Background on the Orbifold Theorem

Many people have obtained partial results and developed related ideas. The following is a selective list. Some of this work is used in our approach, and other parts are used in the approach of Boileau and Porti. We have included some hearsay concerning the events surrounding the Orbifold theorem.

1978 The Smith Conjecture is proved [64].
This was a culmination of the work of many people and used a major part of the theory of 3-manifolds, in particular the work of Bass, Culler & Shalen, Gordon & Litherland, Meeks & Yau, Haken, Waldhausen, and Thurston. It is now a (very special) consequence of the Orbifold theorem.

1981 Thurston announces the Orbifold Theorem [81], [83].

Theorem A. [83] *Let M^3 be a prime, $\mathbb{P}^2$-irreducible, compact 3-manifold which admits a diffeomorphism ϕ ($\neq 1$) of finite order whose fixed point set is more than a finite set of points. Then M has a geometric decomposition.*

Theorem B. [83] *Suppose F is a finite group of diffeomorphisms of a compact 3-manifold M, of which some element $\phi \in F$ ($\phi \neq 1$) has more than a finite set of fixed points. Let $O = M/F$ and $\Sigma \subset O$ be the image of the union of fixed point sets of all elements $\phi \neq 1$ of F. Suppose that the 3-manifold $O - \Sigma$ is prime, and that any 2-sided projective plane in $O - \Sigma$ is homotopic to the boundary of a regular neighbourhood of an isolated point of Σ. Then M has a geometric decomposition which is invariant by F. More precisely, there is a collection of disjoint embedded spheres, projective planes, incompressible tori and incompressible Klein bottles, whose union is invariant by F and geometric structures on the pieces obtained by decomposition along the surfaces on which F acts isometrically.*

Interestingly, Thurston's original theorem pre-dated Hamilton's announcement of the results in [35] by a couple of months. Thurston's first version concluded that either an orbifold has a geometric decomposition or else it admits a metric of positive Ricci curvature. Two months later, Thurston heard of Hamilton's result and was thus able to complete his proof of the geometrization theorem for 3-orbifolds with one-dimensional singular locus. Thurston outlined his proof on two occasions in courses at Princeton; in 1982 and again in 1984. On both occasions, due to running out of time, the outline was incomplete in certain aspects at the end of the proof in the collapsing case. In particular the Euclidean/spherical transition in the case of vertices was treated in a few sentences.

1982 Tollefson [79] showed that two involutions of a Haken 3-manifold that are homotopic are in fact conjugate by a diffeomorphism isotopic to the identity provided that the manifold is not a Seifert fibre space and $H_1(M)$ is infinite.

1982 Hamilton [35] classifies 3-manifolds with positive Ricci curvature. At present all known proofs of the orbifold theorem make use of either this result, or else the strengthened 1986 version. It remains an interesting and important question whether there is a proof that does not rely on PDE techniques.

1983,1985 Bonahon & Siebenmann [9],[10] classify orbifold Seifert fibre spaces (OSFS).

1984 Hamilton [37] distributes a preprint giving an orbifold version of his positive Ricci curvature theorem. This version had been suggested by Thurston as a way of completing his proof of the orbifold theorem. Thurston later claimed a proof which avoided appealing to this result. Hamilton's preprint has not yet appeared.

1985 Soma, Ohshika & Kojima [76] give some details of the proof of the Orbifold theorem. In particular they give a somewhat different proof of the classification of non-compact Euclidean 3-dimensional cone-manifolds from the one outlined by Thurston.

1986 Hamilton [36] classifies 3-manifolds with non-negative Ricci curvature.

1986 [43] Hodgson's thesis gives many examples and develops the theory of deformations and change of geometry.

1986 Meeks & Scott [61] show that if a finite group acts on a closed P^2-

irreducible Seifert fibre space M with infinite fundamental group then there is a homogeneous metric (i.e. geometric structure) preserved by this action. If the action preserves a Seifert fibration up to homotopy then M has an invariant Seifert fibration.

1986 Cheeger & Gromov [17],[18], Fukaya [29] study collapse with bounded curvature and introduce F-structures. The collapsing theorem they prove works in all dimensions. The statement is very similar to that of the collapsing theorem for cone-manifolds used in the proof of the orbifold theorem. However the fact that the local fundamental group of a cone-manifold is not virtually abelian means that different techniques must be used.

1987 Bonahon & Siebenmann [11] develop a JSJ decomposition for orbifolds. This is a characteristic splitting of a 3-orbifold by incompressible Euclidean 2-suborbifolds.

1987 Hodgson [44] gives background and outlines Thurston's proof of the Orbifold theorem. The treatment of the collapsing case was somewhat incomplete, reflecting Thurston's presentation.

1987 McCullough & Miller [59] show that a 3-orbifold with a geometric decomposition has a finite orbifold covering which is a manifold. They deduce that such orbifold fundamental groups are residually finite. They also deduce that an isomorphism between *sufficiently large* 3-orbifold fundamental groups preserving the peripheral structure is induced by an orbifold isomorphism provided the boundary consists of incompressible Euclidean orbifolds.

1988 Dunbar [26], [28] classifies non-hyperbolic geometric 3-orbifolds with underlying space S^3. In [27] he develops hierarchies for 3-orbifolds.

1990 Zhou's thesis [90] gives some details of the proof of the Orbifold theorem, for the case where the singular locus is a 1-manifold.

1992-1998 Hodgson & Kerckhoff [45], [49], [46] develop a rigidity theory for cone-manifolds, using harmonic deformations.

1995 Kirby [50] lists the Orbifold theorem as a conjecture (Problem 3.46) in his problem list.

1998 Kojima [53] establishes global rigidity for hyperbolic 3-cone-manifolds with cone angles at most π.

1998 Boileau & Porti [8] distribute a preprint with a proof for the case of

an orbifold of the form $Q = M/G$ where M is an irreducible 3-manifold and G a finite group, and the singular locus of Q is a 1-*manifold*.
They handle the collapsing case by showing that there is a geometric structure on M and using Meeks and Scott to deduce there is a G-equivariant structure, which therefore descends to a geometric structure on Q. They find a curve $\gamma \subset M$ (which, for example, may be a regular fibre of a Seifert fibre space) and show that the Haken manifold $N = M - \mathcal{N}(\gamma)$ is a graph manifold by showing it has Gromov norm zero. This implies there are no hyperbolic pieces. Then by Thurston's theorem for Haken manifolds, N is a graph manifold. The geometry of the collapse is used to give γ and to construct an open cover of N which is used to show the Gromov norm of N (hence of M) is zero.

1998 Cooper, Hodgson & Kerckhoff announce a proof in the case the singular locus is a 1-manifold and the boundary is Euclidean. They outline the proof in a series of 15 lectures at a meeting of the MSJ in Tokyo and distribute a preprint.

The heart of the argument is the **collapsing case** when the injectivity radius goes to zero everywhere. There are two subcases. If one rescales the metrics by multiplying by $max(1, 1/diameter)$ either the injectivity radius goes to zero somewhere, or else it does not. In the latter case one obtains a Euclidean cone-manifold in the limit. This may be the final orbifold structure. Otherwise one uses Hamilton's theorem to show the final orbifold is spherical. The remaining case is that, even after the rescaling, the injectivity radius goes to zero everywhere. Then every point in one of these almost-collapsed manifolds has a neighbourhood of simple topology. This case is handled by the collapsing theorem 7.13 (Collapsing theorem) where we construct an orbifold Seifert fibration, or orbifold bundle structure on these pieces and fit them all together.

The collapsing case was handled somewhat differently by Thurston, who described the geometry of the collapse in more detail, (in the spirit of the F-structures subsequently developed by Gromov and Cheeger in their proof of a collapsing theorem for manifolds of bounded curvature) and produced the Seifert fibration or Solv structure directly from this geometry. Some of the main ideas are already present in chapter 4 of the original notes of Thurston, [84]. We hope to recover this description from our approach in a future paper.