective in any neighborhood of zero. Denote by $R_{2\pi/n}$ the rotation of S^1 by the angle $\frac{2\pi}{n}$ and note that this rotation may be chosen arbitrary close to the identity map $\mathrm{id} \in \mathrm{Diff}_+(S^1)$ for sufficiently large n.

Consider 1-parameter subgroups of $\mathrm{Diff}_+(S^1)$ of the form $f \circ S^1 \circ f^{-1}$, where $f \in \mathrm{Diff}_+(S^1)$ and S^1 is identified with the subgroup of rotations in $\mathrm{Diff}_+(S^1)$. Denote by Γ_n the subgroup in $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms f, commuting with the rotation $R_{2\pi/n}$:

$$R_{2\pi/n}^{-1} \circ f \circ R_{2\pi/n} = f \ .$$

In other words, it is the subgroup of $(2\pi/n)$-periodic diffeomorphisms in $\mathrm{Diff}_+(S^1)$. An element $f \in \Gamma_n$ can be written in the form

$$f(\theta) = \theta + h(\theta) \bmod 2\pi \ ,$$

where h is a smooth $(2\pi/n)$-periodic function on $\mathbb{R}$ and S^1 is identified with $\mathbb{R}/2\pi\mathbb{Z}$. If $f \in \Gamma_n$, then the 1-parameter subgroup $f \circ S^1 \circ f^{-1}$ contains $R_{2\pi/n}$, since

$$f^{-1} \circ R_{2\pi/n} \circ f = R_{2\pi/n} \in S^1 \Longrightarrow R_{2\pi/n} \in f \circ S^1 \circ f^{-1} \ .$$

Hence, *all 1-parameter subgroups* Γ_n *of the above form intersect in* $R_{2\pi/n}$, so the exponential is not injective near zero.

To prove that the exponential is not a surjection onto a neighborhood of id in $\mathrm{Diff}_+(S^1)$, we use the diffeomorphisms from Γ_n, which are small deformations of the rotation $R_{2\pi/n}$. Such a diffeomorphism $f \in \Gamma_n$ can be given by the formula

$$f(\theta) = \theta + \frac{2\pi}{n} + \epsilon \sin(n\theta) \bmod 2\pi \ .$$

For sufficiently large n and sufficiently small $\epsilon > 0$ this diffeomorphism may be made arbitrary close to the identity. The point $\theta = 0$ is a *periodic point of this diffeomorphism of order* n, i.e.

$$f^n(0) = \underbrace{f \circ \cdots \circ f}_{n\,\text{times}}(0) = 0 \bmod 2\pi \ ,$$

but $f^n \neq \mathrm{id}$, since the derivative of f^n at zero is equal (by the composition law) to $(1+\epsilon n)^n$. Moreover, for a sufficiently small ϵ the diffeomorphism f is close to the rotation and therefore has no fixed points.

It follows that f *cannot be the exponential of any vector field* $v \in \mathrm{Vect}(S^1)$. Indeed, assuming the opposite, let $f = \exp v$ for some $v \in \mathrm{Vect}(S^1)$. The vector field $v = v(\theta)\frac{d}{d\theta}$ does not vanish, since f has no fixed points. Hence, the vector field $v(\theta)\frac{d}{d\theta}$ may be transformed into a constant field $c\frac{d}{d\theta}$ with the help of a smooth change of variable $\chi = \chi(\theta)$ of the form

$$\chi(\theta) = c \int_0^\theta \frac{dt}{v(t)} \ , \quad 0 \leq \theta \leq 2\pi \ ,$$

where the normalizing constant $c = 2\pi \left(\int_0^{2\pi} \frac{dt}{v(t)} \right)^{-1}$ is chosen from the condition: $\chi(2\pi) = 2\pi$. This argument shows that the 1-parameter subgroup, generated by the vector v, is conjugate to a rotation R:

$$f = \chi^{-1} \circ R \circ \chi \ .$$

Then $f^n = \chi^{-1} \circ R^n \circ \chi$ and, since $f^n(0) = 0$, the rotation R^n has a fixed point, i.e. $R^n = \mathrm{id}$, which contradicts the relation $f^n \neq \mathrm{id}$. □

Remark 3. The last Proposition asserts that there exist diffeomorphisms in $\mathrm{Diff}_+(S^1)$, which cannot be represented as the exponential of a smooth vector field on the circle. One can ask if there exist diffeomorphisms in $\mathrm{Diff}_+(S^1)$, which cannot be represented as the nth power (with respect to the composition) of a diffeomorphism from $\mathrm{Diff}_+(S^1)$? It's clear that such diffeomorphisms, if they exist, also cannot be represented as the exponentials of smooth vector fields. We try to construct these diffeomorphisms again in the form

$$f(\theta) = \theta + \frac{2\pi}{n} + \epsilon \tilde{h}(\theta) \bmod 2\pi \ , \tag{2.1}$$

where $\epsilon > 0$ is sufficiently small (the map f constitutes a diffeomorphism of S^1, when ϵ is less than $1/\max|\tilde{h}'|$). The function $\tilde{h}$, $0 \leq \tilde{h} \leq 1$, is a smooth $2\pi/n$-periodic function on the real line, whose restriction to the interval $[0, 2\pi/n)$ is denoted by h.

Note that the zeros of the function $\tilde{h}$ are n-periodic points of the diffeomorphism f. Then the following assertion is true.

Suppose that h *vanishes on the interval* $[0, 2\pi/n)$ *in a finite number of points, and this number is not divisible by* n. *Then for a sufficiently small* ϵ *the diffeomorphism* f, *given by the formula* (2.1) *above, can not be represented as the* n*th power of any diffeomorphism from* $\mathrm{Diff}_+(S^1)$.

To prove this assertion, we note that *if* g *is a diffeomorphism from* $\mathrm{Diff}_+(S^1)$, *then the number of orbits of* n*-periodic points of* g^n *is a multiple of* n. The latter statement is a corollary of the following combinatorial fact: *the number of orbits of* k*-periodic points of* g^n *is a multiple of the largest common divisor of* n *and* k, *denoted by* (n, k), which is easy to check by direct calculation.

To deduce our assertion from the statement on the number of n-orbits of g^n, it is sufficient to prove that *our diffeomorphism* f *has no other* n*-periodic points apart from those, given by zeros of* $\tilde{h}$. Indeed, suppose for a moment that we have proved already that the set of n-periodic points of f coincides with the set of zeros of $\tilde{h}$. The number of orbits of n-periodic points is equal to the number of zeros of h on the interval $[0, 2\pi/n)$, which is not divisible by n by the assumption. Hence, by the above statement, f cannot be represented in the form g^n for any $g \in \mathrm{Diff}_+(S^1)$.

To prove that the diffeomorphism f has no other n-periodic points apart from the zeros of $\tilde{h}$, suppose, on the contrary, that there exists an n-periodic point θ_0, in which $h(\theta_0) > 0$. Consider the orbit $\{\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n = \theta_0\}$ of this point on S^1 under f. Then θ_n may be written in the form

$$\theta_n = f^n(\theta_0) = \theta_0 + \epsilon \left(\tilde{h}(\theta_0) + \tilde{h}(\theta_1) + \cdots + \tilde{h}(\theta_{n-1}) \right) \bmod 2\pi \ .$$

If $f^n(\theta_0) = \theta_0 \bmod 2\pi$, then $\epsilon \left(h(\theta_0) + \cdots + h(\theta_{n-1}) \right) = 0 \bmod 2\pi$. The coefficient of ϵ in the latter relation is positive and does not exceed n, since $0 \le h \le 1$. Hence, for $\epsilon < \frac{2\pi}{n}$ this relation cannot be true, i.e. $f^n(\theta_0)$ cannot be equal to θ_0 modulo 2π. This contradiction proves that the only n-periodic points of f are those, given by zeros of $\tilde{h}$, which implies that f cannot be represented in the form g^n for any $g \in \mathrm{Diff}_+(S^1)$.

Using the above assertion, one can easily construct concrete examples of diffeomorphisms $f \in \mathrm{Diff}_+(S^1)$, which cannot be represented as the nth power $(n > 1)$ of any diffeomorphism from $\mathrm{Diff}_+(S^1)$. For instance, one can take a diffeomorphism f of the type (2.1) with

$$h(\theta) = \sin^2(n\frac{\theta}{2}) \quad \text{for } 0 \le \theta < 2\pi/n \ .$$

Or, take $h(\theta) = h_0\left(\frac{\pi}{n}(\theta + 1)\right)$, where h_0 is a smooth function on $[-1, 1)$ of the form

$$h_0(t) = (t-1)^2(t+1)^2 \quad \text{or} \quad h_0(t) = e^{1/(t^2-1)} \quad \text{for } -1 \le t < 1 \ .$$

All these diffeomorphisms f cannot be represented as the nth power of any diffeomorphism from $\mathrm{Diff}_+(S^1)$.

2.2.3 Simplicity of $\mathrm{Diff}_+(S^1)$

One of the remarkable properties of the group $\mathrm{Diff}_+(S^1)$ is its simplicity, which means that the only normal subgroups in $\mathrm{Diff}_+(S^1)$ are the identity and the group itself. This fact (which can be anticipated from Prop.2 in Subsec. 2.2.1) was proved by M.R.Herman in [33, 34]. We shall present in this Subsection an idea how to prove the following, somewhat weaker, statement, contained in [33].

Proposition 4. *Any normal subgroup in $\mathrm{Diff}_+(S^1)$, containing the rotation subgroup S^1, coincides with the whole group $\mathrm{Diff}_+(S^1)$.*

The simplicity property of the group $\mathrm{Diff}_+(S^1)$ is closely related to the following problem, going back to Poincaré and Denjoy: *when a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ is conjugate to a rotation*? We have already touched upon this problem in the proof of Prop.3 in Subsec. 2.2.2. We shall discuss it in more detail after a short digression on the Poincaré rotation number.

Digression 1 (Poincaré rotation number). Let f be an arbitrary diffeomorphism from the group $\mathrm{Diff}_+(S^1)$. Denote by $\tilde{f} : \mathbb{R} \to \mathbb{R}$ its pull-back to $\mathbb{R}$, induced by the universal covering map

$$\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \approx S^1 \ .$$

Then $\tilde{f}$ is a diffeomorphism of $\mathbb{R}$ of the form $\tilde{f} = \mathrm{id} + h$ with h being a smooth periodic function on the real line with period 1. Denote the set of diffeomorphisms of $\mathbb{R}$ of this form by $\mathrm{Diff}_1(\mathbb{R})$. (Recall that $\tilde{f}$ is determined by f up to an integer additive constant). Note that any shift $\tilde{R}_\lambda : x \mapsto x + \lambda$ of $\mathbb{R}$ by the real number λ projects under the above covering map to the rotation R_α of S^1 by the angle $\alpha \equiv \lambda \bmod 1$.

H.Poincaré has found that any diffeomorphism $\tilde{f} \in \mathrm{Diff}_1(\mathbb{R})$, being iterated sufficiently many times, behaves like a translation $\tilde{R}_\lambda$. More precisely, there exists the uniform limit

$$\frac{\tilde{f}^k - \mathrm{id}}{k} \longrightarrow \lambda \quad \text{for } k \to \infty \ ,$$

where λ is a real number, called the *rotation number* of $\tilde{f}$ and denoted by $\lambda = \tilde{\rho}(\tilde{f})$.

The map $\tilde{\rho} : \mathrm{Diff}_1(\mathbb{R}) \to \mathbb{R}$ is continuous in the C^0-topology. Moreover, for any shift $\tilde{R}_\lambda$ we have the following relations:

$$\tilde{\rho}(\tilde{R}_\lambda) = \lambda \quad \text{and} \quad \tilde{\rho}(\tilde{R}_n \circ \tilde{f}) = n + \tilde{\rho}(\tilde{f}) \quad \text{for any } n \in \mathbb{Z} \ .$$

Therefore, pushing down to S^1, we obtain a correctly defined, continuous map

$$\rho : \mathrm{Diff}_+(S^1) \longrightarrow \mathbb{R}/\mathbb{Z} \approx S^1 \ ,$$

assigning to a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ its *Poincaré number* $\rho(f) \in S^1$. This number is invariant under conjugations.

If the rotation number of a diffeomorphism $\tilde{f} \in \mathrm{Diff}_1(\mathbb{R})$ is rational, i.e. $\tilde{\rho}(\tilde{f}) = \frac{p}{q}$ for coprime integers p and q, then there is a simple criterion of its conjugacy to a shift, namely: *$\tilde{f}$ is conjugate to the shift $\tilde{R}_{p/q}$ if and only if $\tilde{f}^q = \tilde{R}_p$.*

The situation in the case of an irrational Poincaré number is much more delicate — everything depends on the arithmetic properties of this number. V.I.Arnold (cf.

[4]) gave an example of a diffeomorphism with an irrational Poincaré number, which is not conjugate to a shift, and conjectured that there exists a set $A \subset S^1 \setminus (\mathbb{Q}/\mathbb{Z})$ of a full Haar measure on S^1, such that any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ with the Poincaré number $\alpha \in A$ is conjugate to the shift R_α. This conjecture was proved by M.R.Herman in [34]. As it was anticipated, the set A in the Herman's theorem has a Diofantine nature and may be described in terms of the decomposition of α into the continuous fraction.

We shall describe here a simpler result by Herman of a similar character, sufficient for the proof of the above Prop. 4.

Recall that, according to the Dirichlet principle, *any irrational number λ may be approximated by rationals so that the following relation holds*

$$\left|\lambda - \frac{p}{q}\right| < \frac{1}{q^2}$$

where $\frac{p}{q} \in \mathbb{Q}$ is an irreducible fraction.

We say that a number λ satisfies the *Diofantine condition* (B_ϵ) with some $\epsilon > 0$, if there exists a constant $C_\epsilon > 0$, such that for all rational numbers p/q the following inequality holds

$$\left|\lambda - \frac{p}{q}\right| \geq \frac{C_\epsilon}{q^{2+\epsilon}} \ .$$

If a number λ satisfies to the Diofantine condition (B_ϵ) for any ϵ (with a constant C_ϵ, depending on ϵ), then λ is called the *Roth number*, and the corresponding $\alpha \in S^1$ form a set of a full Haar measure on the circle. (The numbers, which do not satisfy the condition (B_ϵ) for any $\epsilon > 0$, are called the *Liouville numbers.*)

Lemma 1 (cf. [33]). *Suppose that $\alpha \in S^1 \setminus (\mathbb{Q}/\mathbb{Z})$ satisfies the condition (B_ϵ) for some $\epsilon > 0$. Then there exists a neighborhood U of the rotation R_α in $\mathrm{Diff}_+(S^1)$ such that any diffeomorphism $f \in U$ is represented in the form*

$$f = R_\beta \circ \left(g \circ R_\alpha \circ g^{-1}\right)$$

for some $g \in \mathrm{Diff}_+(S^1)$ and $\beta \in S^1$.

The proof of this Lemma can be found in [33], we shall only demonstrate how it implies the Prop. 4.

Proof of Proposition 4. Let H be a normal subgroup in $\mathrm{Diff}_+(S^1)$, containing S^1. Take $\alpha \in S^1 \setminus (\mathbb{Q}/\mathbb{Z})$, satisfying the Diofantine condition (B_ϵ) for some $\epsilon > 0$. The rotation $R_\alpha \in H$ (since $H \supset S^1$), and Lemma 1 implies that the whole neighborhood U of R_α belongs to H, due to the normality of H. Hence, the subgroup H is open and so contains a neighborhood of the identity in the group $\mathrm{Diff}_+(S^1)$. It implies that H is also closed, hence it should coincide with the whole group $\mathrm{Diff}_+(S^1)$, due to the connectedness of $\mathrm{Diff}_+(S^1)$. The Proposition is proved. $\square$

Remark 4. We have proved in Prop. 3 from Subsec. 2.2.2 that there are diffeomorphisms from $\mathrm{Diff}_+(S^1)$, which cannot be represented as the exponentials of smooth vector fields on the circle. Using Prop. 4, it's easy to prove that, nevertheless, the exponentials of smooth vector fields generate the whole group $\mathrm{Diff}_+(S^1)$. More precisely, any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ may be written as the composition

$$f = \exp v_1 \circ \cdots \circ \exp v_k$$

for some vector fields $v_1, \ldots, v_k \in \mathrm{Vect}(S^1)$.

Another non-trivial corollary of Prop. 4 is that the group $\mathrm{Diff}_+(S^1)$ *does not admit the complexification.* In other words, *there is no complex Lie group, having the complexified algebra* $\mathrm{Vect}^{\mathbb{C}}(S^1)$ *as its Lie algebra.*

This statement is the corollary of the following Proposition.

Proposition 5. *There are no non-trivial homomorphisms from the group* $\mathrm{Diff}_+(S^1)$ *into a connected complex Lie group.*

Proof. Take the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ of fractional linear automorphisms of the unit disc, which can be considered as a subgroup of $\mathrm{Diff}_+(S^1)$. Denote by $G_n := \mathrm{PSL}^{(n)}(2,\mathbb{R})$ the n-fold covering group of $\mathrm{PSL}(2,\mathbb{R})$. More precisely, denote by λ the n-fold covering map of S^1, given by $\lambda : z \mapsto z^n$. Then, by definition, G_n consists of the diffeomorphisms of S^1, which are the n-fold coverings of diffeomorphisms from $\mathrm{PSL}(2,\mathbb{R})$. It means that for any $\varphi \in G_n$ there exists an element $\psi \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$\lambda \circ \varphi = \psi \circ \lambda \ .$$

On the level of Lie algebras, the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ is generated by the vector fields $\frac{d}{d\theta}$, $\sin\theta\frac{d}{d\theta}$, $\cos\theta\frac{d}{d\theta}$, and the Lie algebra of the group G_n (isomorphic to $\mathrm{sl}(2,\mathbb{R})$) is generated by the vector fields $\frac{d}{d\theta}$, $\sin(n\theta)\frac{d}{d\theta}$, $\cos(n\theta)\frac{d}{d\theta}$.

The center of the group G_n consists of rotations $\{R_{2\pi k/n} : k = 0, 1, \ldots, n-1\}$. And it can be proved, as in Subsec.2.1.3, that any homomorphism from G_n to a complex connected Lie group should factor through $\mathrm{PSL}(2,\mathbb{R})$. In other words, its kernel contains all rotations from the centre of G_n. It follows that the kernel of any homomorphism from $\mathrm{Diff}_+(S^1)$ into a complex connected Lie group should contain all rotations of the form $\{R_{2\pi k/n} : k = 0, 1, \ldots, n-1\}$ for any n, hence, all rotations from S^1. But this kernel is a normal subgroup in $\mathrm{Diff}_+(S^1)$, and any normal subgroup in $\mathrm{Diff}_+(S^1)$, containing S^1, should coincide, according to Prop. 4, with the whole group $\mathrm{Diff}_+(S^1)$. This proves that there are no non-trivial homomorphisms from $\mathrm{Diff}_+(S^1)$ into a connected complex Lie group. □

Bibliographic comments

General properties of Frechet Lie groups and algebras are described in Hamilton's paper [32], already mentioned in the bibliographic comments to Ch.1, and Milnor's paper [55]. (Cf. also [47] for the case of Banach Lie groups.)

Key references to Secs. 2.1 and 2.2 are the Pressley–Segal book [65] and Hamilton's paper [32]. In particular, Prop. 1 is formulated in Sec.3.2 of [65], as well as the example before this Proposition (the proof of Prop.1, proposed by K.A.Trushkin, is borrowed from [81]). The Prop.2 is proved, e.g., in [72] (cf. also [81]). The property of the exponential map of the Lie algebra $\mathrm{Vect}(S^1)$, stated in Prop.3, was pointed out by R.S.Hamilton in [32]. A solution to the problem, considered in Rem.3, was proposed by V.V.Kruglov in [81]. The simplicity of the group of diffeomorphisms $\mathrm{Diff}_+(S^1)$, as well as its weaker version, stated in Prop.4, is due to Herman [33, 34]. Prop.5 on the non-existence of the complexification of $\mathrm{Diff}_+(S^1)$ is proved in [65].

Chapter 3

Flag manifolds and representations

Flag manifolds are compact Kähler manifolds, homogeneous with respect to a Lie group action. They can be characterized by the existence of two kinds of homogeneous space representations, namely, a "real" one, as a quotient of a compact Lie group G, and a "complex" one, as a quotient of the complexified Lie group $G^{\mathbb{C}}$. The real representation implies that the flag manifold is compact and homogeneous with respect to the G-action by left shifts, and the complex representation implies that it is a complex Kähler manifold.

Flag manifolds are closely related to the representation theory of the group G via the Borel–Weil construction. We present this construction in Subsec. 3.2.2 together with a necessary background from the representation theory of semisimple Lie groups, given in Subsec. 3.2.1. In the last Subsec. 3.2.3 we give an outline of the orbit method, related to the coadjoint representation of G, which stands behind many constructions in this book.

3.1 Flag manifolds

3.1.1 Geometric definition of flag manifolds

To define flag manifolds in $\mathbb{C}^n$, we fix a decomposition of n into the sum of natural numbers

$$n = k_1 + \cdots + k_r$$

and denote $\mathbf{k} = (k_1, \dots, k_r)$.

Definition 21. A *flag manifold of type* $\mathbf{k}$ in $\mathbb{C}^n$ is the space

$$\begin{aligned}\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \{\text{flags } \mathbf{E} = (E_1, \dots, E_r) :\ & E_i \text{ are linear subspaces} \\ & \text{in } \mathbb{C}^n\text{:}\ \ E_1 \subset \ldots \subset E_r \text{ with } \dim E_i = k_1 + \ldots + k_i\}\ .\end{aligned} \tag{3.1}$$

In particular, for $\mathbf{k} = (k, n-k)$ we obtain

$$\mathrm{Fl}_{(k,n-k)}(\mathbb{C}^n) = \{\text{subspaces } E \subset \mathbb{C}^n \text{ of dimension } k\} = \mathrm{Gr}_k(\mathbb{C}^n)\ ,$$

i.e. the flag manifold in this case is the same as the *Grassmann manifold* of k-dimensional subspaces in $\mathbb{C}^n$. For $k = 1$ it coincides with the $(n-1)$-dimensional complex projective space $\mathrm{Fl}_{(1,n-1)}(\mathbb{C}^n) = \mathbb{CP}^{n-1}$.

For $\mathbf{k} = (1, \ldots, 1)$ the manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) =: \mathrm{Fl}(\mathbb{C}^n)$ is called the *full flag manifold*

$$\mathrm{Fl}(\mathbb{C}^n) = \{E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n : \dim E_i = i\} .$$

The unitary group $\mathrm{U}(n)$ acts transitively on the flag manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$, so that $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ coincides with a homogeneous space of this group. In more detail, fix a basis in $\mathbb{C}^n$ and denote by $\mathbf{E}^0$ the *standard flag* in $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ with $\mathbf{E}^0 = (E_1^0, \ldots, E_r^0)$, where E_i^0 is the subspace in $\mathbb{C}^n$, generated by the first $k_1 + \cdots + k_i$ vectors of our basis. The isotropy subgroup of $\mathrm{U}(n)$ at the point $\mathbf{E}^0$ coincides with the direct product

$$\mathrm{U}_{\mathbf{k}}(n) = \mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r) ,$$

so that the flag manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ is a homogeneous space of $\mathrm{U}(n)$ of the form

$$\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}_{\mathbf{k}}(n) = \mathrm{U}(n)/\,\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r) . \tag{3.2}$$

On the other hand, the complex general linear group $\mathrm{GL}(n, \mathbb{C})$ is also acting on $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ transitively. The isotropy subgroup at the standard flag $\mathbf{E}^0 \in \mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ coincides in this case with the subgroup $P_{\mathbf{k}}$ of blockwise upper-triangular matrices of the form

$$\begin{pmatrix} \underset{r_1}{*}\;{}^{r_1} & * & * & \cdots & * \\ 0 & \underset{r_2}{\boxed{*}}\;{}^{r_2} & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & {}^{r_n}\;\overset{r_n}{*} \end{pmatrix}$$

So, along with the "real" homogeneous representation (3.2), we obtain for $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ a "complex" representation as a homogeneous space of the group $\mathrm{GL}(n, \mathbb{C})$:

$$\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})/\, P_{\mathbf{k}} . \tag{3.3}$$

In the particular cases $\mathbf{k} = (k, n-k)$ and $\mathbf{k} = (1, \ldots, 1)$ we get the well known homogeneous representations for the Grassmann manifold

$$\mathrm{Gr}_k(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(k) \times \mathrm{U}(n-k) = \mathrm{GL}(n, \mathbb{C})/P_{(k,n-k)}$$

and the full flag manifold

$$\mathrm{Fl}(\mathbb{C}^n) = \mathrm{U}(n)/T^n = \mathrm{GL}(n, \mathbb{C})/B_+ ,$$

where $T^n = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$ is the n-dimensional torus, and $B_+ = P_{(1,\ldots,1)}$ is the standard Borel subgroup of upper-triangular matrices.

Note that the flag manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ can be represented also as a homogeneous space of a complex semisimple Lie group by replacing the group $\mathrm{GL}(n, \mathbb{C})$ with $\mathrm{SL}(n, \mathbb{C})$. The corresponding homogeneous representations will take the form

$$\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \mathrm{SU}(n)/\mathrm{SU}_{\mathbf{k}}(n) = \mathrm{SL}(n, \mathbb{C})/\mathrm{S}P_{\mathbf{k}} ,$$

where

$$\mathrm{SU}_{\mathbf{k}}(n) = \mathrm{S}\left[\mathrm{U}(k_1) \times \ldots \times \mathrm{U}(k_n)\right] = \mathrm{U}(k_1) \times \ldots \times \mathrm{U}(k_n) \cap \mathrm{SL}(n, \mathbb{C}),$$
$$\mathrm{S}P_{\mathbf{k}}(n) = P_{\mathbf{k}} \cap \mathrm{SL}(n, \mathbb{C}) \ .$$

3.1.2 Borel and parabolic subalgebras

To give an invariant definition of flag manifolds, we need some basic notions, related to the Borel and parabolic subalgebras. We recall them here, assuming that a reader is familiar with the basics of the theory of semisimple Lie algebras and groups, presented, e.g., in [77, 76, 28, 68].

Let $G_{\mathbb{C}}$ be a complex semisimple Lie group with the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Recall that a *Cartan subalgebra* in $\mathfrak{g}_{\mathbb{C}}$ is a maximal Abelian subalgebra $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, for which all the operators $\mathrm{ad}\, x$, $x \in \mathfrak{h}_{\mathbb{C}}$, are diagonal in $\mathfrak{g}_{\mathbb{C}}$. All Cartan subalgebras in $\mathfrak{g}_{\mathbb{C}}$ are conjugate to each other with respect to the adjoint action of the group $G_{\mathbb{C}}$ on its Lie algebra $\mathfrak{g}_{\mathbb{C}}$. A standard example of the Cartan subalgebra in the case of the general matrix algebra $\mathfrak{g}_{\mathbb{C}} = \mathrm{gl}(n, \mathbb{C})$ is the algebra of all diagonal matrices in $\mathfrak{g}_{\mathbb{C}}$.

We fix now a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ in a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and consider the adjoint action ad of $\mathfrak{h}_{\mathbb{C}}$ on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Note that the operators $\mathrm{ad}\, h$ for different $h \in \mathfrak{h}_{\mathbb{C}}$ commute with each other. The eigenspaces of the adjoint representation, having the form

$$\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g}_{\mathbb{C}} : \mathrm{ad}\, h(\xi) = \alpha(h)\xi\} \ ,$$

where α is a linear functional on $\mathfrak{h}_{\mathbb{C}}$ (i.e. an element of the dual space $\mathfrak{h}^*_{\mathbb{C}}$), are called the *root subspaces*. The linear functionals α, entering into this definition, are called the *roots* of the algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, and the eigenvectors ξ are called the *root vectors*. In particular, the root subspace $\mathfrak{g}_0$, corresponding to the zero functional $\alpha = 0 \in \mathfrak{h}^*$, coincides with the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ itself.

The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ decomposes into the direct sum of its root subspaces

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \ , \tag{3.4}$$

where Δ denotes the set of all nonzero roots of the algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$. This decomposition, called the *root decomposition*, determines a filtration in $\mathfrak{g}_{\mathbb{C}}$, since

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \ .$$

A subset $\Pi \subset \Delta$ is called the *set of simple roots*, if any root $\alpha \in \Delta$ can be represented as a linear combination of roots from Π with integer coefficients, such that all of them are either positive, or (all of them are) negative. Such subsets Π, forming bases in $\mathfrak{h}^*_{\mathbb{C}}$, always exist. It can be shown that all of them are conjugate to each other with respect to the coadjoint action of the group $G_{\mathbb{C}}$.

Fix some set $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_{\mathbb{C}}$. The choice of Π defines on $\mathfrak{h}_{\mathbb{C}}^*$ (hence, on Δ) a *partial ordering*, namely, for $\alpha, \beta \in \mathfrak{h}_{\mathbb{C}}^*$ the relation $\alpha \geq \beta$ means that

$$\alpha - \beta = \sum_{i=1}^{l} a_i \alpha_i \quad \text{with} \quad a_i \geq 0 \ .$$

In particular, a root $\alpha \in \Delta$ is called *positive* (notation: $\alpha \in \Delta^+$), if

$$\alpha = \sum_{i=1}^{l} a_i \alpha_i \quad \text{with} \quad a_i > 0 \ .$$

Using the Killing form $(\cdot, \cdot)$ on $\mathfrak{g}_{\mathbb{C}}$, we can identify the dual space $\mathfrak{h}_{\mathbb{C}}^*$ with $\mathfrak{h}_{\mathbb{C}}$, so that any root α can be considered also as an element α^* of $\mathfrak{h}_{\mathbb{C}}$. We associate with a root α of the algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ the *dual root* or *co-root* $\alpha^\vee$ by the formula

$$\alpha^\vee = 2\frac{\alpha^*}{(\alpha, \alpha)} \ .$$

It is well known that *a system of simple roots* $\Pi = \{\alpha_1, \dots, \alpha_l\}$ *and its Cartan matrix*, defined by:

$$c_{ij} := (\alpha_i, \alpha_j^\vee) \ ,$$

uniquely determine the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Example 23. Consider as an example the complex semisimple Lie algebra $\mathrm{sl}(n, \mathbb{C})$. Choose in $\mathrm{sl}(n, \mathbb{C})$ the standard Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, consisting of diagonal matrices. Denote by E_{ij} the matrix, having 1 at the (i, j)th place, and zeros at all other places. The matrices E_{ij} are the root vectors of the algebra $\mathrm{sl}(n, \mathbb{C})$:

$$\mathrm{ad}(z_1, \dots, z_n) E_{ij} = (z_i - z_j) E_{ij} \ ,$$

where we denote by $(z_1, \dots, z_n)$ the diagonal matrix $\mathrm{diag}(z_1, \dots, z_n)$.

Introduce a functional $\epsilon_i \in \mathfrak{h}_{\mathbb{C}}^*$ by the formula

$$\epsilon_i(z_1, \dots, z_n) = z_i \ .$$

Then the roots of the algebra $\mathrm{sl}(n, \mathbb{C})$ with respect to $\mathfrak{h}_{\mathbb{C}}$ will have the form

$$\Delta = \{\epsilon_i - \epsilon_j \, : \, i \neq j\} \ .$$

The roots

$$\Pi = \{\epsilon_i - \epsilon_{i+1} \, : \, i = 1, \dots, n-1\}$$

form a system of simple roots, so that the set of positive roots is given by

$$\Delta^+ = \{\epsilon_i - \epsilon_j \, : \, i < j\} \ .$$

By analogy with the Borel subalgebra of upper-triangular matrices in $\mathrm{gl}(n, \mathbb{C})$, we can define a *standard Borel subalgebra* $\mathfrak{b}_+$ of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ as

$$\mathfrak{b}_+ = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_+ \ ,$$

where $\mathfrak{n}_+$ is a nilpotent subalgebra of the form

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \ .$$

In the particular case of the algebra $\mathrm{sl}(n,\mathbb{C})$, considered in Ex. 23 above, the subalgebra $\mathfrak{n}_+$ coincides with the subalgebra of above-diagonal matrices, while $\mathfrak{b}_+$ is the subalgebra of upper-triangular matrices.

Definition 22. A *Borel subalgebra* is a subalgebra $\mathfrak{b}$ in $\mathfrak{g}_\mathbb{C}$, conjugate to the standard Borel subalgebra $\mathfrak{b}_+$ with respect to the adjoint action of the group $G_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$. (In a more invariant way, a Borel subalgebra is a maximal solvable subalgebra in $\mathfrak{g}_\mathbb{C}$.) Any subalgebra $\mathfrak{p}$ in $\mathfrak{g}_\mathbb{C}$, containing a Borel subalgebra $\mathfrak{b}$, is called *parabolic*.

As in the case of Borel subalgebras, we could define the parabolic subalgebras $\mathfrak{p}$ as subalgebras in $\mathfrak{g}_\mathbb{C}$, which are conjugate to one of standard parabolic subalgebras. These standard subalgebras (their explicit description is given below) are analogous to the parabolic subalgebras $\mathfrak{p}_\mathbf{k}$ of the algebra $\mathrm{gl}(n,\mathbb{C})$, being the Lie algebras of the parabolic subgroups $P_\mathbf{k}$ from Subsec. 3.1.1.

Now we define the standard parabolic subalgebras in $\mathfrak{g}_\mathbb{C}$ explicitly. For that fix a set $\Pi = \{\alpha_1,\dots,\alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_\mathbb{C}$ and an arbitrary ordered subset π in the set $\{1,\dots,l\}$. We associate with π a subset of simple roots $\Pi_\pi \subset \Pi$ with indices from π. To define the corresponding standard parabolic subalgebra $\mathfrak{p}_\pi$, we denote by Δ_π the linear span of simple roots from Π_π in Δ and introduce a *reductive Levi subalgebra* of the form

$$\mathfrak{l}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Delta_\pi} \mathfrak{g}_\alpha \ .$$

We define also a *nilpotent subalgebra* in $\mathfrak{g}_\mathbb{C}$ by setting

$$\mathfrak{u} = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_\pi} \mathfrak{g}_\alpha \ .$$

The *standard parabolic subalgebra* $\mathfrak{p}_\pi$ is by definition

$$\mathfrak{p}_\mathbf{k} = \mathfrak{l}_\mathbb{C} \oplus \mathfrak{u} \ .$$

It contains the standard Borel subalgebra $\mathfrak{b}_+$ and so is, indeed, parabolic. In the case of the algebra $\mathrm{sl}(n,\mathbb{C})$ the subalgebra $\mathfrak{l}_\mathbb{C}$ coincides with the subalgebra of block-diagonal matrices in $\mathrm{sl}(n,\mathbb{C})$, while $\mathfrak{u}$ is the subalgebra of blockwise above-diagonal matrices.

3.1.3 Algebraic definition of flag manifolds

After this algebraic digression, we can give an invariant definition of flag manifolds of a complex semisimple Lie group $G_\mathbb{C}$.

Definition 23. Let $\mathfrak{p}$ be an arbitrary parabolic subalgebra in $\mathfrak{g}_\mathbb{C}$ and P is the corresponding parabolic subgroup in $G_\mathbb{C}$, having $\mathfrak{p}$ as its Lie algebra. (Otherwise,

P can be defined as the normalizer $N(\mathfrak{p})$ of the subalgebra $\mathfrak{p}$ in $G_{\mathbb{C}}$ with respect to the adjoint representation.) A *flag manifold* of the group $G_{\mathbb{C}}$, associated with the parabolic subalgebra $\mathfrak{p}$, is a homogeneous space of the form

$$F = G_{\mathbb{C}}/P \ . \tag{3.5}$$

Along with the "complex" representation (3.5), taken as the definition of the flag manifold F, there exists also a "real" representation of F as a homogeneous space of a real Lie group. Namely, suppose that the group $G_{\mathbb{C}}$ coincides with the complexification $G^{\mathbb{C}}$ of a compact Lie group G. Then G acts transitively on $G^{\mathbb{C}}/P$ and

$$F = G/G \cap P = G/L \ , \tag{3.6}$$

where the *Levi subgroup* $L = G \cap P$ in the case of the standard parabolic subalgebra $\mathfrak{p}$ has the Lie algebra, given by the real form $\mathfrak{l}$ of the Levi subalgebra $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}}$, introduced above in Subsec. 3.1.2. (In a more invariant way, the subgroup L can be defined as the centralizer of a torus in G.)

Hence, we have obtained for the flag manifold F two different representations as a homogeneous space

$$F = G/L = G^{\mathbb{C}}/P \ .$$

The complex representation (3.5) implies that F is a complex manifold, provided with a G-invariant complex structure. The space of tangent vectors of type $(1,0)$ at the origin with respect to this structure can be identified with the subalgebra $\overline{\mathfrak{u}}$ in the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u} \oplus \overline{\mathfrak{u}} \ , \quad \mathfrak{p} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u} \ ,$$

where the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ has the property that $\overline{\mathfrak{g}} = \mathfrak{g}$.

The real representation (3.6) implies that F is compact and Kähler. We note also that F is simply connected, if the group G is simply connected. It can be shown that flag manifolds F exhaust all simply connected compact Kähler G-manifolds with the transitive action of a compact semisimple Lie group G (cf. [10, 78]).

Remark 5. The real representation (3.6) implies that that the Lie algebra $\mathfrak{p}$ of the parabolic group P has the form

$$\mathfrak{p} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}$$

where $\mathfrak{l}^{\mathbb{C}}$ is the Levi subalgebra and $\mathfrak{u}$ is the nilpotent subalgebra of $\mathfrak{p}$, described in Subsec. 3.1.2 for the case of the standard parabolic subalgebras. The parabolic subalgebras can be defined also in terms of the so called canonical element.

Namely, *for any parabolic subalgebra* $\mathfrak{p}$ *there exists a unique element* ξ *(belonging to the center of the Levi subalgebra* $\mathfrak{l}^{\mathbb{C}}$*), for which the operator* $ad\xi$ *has only imaginary integer eigenvalues, belonging to* $\sqrt{-1}\mathbb{Z}$. *Such an element* ξ *is called the* canonical element *of the parabolic subalgebra* $\mathfrak{p}$. (This fact is proved, e.g., in [15], Theor. 4.4.)

We use this equivalent definition of parabolic subalgebras for the construction of a certain canonical bundle, associated with a flag manifold. The importance of the canonical bundle will become clear in Sec. 7.5, where we show that the loop space ΩG can be considered as a universal flag manifold of the group G.

Denote by $\mathfrak{g}_j$ the eigenspace of the operator $\operatorname{ad}\xi$ with the eigenvalue $\sqrt{-1}j$. In terms of $\mathfrak{g}_j$ the parabolic subalgebra $\mathfrak{p}$ and nilpotent subalgebra $\mathfrak{u}$ can be described as

$$\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i \ , \quad \mathfrak{u} = \bigoplus_{i\geq 1} \mathfrak{g}_i \ .$$

We define now a symmetric space $N = N(F)$, canonically associated with the flag manifold F, by setting

$$N = G/K \ ,$$

where K is a subgroup of G with the Lie algebra

$$\mathfrak{k} = \mathfrak{g} \cap \left[\bigoplus_i \mathfrak{g}_{2i} \right] \ .$$

Since the Lie algebra $\mathfrak{l}$ of the Levi group L is contained in $\mathfrak{g}_0$, there exists a homogeneous bundle

$$F = G/L \longrightarrow G/K = N$$

of the flag manifold F over the associated symmetric space N. So we have constructed for our flag manifold F the associated *symmetric G-space* $N = N(F)$ and *canonical homogeneous bundle* $F \to N$. Note that the symmetric space N is uniquely determined by F, while the canonical bundle $F \to N$ is not uniquely defined, due to the fact that different points of N may have the same stabilizer K. The number of such points is finite, so there exist only a finite number of canonical bundles of the above type.

The importance of flag manifolds is due, in particular, to the fact that all irreducible representations of the group G can be realized in spaces of holomorphic sections of complex line bundles over the flag manifolds of G. This is the Borel–Weil construction, given in Subsec. 3.2.2. To explain this construction, we need some basic facts from the representation theory of complex semisimple Lie groups, collected in the next Subsec. 3.2.1 (cf. for a more detailed exposition [76, 77, 28, 39, 68]).

3.2 Irreducible representations

3.2.1 Irreducible representations of complex semisimple Lie groups

Let $\mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $\rho : \mathfrak{g}_{\mathbb{C}} \to \operatorname{End} V$ is a representation of the algebra $\mathfrak{g}_{\mathbb{C}}$ in a complex vector space V.

A *weight of the representation* ρ is a linear functional $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, for which there exists a vector $v \in V \setminus \{0\}$, called the *weight vector*, such that

$$\rho(h)v = \lambda(h)v \quad \text{for any } h \in \mathfrak{h}_{\mathbb{C}} \ .$$

The linear subspace V_λ, consisting of the vectors $v \in V$, satisfying the relation $\rho(h)v = \lambda(h)v$ for any $h \in \mathfrak{h}_{\mathbb{C}}$, is called the *weight subspace* of weight λ.

Denote by $\Delta_\rho(V) \subset \mathfrak{h}_{\mathbb{C}}^*$ the set of weights of the representation ρ. There is a *weight decomposition* of ρ, analogous to the root decomposition (3.4) for the adjoint representation $\rho = \mathrm{ad}$ from Subsec. 3.1.2. It has the form

$$V = \bigoplus_{\lambda \in \Delta_\rho(V)} V_\lambda \ ,$$

where V_λ is the weight subspace of weight λ.

Fix a system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Among the weights of a representation the special role is played by the *highest weights*, which are the maximal elements in the set of weights of a representation with respect to the partial ordering on $\mathfrak{h}_{\mathbb{C}}^*$, introduced in Subsec. 3.1.2. A highest weight Λ of a representation ρ is characterized by the property that its weight vector v is annihilated by the nilpotent subalgebra $\mathfrak{n}_+$, i.e.

$$\rho(\xi)v = 0 \quad \text{for any } \xi \in \mathfrak{n}_+ \ .$$

We associate with a system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_{\mathbb{C}}$ the dual system of weights $\{\omega_1, \dots, \omega_l\}$, defined by the relation

$$(\omega_i, \alpha_j^\vee) = \delta_{ij} \ ,$$

where $\alpha_j^\vee$ is the co-root, associated with α_j (cf. Subsec. 3.1.2). The elements $\omega_1, \dots, \omega_l \in \mathfrak{h}_{\mathbb{C}}^*$ are called the *fundamental weights* and form a basis in the space of weights, so that any weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ can be written in the form

$$\lambda = \sum_j (\lambda, \alpha_j^\vee)\omega_j$$

and is uniquely determined by the coefficients $(\lambda, \alpha_j^\vee)$. A weight λ is called *dominant*, if all the coefficients $(\lambda, \alpha_j^\vee)$ are non-negative integers.

The highest weight characterizes uniquely an irreducible representation of a complex semisimple Lie algebra. More precisely, we have the following

Theorem 1. *Let ρ be an irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then it has a unique highest weight Λ. This weight is dominant and any other weight $\lambda \in \Delta_\rho(V)$ can be written in the form*

$$\lambda = \Lambda - \alpha_{i_1} - \dots - \alpha_{i_k} \ , \quad \textit{where } \alpha_{i_j} \in \Pi \ .$$

An irreducible representation is uniquely determined (up to equivalence) by its highest weight.

We add a comment on the last statement of the Theorem. An irreducible representation can be reconstructed from its highest weight Λ in the following way. Let v_Λ be the weight vector, corresponding to the weight Λ. Then by definition

$$\rho(\xi)v_\Lambda = 0 \quad \text{for any } \xi \in \mathfrak{n}_+ \ , \tag{3.7}$$

$$\rho(h)v_\Lambda = \Lambda(h)v_\Lambda \quad \text{for any } h \in \mathfrak{h}_{\mathbb{C}} \ . \tag{3.8}$$

Consider the vectors, which can be obtained by the action of elements of the nilpotent subalgebra $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$ on the highest vector v_Λ. More precisely, denote

$$v_{i_1\dots i_k} = \rho(\xi_{-i_k})\cdot\dots\cdot\rho(\xi_{-i_1})v_\Lambda \ ,$$

where $\xi_{-i} \in \mathfrak{g}_{-\alpha_i}$. Then the vectors $\{v_\Lambda, v_{i_1\dots i_k}\}$ generate a subspace $\hat{V}$ with a natural action of the representation ρ. The required representation space V is obtained from $\hat{V}$ by taking the quotient with respect to the maximal invariant subspace in $\hat{V}$ (different from $\hat{V}$) and providing it with the induced action of the representation ρ.

In the representation theory of loop groups LG it is customary to use, instead of the highest and dominant weights, the *lowest* and *antidominant weights*, dual to the introduced highest and dominant weights. The main reason for that is that the Borel–Weil construction of irreducible representations of complex semisimple Lie groups, presented in the next Subsec. 3.2.2, is naturally formulated in terms of the lowest and antidominant weights. In order to switch to the lowest and antidominant weights in the above definitions, it's sufficient to replace the nilpotent subalgebra $\mathfrak{n}_+$ with its counter-part $\mathfrak{n}_-$, defined by

$$\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \ .$$

It follows, in particular, that a weight λ is antidominant if and only if the weight $-\lambda$ is dominant. If V is a representation of an algebra $\mathfrak{g}_\mathbb{C}$ with a highest weight Λ, then the representation of $\mathfrak{g}_\mathbb{C}$ with the lowest weight $-\Lambda$ is realized in the dual vector space V^*. The above Theorem 1 admits an evident reformulation in terms of antidominant lowest weights.

3.2.2 Borel–Weil construction

The Borel–Weil construction, presented in this Subsection, realizes the irreducible representation of a complex semisimple Lie group, associated with a given lowest weight (or a character of the Cartan subgroup), in a space of holomorphic sections of a complex line bundle over the full flag manifold.

Suppose that a Lie group $G^\mathbb{C}$ is the complexification of a compact Lie group G and H is its Cartan subgroup. A *character* of H is a homomorphism $\lambda : H \to \mathbb{C}^*$ into the multiplicative group of nonzero complex numbers $\mathbb{C}^*$. The group $X(H)$ of all characters of H is a free Abelian group of rank, equal to $\dim H$. If $\lambda \in X(H)$ is a character of H, then the map λ_*, tangent to λ, is linear, hence, belongs to the dual space $\mathfrak{h}^*$. This defines a monomorphism of the group $X(H)$ into $\mathfrak{h}^*$, which allows to identify a character λ with the corresponding linear functional λ_*.

Suppose now that the subgroup H is a *maximal torus* (i.e. H is a maximal commutative subgroup in $G^\mathbb{C}$, isomorphic to the product of several copies of the group $\mathbb{C}^*$). Let $R : G^\mathbb{C} \to \mathrm{GL}(V)$ be a linear representation of the group $G^\mathbb{C}$. If $\lambda \in X(H)$ is a character of H, then, by analogy with Subsec. 3.2.1, it is called the *weight of the representation* R, if there exists a vector $v \in V \setminus \{0\}$, called the *weight vector*, such that

$$R(h)v = \lambda(h)v \quad \text{for any } h \in H \ . \tag{3.9}$$

The vectors $v \in V$, satisfying the relation (3.9), form the *weight subspace* V_λ, associated with weight λ.

Any representation $R : G^{\mathbb{C}} \to \mathrm{GL}(V)$ of the group $G^{\mathbb{C}}$ admits a *weight decomposition*

$$V = \bigoplus_{\text{weights } \lambda \text{ of } R} V_\lambda \ ,$$

where the summation is taken over the weights $\lambda \in X(H)$ of the representation R. This decomposition is analogous to the weight decomposition from Subsec. 3.2.1 in the case of Lie algebras. Moreover, the weights of the representation R of the group $G^{\mathbb{C}}$ may be identified with the corresponding weights of the associated representation $R_* : \mathfrak{g}^{\mathbb{C}} \to \mathrm{End}\, V$ of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and the associated weight subspaces coincide.

Assume now that the maximal complex torus H is the complexification of some maximal torus T in G. By analogy with Subsec. 3.1.1, we define the full flag manifold F, associated with T, as

$$F = G/T = G^{\mathbb{C}}/B_+ \ , \tag{3.10}$$

where B_+ is the standard Borel subgroup in $G^{\mathbb{C}}$, having the standard Borel subalgebra $\mathfrak{b}_+$ from Subsec. 3.1.2 as its Lie algebra. On the Lie algebra level the homogeneous representations (3.10) correspond to the decompositions

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- = \mathfrak{b}_+ \oplus \mathfrak{n}_- \ . \tag{3.11}$$

Let $\lambda \in X(H)$ be a character of H, associated with a lowest weight vector of the algebra $\mathfrak{g}^{\mathbb{C}}$. It can be extended to a holomorphic homomorphism $\lambda : B_+ \to \mathbb{C}^*$ of the Borel subgroup B_+, by setting it equal to 1 on the subgroup N_+, having the nilpotent subalgebra $\mathfrak{n}_+$ as its Lie algebra. We define a *complex homogeneous line bundle L_λ over the flag manifold $F = G^{\mathbb{C}}/B_+$, associated with the character λ*:

$$\begin{array}{c} L_\lambda = G^{\mathbb{C}} \times_{B_+} \mathbb{C} \\ \Big\downarrow \\ F = G^{\mathbb{C}}/B_+ \end{array} \quad ,$$

where $G^{\mathbb{C}} \times_{B_+} \mathbb{C}$ is identified with the quotient $G^{\mathbb{C}} \times \mathbb{C}$ modulo the equivalence relation: $(gb, c) \sim (g, \lambda(b)c)$ for any $g \in G^{\mathbb{C}}$, $b \in B_+$ and $c \in \mathbb{C}$. A section of the line bundle L_λ is identified with a function $f : G^{\mathbb{C}} \to \mathbb{C}$, subject to the relation

$$f(gb) = \lambda(b^{-1}) f(g) \quad \text{for all } g \in G^{\mathbb{C}},\ b \in B_+ \ . \tag{3.12}$$

Denote by Γ_λ the *space of holomorphic sections of the bundle L_λ* or, in other words, the space of holomorphic functions on $G^{\mathbb{C}}$, satisfying the condition (3.12). The group $G^{\mathbb{C}}$ acts from the left on L_λ, hence, on the space Γ_λ.

Theorem 2 (Borel–Weil theorem). *If the weight λ is antidominant, then the representation of the group G in the space of holomorphic sections Γ_λ, constructed above, is the irreducible representation with the lowest weight λ and all irreducible representations of the group G can be realized in this way.*

3.2.3 Orbit method and coadjoint representation

In this Subsection we outline briefly another method of constructing irreducible representations of Lie groups, using the orbits of the coadjoint representation of the group on the dual space of its Lie algebra (the details may be found in Kirillov's book [39]). Though we do not use this method for the construction of representations, we found it useful to explain its idea to motivate the study of coadjoint representations of various infinite-dimensional groups in this book.

We recall first some basic facts on the characters of irreducible representations. Let $T : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of a Lie group G. We define its *character* as a function $\chi_T : G \to \mathbb{C}^*$, given by the formula

$$\chi_T(g) := \operatorname{Tr} T(g) \ , \ g \in G \ .$$

This function is constant on conjugacy classes and depends only on the equivalence class of the representation T. Moreover, it is a homomorphism with respect to the tensor product of representations, i.e. $\chi_{T_1 \otimes T_2} = \chi_{T_1}\chi_{T_2}$. A character of an irreducible representation determines it uniquely up to equivalence.

Let G be a compact Lie group and $L^2(G, dg)$ denotes the space of all square integrable functions on G with respect to the Haar measure dg. Then the characters of all its irreducible unitary representations form an orthonormal basis in a subspace of $L^2(G, dg)$, consisting of functions, constant on conjugacy classes.

The definition of the character χ_T, given above, is valid, evidently, only for finite-dimensional representations T. However, for an infinite-dimensional representation it's often possible to define its character as a distribution on the group G. Namely, denote by $\mathcal{D}(G)$ the space of C^∞-smooth functions on G and suppose that all operators of the form

$$T(f) := \int_G f(g)T(g)\, dg \ , \qquad f \in \mathcal{D}(G) \ ,$$

are of trace class (the definition of the trace class is given in Sec. 5.3 below). Then we can define a character of the representation T as a distribution on the space $\mathcal{D}(G)$ of test functions, or, in other words, as a continuous linear functional on $\mathcal{D}(G)$, acting by the formula

$$\chi_T(f) := \operatorname{Tr} T(f) \ , \qquad f \in \mathcal{D}(G) \ .$$

If, in particular, the group G is semisimple, then the character χ_T can be given by the formula

$$\chi_T(f) = \int_G \chi_T(g) f(g)\, dg \ ,$$

where χ_T is some measurable locally integrable function on G. As in the case of finite-dimensional representations, the character $\chi_T(f)$ is constant on conjugacy classes, i.e.

$$\chi_T(f) = \operatorname{Tr} T(f) = \operatorname{Tr}[T(g)T(f)T(g^{-1})]$$

for any $f \in \mathcal{D}(G)$, $g \in G$. Again, an irreducible representation is uniquely determined (up to equivalence) by its character.

We turn now to the *coadjoint representation* of the group G. Let $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ is its dual space. The adjoint action Ad of the group G on its Lie algebra $\mathfrak{g}$ induces by duality the *coadjoint action* Ad^* of the group G on the space $\mathfrak{g}^*$.

Consider an orbit $F = G{\cdot}\varphi$ of an arbitrary point $\varphi \in \mathfrak{g}^*$ in $\mathfrak{g}^*$ under the coadjoint action and denote by G_φ the isotropy subgroup at φ. Let $\mathfrak{g}_\varphi$ be the Lie algebra of the group G_φ. Then the tangent space to the orbit F at φ may be identified with the quotient $\mathfrak{g}/\mathfrak{g}_\varphi$.

The orbits F of the coadjoint representation turn out to be symplectic manifolds, provided with a canonical *Kirillov symplectic form* ω_F. This form is generated by a G_φ-invariant 2-form ω_φ on $\mathfrak{g}$, given by the formula

$$\omega_\varphi(\xi,\eta) := \varphi([\xi,\eta]) \ , \quad \xi,\eta \in \mathfrak{g} \ .$$

The kernel of ω_φ on $\mathfrak{g}$ coincides with $\mathfrak{g}_\varphi$, so the form ω_φ can be pushed down to a form on $\mathfrak{g}/\mathfrak{g}_\varphi$ (denoted by the same letter), which is a non-degenerate G_φ-invariant 2-form on $\mathfrak{g}/\mathfrak{g}_\varphi$. So it can be extended to a non-degenerate G-invariant 2-form ω_F on F, which does not depend on the choice of the point φ on the orbit F. Moreover, the form ω_φ satisfies the Jacobi identity, hence, it is a cocycle on $\mathfrak{g}$. This implies that the induced G-invariant 2-form ω_F is closed on F, and so defines a symplectic structure on F.

It may be proved that any G-homogeneous (with respect to the action of a connected Lie group G by symplectic transformations) symplectic manifold M is locally isomorphic to an orbit of the group G or its central extension $\widetilde{G}$ in the coadjoint representation (cf. [46]).

We explain now the idea of the *orbit method*. We want to construct an irreducible unitary representation T from an orbit of the coadjoint representation in $\mathfrak{g}^*$.

Let $F = G\cdot\varphi$ be such an orbit. We construct from it a one-dimensional unitary representation of the group G_φ. In a neighborhood of the identity of G_φ we define it by the formula

$$\chi(\exp\xi) = e^{2\pi i\varphi(\xi)} \ ,$$

where $\exp : \mathfrak{g}_\varphi \to G_\varphi$ is the exponential map. It extends to a representation of the isotropy group G_φ and induces an irreducible unitary representation T_F of the whole group G, if the orbit F is *integral*, i.e. the canonical symplectic form ω_F is an integral form on F (the precise definition of an integral form is given in the beginning of Sec. 8.1).

The character of the irreducible unitary representation T_F is given by the formula

$$\chi_F(\exp\xi) = \frac{1}{p_F(\exp\xi)}\int_F e^{2\pi i\varphi(\xi)}\beta_F(\varphi) \ , \quad \xi\in\mathfrak{g} \ , \tag{3.13}$$

where β_F is the Liouville volume form on F, generated by the symplectic form ω_F, and p_F is some smooth invariant (with respect to conjugations) function on G, equal to 1 at $e\in G$. The formula (3.13) should be understood in the distributional sense, i.e. for any test function $f\in\mathcal{D}(G)$ the integral

$$\chi_F(f) = \operatorname{Tr} T_F(f) = \int_F \left\{\int_{\mathfrak{g}} \frac{f(\exp\xi)}{p_F(\exp\xi)} e^{2\pi i\varphi(\xi)}\, d\xi\right\}\beta_F(\varphi) \ ,$$

converges (here $d\xi$ is the Lebesgue measure on $\mathfrak{g}$).

In particular, for compact groups G we have $\dim T_F := \chi_F(e) = \mathrm{Vol}\ F < \infty$, and the integral orbits in this case correspond to flag manifolds. In this case the orbit method is equivalent to the Borel–Weil method from the previous Subsec. 3.2.2.

Bibliographic comments

The content of this Chapter is mostly of the reference character and may be found in a number of books. In particular, general properties of flag manifolds are presented in [6, 15, 61]. The basics of the representation theory of semisimple Lie algebras and groups may be found, e.g., in [77, 76, 28, 68]. The Borel–Weil construction is explained, in particular, in the book [6]. The orbit method is presented in [39, 43].

Chapter 4

Central extensions and cohomologies of Lie algebras and groups

In the first Section of this Chapter (Sec. 4.1) we recall the definition and basic properties of central extensions of Lie algebras and groups. In particular, we point out a relation between central extensions of Lie groups and their projective representations. In Sec. 4.2 we introduce the Lie algebra cohomologies and give several important examples of this notion (including the cohomological interpretation of central extensions). The last Sec. 4.3 is devoted to the Lie group cohomologies and their relation to projective representations.

4.1 Central extensions of Lie groups and projective representations

Definition 24. A *central extension of a Lie algebra* $\mathfrak{G}$ (over the field $\mathbb{R}$) is a Lie algebra $\tilde{\mathfrak{G}}$, which can be included into the exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathbb{R} \longrightarrow \tilde{\mathfrak{G}} \longrightarrow \mathfrak{G} \longrightarrow 0 \ , \tag{4.1}$$

where $\mathbb{R}$ is considered as an Abelian Lie algebra and the image of the monomorphism $\mathbb{R} \to \tilde{\mathfrak{G}}$ is contained in the center of the algebra $\tilde{\mathfrak{G}}$. Two central extensions $\tilde{\mathfrak{G}}_1$ and $\tilde{\mathfrak{G}}_2$ of the same Lie algebra $\mathfrak{G}$ are said to be *equivalent*, if there exist a commutative diagram of Lie algebra homomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{\mathfrak{G}}_1 & \longrightarrow & \mathfrak{G} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{\mathfrak{G}}_2 & \longrightarrow & \mathfrak{G} & \longrightarrow & 0 \ .
\end{array}$$

The exact sequence (4.1) implies that the Lie algebra $\tilde{\mathfrak{G}}$, as a vector space, is isomorphic to $\tilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathbb{R}$ and the Lie bracket in $\tilde{\mathfrak{G}}$, due to the centrality of the image of $\mathbb{R} \to \tilde{\mathfrak{G}}$, has the form

$$[(\xi, s), (\eta, t)] = [(\xi, 0), (\eta, 0)] = ([\xi, \eta], \omega(\xi, \eta)) \ ,$$

where ω is a skew-symmetric bilinear form on $\mathfrak{G}$, called the *cocycle of the central extension*.

By analogy with Def. 24, we can define central extensions of Lie groups.

Definition 25. A *central extension of a Lie group* $\mathcal{G}$ is a Lie group $\tilde{\mathcal{G}}$, which can be included into the exact sequence of Lie group homomorphisms

$$1 \longrightarrow S^1 \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G} \to 1 ,$$

where the image of the circle group under the monomorphism $S^1 \to \tilde{\mathcal{G}}$ is contained in the center of the group $\tilde{\mathcal{G}}$.

Topologically, the map $\tilde{\mathcal{G}} \to \mathcal{G}$ is a principal S^1-bundle. Consider the case, when this S^1-bundle is trivial, i.e. $\tilde{\mathcal{G}} \to \mathcal{G}$ admits a global section $\sigma : \mathcal{G} \to \tilde{\mathcal{G}}$. With the help of this section, we can identify $\tilde{\mathcal{G}}$ with the group $\mathcal{G} \times S^1$, provided with the multiplication

$$(g, \lambda) \cdot (h, \mu) = (gh, \lambda\mu c(g, h)) \ ,$$

where $c(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$ is called the *cocycle of the central extension* $\tilde{\mathcal{G}}$.

Central extensions of Lie groups are closely related to their projective representations.

Definition 26. A *projective (unitary) representation of a Lie group* $\mathcal{G}$ is a map

$$\rho : \mathcal{G} \to \mathrm{U}(H)$$

of the group $\mathcal{G}$ into the group of unitary operators, acting in a complex Hilbert space H, satisfying the relation

$$\rho(g_1)\rho(g_2) = c(g_1, g_2)\rho(g_1 g_2) \quad \text{for all } \ g_1, g_2 \in \mathcal{G} \ ,$$

where $c(g_1, g_2)$ is a complex number with modulus 1, which is called the *cocycle of the projective representation*.

Another projective representation $\rho' : \mathcal{G} \to \mathrm{U}(H)$ of the same group $\mathcal{G}$ is *equivalent* to ρ, if

$$\rho'(g) = \lambda(g)\rho(g) \ , \quad g \in \mathcal{G} \ ,$$

for some $\lambda : \mathcal{G} \to S^1$.

Any projective unitary representation ρ of a Lie group $\mathcal{G}$ determines a true unitary representation $\tilde{\rho}$ of some central extension $\tilde{\mathcal{G}}$ of the group $\mathcal{G}$, which is a topologically trivial S^1-bundle with the cocycle, equal to the cocycle of the projective representation. Namely, we define

$$\tilde{\rho}(g, \lambda) := \lambda\rho(g) \quad \text{for all } \ g \in \mathcal{G}, \ \lambda \in S^1 \ .$$

Then we'll have

$$\tilde{\rho}\left((g_1, \lambda_1) \cdot (g_2, \lambda_2)\right) = \lambda_1\lambda_2 c(g_1, g_2)\rho(g_1 g_2) = \lambda_1\lambda_2\rho(g_1)\rho(g_2) = \tilde{\rho}(g_1, \lambda_1)\tilde{\rho}(g_2, \lambda_2)$$

for any $g_1, g_2 \in \mathcal{G}$, $\lambda_1, \lambda_2 \in S^1$.

Conversely, any unitary representation $\tilde{\rho}$ of a topologically trivial central extension $\tilde{\mathcal{G}}$, such that $\tilde{\rho}(\lambda) = \lambda \cdot \mathrm{id}$ for any $\lambda \in S^1$, determines a projective representation ρ of the group $\mathcal{G}$, which is defined in the following way. The cocycle c of the central extension $\tilde{\mathcal{G}}$ is given in terms of the trivializing section $\sigma : \mathcal{G} \to \tilde{\mathcal{G}}$ by the formula

$$c(g_1, g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1} , \quad g_1, g_2 \in \mathcal{G} .$$

Then the map ρ, defined by $\rho(g) := \tilde{\rho}\,(\sigma(g))$, determines a projective representation $\rho : \mathcal{G} \to \mathrm{U}(H)$, since

$$\rho(g_1 g_2) = \tilde{\rho}\,(\sigma(g_1 g_2)) = \tilde{\rho}\left(c(g_1, g_2)^{-1}\sigma(g_1)\sigma(g_2)\right) = c(g_1, g_2)^{-1}\rho(g_1)\rho(g_2)$$

for any $g_1, g_2 \in \mathcal{G}$.

4.2 Cohomologies of Lie algebras

Let $\mathfrak{G}$ be a Lie algebra and $\rho : \mathfrak{G} \to \mathrm{End}\, V$ is a representation of $\mathfrak{G}$ in a vector space V. In other words, V is a $\mathfrak{G}$-module.

Definition 27. A *q-cochain of the algebra $\mathfrak{G}$ with coefficients in V* is a skew-symmetric continuous q-linear functional on $\mathfrak{G}$ with values in V, i.e. a continuous map

$$\alpha : \underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{q} \longrightarrow V ,$$

which is skew-symmetric and q-linear. The set of all such cochains is denoted by $C^q(\mathfrak{G}, V)$.

We define the *differential (coboundary map)*

$$\delta_q : C^q(\mathfrak{G}, V) \longrightarrow C^{q+1}(\mathfrak{G}, V)$$

by the formula

$$\begin{aligned}\delta_q\alpha(\xi_1, \ldots, \xi_{q+1}) = &\sum_{1\le i\le q+1} (-1)^i \xi_i \alpha(\xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_{q+1}) + \\ &+ \sum_{1\le i<j\le q+1} (-1)^{i+j-1}\alpha([\xi_i, \xi_j], \xi_1, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_{q+1})\end{aligned} \tag{4.2}$$

for $\alpha \in C^q(\mathfrak{G}, V)$, $\xi_1, \ldots, \xi_{q+1} \in \mathfrak{G}$.

It's easy to check that the coboundary maps have the property $\delta_q \circ \delta_{q-1} = 0$, so we obtain a complex

$$\ldots \longrightarrow C^{q-1}(\mathfrak{G}, V) \xrightarrow{\delta_{q-1}} C^q(\mathfrak{G}, V) \xrightarrow{\delta_q} C^{q+1}(\mathfrak{G}, V) \longrightarrow \ldots .$$

The cohomologies of this complex are called the *cohomologies of the Lie algebra $\mathfrak{G}$ with coefficients in the $\mathfrak{G}$-module V* and denoted by

$$\begin{aligned}H^q(\mathfrak{G}, V) &:= \mathrm{Ker}\,\delta_q/\,\mathrm{Im}\,\delta_{q-1} = \\ &= \frac{\{\xi \in C^q(\mathfrak{G}, V) : \delta_q\,\xi = 0\}}{\{\xi \in C^q(\mathfrak{G}, V) :\ \xi = \delta_{q-1}\,\eta \quad \text{for some } \eta \in C^{q-1}(\mathfrak{G}, V)\}} .\end{aligned} \tag{4.3}$$

In the particular case, when V is the basic number field $k = \mathbb{R}, \mathbb{C}$, considered as the trivial $\mathfrak{G}$-module, the cohomologies $H^q(\mathfrak{G}, k)$ are denoted by $H^q(\mathfrak{G})$.

The above expression for the coboundary map looks like exterior derivative of a differential form. This is because differential forms on a smooth manifold X may be considered as cochains of the Lie algebra $\mathrm{Vect}(X)$ with coefficients in the module $C^\infty(X)$ of smooth functions on X, considered as a $\mathrm{Vect}(X)$-module.

Here are several particular examples of Lie algebra cohomologies.

Example 24 (cohomology $H^0(\mathfrak{G}, V)$)**.** Setting $C^{-1}(\mathfrak{G}, V) = 0$, we get

$$\begin{aligned} H^0(\mathfrak{G}, V) = \mathrm{Ker}\ &\left\{\delta_0 : C^0(\mathfrak{G}, V) = V \longrightarrow C^1(\mathfrak{G}, V)\right\} \\ &= \{v \in V : \xi v = 0 \text{ for any } \xi \in \mathfrak{G}\} \ . \end{aligned} \tag{4.4}$$

In other words, the cohomology $H^0(\mathfrak{G}, V)$ coincides with the set of invariants of $\mathfrak{G}$-module V.

Example 25 (cohomology $H^1(\mathfrak{G})$)**.** In this case the differential $\delta_0 : C^0(\mathfrak{G}) \to C^1(\mathfrak{G})$ is trivial, since the action of $\mathfrak{G}$ on k is trivial. So

$$\begin{aligned} H^1(\mathfrak{G}) = \mathrm{Ker}\ &\left[\delta_1 : C^1(\mathfrak{G}) = \mathfrak{G}^* \longrightarrow C^2(\mathfrak{G})\right] \\ &= \{\beta \in \mathfrak{G}^* : \beta([\xi, \eta]) = 0 \text{ for all } \xi, \eta \in \mathfrak{G}\} = \left(\mathfrak{G}/\left[\mathfrak{G}, \mathfrak{G}\right]\right)^* \ . \end{aligned} \tag{4.5}$$

Otherwise speaking, the cohomology $H^1(\mathfrak{G})$ consists of continuous linear functionals on $\mathfrak{G}/\left[\mathfrak{G}, \mathfrak{G}\right]$.

Example 26 (cohomology $H^1(\mathfrak{G}; \mathfrak{G})$)**.** Consider a Lie algebra $\mathfrak{G}$ as a $\mathfrak{G}$-module with respect to the adjoint action ad of $\mathfrak{G}$ on itself. The cohomology $H^1(\mathfrak{G}, \mathfrak{G})$ is interpreted as the set of outer derivations of the algebra $\mathfrak{G}$. Recall that a homomorphism $\phi : \mathfrak{G} \to \mathfrak{G}$ is called the *derivation* of $\mathfrak{G}$, if

$$\phi([\xi, \eta]) = [\phi(\xi), \eta] + [\xi, \phi(\eta)] \ .$$

The *inner derivations*, defined by

$$\xi \longmapsto [\xi, \xi_0] = \mathrm{ad}_{\xi_0}(\xi) \ ,$$

where ξ_0 is a fixed element of $\mathfrak{G}$, may serve as an example.

The set of *outer derivations* coincides, by definition, with the quotient of the set of all derivations of the algebra $\mathfrak{G}$ modulo inner derivations.

Let us show that the cohomology $H^1(\mathfrak{G}, \mathfrak{G})$ coincides with the set of outer derivations of the algebra $\mathfrak{G}$.

Indeed, cochains from $C^1(\mathfrak{G}, \mathfrak{G})$ are given by linear maps $\phi : \mathfrak{G} \to \mathfrak{G}$. The condition $\delta_1\phi = 0$ means that ϕ is a derivation, since

$$\delta_1\phi(\xi, \eta) = \phi([\xi, \eta]) - \xi\phi(\eta) + \eta\phi(\xi) = \phi([\xi, \eta]) - [\xi, \phi(\eta)] - [\phi(\xi), \eta] \ .$$

The cochains from $C^1(\mathfrak{G}, \mathfrak{G})$, belonging to the image of the map $\delta_0 : C^0(\mathfrak{G}, \mathfrak{G}) \to C^1(\mathfrak{G}, \mathfrak{G})$, are inner derivations of the algebra $\mathfrak{G}$, since

$$\xi \in \mathfrak{G} = C^0(\mathfrak{G}, \mathfrak{G}) \quad \Longrightarrow \quad \delta_0\xi(\eta) = -\xi \cdot \eta = [-\xi, \eta] \ .$$

Example 27 (cohomology $H^2(\mathfrak{G})$)**.** The cohomology $H^2(\mathfrak{G})$ may be identified with set of equivalence classes of central extensions of the Lie algebra $\mathfrak{G}$, considered in the previous Sec. 4.1.

Indeed, associate with a cocycle $\omega \in C^2(\mathfrak{G})$ the central extension

$$0 \longrightarrow k \longrightarrow k \oplus \mathfrak{G} \longrightarrow \mathfrak{G} \longrightarrow 0 \ ,$$

where the map $k \to k \oplus \mathfrak{G}$ is an embedding $s \mapsto (s, 0)$, and the map $k \oplus \mathfrak{G} \to \mathfrak{G}$ coincides with the projection $(s, \xi) \mapsto \xi$. The bracket in the algebra $\widetilde{\mathfrak{G}} = k \oplus \mathfrak{G}$ is given by the formula

$$[(s, \xi), (t, \eta)] = (\omega(\xi, \eta), [\xi, \eta]) \ .$$

The Jacoby identity in the algebra $\widetilde{\mathfrak{G}}$ is equivalent to the cocyclicity of ω. Moreover, cohomologous cocycles correspond to equivalent central extensions, and the zero in $H^2(\mathfrak{G})$ corresponds to the trivial central extension $\widetilde{\mathfrak{G}} = k \oplus \mathfrak{G}$.

Example 28 (cohomology $H^3(\mathfrak{G})$)**.** The cohomology $H^3(\mathfrak{G})$ of a semisimple Lie algebra $\mathfrak{G}$ is interpreted as the set of invariant symmetric bilinear forms on $\mathfrak{G}$.

Indeed, with any such form $\langle \cdot \, , \cdot \rangle$ we can associate an element of $H^3(\mathfrak{G})$, given by the 3-cocycle of the form

$$\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \ni (\xi, \eta, \zeta) \longmapsto \langle \xi, [\eta, \zeta] \rangle \ .$$

Apart from the above examples, demonstrating the importance of the cohomologies of Lie algebras, there is one more motivation to introduce such an object. Namely, the cohomologies of a Lie algebra $\mathfrak{G}$ are closely related to the de Rham cohomologies of the corresponding Lie group $\mathcal{G}$, considered as a smooth manifold. Let us denote the latter cohomology groups by $H^q_{\text{top}}(\mathcal{G}, k)$. A relation between $H^q_{\text{top}}(\mathcal{G}, k)$ and the cohomologies of the Lie algebra $\mathfrak{G}$ is established in the following way.

Construct first a map of the cochain complex $C^\bullet(\mathfrak{G})$ into the de Rham complex $\Omega^\bullet(\mathcal{G})$ of the group $\mathcal{G}$. Denote by $\Omega^q_{\text{inv}}(\mathcal{G})$ the subspace of differential forms of degree q in $\Omega^q(\mathcal{G})$, invariant under the right translations on $\mathcal{G}$. A form in $\Omega^q_{\text{inv}}(\mathcal{G})$ is uniquely determined by its restriction to the tangent space $T_e\mathcal{G} = \mathfrak{G}$, i.e. there is an isomorphism

$$\Omega^q_{\text{inv}}(\mathcal{G}) \overset{\approx}{\longleftrightarrow} \Lambda^q(\mathfrak{G}) = C^q(\mathfrak{G}) \ .$$

Moreover, the differential $\delta_q : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ coincides with the restriction of the exterior differential $d_q : \Omega^q(\mathcal{G}) \to \Omega^{q+1}(\mathcal{G})$ to $\Omega^q_{\text{inv}}(\mathcal{G})$. So there is a canonical map

$$H^q(\mathfrak{G}) \longrightarrow H^q_{\text{top}}(\mathcal{G}, k) \ . \tag{4.6}$$

This homomorphism is an isomorphism, when $k = \mathbb{R}$ and $\mathcal{G}$ is a compact Lie group (in this case one can associate with any form on $\mathcal{G}$ a right-invariant form by averaging the original form over $\mathcal{G}$). In the complex case $k = \mathbb{C}$ the above homomorphism is an isomorphism, if $\mathcal{G}$ is a complex semisimple Lie group. The isomorphism (4.6) extends also to some infinite-dimensional Lie groups, in particular, to the loop group $LG = C^\infty(S^1, G)$ of a compact Lie group G ($k = \mathbb{R}$ in this case).

4.3 Cohomologies of Lie groups

Let $\mathcal{G}$ be a Lie group and V is a $\mathcal{G}$-module, i.e. we have a representation $\rho : \mathcal{G} \to \mathrm{GL}(V)$ of the group $\mathcal{G}$ in the vector space V. There are two natural definitions of the cochain complex with values in the $\mathcal{G}$-module V. In the first definition cochains are given by equivariant functions on $\mathcal{G}$ with values in V.

Definition 28. A *q-cochain of the group $\mathcal{G}$* with values in V is a function

$$\varphi : \underbrace{\mathcal{G} \times \cdots \times \mathcal{G}}_{q+1} \longrightarrow V \ ,$$

which has the following equivariance property

$$\varphi(gg_0, \ldots, gg_q) = g \cdot \varphi(g_0, \ldots, g_q) \ ,$$

where "$\cdot$" in the right hand side denotes the action of the group $\mathcal{G}$ on V, given by the representation ρ. The space of all q-cochains is denoted by $C^q(\mathcal{G}, V)$ and the *differential*

$$\delta_q : C^q(\mathcal{G}, V) \to C^{q+1}(\mathcal{G}, V)$$

is given by the formula

$$\delta_q\varphi(g_0, \ldots, g_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(g_0, \ldots, \widehat{g_i}, \ldots, g_{q+1}) \ .$$

In the second definition cochains are given by arbitrary functions on $\mathcal{G}$ with values in V.

Definition 29. A *q-cochain on the group $\mathcal{G}$* with values in V is a function

$$\psi : \underbrace{\mathcal{G} \times \cdots \times \mathcal{G}}_{q+1} \longrightarrow V \ .$$

The space of all q-cochains on $\mathcal{G}$ with values in V is denoted again by $C^q(\mathcal{G}, V)$, but the *differential*

$$\delta_q : C^q(\mathcal{G}, V) \to C^{q+1}(\mathcal{G}, V)$$

is given in this case by the formula

$$\begin{aligned} \delta_q\psi(g_1, \ldots, g_{q+1}) &= g_1 \cdot \psi(g_2, \ldots, g_{q+1}) + \\ &+ \sum_{i=1}^{q} (-1)^i \psi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{q+1}) + (-1)^{q+1} \psi(g_1, \ldots, g_q) \ . \end{aligned} \tag{4.7}$$

A relation $\varphi \leftrightarrow \psi$ between these two definitions of cochains is established via the formulas

$$\varphi(g_0, \ldots, g_q) = g_0 \cdot \psi(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{q-1}^{-1} g_q) \ , \tag{4.8}$$

$$\psi(g_1, \ldots, g_q) = \varphi(1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdot \ldots \cdot g_q) \ . \tag{4.9}$$

The *cohomologies of the group $\mathcal{G}$ with values in* the *$\mathcal{G}$-module* V in both cases are defined as the cohomologies of the complex $\{C^q(\mathcal{G},V),\delta_q\}$, i.e.

$$H^q(\mathcal{G},V)=\frac{\mathrm{Ker}\,[\delta_q : C^q(\mathcal{G},V)\to C^{q+1}(\mathcal{G},V)]}{\mathrm{Im}\,[\delta_{q-1} : C^{q-1}(\mathcal{G},V)\to C^q(\mathcal{G},V)]}\ .$$

We consider now a relation between 2-dimensional cohomologies of the group $\mathcal{G}$ with its projective representations and central extensions (cf. Sec. 4.1).

Let $\rho:\mathcal{G}\to \mathrm{U}(V)$ be a projective representation of the Lie group $\mathcal{G}$, satisfying the relation

$$\rho(g_1)\rho(g_2)=c(g_1,g_2)\rho(g_1g_2) \quad \text{for any}\ g_1,g_2\in\mathcal{G}\ ,$$

where $c(g_1,g_2)$ is the cocycle of the representation ρ. The associativity of the multiplication in $\mathcal{G}$ and $\mathrm{U}(V)$ implies that c is a 2-cocycle of the group $\mathcal{G}$ with values in the multiplicative group S^1 with the trivial action of the group $\mathcal{G}$, given by $\rho:\mathcal{G}\to 1$. In other words, for any three elements g_1,g_2,g_3 of the group $\mathcal{G}$ we have the relation

$$c(g_2,g_3)c(g_1g_2,g_3)^{-1}c(g_1,g_2g_3)c(g_1,g_2)^{-1}=1\ ,$$

which means that $\delta_2 c=1$ (we use here the multiplicative analog of δ_2 from Def. 29).

On the other hand, an equivalent projective representation of the form

$$\rho'(g)=\lambda(g)\rho(g)$$

with $\lambda:\mathcal{G}\to S^1$, corresponds to the cocycle

$$c'(g_1,g_2)=c(g_1,g_2)\lambda(g_1g_2)\lambda(g_1)^{-1}\lambda(g_2)^{-1}\ ,$$

i.e. to the cocycle $c'\in C^2(\mathcal{G},S^1)$, cohomologous to the cocycle $c\in C^2(\mathcal{G},S^1)$. So the class $[c]$ of the cocycle c in the cohomologies $H^2(\mathcal{G},S^1)$ depends only on the equivalence class of the projective representation ρ. Hence, *the equivalence classes of projective representations of the Lie group $\mathcal{G}$ in a Hilbert space V can be identified with the cohomologies $H^2(\mathcal{G},S^1)$.*

On the other hand, in Sec. 4.1 we have assigned to any topologically trivial central extension $\tilde{\mathcal{G}}\to\mathcal{G}$ of the group $\mathcal{G}$ its cocycle c, which is the same as a 2-cocycle of the group $\mathcal{G}$ with values in the trivial $\mathcal{G}$-module S^1. Moreover, equivalent central extensions of the group $\mathcal{G}$ correspond to cohomologous cocycles in $H^2(\mathcal{G},S^1)$. So, the class $[c]$ of the cocycle c in $H^2(\mathcal{G},S^1)$ depends only on the equivalence class of the central extension $\tilde{\mathcal{G}}$ and we can identify the *set of equivalence classes of (topologically trivial) central extensions of the Lie group $\mathcal{G}$ with the cohomology $H^2(\mathcal{G},S^1)$.*

Bibliographic comments

The content of this Chapter is also of reference character and may be found in [31, 21, 22]. Central extensions and projective representations, together with cohomologies of Lie algebras and groups, will play an important role in the study of loop groups and diffeomorphism groups in Parts II and III.

Chapter 5

Grassmannians of a Hilbert space

In this Chapter we introduce infinite-dimensional Grassmann manifolds of closed subspaces in a Hilbert space H. We assume that H is polarized, i.e. decomposed into the direct sum of closed (infinite-dimensional) subspaces $H = H_+ \oplus H_-$, and consider Grassmannians, consisting of subspaces, "close" to H_+ in different senses. The most important case is the so called *Hilbert–Schmidt Grassmannian*, introduced in Sec. 5.2. It is a Hilbert Kähler manifold, which has many features of standard finite-dimensional Grassmannians. In particular, it is the homogeneous space of a Hilbert Lie group and can be provided with a natural determinant bundle, constructed in Sec. 5.3.

5.1 Grassmannian $\mathrm{Gr}_b(H)$

Let H be a complex (separable) Hilbert space. Suppose that H is *polarized*, i.e. it is provided with a decomposition into the direct orthogonal sum

$$H = H_+ \oplus H_- \tag{5.1}$$

of closed infinite-dimensional subspaces. Denote by pr_+ (resp. pr_-) the orthogonal projection $\mathrm{pr}_+ : H \to H_+$ (resp. $\mathrm{pr}_- : H \to H_-$).

We usually have in mind a standard example of such a polarized Hilbert space H, given by the Hilbert space $L^2_0(S^1, \mathbb{C})$ of L^2-functions on the unit circle S^1 with zero average value. Functions $f \in L^2_0(S^1, \mathbb{C})$ have Fourier decompositions, converging in L^2-sense, of the form

$$f(z) = \sum_{k=-\infty}^{+\infty} f_k z^k \ , \quad f_0 = 0 \ ,$$

where $z = e^{i\theta}$. For this particular realization of H we take for H_+ (resp. H_-) the subspace, consisting of the functions $f \in L^2_0(S^1, \mathbb{C})$, which have vanishing Fourier coefficients with negative (resp. positive) indices:

$$H_+ = \{f \in H : f(z) = \sum_{k=1}^{\infty} f_k z^k\} \ , \quad H_- = \{f \in H : f(z) = \sum_{k=-\infty}^{-1} f_k z^k\} \ .$$

Definition 30. The Grassmannian $\mathrm{Gr}_b(H)$ consists of all closed subspaces $W \subset H$, such that the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ is a Fredholm operator.

Recall that a linear operator $T : H_1 \to H_2$, mapping a Hilbert space H_1 into a Hilbert space H_2, is called *Fredholm*, if it has finite-dimensional kernel and cokernel. For such an operator one can define its *Fredholm index* by the formula

$$\operatorname{ind} T := \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T) \ .$$

The Fredholm index of T is a topological invariant of T, i.e. it does not change under bounded continuous deformations of T. An equivalent definition: an operator T is Fredholm, if it is invertible modulo compact operators, i.e. if there exists an operator $S : H_2 \to H_1$ such that the operators $\mathrm{id}_{H_1} - ST$ and $\mathrm{id}_{H_2} - TS$ are compact.

We can reformulate Def. 30 in an equivalent way as follows: a subspace $W \in \mathrm{Gr}_b(H)$ iff it coincides with the image of a bounded linear operator

$$w : H_+ \longrightarrow H \ ,$$

such that the operator $w_+ := \mathrm{pr}_+ \circ w$ is Fredholm.

With respect to the polarization $H = H_+ \oplus H_-$ any linear operator $w \in \operatorname{End} H$ can be written in the block form

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a : H_+ \to H_+ \ , & b : H_- \to H_+ \\ c : H_+ \to H_- \ , & d : H_- \to H_- \end{pmatrix} \ .$$

In these terms the subspace $W \in \mathrm{Gr}_b(H)$ iff a is Fredholm.

For any $W \in \mathrm{Gr}_b(H)$ denote by

$$U_W = \{W' \in \mathrm{Gr}_b(H) : \text{ the orthogonal projection } W' \to W \text{ is an isomorphism}\} \ .$$

We want to define the structure of a complex Banach manifold on $\mathrm{Gr}_b(H)$, for which the sets U_W will play the role of coordinate neighborhoods. More precisely, we have the following

Proposition 6. *$Gr_b(H)$ is a complex Banach manifold, having for its local model the Banach space $B(H_+, H_-)$ of bounded linear operators $w : H_+ \to H_-$. The coordinate neighborhoods*

$$U_W = \{W' \in Gr_b(H) : \textit{ the orthogonal projection } W' \to W \textit{ is an isomorphism}\} \ ,$$

introduced above, form an atlas of $Gr_b(H)$ and coordinate charts are given by the maps

$$U_W \ni W' \longmapsto w' \in B(W, W^\perp) \ .$$

Proof. The proof follows the proof of Proposition 7.1.2 from [65]. In order to show that the atlas $\{U_W\}$ with given charts does define on $\mathrm{Gr}_b(H)$ the structure of a complex Banach manifold, consider the intersection $U_{W_1} \cap U_{W_2} \neq \emptyset$ of two coordinate neighborhoods. The coordinate change in H, transforming the decomposition $H = W_1 \oplus W_1^\perp$ into the decomposition $H = W_2 \oplus W_2^\perp$, is given by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : W_1 \oplus W_1^\perp \to W_2 \oplus W_2^\perp \ ,$$

in which the operators a and d are Fredholm, while b and c are bounded. If a subspace $W \in U_{W_1} \cap U_{W_2}$, then it can be represented, on one hand, as the graph of a bounded operator $w_1 : W_1 \to W_1^\perp$, and, on the other hand, as the graph of a bounded operator $w_2 : W_2 \to W_2^\perp$. The orthogonal projection of W onto the subspaces W_1 and W_2 is an isomorphism, which defines an isomorphism $v : W_1 \to W_2$, so that W is the graph of the operator $w_2 \circ v : W_1 \to W_2^\perp$. It implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ w_1 \end{pmatrix} = \begin{pmatrix} v \\ w_2 \circ v \end{pmatrix}$$

as operators from W_1 to $W_2 \oplus W_2^\perp$. In other words, the coordinate change

$$B(W_1, W_1^\perp) \longrightarrow B(W_2, W_2^\perp) , \quad w_1 \longmapsto w_2 ,$$

which is given by the formula

$$w_2 = (c + dw_1)(a + bw_1)^{-1} ,$$

determines a holomorphic map, defined on the open subset $U_{W_1} \cap U_{W_2}$, identified with the subset $\{w_1 \in B(W_1, W_1^\perp) : a + bw_1 \text{ is invertible}\}$. □

Note that the manifold $\mathrm{Gr}_b(H)$ *has a countable number of connected components, numerated by the index of the Fredholm operator* w_+ for a subspace $W \in \mathrm{Gr}_b(H)$, coinciding with the image of a linear operator $w : H_+ \to H$. We say that the subspace W has the *virtual dimension* d, if the index of w_+ is equal to d.

5.2 Hilbert–Schmidt Grassmannian $\mathbf{Gr_{HS}}(H)$

Recall that a linear operator $T : H_1 \to H_2$, acting from a complex Hilbert space H_1 into another complex Hilbert space H_2, is called a *Hilbert–Schmidt operator*, if for some orthonormal basis $\{e_i\}$ in H_1 the series

$$\sum_i \|Te_i\| < \infty$$

is converging. Note that this condition is satisfied for any orthonormal basis in H_1, if it is satisfied for some orthonormal basis $\{e_i\}$ in H_1. We define the *Hilbert–Schmidt norm* of the operator T by the formula

$$\|T\|_2 = \left(\sum_{i=1}^{\infty} \|Te_i\|^2 \right)^{1/2} .$$

The *Hilbert–Schmidt operators* $T : H_1 \to H_2$ *form a complex Hilbert space* $\mathrm{HS}(H_1, H_2)$ with respect to the introduced norm. Moreover, the space $\mathrm{HS}(H, H)$ of Hilbert–Schmidt operators, acting in a Hilbert space H, is a two-sided ideal in the algebra $B(H)$ of all bounded linear operators in H.

Denote by $\mathrm{GL}(H)$ the group of all linear bounded operators in H, having a bounded inverse.

Definition 31. The *general linear Hilbert–Schmidt group* $\mathrm{GL_{HS}}(H)$ consists of linear operators $A \in \mathrm{GL}(H)$, such that in their block representation (with respect to polarization $H = H_+ \oplus H_-$)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the "off-diagonal" terms b and c are Hilbert–Schmidt operators (for brevity: HS-operators). We denote by $\mathrm{U_{HS}}(H)$ the intersection of the group $\mathrm{GL_{HS}}(H)$ with the group $\mathrm{U}(H)$ of all unitary operators in H.

In other words, the group $\mathrm{GL_{HS}}(H)$ consists of operators $A \in \mathrm{GL}(H)$, for which the "off-diagonal" terms b and c are "small" with respect to the "diagonal" terms a and d.

We introduce now the structure of a Banach Lie group on $\mathrm{GL_{HS}}(H)$. Namely, consider a subalgebra $B_{\mathrm{HS}}(H)$ of the algebra $B(H)$, consisting of operators of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(H) ,$$

for which the operators b and c are Hilbert–Schmidt. The algebra $B_{\mathrm{HS}}(H)$ is a Banach algebra with the norm, given by the formula

$$|||A||| := \|A\| + \|b\|_2 + \|c\|_2 .$$

The group $\mathrm{GL_{HS}}(H)$ coincides with the group of invertible elements of the algebra $\mathrm{B_{HS}}(H)$ and is a complex *Banach Lie group.* Accordingly, $\mathrm{U_{HS}}(H)$ is a real Banach Lie group, whose complexification coincides with $\mathrm{GL_{HS}}(H)$.

There is a Grassmann manifold $\mathrm{Gr_{HS}}(H)$, associated with the group $\mathrm{GL_{HS}}(H)$.

Definition 32. The *Hilbert–Schmidt Grassmannian* $\mathrm{Gr_{HS}}(H)$ is the set of all closed subspaces $W \subset H$, such that the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ is a Fredholm operator, and the orthogonal projection $\mathrm{pr}_- : W \to H_-$ is a Hilbert–Schmidt operator.

In other words, $\mathrm{Gr_{HS}}(H)$ consists of the subspaces $W \subset H$, which differ "little" from the subspace H_+ in the sense that $\mathrm{pr}_+ : W \to H_+$ is an "almost isomorphism" (recall that Fredholm operators are invertible modulo compact operators, cf. Sec. 5.1), and $\mathrm{pr}_- : W \to H_-$ is "small".

Equivalently, a subspace $W \in \mathrm{Gr_{HS}}(H)$ iff it coincides with the image of a linear operator

$$w : H_+ \longrightarrow H ,$$

such that the operator $w_+ := \mathrm{pr}_+ \circ w$ is Fredholm, and $w_- := \mathrm{pr}_- \circ w$ is Hilbert–Schmidt.

It's easy to see that if $W \in \mathrm{Gr_{HS}}(H)$, then the graph of any HS-operator $w' : W \to W^\perp$ also belongs to $\mathrm{Gr_{HS}}(H)$. We denote the set of all such subspaces by U_W:

$$U_W = \{W' \in \mathrm{Gr_{HS}}(H) : W' \text{ is the graph of an HS-operator } w' : W \to W^\perp\} .$$

As in Sec. 5.1, this definition can be rewritten in the form

$$U_W = \{W' \in \mathrm{Gr_{HS}}(H) : \text{the orthogonal projection } W' \to W \text{ is an isomorphism}\} .$$

The group $\mathrm{GL}_{\mathrm{HS}}(H)$, introduced above, acts in a natural way on $\mathrm{Gr}_{\mathrm{HS}}(H)$. Consider, in particular, the action of its unitary subgroup $\mathrm{U}_{\mathrm{HS}}(H)$ on $\mathrm{Gr}_{\mathrm{HS}}(H)$ and show that it is transitive. It will allow us to obtain a realization of $\mathrm{Gr}_{\mathrm{HS}}(H)$ as a homogeneous space of the group $\mathrm{U}_{\mathrm{HS}}(H)$, analogous to the realization of the finite-dimensional Grassmannian as a homogeneous space of the unitary group.

To prove that the action of $\mathrm{U}_{\mathrm{HS}}(H)$ on $\mathrm{Gr}_{\mathrm{HS}}(H)$ is transitive, we should construct for a given subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ an operator $A \in \mathrm{U}_{\mathrm{HS}}(H)$ such that $A(H_+) = W$. Consider an isometric operator $w : H_+ \to H$, which has the image, equal to W, and denote by $w^\perp : H_- \to H$ an isometric operator with the image $W^\perp$. Then the operator

$$A = w \oplus w^\perp : H = H_+ \oplus H_- \to H = W \oplus W^\perp$$

defines an isometry of H onto itself and so is unitary. Moreover, it maps H_+ onto W and has the block representation of the form

$$A = \begin{pmatrix} w_+ & w_+^\perp \\ w_- & w_-^\perp \end{pmatrix} .$$

Here, the operator w_+ is Fredholm, and w_- is Hilbert–Schmidt, because $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$. Since A is also unitary, it follows that $A \in \mathrm{U}_{\mathrm{HS}}(H)$.

The isotropy subgroup of $\mathrm{U}_{\mathrm{HS}}(H)$ at $H_+ \in \mathrm{Gr}_{\mathrm{HS}}(H)$ coincides with $\mathrm{U}(H_+) \times \mathrm{U}(H_-)$, hence we have the following

Proposition 7. *The Grassmannian $Gr_{HS}(H)$ is a homogeneous space of the group $U_{HS}(H)$ of the form*

$$Gr_{HS}(H) = U_{HS}(H)/\, U(H_+) \times U(H_-) .$$

The Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ has the structure of a complex Hilbert manifold, defined in the following way.

Proposition 8. *The Grassmannian $Gr_{HS}(H)$ is a complex Hilbert manifold, having for its local model the Hilbert space of Hilbert Schmidt operators $HS(H_+, H_-)$. The coordinate neighborhoods*

$$U_W = \{W' \in Gr_{HS}(H) : W' \text{ is the graph of an HS-operator } w' : W \to W^\perp\}$$

form an atlas for $Gr_{HS}(H)$, and the coordinate charts are given by the maps

$$U_W \ni W' \longmapsto w' \in HS(W, W^\perp) .$$

This Proposition is proved in the same way, as Prop. 6 from Sec. 5.1.

There is another atlas on $\mathrm{Gr}_{\mathrm{HS}}(H)$, which is more natural in some sense. To construct it, we identify H with the Hilbert space $L^2(S^1, \mathbb{C})$. This space has a canonical basis, given by $\{z^k\}$, $k \in \mathbb{Z}$. The subspace H_+ is generated by the elements $\{z^k\}$, $k \in \mathbb{Z}_+$, and H_- by the elements $\{z^k\}$, $k \in \mathbb{Z}_-$, where we denote by $\mathbb{Z}_+$ the subset of nonnegative integers in $\mathbb{Z}$, and by $\mathbb{Z}_-$ its complement in $\mathbb{Z}$.

We take for "coordinate" subspaces in H the closed linear subspaces $H_S \subset H$, generated by vectors $\{z^s\}$, $s \in S$, which are numerated by the subsets $S \subset \mathbb{Z}$, comparable with $\mathbb{Z}_+$. We say that a subset $S \subset \mathbb{Z}$ is *comparable with* $\mathbb{Z}_+$, if the sets $S - \mathbb{Z}_+$ and $\mathbb{Z}_+ - S$ consist of finite number of points. The ensemble of all such subsets $S \subset \mathbb{Z}$ is denoted by $\mathcal{S}$, and the number $|S - \mathbb{Z}_+| - |\mathbb{Z}_+ - S|$ is called the *virtual cardinality* of S. Note that the virtual dimension of the subspace H_S is equal precisely to the virtual cardinality of S.

Lemma 2. *For any $W \in Gr_{HS}(H)$ there exists a subset $S \in \mathcal{S}$, such that the orthogonal projection*

$$pr_S : W \longrightarrow H_S$$

is an isomorphism.

Proof. Indeed, if $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$, then the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ has finite-dimensional kernel and cokernel, so there exists a subset $S_0 \in \mathcal{S}$, containing $\mathbb{Z}_+$, for which the orthogonal projection

$$\mathrm{pr} : W \longrightarrow H_{S_0}$$

is injective. If it's not surjective, then one can find an $s \in S_0$, such that z^s does not belong to $\mathrm{pr}(W)$. In this case we replace S_0 with $S_1 := S_0 \setminus \{s\}$. The projection $\mathrm{pr} : W \to H_{S_1}$ still remains injective. If it's not surjective, we repeat the described procedure. Since the complement of $\mathrm{pr}_+(W)$ in H_+ is finite-dimensional, after a finite number of steps we shall arrive to a subset S, for which the projection $\mathrm{pr}_S : W \to H_S$ is an isomorphism. □

Based on the above Lemma, we can define an atlas on $\mathrm{Gr}_{\mathrm{HS}}(H)$, formed by the open sets $\{U_S\}_{S\in\mathcal{S}}$, where the coordinate neighborhood $U_S = U_{H_S}$ consists of the subspaces, which are the graphs of Hilbert–Schmidt operators $H_S \to H_S^\perp = H_{S^\perp}$ with $S^\perp = \mathbb{Z} - S$.

Since $\mathrm{U}_{\mathrm{HS}}(H)$ acts transitively on the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, one can construct an $\mathrm{U}_{\mathrm{HS}}(H)$-invariant Kähler metric on $\mathrm{Gr}_{\mathrm{HS}}(H)$ from an inner product on the tangent space $T_{H_+}\mathrm{Gr}_{\mathrm{HS}}(H)$ at the origin $H_+ \in \mathrm{Gr}_{\mathrm{HS}}(H)$, invariant under the action of the isotropy subgroup $\mathrm{U}(H_+) \times \mathrm{U}(H_-)$.

The tangent space $T_{H_+}\mathrm{Gr}_{\mathrm{HS}}(H)$ coincides with the Hilbert space of Hilbert–Schmidt operators $\mathrm{HS}(H_+, H_-)$, and an invariant inner product on it can be given by the formula

$$(A, B) \longmapsto \mathrm{Re}\,\{\mathrm{tr}(AB^*)\}\,, \quad A, B \in \mathrm{HS}(H_+, H_-)\ .$$

The imaginary part of the complex inner product $\mathrm{tr}(AB^*)$:

$$\omega(A, B) := \mathrm{Im}\,\{\mathrm{tr}(AB^*)\}$$

defines a non-degenerate invariant 2-form on $T_{H_+}\mathrm{Gr}_{\mathrm{HS}}(H)$, which extends to an $\mathrm{U}_{\mathrm{HS}}(H)$-invariant symplectic form on $\mathrm{Gr}_{\mathrm{HS}}(H)$.

This defines on $\mathrm{Gr}_{\mathrm{HS}}(H)$ a *Kähler structure*, making $\mathrm{Gr}_{\mathrm{HS}}(H)$ into a *Kähler Hilbert manifold.*

We shall use in Ch. 9 the "smooth" part $\mathrm{Gr}^\infty(H)$ of the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, which can be defined in terms of the open covering $\{U_S\}_{S\in\mathcal{S}}$ in the following way.

Definition 33. The Grassmannian $\mathrm{Gr}^\infty(H)$ consists of the graphs of all bounded linear operators $w : H_S \to H_S^\perp$, $S \in \mathcal{S}$, whose matrix components w_{pq} with $p \in \mathbb{Z} - S$, $q \in S$ are rapidly decreasing, i.e. the quantities $|p - q|^r w_{pq}$ are bounded for each r.

5.3 Plücker embedding and determinant bundle

As in the finite-dimensional case, the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ may be realized, with the help of the Plücker embedding, as a submanifold in a projective Hilbert space.

In order to define this Plücker embedding, we introduce a notion of an admissible basis for a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$. Suppose that $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ has the virtual dimension d. A model example for such a subspace in the case of $H = L^2_0(S^1, \mathbb{C})$ is the subspace $W = z^{-d}H_+$.

Definition 34. A basis in W, consisting of elements $\{w_k\}_{k\geq -d}$, is called *admissible* if:

1. the linear map
$$w : z^{-d}H_+ \longrightarrow W \ ,$$
defined on the basis elements $\{z^k\}_{k\geq -d}$ by the formula $z^k \mapsto w_k$, is a continuous isomorphism;

2. the composition of w with the orthogonal projection onto the subspace $z^{-d}H_+$:
$$\mathrm{pr} \circ w : z^{-d}H_+ \longrightarrow z^{-d}H_+$$
is an operator with determinant.

We recall the definitions of the class Tr of operators with trace and related class $\mathrm{Det} = 1 + \mathrm{Tr}$ of operators with determinant. A linear operator $T : H_1 \to H_2$, acting from a Hilbert space H_1 into a Hilbert space H_2, is called an *operator with trace* or an *operator of trace class*, if for some orthonormal bases $\{e_i\}$ in the space H_1 and $\{f_i\}$ in the space H_2 the series
$$\sum_i (Te_i, f_i)$$
converges. If this condition is satisfied for some orthonormal bases in H_1 and H_2, then it is fulfilled also for any orthonormal bases $\{e_i\}$ in H_1 and $\{f_i\}$ in H_2 and the sum
$$\sum_i (Te_i, f_i)$$
does not depend on the choice of bases. It is called the *trace* of the operator T and denoted by $\mathrm{Tr}\, T$. Operators $T : H \to H$ of trace class, acting in a Hilbert space H, form a two-sided ideal $\mathrm{Tr}(H, H)$ in the algebra $B(H)$ of all bounded linear operators in H, which is contained in the ideal $\mathrm{HS}(H, H)$ of Hilbert–Schmidt operators. Moreover, it's easy to see that the product of two Hilbert–Schmidt operators from $\mathrm{HS}(H, H)$ belongs to $\mathrm{Tr}(H, H)$, being an operator of trace class. The trace of an operator $T \in \mathrm{Tr}(H, H)$ coincides with the sum of its eigenvalues
$$\mathrm{tr}\, T = \sum_i \lambda_i(T) \ .$$
and behaves like the matrix trace.

If $T : H \to H$ is an operator of the trace class, then one can define for the operator $I - T$, where I is the identity operator, its *determinant* by

$$\det(I - T) := \prod_i (1 - \lambda_i(T)) \ .$$

The product in the right hand side is converging, since for an operator $T : H \to H$ of the trace class the sum $\sum_i |\lambda_i(T)|$ is always finite. Operators of the form $A = I - T$, where $T \in \mathrm{Tr}(H, H)$, are called the *operators with determinant* or *operators of determinant class*, and the set of such operators is denoted by $\mathrm{Det}(H, H)$. It's clear that the class $\mathrm{Det}(H, H)$ is closed under the product of operators.

Coming back to the Def. 34, note that the second condition in this definition means that the isomorphism w is "sufficiently close" to the identity. Moreover, it implies that the orthogonal projection $\mathrm{pr}_S \circ w : z^{-d}H_+ \to H_S$ onto any subspace H_S of virtual dimension d has a determinant, and any two admissible bases in a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ are related by the change of variables, which has a determinant.

Using the notion of the admissible basis, we can define the Plücker coordinate of a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$.

Definition 35. Let W be a subspace of virtual dimension d, having an admissible basis w. The *Plücker coordinate* of W is a function of $S \in \mathcal{S}$ of the following form

$$\pi_S(w) = \begin{cases} \det(\mathrm{pr}_S \circ w) & \text{for } S \in \mathcal{S} \text{ of virtual cardinality } d \ , \\ 0 & \text{for } S \in \mathcal{S} \text{ of any virtual cardinality, other than } d \ . \end{cases}$$

If w' is another admissible basis in W, then

$$\pi_S(w') = \Delta_{ww'} \pi_S(w) \ ,$$

where $\Delta_{ww'}$ is the determinant of the change of variables, relating w with w'. Hence, the *projective class* $[\pi_S(w)]$ *does not depend on the choice of an admissible basis* w *in the subspace* W *and is uniquely determined by the subspace itself.*

In terms of the Plücker coordinate the neighborhoods U_S may be redefined as follows:

$$W \in U_S \Longleftrightarrow \pi_S(w) \neq 0 \text{ for any admissible basis } w \text{ in } W \ .$$

Proposition 9. *The Plücker map*

$$\pi : Gr_{HS}(H) \longrightarrow P(\mathcal{H}) \ , \quad W \longmapsto [\pi_S(w)]_{S \in \mathcal{S}} \ ,$$

determines a holomorphic embedding of the Grassmannian $Gr_{HS}(H)$ *into the projectivization of the Hilbert space* $\mathcal{H} = l^2(\mathcal{S})$.

We omit the proof of this assertion (it may be found in [65], Prop. 7.5.2), and only note that it is based on the relation

$$\sum_{S \in \mathcal{S}} |\pi_S(w)|^2 = \det(w^* w) < \infty \ , \tag{5.2}$$

satisfied for any admissible basis w in $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$.

We shall construct now a holomorphic line bundle over $\mathrm{Gr_{HS}}(H)$, being an analogue of the determinant bundle over the finite-dimensional Grassmannian.

Let a subspace $W \in \mathrm{Gr_{HS}}(H)$ has the virtual dimension d. Consider the linear space, consisting of formal semi-infinite forms of the type

$$[\lambda, w] := \lambda w_{-d} \wedge w_{-d+1} \wedge \dots \ ,$$

where $\lambda \in \mathbb{C}$, $w = \{w_k\}_{k \geq -d}$ is an admissible basis in W. If w' is another admissible basis in W, then we shall identify the pair $[\lambda', w']$ with the pair $[\lambda, w]$, if $\lambda' = \lambda \Delta_{ww'}$, where $\Delta_{ww'}$ is the determinant of the change of variables, relating w with w'.

The linear space $\mathrm{Det}\, W$, obtained by taking the quotient of the space of semi-infinite forms of the type $[\lambda, w]$ with respect to the above equivalence relation, is a complex line.

We denote by Det the union of spaces $\mathrm{Det}\, W$ over all $W \in \mathrm{Gr_{HS}}(H)$.

Proposition 10. *The natural projection*

$$Det \longrightarrow Gr_{HS}(H)$$

is a holomorphic line bundle.

This Proposition follows from the fact that the restriction of Det to any coordinate neighborhood U_S is trivial and the transition function for $U_{S_1} \cap U_{S_2} \neq \emptyset$ is given (in the notation of Sec. 5.1) by the formula

$$[\lambda_1, w_1] \longmapsto [\lambda_2, w_2] \ ,$$

where

$$w_2 = (c + dw_1)(a + bw_1)^{-1}, \quad \lambda_2 = \lambda \det(a + bw_1) \ .$$

This defines the structure of a holomorphic line bundle on Det.

We add several comments on the Plücker embedding and determinant bundle.

Remark 6. The bundle Det can be provided with a natural *Hermitian metric*, given by

$$\|[\lambda, w]\|^2 := |\lambda|^2 \det(w^* w)^2 \ .$$

Remark 7. The Plücker embedding $\pi : \mathrm{Gr_{HS}}(H) \to P(\mathcal{H})$ may be pulled up to a holomorphic map

$$\tilde{\pi} : \mathrm{Det} \to \mathcal{H} \ ,$$

which is linear on the fibres, so that the *bundle Det will coincide with the inverse image of the tautological line bundle over $P(\mathcal{H})$ with respect to the embedding π.* Moreover, the pulled back map $\tilde{\pi} : \mathrm{Det} \to \mathcal{H}$ will preserve the norms (it follows from the relation (5.2) above).

Remark 8. The holomorphic line bundle Det has no non-trivial (global) sections, on the contrary, the dual bundle Det^* has many such sections. For example, all Plücker coordinates π_S determine holomorphic sections of Det^*. Indeed, the formula $[\lambda, w] \mapsto \lambda \pi_S(w)$ defines a holomorphic function $\mathrm{Det} \to \mathbb{C}$, which is linear on fibres, and induces a global holomorphic section of Det^*.

Note also that the symplectic form of the manifold $\mathrm{Gr_{HS}}(H)$, constructed in Sec. 5.2, represents the Chern class of the complex line bundle $\mathrm{Det} \to \mathrm{Gr_{HS}}(H)$. Otherwise speaking, it is *induced by the Fubini-Study form on $P(\mathcal{H})$ under the Plücker embedding $\pi : \mathrm{Gr_{HS}}(H) \to P(\mathcal{H})$.*

Bibliographic comments

A key reference to this Chapter is the Pressley–Segal book [65]. Most of the assertions are taken from Ch.7 of [65].

Chapter 6

Quasiconformal maps

In this Chapter we introduce quasiconformal maps and prove main existence and uniqueness theorems for such maps. The quasiconformal maps will play a crucial role in Ch. 11, where we study the universal Teichmüller space. For a detailed exposition of the theory of quasiconformal maps cf. [1, 49].

6.1 Definition and basic properties

Let $w : D \to w(D)$ be a homeomorphism, mapping a domain D in the Riemann sphere $\overline{\mathbb{C}}$ onto another domain $w(D)$ in $\overline{\mathbb{C}}$.

Definition 36. Suppose that $w : D \to w(D)$ is a homeomorphism and w has locally L^1-integrable derivatives (in the generalized sense) in D. Then w is called *quasiconformal*, if there exists a measurable complex-valued function $\mu \in L^\infty(D)$ with

$$\|\mu\|_\infty := \text{ess sup}_{z\in D}|\mu(z)| =: k < 1 , \tag{6.1}$$

such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{6.2}$$

holds for almost all $z \in D$.

The function $\mu = \mu_w$ is called the *Beltrami differential* or the *complex dilatation of* w, and the constant k is often indicated in the name of the k-*quasiconformal maps.*

In particular, for $k = 0$ the homeomorphism w determines a conformal map from D onto $w(D)$. For diffeomorphisms w the quasiconformality of w means that infinitesimally it transforms small circles into ellipses, whose eccentricities (the ratio of the large axis to the small one) are bounded by a common constant $K < \infty$, related to the above constant $k = \|\mu\|_\infty$ by the formula

$$K = \frac{1+k}{1-k} \ .$$

The least possible constant K is called the *maximal dilatation* of w and is often included in the name of the K-*quasiconformal maps.*

The term "Beltrami differential" for the complex dilatation μ is motivated by the behavior of μ under conformal changes of variables. Namely, it follows from (6.2) that for a conformal change of variables f we should have

$$\mu(f(z)) = \mu(z)\frac{\overline{f_z(z)}}{f_z(z)}$$

for almost all $z \in D$. In general, we call a functional φ_w, defined on complex-valued functions w, a *differential of type* (m,n) with $m,n \in \mathbb{Z}$, if the quantity $\varphi_w(z)dz^m d\bar{z}^n$ remains invariant under conformal changes of variables. In the sense of this definition the complex dilatation μ_w is a differential of type $(-1,1)$.

The inverse of a K-quasiconformal map f is again K-quasiconformal. The composition of a K_1-quasiconformal map f with a K_2-quasiconformal map g is a (K_1K_2)-quasiconformal map. This composition property may be deduced from the chain rule for Beltrami differentials. Namely, if f and g are two quasiconformal maps of a domain D with Beltrami differentials μ_f and μ_g respectively, then the following chain rule holds

$$\mu_{f\circ g^{-1}}(g(z)) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \cdot \frac{g_z(z)}{\overline{g_z(z)}} \ , \tag{6.3}$$

for almost all $z \in D$. In particular,

$$\mu_{g^{-1}}(g(z)) = -\mu_g(z) \cdot \frac{g_z(z)}{\overline{g_z(z)}} \ ,$$

so $|\mu_{g^{-1}}(g(z))| = |\mu_g(z)|$ for almost all $z \in D$.

From the chain rule (6.3) we can deduce the following transformation property of Beltrami differentials μ_w with respect to compositions of w with conformal maps f. If f is a conformal map (i.e. $\mu_f \equiv 0$), then

$$\mu_{f\circ w}(z) \equiv \mu_w(z) \ , \quad \mu_{w\circ f} = (\mu_w \circ f)\,\frac{\overline{f_z}}{f_z} \ .$$

These transformation rules for Beltrami differentials imply the following uniqueness property of solutions of the equation (6.2).

Proposition 11. *Suppose that two quasiconformal homeomorphisms w_1 and w_2 in a domain D satisfy the same Beltrami equation*

$$w_{\bar{z}} = \mu w_z$$

for almost all $z \in D$, where μ is a Beltrami differential in D, satisfying the condition (6.1). Then $w_1 \circ w_2^{-1}$ and $w_2 \circ w_1^{-1}$ are conformal. Conversely, the composition $f \circ w_1$ with any conformal map f, defined on $w_1(D)$, satisfies the same Beltrami equation, as w_1.

Quasiconformal homeomorphisms have a good behavior at the boundary, according to the following

Theorem 3 (Mori (cf. [1])). *Let $w : \Delta \to \Delta$ be a K-quasiconformal homeomorphism of the unit disc onto itself, normalized by the condition: $w(0) = 0$. Then the following sharp estimate*

$$|w(z_1) - w(z_2)| < 16|z_1 - z_2|^{1/K}$$

holds for any $z_1 \neq z_2 \in \Delta$. In other words, the homeomorphism w satisfies the Hölder condition of order $1/K$ in the disc Δ.

Mori's theorem implies, in particular, that w extends to a homeomorphism of the closed unit disc $\overline{\Delta}$. Another corollary of Mori's theorem is that K-quasiconformal homeomorphisms w of the unit disc Δ onto itself, normalized by the condition $w(0) = 0$, form a compact family with respect to the topology of normal convergence (i.e. uniform convergence on compact subsets). This result easily extends to general domains $D \subset \overline{\mathbb{C}}$.

Proposition 12. *Consider the family of all K-quasiconformal maps in D, normalized by the condition that any map in the family sends two fixed distinct points $z_1, z_2 \in D$ to another two fixed distinct points ζ_1, ζ_2. Then this family is compact with respect to the topology of normal convergence and any map w in this family satisfies the Hölder condition*

$$|w(z_1) - w(z_2)| < A|z_1 - z_2|^{1/K}$$

on any compact subset in D, where the constant A depends only on K and the compact subset.

In particular, any quasiconformal homeomorphism $w : D_1 \to D_2$ extends to a homeomorphism $w : \bar{D}_1 \to \bar{D}_2$ of the closures and so defines a homeomorphism of the boundaries.

We can ask the converse question: when a given homeomorphism $w : \partial D_1 \to \partial D_2$ extends to a quasiconformal homeomorphism $D_1 \to D_2$. It's convenient to study this problem first in the partial case, when both domains coincide with the upper half-plane: $D_1 = D_2 = H$.

Suppose that $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a monotone-increasing homeomorphism of the extended real line $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$, satisfying the condition: $f(\infty) = \infty$. We call it *quasisymmetric*, if there exists a constant $A > 0$, such that the following finite-difference condition

$$\frac{1}{A} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq A \tag{6.4}$$

is satisfied for all $x \in \mathbb{R}$ and all $t > 0$.

This condition can be considered as a variant of the cross ratio condition for quadruples of points. Recall that the *cross ratio* of four different points z_1, z_2, z_3, z_4 on the complex plane is given by the quantity

$$\rho = \rho(z_1, z_2, z_3, z_4) := \frac{z_4 - z_1}{z_4 - z_2} : \frac{z_3 - z_1}{z_3 - z_2} \, .$$

The equality of two cross ratios $\rho(z_1, z_2, z_3, z_4) = \rho(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is a necessary and sufficient condition for the existence of a fractional-linear map of the complex plane, transforming the quadruple z_1, z_2, z_3, z_4 into the quadruple $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. In the case of quasiconformal maps the cross ratios of quadruples may change but in a controlled way. The quasisymmetricity condition (6.4) expresses this control in a convenient form. Namely, we choose for a given homeomorphism $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ a quadruple of points on $\overline{\mathbb{R}}$ in the form $\vec{x} := (x-t, x, x+t, \infty)$ with the cross ratio $\rho(\vec{x}) =: \rho$ and associate with it the quantity

$$M(\rho(\vec{x})) = M(\rho) := \frac{\rho}{1-\rho} .$$

If, in particular, $\rho = 1/2$, then $M(\rho) = 1$. In this case the condition (6.4) means that the corresponding cross ratio of the quadruple $f(\vec{x}) := (f(x-t), f(x), f(x+t), \infty)$ satisfies the inequality

$$\frac{1}{A} \leq M(\rho(f(\vec{x})) \leq A .$$

The same condition in terms of $\rho(f(\vec{x}))$ can be rewritten as

$$\frac{1}{A+1} \leq \rho(f(\vec{x})) \leq \frac{A}{A+1}$$

or as

$$\frac{1}{2} - \epsilon \leq \rho(f(\vec{x})) \leq \frac{1}{2} + \epsilon ,$$

where $\epsilon = \epsilon(A) := \frac{1}{2} - \frac{1}{A+1}$.

Theorem 4 (Beurling–Ahlfors (cf. [1, 49])). *Suppose that $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a monotone-increasing homeomorphism of the extended real line $\overline{\mathbb{R}}$ onto itself, satisfying the condition: $f(\infty) = \infty$. Then it can be extended to a quasiconformal homeomorphism $w : H \to H$ if and only if f is quasisymmetric, i.e. if there exists a constant $A > 0$, such that*

$$\frac{1}{A} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq A$$

for all $x \in \mathbb{R}$, $t > 0$.

We have already explained above, where the necessity of the condition (6.4) comes from. The sufficiency of this condition is based on the following remarkable *Beurling–Ahlfors formula*, which gives a quasiconformal extension w to H of the quasisymmetric homeomorphism f:

$$w(x+iy) = \frac{1}{2} \int_0^1 (f(x+ty) + f(x-ty))\, dt + i \int_0^1 (f(x+ty) - f(x-ty))\, dt$$

for $x + it \in H$.

We formulate also an analogue of the above Theorem for the case of the circle S^1. We say that an orientation-preserving homeomorphism $f : S^1 \to S^1$ is *quasisymmetric*, if it satisfies for some $0 < \epsilon < 1$ the inequality

$$\frac{1}{2}(1-\epsilon) \leq \rho(f(z_1), f(z_2), f(z_3), f(z_4)) \leq \frac{1}{2}(1+\epsilon) \tag{6.5}$$

for any quadruple $z_1, z_2, z_3, z_4 \in S^1$ with cross ratio $\rho(z_1, z_2, z_3, z_4) = \frac{1}{2}$.

An analogue of the Beurling–Ahlfors theorem for S^1 asserts that *an orientation-preserving homeomorphism $f : S^1 \to S^1$ can be extended to a quasiconformal homeomorphism $w : \Delta \to \Delta$ if and only if it is quasisymmetric.* Douady and Earle (cf. [19]) have found an explicit extension operator E, which assigns to a quasisymmetric homeomorphism f its extension to a quasiconformal homeomorphism w of Δ and is conformally invariant in the sense that $E(w \circ f) = w \circ E(f)$ for any fractional-linear automorphism of Δ.

The image C of the circle S^1 under a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ is called a *quasicircle* and the domains D_1, D_2, complementary to C in $\overline{\mathbb{C}}$, are called *quasidiscs*. All quasicircles have zero area and their Hausdorff dimension is always less than 2. However, it can be equal to any λ with $1 \leq \lambda < 2$ (cf. [24]).

Remark 9. There is a natural description of quasicircles in terms of quasiconformal reflections. Recall that a *reflection* across a Jordan curve C on $\overline{\mathbb{C}}$, dividing $\overline{\mathbb{C}} \setminus C$ into two domains D_1, D_2, is an orientation-preserving involutive homeomorphism φ of $\overline{\mathbb{C}}$, which maps D_1 onto D_2 (and vice versa) and fixes every point of C. The quasicircles are characterized by the following

Proposition 13. *A Jordan curve C on $\overline{\mathbb{C}}$ is a quasicircle if and only if it admits a quasiconformal reflection across it.*

We omit the proof of this Proposition, referring to the book [49], Theor. 6.1.

There is a simple geometric criterion for the quasicircles, passing through $\infty \in \overline{\mathbb{C}}$. Namely, a Jordan curve C, passing through ∞, is a quasicircle if and only if there exists a constant $c > 0$, for which the following condition is satisfied: for any three finite points z_1, z_2, z_3 on C, such that z_2 lies between z_1 and z_3, we have an inequality

$$|z_1 - z_2| + |z_2 - z_3| < c|z_1 - z_3|$$

(cf. [1, 49]).

6.2 Existence of quasiconformal maps

A key role in the theory of quasiconformal maps is played by the following existence theorem for solutions of the Beltrami equation (6.2).

Theorem 5 (Existence theorem). *For any measurable function μ in a domain $D \subset \overline{\mathbb{C}}$, such that $\|\mu\|_\infty = k < 1$, there exists a quasiconformal map on D, whose complex dilatation agrees with μ almost everywhere on D. In other words, there exists a solution w of the Beltrami equation*

$$w_{\bar{z}} = \mu w_z \ ,$$

satisfied for almost all $z \in D$.

As we have already pointed out earlier (cf. Prop. 11 in Sec. 6.1), any other solution $\tilde{w}$ of the above Beltrami equation has the form

$$\tilde{w} = w \circ f \ ,$$

where f is a conformal map.

The existence theorem implies the following generalization of the Riemann mapping theorem: *Let D_1 and D_2 be two domains in $\overline{\mathbb{C}}$, whose boundaries consist of more than one point. If μ is a measurable function on D_1 with $\|\mu\|_\infty < 1$, then there exists a quasiconformal map of D_1 onto D_2, whose complex dilatation agrees with μ almost everywhere.*

Proof. A detailed proof of Theorem 5 is given in [1], here we only point out its main points. First of all, it's sufficient to prove the existence theorem for the whole plane, since any $\mu \in L^\infty(D)$ with $\|\mu\|_\infty < 1$ can be extended (by setting it equal to zero outside D) to the whole plane, preserving the estimate $\|\mu\|_\infty < 1$.

Starting the proof of the existence theorem for the complex plane, we restrict first to the case, when the complex dilatation μ has a compact support.

We show under this hypothesis that *there exists a unique solution of the Beltrami equation* (6.2)*:*

$$w_{\bar{z}} = \mu w_z \ ,$$

satisfying the conditions:

$$w(0) = 0 \quad \textit{and} \quad w_z - 1 \in L^p \ ,$$

where $p > 2$ is a number, sufficiently close to 2, which will be chosen later.

Introduce the *Cauchy–Green operator*

$$Ph(\zeta) := -\frac{1}{\pi} \int h(z) \left(\frac{1}{z-\zeta} - \frac{1}{z} \right) dx\, dy \ ,$$

where the integral is taken over the complex plane. This operator is correctly defined for functions $h \in L^p$ with $p > 2$ and determines a continuous function (the function $Ph(\zeta)$ is even Hölder-continuous in ζ with Hölder exponent $1 - \frac{2}{p}$).

The partial derivatives of Ph (in the generalized sense) satisfy the equations

$$(Ph)_{\bar{z}} = h \ , \quad (Ph)_z = Th \ ,$$

where T is the *Calderon–Zygmund integral operator*, defined by

$$Th(\zeta) := -\frac{1}{\pi} P.V. \int h(z) \frac{1}{(z-\zeta)^2} dx\, dy \ .$$

Here the integral is taken in the *principal value sense*, i.e.

$$Th(\zeta) := -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|z-\zeta| > \epsilon} h(z) \frac{1}{(z-\zeta)^2} dx\, dy \ .$$

The operator Th is correctly defined on functions h of class C_0^2 (i.e. C^2-smooth with compact supports). For such h, the function $Th(\zeta)$ is C^1-smooth. The operator T is also isometric in L^2-sense, i.e.

$$\|Th\|_2 = \|h\|_2 \ .$$

It follows that it can be extended to a bounded linear operator on L^2. Moreover, it can be proved, using the Calderon–Zygmund inequality, that T is bounded on functions $h \in L^p$ with $p > 1$:

$$\|Th\|_p \leq C_p \|h\|_p \ ,$$

and $C_p \to 1$ for $p \to 2$. We choose now $p > 2$ in such a way that the inequality $\|\mu\|_\infty C_p < 1$ is satisfied.

We return to the construction of a solution w of the Beltrami equation (6.2), satisfying the conditions: $w(0) = 0$ and $w_z - 1 \in L^p$.

We show first that there could be only one such solution. Suppose that w is such a solution and consider the function

$$W := w - P(w_{\bar{z}}) \ .$$

Then its partial derivative with respect to $\bar{z}$ is equal to zero, hence W is an entire function. On the other hand, the condition $w_z - 1 \in L^p$ implies that the derivative of W, equal to $W' = w_z - T(w_{\bar{z}})$, satisfies the condition $W' - 1 \in L^p$, since $w_{\bar{z}} = \mu w_z$ belongs to L^p. This is possible only if $W' \equiv 1$, i.e. $W(z) \equiv z + \text{const}$. The constant is equal to zero because of the normalization, so $W(z) \equiv z$ and

$$w = P(w_{\bar{z}}) + z \ .$$

By differentiating this equality in z, we get for w_z an integral equation

$$w_z = T(w_{\bar{z}}) + 1 = T(\mu w_z) + 1 \ ,$$

in which the operator $h \mapsto T(\mu h)$ is contractible, since

$$\|T \circ \mu\|_p \leq \|\mu\|_\infty C_p < 1 \ .$$

Suppose now that $\tilde{w}$ is another solution of (6.2), satisfying the conditions $\tilde{w}(0) = 0$ and $\tilde{w}_z - 1 \in L^p$. Then $\tilde{w} - w$ satisfies the equation

$$\tilde{w}_z - w_z = T(\mu(\tilde{w}_z - w_z))$$

which implies, because of the uniqueness of its solution, that $\tilde{w}_z = w_z$ almost everywhere. It follows from the Beltrami equation that also $\tilde{w}_{\bar{z}} = w_{\bar{z}}$ almost everywhere. Hence, $\tilde{w} - w$ is a constant, which is equal to zero, due to the normalization.

To prove the existence of a solution w of (6.2), satisfying the conditions $w(0) = 0$ and $w_z - 1 \in L^p$, we use the integral equation

$$h = T(\mu h) + T\mu \ .$$

Its unique L^p-solution yields a desired solution of the Beltrami equation (6.2), given by the formula

$$w = P(\mu(h + 1)) + z \ . \tag{6.6}$$

Indeed, since $\mu(h + 1) \in L^p$ (recall that μ has a compact support), the function $P(\mu(h+1))$ is correctly defined and continuous. The derivatives of w (in the generalized sense) are equal to

$$w_{\bar{z}} = \mu(h + 1) \ , \quad w_z = T(\mu(h + 1)) + 1 = h + 1$$

and $w_z - 1 = h \in L^p$. Hence, w, given by (6.6), satisfies the equation (6.2) and additional conditions $w(0) = 0$ and $w_z - 1 \in L^p$. According to the uniqueness assertion in Prop. 11, the constructed solution w of (6.2) will be uniquely defined, if we suppose additionally that it fixes not only the origin 0, but also two other points, say, $z = 1$ and $z = \infty$. We denote such a normalized solution by $w[\mu]$.

To end the proof, we should get rid of the compactness of the support of the complex dilatation μ. This can be done, using the following trick from [1], Sec. VB.

Note that the case, when $\mu \equiv 0$ in a neighborhood of 0, which is opposite to the case, when μ has a compact support, can be settled down by the reflection with respect to the unit circle S^1. More precisely, given a μ, vanishing in a neighborhood of 0, we set

$$\tilde{\mu}(z) := \mu(\frac{1}{z}) \cdot \frac{z^2}{\bar{z}^2} .$$

Then $\tilde{\mu}$ has a compact support, so we can find a normalized solution $\tilde{w} = w[\tilde{\mu}]$ of the Beltrami equation with the complex dilatation $\tilde{\mu}$, satisfying the additional conditions, indicated in the proof above. Then the "reflected" function

$$w(z) := \frac{1}{\tilde{w}(\frac{1}{z})}$$

will coincide with the normalized solution $w[\mu]$ of the Beltrami equation (6.2).

In the general case we decompose a given complex dilatation μ into the sum $\mu = \mu_\infty + \mu_0$ of complex dilatations μ_∞, having a compact support, and μ_0, equal to zero in a neighborhood of 0. We would like to write $w[\mu]$ as the composition $w[\mu_\infty] \circ w[\mu_0]$ of the corresponding normalized solutions $w[\mu_\infty]$ and $w[\mu_0]$. But this is not possible, unfortunately, due to the composition formula (6.3) for complex dilatations. However, taking into account the formula (6.3), we can write $w[\mu]$ as the composition

$$w[\mu] = w[\lambda] \circ w[\mu_0] ,$$

where the complex dilatation

$$\lambda := \left[\left(\frac{\mu - \mu_0}{1 - \mu\bar{\mu}_0} \right) \left(\frac{w[\mu_0]_z}{\bar{w}[\mu_0]_{\bar{z}}} \right) \right] \circ (w[\mu_0])^{-1}$$

still has a compact support. This concludes the proof of the existence theorem. $\square$

Due to the uniqueness theorem (Prop. 11 in Sec. 6.1), we have the following

Corollary 1. *For any measurable function μ on extended complex plane $\overline{\mathbb{C}}$ with $\|\mu\|_\infty < 1$, there exists a unique normalized quasiconformal map on $\overline{\mathbb{C}}$, fixing the points $0, 1, \infty$, whose complex dilatation agrees with μ almost everywhere on $\overline{\mathbb{C}}$.*

Using the existence Theor. 5, it's easy to construct a solution of the Beltrami equation (6.2) in the upper half-plane $H = H_+$, preserving H. For that it's sufficient to extend the dilatation μ to the lower half-plane $H^* = H_-$ by symmetry, setting

$$\hat{\mu}(z) := \overline{\mu(\bar{z})} \quad \text{for } z \in H_- . \tag{6.7}$$

Then, applying the existence theorem to the Beltrami equation with the dilatation $\hat{\mu}$, we obtain a unique solution w_μ of this equation, fixing the points $0, 1, \infty$. It follows from the uniqueness of the solution that w_μ satisfies the relation

$$w_\mu(\bar{z}) = \overline{w_\mu(z)} .$$

So w_μ maps the real axis onto itself and, consequently, preserves the upper half-plane H_+.

On the other hand, one can use the method, proposed in the beginning of the proof of existence Theor. 5, extending the given potential μ to the whole plane $\mathbb{C}$ by zero outside H:

$$\check{\mu}(z) = 0 \quad \text{for } z \in H_- \ .$$

Applying the existence theorem to the Beltrami equation with the dilatation $\check{\mu}$, we obtain a solution w^μ, which is conformal in the lower half-plane H_- and fixes the points $0, 1, \infty$.

The first method of constructing the solution w_μ of the Beltrami equation in H_+ is called *real-analytic*, since in this case w_μ depends real-analytically on μ. Respectively, the second method is called *complex-analytic*, since w^μ depends on μ holomorphically (cf. [56], Ch. 1.2, for a rigorous proof of these assertions).

Both methods are naturally transferred to the Beltrami equation in the unit disc Δ. For that in the first method one should substitute the symmetry transformation (6.7) by the reflection with respect to the unit circle $S^1 := \partial\Delta$. In other words, the dilatation μ, defined in the unit disc $\Delta = \Delta_+$, is extended to its exterior Δ_- by the formula

$$\hat{\mu}(\frac{1}{\bar{z}}) := \overline{\mu(z)} \cdot \frac{z^2}{\bar{z}^2} \quad \text{for } z \in \Delta \ .$$

The existence theorem for the extended dilatation $\hat{\mu}$ yields a quasiconformal homeomorphism $w_\mu : \mathbb{C} \to \mathbb{C}$, which preserves Δ_+ and Δ_- and fixes the points $\pm 1, -i$. The second method provides a quasiconformal homeomorphism $w^\mu : \mathbb{C} \to \mathbb{C}$, which is conformal on Δ_- and fixes the points $\pm 1, -i$.

Remark 10. There is an interesting assertion, due to Mañé, Sad and Sullivan, characterizing quasiconformal homeomorphisms as holomorphic perturbations of the identity. More precisely, we say that a homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic perturbation of the identity, if it can be included into a family of homeomorphisms $f_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, depending on a parameter $\lambda \in \Delta$, such that for every fixed $z_0 \in \mathbb{C}$ the function $f_\lambda(z_0)$ is holomorphic in $\lambda \in \Delta$, and

$$f_0 = \mathrm{id} \ , \quad f_{\lambda_0} = f \quad \text{for some } \lambda_0 \in \Delta \ .$$

It is proved in [52] that any member f_λ of such a family necessarily extends to a quasiconformal homeomorphism $\tilde{f}_\lambda$ of the extended complex plane $\overline{\mathbb{C}}$ with complex dilatation, not exceeding $(1 + |\lambda|)/(1 - |\lambda|)$.

Conversely, any quasiconformal homeomorphism f of the extended complex plane $\overline{\mathbb{C}}$ is a holomorphic perturbation of the identity. Indeed, if $f = w^\mu$ for some Beltrami differential μ with $\|\mu\|_\infty = k < 1$, then we can include f into a holomorphic family of quasiconformal homeomorphisms, defined by

$$f_\lambda := w^{\lambda\mu/k} \ .$$

Bibliographic comments

Key references to this Chapter are the Ahlfors lectures on quasiconformal mappings [1] and Lehto's book [49]. Most of the assertions can be found in these sources.

In particular, we follow Ahlfors' lectures in proving the main existence theorem for quasiconformal maps.