

# LECTURES ON KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND HECKE ALGEBRAS

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ABSTRACT. This paper is the course of lectures delivered by the first author in Kyoto in 1996-97 and recorded by the others. We tried to follow closely the notes of the lectures not yielding to the temptation of giving more examples and names. The focus is on the relations of the Knizhnik-Zamolodchikov equations and Kac-Moody algebras to a new theory of spherical and hypergeometric functions based on affine and double affine Hecke algebras. Here mathematics and physics are closer than Siamese twins. We did not try to separate them, but the course turned out to be mainly about the mathematical issues. However we hope that the paper will be understandable for both physicists and mathematicians, for those who want to master the new Hecke algebra technique.

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## 1. INTRODUCTION : HECKE ALGEBRAS IN REPRESENTATION THEORY

Before a systematic exposition, I will try to outline the connections of the representation theory of Lie groups, Lie algebras, and Kac-Moody algebras with Hecke algebras and the Macdonald theory.

A couple of remarks about the growth of Mathematics. It can be illustrated (with all butts and ifs) by Fig1.

It is extremely fast in the imaginary (conceptual) direction but very slow in the real direction. Mainly I mean modern mathematics, but it may be more general. For instance, ancient Greeks created a highly conceptual axiomatic geometry with a modest 'real output'. I do not think that the ratio *Real/Imaginary* is much higher now. There are many theories and a very limited number of functions which are really special. Let us try to project the representation theory on the real axis (Fig1). We focus on Lie groups(algebras) and Kac-Moody algebras, ignoring the arithmetic direction (adèles and automorphic forms). Look at Fig2.

- 1) By this I mean the zonal spherical functions on  $K \backslash G / K$  for maximal compact  $K$  in a semi-simple Lie group  $G$ . The theory was started by Gelfand et al. in the early 50's and completed by Harish-Chandra and many others. It generalized quite a few classical special functions. Lie groups greatly

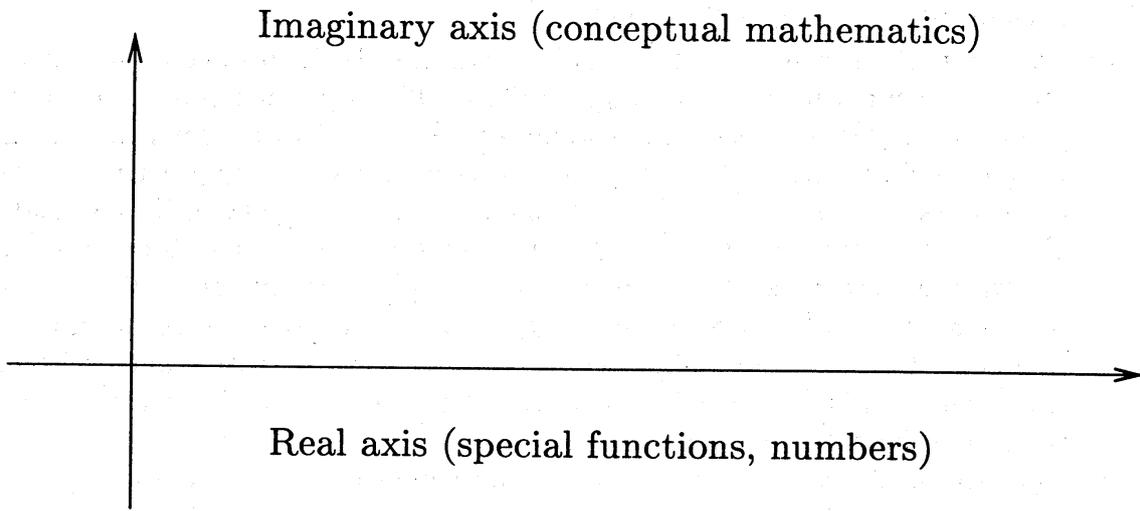


FIGURE 1. Real and Imaginary

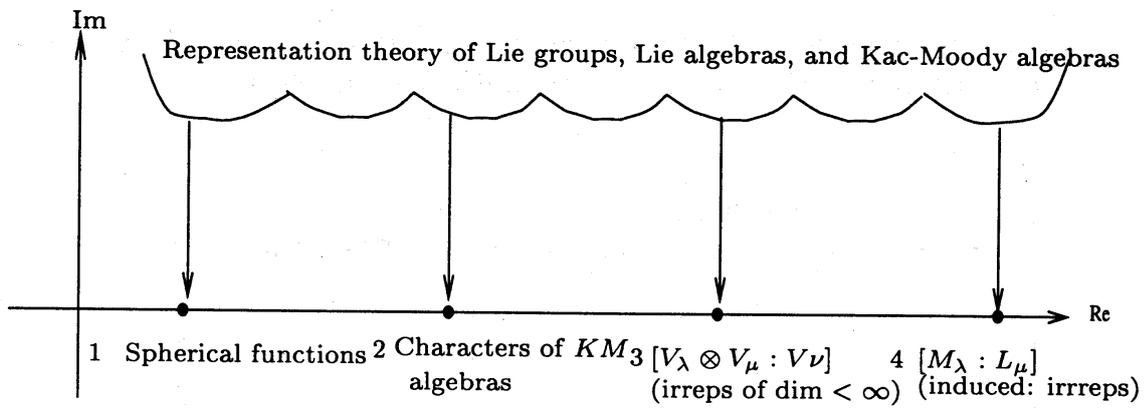


FIGURE 2. Representation Theory

helped to elaborate a systematic approach, although much can be done without them, as we will see below.

- 2) The characters of Kac-Moody algebras can also be introduced without any representation theory (Looijenga, Saito). They are not too far from the products of classical one-dimensional  $\theta$ -functions. However it is a new and very important class of special functions with various applications. The representation theory explains well some of their properties (but not all).
- 3) This construction gives a lot of remarkable combinatorial formulas, and generating functions. Decomposing tensor products of finite dimensional representations of compact Lie groups was in the focus of representation theory in the 70's and early 80's, as well as various restriction problems. This direction is still very important, but the representation theory moved towards infinite-dimensional objects.
- 4) Here the problem is to calculate the multiplicities of irreducible representations of Lie algebras in the Verma modules or other induced representations. It is complicated. It took time to realize that these multiplicities are 'real' and to establish relations to the theory of special functions. The Verma modules and the BGG-resolutions were designed as a technical tool for the Weyl character formula and seemed too algebraic for real applications when they appeared.

Let us update the picture adding the results which were obtained in the 80's and 90's mainly inspired by a breakthrough in mathematical physics.

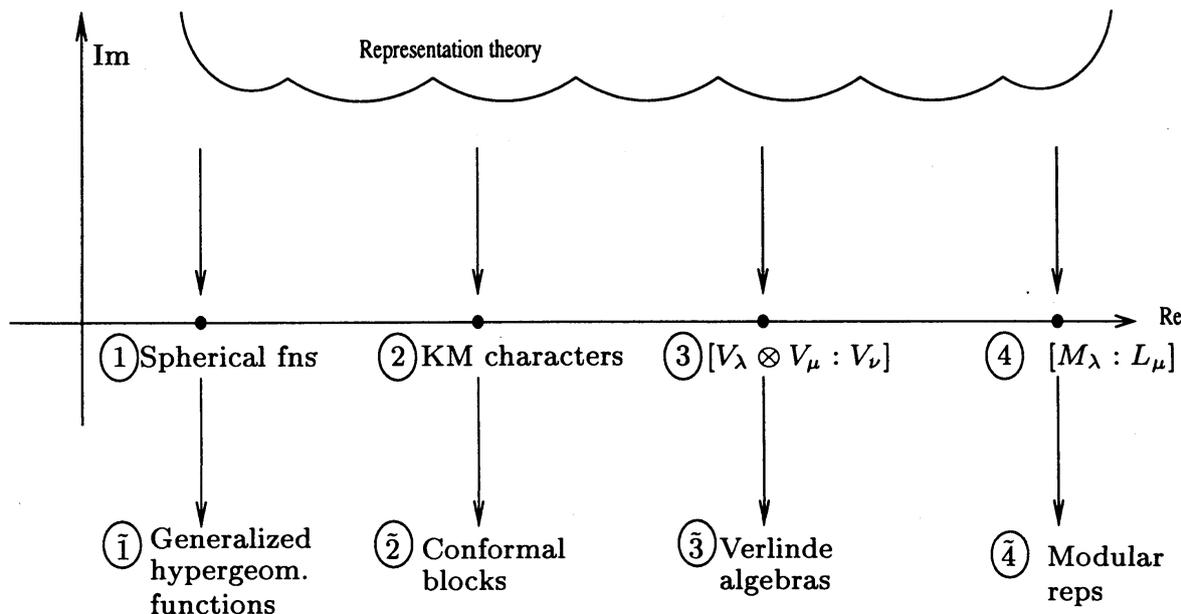


FIGURE 3. New Vintage

- 1) These functions will be the subject of my mini-course. We will study them in the differential and difference cases. The interpretation and generalization of hypergeometric functions was an important problem of the representation theory (the Gelfand program).
- 2) Actually conformal blocks belong to the imaginary axis (conceptual mathematics). Only some of them can be considered as 'real' functions. Mostly it happens in the case of rational and elliptic KZ-equations.
- 3) The Verlinde algebras are formed by integrable representations of Kac-Moody algebras of a given level with the fusion instead of tensoring. They can be also defined using quantum groups at roots of unity.
- 4) Whatever you think about the 'reality' of  $[M_\lambda : L_\mu]$ , these multiplicities are connected with modular representations including the representations of the symmetric group over fields of finite characteristic. Nothing can be more real!

**Conjecture 1.1.** *The real projection of the representation theory goes through Hecke-type algebras.*

As to the examples under discussion the picture is as follows:

- a) This arrow is the most recognized now. Several questions in the Harish-Chandra theory (the zonal case) were covered by the representation theory of the degenerate (graded) affine Hecke algebras defined by Lusztig [52]. For instance, the operators from [14] give a simple approach to the radial parts of invariant differential operators on symmetric spaces and are useful for the Harish-Chandra transform. The hypergeometric functions (the arrow ( $\bar{a}$ )) appear naturally in this way [38],[61]. Still, I believe, the difference theory is more promising. It was demonstrated in [20] that the difference Fourier transform is self-dual (it is not in the differential case). It could simplify and generalize the Harish-Chandra theory. The same program was started in the  $p$ -adic representation theory (see [21, 24]). Note that certain Macdonald polynomials can be interpreted via the quantum groups (Noumi and others, see[57]). However the Hecke algebra technique seems to be more relevant (especially for arbitrary root systems).
- b) The double Hecke algebras lead to a certain elliptic generalization of the Macdonald polynomials [22, 23, 24]. In the differential case there is also the so-called parabolic operator (see [30] and [22]). As to ( $\bar{b}$ ), the conformal blocks are more general than the characters. Obviously Hecke algebras are not enough to obtain all of them. On the other hand, the conformal blocks are defined for the configuration spaces of type  $A$  only. Double affine Hecke algebras work well for all root systems.
- c) Here one can rediscover the same combinatorial formulas for the characters (mostly based on the so-called Kostant partition function) and find some new ones. I do not expect anything brand new. However if you switch

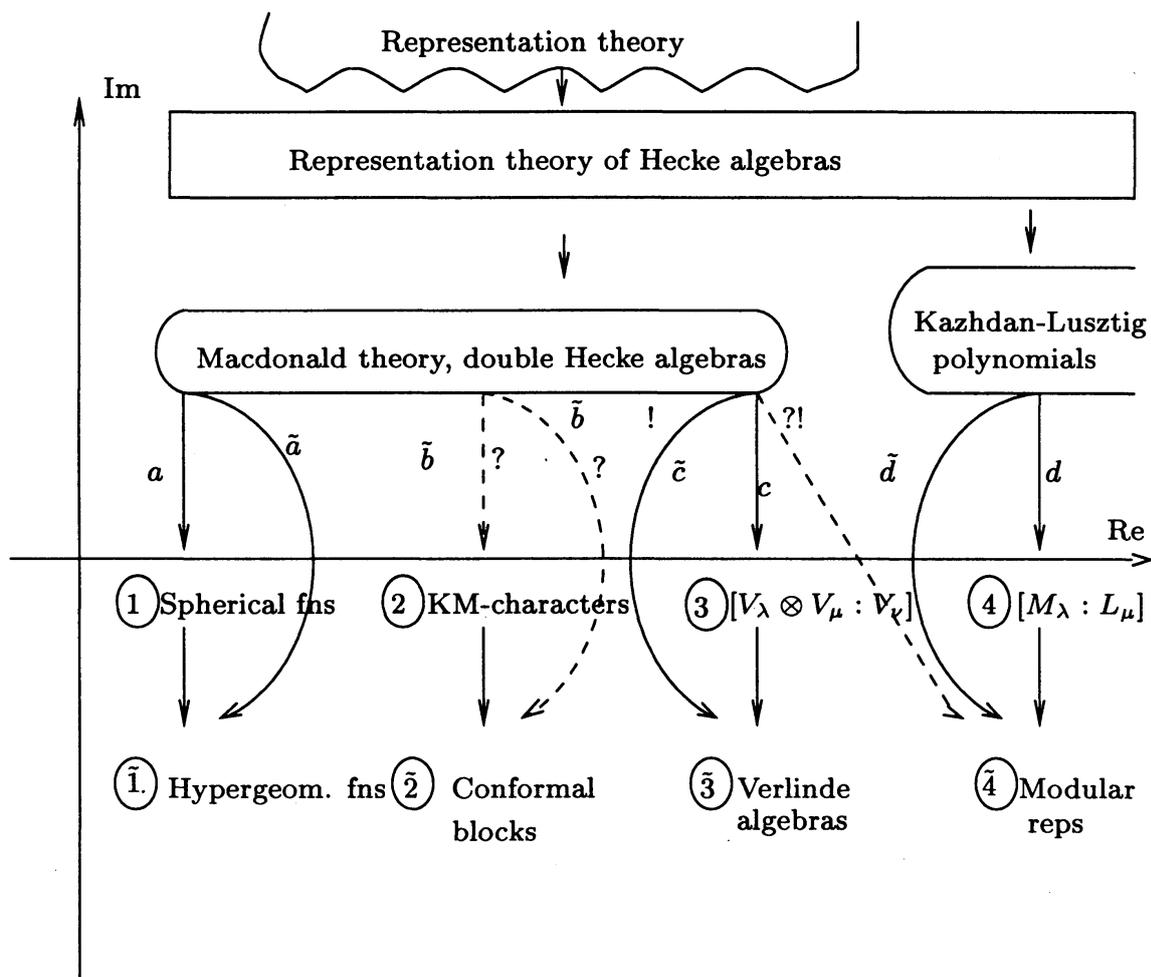


FIGURE 4. Hecke Algebras

to the spherical functions (instead of the characters) then the new theory results in the formulas for the products of spherical functions, which cannot be obtained in the classical theory (they require the difference setting). The multiplicities  $[V_\lambda \otimes V_\mu : V_\nu]$  govern the products of the characters and vice versa.

Concerning  $(\tilde{c})$ , the Macdonald theory at roots of unity gave a simple approach to the Verlinde algebras. I mean [47], and my two papers [20, 21]. A. Kirillov Jr. found a  $k$ -deformation of the Verlinde algebras in the case of  $GL_n$  using quantum groups at roots of unity. My technique is applicable to all root systems and works well for the nonsymmetric Macdonald polynomials (the Kac-Moody characters are symmetric in contrast to the main classical elliptic functions).

- d) This arrow is the Kazhdan-Lusztig conjecture proved by Brylinski-Kashiwara and Beilinson-Bernstein and then generalized to the Kac-Moody case by Kashiwara-Tanisaki. By  $(\tilde{d})$ , I mean the modular Lusztig conjecture (partially) proved by Anderson, Jantzen, and Soergel. The arrow from the Macdonald theory to modular representations is marked by '!'. It seems the most challenging now. I hope to extend my results on the Macdonald polynomials at roots of unity from the restricted case (alcove) to arbitrary weights (parallelogram). It might give a  $k$ -generalization of the classical theory, formulas for the modular characters (not only those for the multiplicities), and a description of modular representations of arbitrary Weyl groups. However now it looks difficult.

Let me comment on the role of the Kac-Moody algebras and their relations (real and imaginary) to the spherical functions and the Hecke algebras. I will give some arguments for and against the existence of this relation, connected with the contents of these lectures.

*Remark 1.1. Fusion procedure.* I think that the penetration of double Hecke algebras into the fusion procedure and related problems of the theory of Kac-Moody algebras is a very convincing demonstration. I had certain personal reasons to develop double Hecke algebras in this direction. The fusion procedure appeared in my paper [8]. Given an integrable representation of the  $n$ -th power of a Kac-Moody algebra (where all  $n$  central elements are identified) and two sets of points on a Riemann surface ( $n$  and  $m$  points), I constructed an integrable representation of the  $m$ -th power of the same Kac-Moody algebra. The procedure does not change the central charge. I missed that in the special case when  $n = 2$ ,  $m = 1$  the multiplicities of irreducibles in the resulting representation are the structural constants of a certain commutative algebra, the Verlinde algebra [70].

Now we know that these multiplicities can be extracted from the simplest (polynomial) representation of the double affine Hecke algebra at roots of unity in the so-called group case. The Macdonald polynomials turn into the finite dimensional characters and do not depend on  $q, t$  in this case, so it is very special. Double Hecke algebras simplify the inner product on the Verlinde algebra, the (projective) action

of  $PSL_2(\mathbb{Z})$ , and give more. These structures exist in the general theory (with  $q, t$ ) as well. Only the integrality and positivity of the structural constants of the Verlinde algebras is missing. So the Kac-Moody algebras are undoubtedly connected with the new theory of difference spherical functions (including the Macdonald polynomials). Let me comment on relations to the classical differential theory.

I'd like to add that I borrowed the fusion procedure from arithmetics. I had known Ihara's papers "On congruence monodromy problem" well. A similar procedure is a foundation of his theory. Of course I changed something, but the procedure is basically the same. Can we go back and define Verlinde algebras in arithmetics?

*Remark 1.2. Kac-Moody algebras and spherical functions.* The classification of the Kac-Moody algebras resembles very much that of the symmetric spaces. See [42], [41]. It is not surprising because the key point is the same in both theories: the description of involutions and more general automorphisms of finite dimensional reductive Lie algebras. The classification lists are similar but do not coincide. The  $BC_n$ -symmetric spaces have no Kac-Moody counterparts. Vice versa, the KM-algebra of type, say,  $D_4^{(3)}$  is not associated (even formally) with either symmetric space. Still one could hope that this parallelism is not incidental.

To try to establish a connection we need to switch from the genuine Harish-Chandra theory of zonal spherical functions to the quantum many-body problem, where the root multiplicities  $k$  can be arbitrary complex numbers. The classical ones are  $k = 1/2$  for  $Sl_2(\mathbb{R})/SO_2$ ,  $k = 1$  in the so-called group case  $Sl_2(\mathbb{C})/SU_2$ ,  $k = 2$  for the  $Sp_2$ , etc. The corresponding generalized spherical functions (due to Heckman and Opdam) can be viewed as deformations of the classical characters of finite dimensional representations of Lie groups. The KM-characters also depend on a new parameter, the central charge  $c$  of representations (the level), and extend the classical characters. Could we expect any  $c \Leftrightarrow k$  correspondence?

Generally speaking, the answer is negative. Indeed, the number of independent  $k$ -parameters can be from 1 ( $A, D, E$ ) to 5 ( $C^\vee C$ , the so-called Koornwinder case), but we have only one  $c$  in the Kac-Moody theory. Moreover, the  $k$ -spherical functions are eigenfunctions of differential operators generalizing the radial parts of the invariant operators on symmetric spaces, and are pairwise orthogonal for different eigenvalues. This has no counterpart for the Kac-Moody characters. After all, the latter are elliptic functions. The spherical functions are not.

We will discuss in the course the elliptic quantum many-body problem. It is a candidate for a theory of spherical functions in the Kac-Moody setup at critical level and leads to a definition of elliptic spherical polynomials. However this is rather about unification of  $c$  and  $k$  than about the connection between them. The elliptic QMBP in the  $GL_n$ -case was introduced by Olshanetsky-Perelomov. The classical root systems were considered in [58]. Finally, the operators for arbitrary root systems were constructed in [22].

In spite of all this, a map from the Kac-Moody algebras to spherical functions exists. It is for  $GL_n$  only and not exactly for the KM-characters, but it exists!

*Remark 1.3. Integral formulas for KZ.* The KZ-equation is the system of differential equations for the matrix elements (correlation functions) of the representations of the Kac-Moody algebras in the  $n$ -point case. The matrix elements are simpler to deal with than the characters. We differentiate them with respect to the positions of the points. A natural setup here is the so-called  $r$ -matrix Kac-Moody algebras [7] and  $r$ -matrix KZ equations introduced in [10]. The points are taken on  $\mathbb{P}^1$  or on elliptic curves, and the theory is “integrable”. The Knizhnik-Zamolodchikov-Bernard (KZB) equation [5, 35] and the double affine KZ from [22] are also of this type (with certain reservations).

The parameter  $k$  which appears in KZ is given by the formula:

$$k = 1/(c + g), \quad g = \text{dual Coxeter number}, \quad c = \text{central charge}.$$

The KZ equations for arbitrary curves involve the derivatives with respect to the moduli of curves and vector bundles and are much more complicated (“non-integrable”).

In the following cases we know explicit integral representations for the solutions of KZ:

- (a) the Yang rational  $r$ -matrix (see [65]),
- (b) the basic trigonometric (or hyperbolic) one [13],
- (c) KZB [5, 35].

Given a Lie algebra  $\mathfrak{g}$  (simple, finite dimensional, or even abstract associated with a Cartan matrix), one may define the integrand using the coinvariant of  $U(\widehat{\mathfrak{g}})$  in the Weil representations [13]. The contours of integration are governed by the quantum  $U_q(\mathfrak{g})$  for  $q = \exp(\pi i k)$  and the above  $k$ . See [36], [69] and references therein.

Here  $q$  appears because the configuration space is endowed with a scalar local system depending on  $k$ . This is closely connected with the equivalence of the  $U(\widehat{\mathfrak{g}})_c$  and  $U_q(\mathfrak{g})$  due to Kazhdan, Lusztig, and Finkelberg (see [46]). In these notes we will discuss the integrands only, which are uniform for any  $r$ -matrix KZ. As to the action of  $U_q(\mathfrak{g})$  on the space of the contours, it is more subtle. Certain affine and elliptic extensions are necessary in the trigonometric and elliptic cases. What does it give for the spherical functions?

*Remark 1.4. From KZ to spherical functions.* Since the configuration space for KZ consists of sets of  $n$  points on  $\mathbb{P}^1$  or elliptic curves, the connection could be expected with the spherical functions of type  $A$  only (for either  $\mathfrak{g}$ ). The corresponding differential operators can be rational, trigonometric (hyperbolic), or elliptic. Let us consider the hyperbolic case, to make the consideration compatible with the Harish-Chandra theory. One may start with any  $\mathfrak{g}$ . However only  $\mathfrak{gl}_N$  leads to scalar differential operators due to Etingof and Kirillov Jr. So let  $\mathfrak{g} = \mathfrak{gl}_N$ .

Now we can apply the isomorphism of KZ with the quantum many-body problem from [56, 19], where  $k$  is the same as for KZ. It results in further constraints. We take  $N = n$  and consider KZ with the values in the the 0-weight component of  $(\mathbb{C}^n)^{\otimes n}$ , which is isomorphic to the group algebra  $\mathbb{C}[\mathbb{S}_n]$ . It readily gives that the

dimension of the contours of integration must be  $n(n-1)/2$ . At least, it is the same as in the Harish-Chandra integral representations for spherical functions of type  $A_{n-1}$ . His integrals are over  $K = SO_n \subset SL_n$ .

It is likely that the Harish-Chandra formula is a particular case of the integral formulas for the hyperbolic KZ subject to the above constraints. Certain calculations due to Mimachi, Varchenko, Felder and others confirm this. If it is true, then we will have a solid relation between spherical functions and Kac-Moody algebras in the case of  $A_{n-1}$ .

It is necessary to note that the integral formulas can be justified without Kac-Moody algebras. A straightforward analysis is somewhat complicated but possible [65]. A simple proof presented below is based on the Kac-Moody coinvariant [13]. However it does not clarify the relation  $k = 1/(c+g)$ . I used Kac-Moody at critical level to get a certain algebraic relation, which contains no  $c, g, k$ . Adding a scalar local system depending on  $k$  (no relation to  $c!$ ), one readily arrives at the integral formulas. So in my approach, Kac-Moody algebras are used as a technical tool (and at critical level only).

There is another justification of the (same) integral formulas based on the coinvariant for the Wakimoto modules instead of that for the Weil ones [32]. It is not simpler than my proof (the calculations with the coinvariant are similar), but requires almost no combinatorial considerations.

Still, I think, we should have more evidence to conclude that the classical theory of spherical functions is related to the Kac-Moody algebras. Anyway, what we know is mainly for type  $A$ , not too convincing without the other root systems.

## 2. THE AFFINE KNIZHNIK-ZAMOŁODCHIKOV EQUATION

We discuss the degenerate affine Hecke algebra and the corresponding affine Knizhnik-Zamolodchikov equation. We show how the former appears as the consistency and invariance conditions for the latter.

**2.1. The algebra  $\mathcal{H}'_{A_1}$  and the hypergeometric equation.** In this section, we introduce the affine Knizhnik-Zamolodchikov (AKZ) equation associated with the root system of type  $A_1$ . It is a first-order differential equation for  $\Phi$ , where  $\Phi$  depends on a single variable  $u$  and takes values in an infinite-dimensional algebra called the degenerate affine Hecke algebra.

The equation is as follows:

$$\frac{\partial \Phi}{\partial u} = \left( k \frac{s}{e^u - 1} + x \right) \Phi. \quad (2.1)$$

Here  $k \in \mathbb{C}$  is a parameter, and  $s$  and  $x$  are operators acting on a vector space where  $\Phi$  takes its value. We impose the following two relations.

$$s^2 = 1, \quad (2.2)$$

$$sx + xs = k. \quad (2.3)$$

These relations make (2.1) invariant. Namely, if  $\Phi$  solves (2.1), then

$$\tilde{\Phi}(u) = s\Phi(-u) \quad (2.4)$$

also is a solution of (2.1). We claim that (2.1) is integrable in terms of the classical hypergeometric functions. At least this statement is valid under a certain irreducibility condition.

The AKZ is the equation (2.1) with values in the *degenerate affine Hecke algebra*  $\mathcal{H}'_{A_1}$  of type  $A_1$  generated by the elements  $s$  and  $x$  satisfying the defining relations (2.2) and (2.3):

$$\mathcal{H}'_{A_1} = \langle s, x \rangle / \{(2.2), (2.3)\}. \quad (2.5)$$

Let  $\Phi(u)$  be a function of  $u$  with values in  $\mathcal{H}'_{A_1}$ . Note that one can multiply  $\Phi(u)$  by an arbitrary constant element on the right, i.e. if  $\Phi(u)$  is a solution, then  $\Phi(u)a$  ( $a \in \mathcal{H}'_{A_1}$ ) is also a solution. Let us check the invariance of AKZ (see (2.4)).

We plug in

$$\begin{aligned} -\frac{\partial \tilde{\Phi}(u)}{\partial u} &= s \left( k \frac{s}{e^{-u} - 1} + x \right) \Phi(-u) \\ &= \left( k \frac{s}{e^{-u} - 1} + sxs \right) \tilde{\Phi}(u) \end{aligned}$$

and use

$$\frac{1}{1 - e^{-u}} = \frac{1}{e^u - 1} + 1.$$

Finally,

$$\frac{\partial \tilde{\Phi}(u)}{\partial u} = \left( k \frac{s}{e^u - 1} + ks - sxs \right) \tilde{\Phi}(u),$$

where  $ks - sxs = x$ .

Now we will integrate (2.1). More generally, let us first consider the equation

$$\frac{\partial \Phi}{\partial z} = \left( \frac{A}{1-z} + \frac{B}{z} \right) \Phi. \quad (2.6)$$

It is (2.1) for  $z = e^{-u}$ ,  $A = -ks$ , and  $B = -x$ . The equation (2.6) is much more complicated than the AKZ. However, if  $A, B$  are  $2 \times 2$  matrices acting on the 2-component vector  $\Phi$ , this equation is nothing but the hypergeometric differential equation. It readily gives the formulas when  $\Phi$  takes values in irreducible representations of  $\mathcal{H}'_{A_1}$ , because the latter exist only in dimensions 1 or 2.

Indeed, a generic 2-dimensional representation  $\rho$  of  $\mathcal{H}'_{A_1}$  is given by

$$\rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(x) = k \left( \frac{s}{2} + \begin{pmatrix} 0 & \zeta \\ \xi & 0 \end{pmatrix} \right). \quad (2.7)$$

Because of the gauge transformation  $\zeta \rightarrow c\zeta$ ,  $\xi \rightarrow c^{-1}\xi$ , it is characterized by  $\zeta\xi$ , or by

$$\mu = \left( \zeta\xi + \frac{1}{4} \right)^{1/2}. \quad (2.8)$$

Then a solution  $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$  for  $A = -k\rho(s)$ ,  $B = -\rho(x)$  is given in terms of the Gauss hypergeometric function. The first component is

$$\Phi_1(u) = z^{-k\mu} (1-z)^k F(k(1-2\mu), k, 1-2k\mu; z), \quad (2.9)$$

where  $z = e^{-u}$  and  $F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n$  with  $(x)_n = x(x+1)\cdots(x+n-1)$ .

If  $\zeta\xi = 0$ , then the representation  $\rho$  (2.7) is reducible. In this case, the solutions are in terms of elementary functions. We note that the parameters  $\alpha, \beta, \gamma$  in (2.9) are not arbitrary but obey the constraint  $\alpha + 1 = \beta + \gamma$ .

**2.2. The AKZ equation for  $GL_n$ .** In this section, we introduce the AKZ equation of type  $GL_n$ . It can be obtained as a specialization of the standard Knizhnik-Zamolodchikov (KZ) equation from the conformal field theory. The consistency and invariance conditions give rise to the defining relations of the degenerate affine Hecke algebra  $\mathcal{H}'_{GL_n}$  introduced by Drinfeld.

Recall that the KZ equation reads

$$\frac{\partial \Phi}{\partial z_i} = k \left( \sum_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{\Omega_{ij}}{z_i - z_j} \right) \Phi \quad (0 \leq i \leq n). \quad (2.10)$$

In the less sophisticated case  $\Omega_{ij}$  are the permutation matrices [48]. Let us assume that  $\Omega_{ij}$  are any constant elements (operators) and  $\Omega_{ij} = \Omega_{ji}$ . We consider  $\Phi(z)$  ( $z = (z_0, \dots, z_n)$ ) taking values in the abstract algebra generated by the elements  $\Omega_{ij}$ . The self-consistency of the system of equations (2.10) means that

$$\frac{\partial A_j}{\partial z_i} - \frac{\partial A_i}{\partial z_j} = [A_i, A_j], \quad (2.11)$$

where

$$A_i = k \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}. \quad (2.12)$$

It holds for all values of the complex parameter  $k$  if and only if

$$[\Omega_{ij}, \Omega_{kl}] = 0, \quad (2.13)$$

$$[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0, \quad (2.14)$$

where the indices  $i, j, k, l$  are pairwise distinct. The KZ in this form is due to Aomoto [1] (it was also studied by Kohno [49]).

The trigonometric KZ (and the elliptic ones) were introduced for the first time in [10]. Actually the paper was about a more general class of the equations which I called *the  $r$ -matrix KZ*. I established their relation to Kac-Moody algebras and calculated the monodromy (see below). The equations corresponding to the simplest trigonometric  $r$ -matrices (there are many of them due to Belavin and Drinfeld) are closely connected with the AKZ. The physical interpretation of the trigonometric KZ was not clear when they appeared (my approach was mathematical). Now they are quite common for both mathematicians and physicists.

Consider the group algebra  $\mathbb{C}[\mathbb{S}_n]$  of the permutation group  $\mathbb{S}_n$  of the set  $\{1, \dots, n\}$ . We denote by  $s_{ij}$  the transposition of  $i$  and  $j$ . If we put  $\Omega_{ij} = s_{ij}$  ( $0 \leq i, j \leq n$ ), the relations (2.13)-(2.14) are satisfied.

Setting

$$z_0 = 0, \quad (2.15)$$

$$\Omega_{ij} = s_{ij} \quad (i, j \neq 0), \quad (2.16)$$

$$\Omega_{0i} = k^{-1} \Omega_i, \quad (2.17)$$

the equation (2.10) turns into

$$\frac{\partial \Phi}{\partial z_i} = \left[ k \left( \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{s_{ij}}{z_i - z_j} \right) + \frac{\Omega_i}{z_i} \right] \Phi \quad (1 \leq i \leq n), \quad (2.18)$$

and the relations (2.13),(2.14) read as follows:

$$[s_{ij}, \Omega_i + \Omega_j] = 0, \quad (2.19)$$

$$[ks_{ij} + \Omega_i, \Omega_j] = 0, \quad (2.20)$$

$$[s_{ij}, \Omega_l] = 0, \quad (2.21)$$

where the indices  $i, j, l$  are pairwise distinct.

Substituting

$$z_i = e^{v_i}, \quad (2.22)$$

we come to

$$\frac{\partial \Phi}{\partial v_i} = \left( k \sum_{j \neq i} \frac{s_{ij}}{1 - e^{v_j - v_i}} + \Omega_i \right) \Phi. \quad (2.23)$$

Using the elements

$$y_i = \Omega_i + k \sum_{j > i} s_{ij}, \quad (2.24)$$

$$\frac{\partial \Phi}{\partial v_i} = \left( k \sum_{j > i} \frac{s_{ij}}{e^{v_i - v_j} - 1} - k \sum_{j < i} \frac{s_{ij}}{e^{v_j - v_i} - 1} + y_i \right) \Phi. \quad (2.25)$$

The elements  $\{y\}$  are convenient since in the limit

$$v_1 \gg v_2 \gg \cdots \gg v_n,$$

we get the system

$$\frac{\partial \Phi}{\partial v_i} = y_i \Phi. \quad (2.26)$$

The consistency of these equations is equivalent to the commutativity

$$[y_i, y_j] = 0. \quad (2.27)$$

We claim that (2.27), a ‘limiting self-consistency’, together with the relations

$$[s_i, y_j] = 0 \quad \text{if } j \neq i, i+1, \quad (2.28)$$

$$s_i y_i - y_{i+1} s_i = k, \quad (2.29)$$

where  $s_i = s_{i i+1}$  ( $1 \leq i \leq n-1$ ), ensure (2.19)-(2.21).

It can be put in the following way. Let us introduce the *degenerate affine Hecke algebra of type  $GL_n$*  as an algebraic span of  $\mathbb{C}[\mathbb{S}_n]$  and  $y_i$  ( $1 \leq i \leq n$ ) with the relations (2.27), (2.28) and (2.29), denoting it by  $\mathcal{H}'_{GL_n}$ , or simply by  $\mathcal{H}'_n$ . We call the system (2.25) with the values in  $\mathcal{H}'_n$  the *AKZ of type  $GL_n$* . It is well-defined, i.e. self-consistent.

Actually the relations for  $\{y\}$  give more than the self-consistence. They also ensure the  $\mathbb{S}_n$ -invariance of the AKZ equation.

The group  $\mathbb{S}_n$  acts on  $\mathbb{C}^n$  naturally by

$$v = (v_1, \dots, v_n) \in \mathbb{C}^n \mapsto w(v) = (v_{i_1}, \dots, v_{i_n}) \in \mathbb{C}^n$$

for  $w^{-1} = (i_1, i_2, \dots, i_n) \in \mathbb{S}_n$ . Given a function  $\Phi(v)$  of  $v \in \mathbb{C}^n$  with values in  $\mathcal{H}'_n$ , we define the action of  $w \in \mathbb{C}[\mathbb{S}_n]$  on  $\Phi(v)$  by

$$(w(\Phi))(v) = w \cdot \Phi(w^{-1}(v)). \quad (2.30)$$

Here the dot means the product in  $\mathcal{H}'_n$ . It follows from (2.28) and (2.29) that if  $\Phi$  solves (2.25), so does  $w(\Phi)$ . Just conjugate the equations by  $\{s_i\}$ . Moreover, the invariance is exactly equivalent to the relations (2.28) and (2.29). Thus the invariance and the limiting self-consistency (2.27) give the self-consistency of our system for all  $k$ .

**Example 2.1.** The basic variant of the Knizhnik-Zamolodchikov equation is as follows. Let  $\mathfrak{gl}_N$  be the matrix algebra with the standard set of generators  $\{e_{lm}, 1 \leq l, m \leq n\}$  with the entries  $e_{lm}^{ab} = \delta_{la}\delta_{mb}$ . Given any representations  $V_0, V_1, \dots, V_n$  of  $\mathfrak{gl}_N$ , the tensor product  $V = V_0 \otimes V_1 \otimes \dots \otimes V_n$  has a natural structure of  $\mathfrak{gl}_N$ -module. We set  $\Omega_{ij} = \sum_{lm} e_{lm}^{(i)} e_{ml}^{(j)}$ , where  $e_{lm}^{(i)}$  act in  $V_i$ ,  $i \neq j$ . Then (2.14) and (2.13) are fulfilled and we get a self-consistent KZ equation. This example was generalized by Kohno to arbitrary reductive Lie algebras and extensively used by Drinfeld in his theory of quasi-Hopf algebras.

If  $V_i$  coincide with the fundamental  $n$ -dimensional representation for all  $i > 0$ , then  $\Omega_{ij}$  can be identified with the transposition  $s_{ij}$  for  $i, j > 0$ . The representation  $V_0$  is still arbitrary. We arrive at a  $\mathbb{S}_n$ -invariant self-consistent equation in the form (2.18). Hence we have a representation of  $\mathcal{H}'_n$  in  $V$ . Moreover, given any weight of  $\mathfrak{gl}_N$ , the subspace of the corresponding highest vectors in  $V$  is a  $\mathcal{H}'_n$ -submodule. The formulas for the action of  $\mathcal{H}'_n$  are the same as above:

$$s_{ij} \mapsto \Omega_{ij}, \quad y_i \mapsto \Omega_{0i} + k \sum_{j>i} \Omega_{ij}, \quad 1 \leq i, j \leq n.$$

**2.3. Degenerate affine Hecke algebras.** In this section, we fix notations for root systems and define the degenerate affine Hecke algebra for an arbitrary root system.

Let  $\Sigma$  be a root system in  $\mathbb{R}^n$  with the inner product  $(\ , \ )$ . Choose a system of simple roots  $\alpha_1, \dots, \alpha_n$  of  $\Sigma$  and denote by  $\Sigma_+$  the set of positive roots. For a root  $\alpha \in \Sigma$ , define the coroot  $\alpha^\vee$  by

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$$

and the reflection  $s_\alpha$  by

$$s_\alpha(u) = u - (\alpha^\vee, u)\alpha \quad (u \in \mathbb{R}^n).$$

We will denote  $s_{\alpha_i}$  simply by  $s_i$ . The fundamental coweights  $b_i$  are as follows:

$$(b_i, \alpha_j) = \delta_{ij}.$$

We also use the notation  $a_i = \alpha_i^\vee$ . For  $u \in \mathbb{R}^n$ , the coordinates will be  $u_i = (u, \alpha_i)$ . We also set  $u_\alpha = (u, \alpha)$  for  $\alpha \in \Sigma$ . Check that

$$\frac{\partial u_\alpha}{\partial u_i} = \nu_\alpha^i = (b_i, \alpha) = \text{the multiplicity of } \alpha_i \text{ in } \alpha.$$

Let  $W$  be the Weyl group of  $\Sigma$ :  $W = \langle s_\alpha, \alpha \in \Sigma \rangle = \langle s_1, \dots, s_n \rangle$ . Define the action of  $W$  on functions on  $\mathbb{R}^n$  by

$${}^w f(u) = f(w^{-1}(u)) \quad (u \in \mathbb{R}^n). \quad (2.31)$$

Then we have

$${}^w u_\alpha = (w^{-1}(u), \alpha) = u_{w(\alpha)}.$$

Now we can define the degenerate affine Hecke algebra  $\mathcal{H}'_\Sigma$  associated with  $\Sigma$ . This definition is due to Lusztig [52] (he calls it the graded affine Hecke algebra, considering  $k$  as a formal parameter). Drinfeld introduced this algebra in the  $GL_n$ -case in [27] prior to Lusztig. These algebras are natural degenerations of the corresponding  $p$ -adic ones.

**Definition 2.1.** Let  $\mathcal{H}'_\Sigma$  be the associative algebra generated by  $\mathbb{C}[W]$  and  $x_1, \dots, x_n$  with the following relations

$$[x_i, x_j] = 0, \quad \forall i, j, \quad (2.32)$$

$$[s_i, x_j] = 0, \quad \text{if } i \neq j, \quad (2.33)$$

$$s_i x_i - \hat{x}_i s_i = k. \quad (2.34)$$

Here  $k$  is a complex number and

$$\hat{x}_i = x_i - \sum_{j=1}^n (\alpha_i^\vee, \alpha_j) x_j. \quad (2.35)$$

Introducing

$$x_b = \sum_{i=1}^n (b, \alpha_i) x_i = \sum_{i=1}^n k_i x_i \quad \text{for } b = \sum_{i=1}^n k_i b_i, \quad (2.36)$$

we can express the right hand side of (2.35) as  $x_{s_i(b_i)} = x_{b_i} - x_{a_i}$ . More generally,

$$s_i x_b - x_{s_i(b)} s_i = x_b s_i - s_i x_{s_i(b)} = k(b, \alpha_i). \quad (2.37)$$

Later we will use the following partial derivatives

$$\partial_b(u_\alpha) = (\alpha, b). \quad (2.38)$$

For instance,

$$\frac{\partial}{\partial u_i} = \partial_i = \partial_{b_i}$$

**2.4. The AKZ equation associated with  $\mathcal{H}'_\Sigma$ .** In this section we introduce the AKZ equation associated with the root system  $\Sigma$ , and give several examples.

Let us consider the following system of partial differential equations

$$\frac{\partial \Phi}{\partial u_i} = \left( k \sum_{\alpha \in \Sigma_+} \nu_\alpha^i \frac{s_\alpha}{e^{u_\alpha} - 1} + x_i \right) \Phi \quad (1 \leq i \leq n). \quad (2.39)$$

Here  $k$  is a complex number. We denote the right hand side of (2.39) as  $A_i \Phi$ . First we assume that  $\Phi$  takes values in an associative algebra generated by  $\mathbb{C}[W]$  and  $x_1, \dots, x_n$ . We say that the system (2.39) is *self-consistent* provided that

$$\left[ \frac{\partial}{\partial u_i} - A_i, \frac{\partial}{\partial u_j} - A_j \right] = 0. \quad (2.40)$$

It is called *invariant* if, for any solution  $\Phi$  of (2.39) and any element  $w$  of  $W$ ,  $w(\Phi)$  (see 2.30) is again a solution of (2.39).

**Theorem 2.1.** *The system (2.39) is self-consistent and invariant if and only if  $s_1, \dots, s_n, x_1, \dots, x_n$  satisfy the relations (2.32), (2.33), (2.34) defining  $\mathcal{H}'_\Sigma$ .*

We introduce the AKZ equation associated with  $\Sigma$  to be the system (2.39) for functions  $\Phi$  with values in  $\mathcal{H}'_\Sigma$ .

Using the notation  $x_b$  and  $\partial_b$  from (2.36) and (2.38), the system (2.39) can be expressed as

$$\partial_b \Phi = \left( k \sum_{\alpha \in \Sigma_+} (b, \alpha) \frac{s_\alpha}{e^{u_\alpha} - 1} + x_b \right) \Phi. \quad (2.41)$$

*Remark 2.1.* The parameter  $k$  may depend on the lengths of roots. Generally speaking the AKZ equation is as follows:

$$\frac{\partial \Phi}{\partial u_i} = \left( \sum_{\alpha \in \Sigma_+} k_{|\alpha|} \nu_\alpha^i \frac{s_\alpha}{e^{u_\alpha} - 1} + x_i \right) \Phi. \quad (2.42)$$

Let us write down the explicit forms of the AKZ equation in the simplest cases.

**Example 2.2.** When  $\Sigma = A_1$ , the AKZ equation is exactly (2.1).

**Example 2.3.** For  $A_2$ , the AKZ equation is

$$\begin{aligned}\frac{\partial\Phi}{\partial u_1} &= \left\{ k \left( \frac{s_{12}}{e^{u_1} - 1} + \frac{s_{13}}{e^{u_1+u_2} - 1} \right) + x_1 \right\} \Phi, \\ \frac{\partial\Phi}{\partial u_2} &= \left\{ k \left( \frac{s_{23}}{e^{u_2} - 1} + \frac{s_{13}}{e^{u_1+u_2} - 1} \right) + x_2 \right\} \Phi,\end{aligned}$$

where  $s_{ij}$  denotes the transposition of  $i$  and  $j$ . In this case  $\hat{x}_1 = x_2 - x_1$ ,  $\hat{x}_2 = x_1 - x_2$ .

**Example 2.4.** The root system  $B_2$  is realized in the following way. Let  $\epsilon_1$  and  $\epsilon_2$  form an orthonormal basis of  $\mathbb{R}^2$ . Then the set of positive roots consists of the following vectors:

$$\begin{aligned}\alpha_1 &= \epsilon_1 - \epsilon_2, \\ \alpha_2 &= \epsilon_2, \\ \alpha_1 + \alpha_2 &= \epsilon_1, \\ \alpha_1 + 2\alpha_2 &= \epsilon_1 + \epsilon_2.\end{aligned}$$

Let  $s = s_1$  and  $t = s_2$ . Then  $s$  and  $t$  satisfy  $tsts = stst$  (the Coxeter relation for  $W_{B_2} = W_{C_2}$ ) and  $s^2 = 1$ ,  $t^2 = 1$ . In this case  $\hat{x}_1 = x_2 - x_1$ ,  $\hat{x}_2 = 2x_1 - x_2$ . The AKZ equation reads as follows:

$$\begin{aligned}\frac{\partial\Phi}{\partial u_1} &= \left\{ k \left( \frac{s}{e^{u_1} - 1} + \frac{sts}{e^{u_1+u_2} - 1} + \frac{tst}{e^{u_1+2u_2} - 1} \right) + x_1 \right\} \Phi \\ \frac{\partial\Phi}{\partial u_2} &= \left\{ k \left( \frac{t}{e^{u_2} - 1} + \frac{sts}{e^{u_1+u_2} - 1} + 2\frac{tst}{e^{u_1+2u_2} - 1} \right) + x_2 \right\} \Phi.\end{aligned}$$

Note the appearance of the coefficient 2 in the latter. In the case of  $E_8$  the coefficients are from 1 to 6 (otherwise they are less than 6).

**2.5. The  $A_{n-1}$  case.** In this section, we will show that the AKZ equation of type  $GL_n$  discussed in §2.2 reduces to the AKZ equation for the root system  $\Sigma \subset \mathbb{R}^{n-1}$  of type  $A_{n-1}$ .

First note that

$$x = y_1 + \dots + y_n \tag{2.43}$$

is central in the algebra  $\mathcal{H}'_n$ . By setting

$$x_i = y_1 + \dots + y_i - \frac{i}{n}x, \tag{2.44}$$

we have an embedding of  $\mathcal{H}'_\Sigma$ , where  $\Sigma$  is the root system of type  $A_{n-1}$ , into  $\mathcal{H}'_n$ . We put

$$u_i = v_i - v_{i+1} \quad (1 \leq i \leq n-1). \tag{2.45}$$

The space  $\mathbb{R}^{n-1}$  is identified with the quotient space of  $\mathbb{R}^n = \{\sum_{i=1}^n v_i \epsilon_i \mid v_i \in \mathbb{R}\}$

$$\mathbb{R}^{n-1} \simeq \bigoplus_{i=1}^n \mathbb{R} \epsilon_i / \mathbb{R} \epsilon$$

where  $\{\epsilon_i\}_{1 \leq i \leq n}$  is the orthonormal basis and  $\epsilon = \epsilon_1 + \dots + \epsilon_n$ . From (2.25) we have

$$\sum_{i=1}^n \frac{\partial \Phi}{\partial v_i} = x \Phi.$$

Therefore, the function

$$\Phi'(v) = e^{-x \cdot \frac{1}{n}(v_1 + \dots + v_n)} \Phi(v)$$

is well-defined on the quotient space  $\mathbb{R}^{n-1}$ . Now it is straightforward to see that (2.25) reduces to the AKZ equation of type  $A_{n-1}$  for  $\Phi'(v)$

$$\frac{\partial \Phi'}{\partial u_i} = \left( k \sum_{j \leq i < l} \frac{s_{jl}}{e^{u_j + \dots + u_{l-1}} - 1} + x_i \right) \Phi' \quad (1 \leq i \leq n-1).$$

### 3. ISOMORPHISM THEOREMS FOR THE AKZ EQUATION

We introduce the affine Hecke algebra  $\mathcal{H}'_\Sigma$  and connect them with the degenerate affine Hecke algebra  $\mathcal{H}'_\Sigma$  using the monodromy of the AKZ equation. We also establish an isomorphism between the solution space of the AKZ equation and that of a quantum many body problem.

**3.1. Representations of  $\mathcal{H}'_\Sigma$ .** In this section we define induced representations of  $\mathcal{H}'_\Sigma$ .

For  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ , the character of  $\mathbb{C}[x_1, \dots, x_n]$  (i.e. a ring homomorphism  $\mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}$ ) is an assignment  $x_i \mapsto \lambda_i$ . We denote it by  $\lambda$ .

**Definition 3.1.** We define an  $\mathcal{H}'_\Sigma$ -module  $I_\lambda$  as the representation induced from  $\lambda$ :

$$I_\lambda = \text{Ind}_{\mathbb{C}[x_1, \dots, x_n]}^{\mathcal{H}'_\Sigma}(\lambda) = \mathcal{H}'_\Sigma \otimes_{\mathbb{C}[x_1, \dots, x_n]} \mathbb{C}_\lambda. \quad (3.1)$$

Here  $\mathbb{C}_\lambda$  is endowed with the  $\mathbb{C}[x_1, \dots, x_n]$ -module structure by the character  $\lambda$ .

We have the Poincaré-Birkhoff-Witt type theorem for  $\mathcal{H}'_\Sigma$ . Namely any  $h \in \mathcal{H}'_\Sigma$  is expressed uniquely in either of the following ways:

$$h = \sum_{w \in W} p_w(x)w = \sum_{w \in W} wq_w(x) \quad (3.2)$$

with  $p_w, q_w \in \mathbb{C}[x_1, \dots, x_n]$ . The existence results from the relations (2.32)–(2.34) in  $\mathcal{H}'_\Sigma$ . Hence

$$I_\lambda = \mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w. \quad (3.3)$$

Thus  $I_\lambda$  is  $\mathbb{C}[W]$  as a  $W$ -module, where the action of  $x_i$  is determined by  $x_i(e) = \lambda_i e$  for the identity  $e \in W$ . The action of  $x_i$  on other elements of  $\mathbb{C}[W]$  have to be determined using the defining relation (similar to the calculations in the Fock representation).

We also need another construction. Let  $J$  be induced from the trivial character  $+: W \rightarrow \mathbb{C}, w \rightarrow 1$ . Then

$$J = \text{Ind}_{\mathbb{C}[W]}^{\mathcal{H}'_\Sigma}(+), \quad (3.4)$$

is isomorphic to  $\mathbb{C}[x_1, \dots, x_n]$  as a vector space and moreover as a  $\mathbb{C}[x_1, \dots, x_n]$ -module. To get finite-dimensional representations from  $J$ , we use the coincidence of the center of  $\mathcal{H}'_\Sigma$  with the algebra of  $W$ -invariant polynomials in  $x_i$ . This theorem is due to Bernstein. The procedure is as follows. Let us fix an element  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  and introduce the ideal  $L_\lambda$  in  $\mathbb{C}[x_1, \dots, x_n]$  generated by  $p(x) - p(\lambda)$  for all  $W$ -invariant polynomials  $p$ . Set  $J_\lambda = J/L_\lambda$ . Then  $J_\lambda$  has a structure of  $\mathcal{H}'_\Sigma$ -module by virtue of the Bernstein theorem.

We will also use the anti-involution  $^\circ$  on  $\mathcal{H}'_\Sigma$ :

$$x_i^\circ = x_i, \quad s_i^\circ = s_i, \quad (ab)^\circ = b^\circ a^\circ, \quad k^\circ = k. \quad (3.5)$$

Since the relations of  $\mathcal{H}'_\Sigma$  are self-dual it is well-defined. For an  $\mathcal{H}'_\Sigma$ -module  $V$ , we consider its dual  $\text{Hom}_\mathbb{C}(V, \mathbb{C})$ . The dual has an anti-action (a right action) of  $\mathcal{H}'_\Sigma$ . Composing it with the anti-automorphism  $^\circ$ , we get a natural (left) action of  $\mathcal{H}'_\Sigma$ . We denote the resulting module by  $V^\circ$ .

We write  $\lambda_b = \sum k_i \lambda_i$  for  $b = \sum k_i b_i$ .

**Theorem 3.1.** (a)  $I_\lambda$  is irreducible if and only if  $\lambda_{\alpha^\vee} \neq \pm k$  for any  $\alpha \in \Sigma_+$ .  
 (b) There exists a permutation  $\lambda'$  of  $\lambda$  (i.e.  $\lambda' = w(\lambda)$  for  $w \in W$ ) such that  $\lambda'_{\alpha^\vee} \neq -k$  for any  $\alpha \in \Sigma_+$ . Then

$$J_\lambda \simeq I_{\lambda'}. \quad (3.6)$$

(c) For the longest element  $w_0$  in  $W$ ,

$$I_\lambda^\circ = I_{w_0(\lambda)} \quad (3.7)$$

A key lemma in proving Theorem 3.1 is

**Lemma 3.2.**  $I_{(0, \dots, 0)}$  is irreducible.

The proof from [20] is based on the intertwining operators of degenerate affine Hecke algebras (to be defined below). See also [45, 44, 62] and the references therein (the non-degenerate case).

**Definition 3.2.** For  $1 \leq i \leq n$  we set

$$f_i = f_{s_i} = s_i - \frac{k}{x_{a_i}}. \quad (3.8)$$

For  $w \in W$  with a reduced decomposition  $w = s_{i_n} \cdots s_{i_1}$ ,  $f_w = f_{i_n} \cdots f_{i_1}$ . We call the elements  $f_w$  *intertwiners*.

The elements  $f_w$  belong to the localization of the degenerate affine Hecke algebra  $\mathcal{H}'_\Sigma$  by the  $W$ -invariant polynomials. They give a certain ‘baxterization’ of  $w$ , and are closely related to the Yang’s  $R$ -matrix. Let us show that  $f_w$  does not depend on the choice of the reduced decomposition of  $w$ .

We have

$$f_{s_i} x_b = x_{s_i(b)} f_{s_i}, \quad (3.9)$$

$$f_w x_b = x_{w(b)} f_w. \quad (3.10)$$

Indeed,

$$\left( s_i - \frac{k}{x_{a_i}} \right) x_b = x_{s_i(b)} \left( s_i - \frac{k}{x_{a_i}} \right), \quad (3.11)$$

which can be rewritten as follows:

$$s_i x_b - x_{s_i(b)} s_i = -k \frac{x_{s_i(b)} - x_b}{x_{a_i}}. \quad (3.12)$$

Using the definition of  $x_b$  (2.36), the right hand side of (3.12) is  $k(b, \alpha_i)$ . So we come to (2.37). The relations (3.10) fix  $f_w$  uniquely up to the multiplication on the right by functions in  $x$ . The leading terms of  $f_w$  being  $w$ , they coincide for any reduced decompositions.

To demonstrate the role of intertwiners, let us check the irreducibility of  $I_\lambda$  for generic  $\lambda$ . First note that the vectors  $\{f_w(e) \in I_\lambda\}$  are common eigenvectors of  $x_b$ , because

$$x_b f_w(e) = f_w x_{w^{-1}(b)}(e) = \lambda_{w^{-1}(b)} f_w(e),$$

For a generic  $\lambda$  the eigenvalues are simple, hence these vectors are linearly independent. Now, any nonzero  $\mathcal{H}'_\Sigma$ -submodule  $A$  of  $I_\lambda$  contains at least one eigenvector of  $x_b$ . By the simplicity of eigenvalues, such an eigenvector must be in the form  $f_w(e)$  for some  $w \in W$ . On the other hand,  $f_w$  are invertible elements. Indeed,

$$f_i^{-1} = \left(1 - \frac{k^2}{x_{a_i}^2}\right)^{-1} f_i.$$

Therefore  $e \in A$ . Since  $I_\lambda$  is generated by  $e$ , we conclude that  $A = I_\lambda$ .

Actually this very reasoning leads to the proof of the Theorem (a),(b). However if  $\lambda$  is arbitrary one must operate with the intertwiners much more carefully. It is necessary to multiply them by the denominators and remember that the invertibility does not hold for special  $\lambda$ .

*Remark 3.1.* The  $\mathcal{H}'$ -quotients  $A$  of  $J_\lambda^\circ$  will be interpreted below as certain quotients of the  $D$ -module representing the quantum many-body eigenvalue problem. A solution of the AKZ in  $J_\lambda^\circ$  induces solutions in any of its  $\mathcal{H}'$ -quotients (if  $I$  is reducible). It gives a one-to-one correspondence between the  $\mathcal{H}'$ -submodules (quotients, constituents) of  $J$  and those of the  $D$ -modules representing the quantum many-body eigenvalue problem. The description of the latter is an analytical problem. The classification of the former is a difficult question in the representation theory of Hecke algebras. For instance, the multiplicities of the irreducible constituents are described in terms of the Kazhdan-Lusztig polynomials. It is very interesting to combine the two approaches.

**3.2. The monodromy of the AKZ equation.** In this section we discuss the monodromy of the AKZ equation, which is a key ingredient in establishing the isomorphism between the AKZ equation in the representation  $J_\lambda^\circ$  and the *quantum many-body problem* (QMBP) with the eigenvalue  $\lambda$ .

Let  $U'$  be the open subset of  $\mathbb{C}^n$  given by

$$U' = \{u \in \mathbb{C}^n \mid \prod_{\alpha \in \Sigma_+} (e^{u_\alpha} - 1) \neq 0\}. \quad (3.13)$$

The lattice generated by  $b_1, \dots, b_n$  will be denoted by  $B$ . It is isomorphic to  $\mathbb{Z}^n$  and acts on  $\mathbb{C}^n$  by translations. Namely,  $b(u) = u + 2\pi\sqrt{-1}b$ , where  $b \in B$ . The

semi-direct product  $\widetilde{W} = W \ltimes B$  is the so-called extended affine Weyl group, acting on  $\mathbb{C}^n$  and leaving  $U'$  invariant. Picking  $u^0 \in U'$ , we set

$$\pi_1 = \pi_1(U'/\widetilde{W}, u^0).$$

The group structure of  $\pi_1$  is described as follows. Given an element  $w \in \widetilde{W}$ , let  $\gamma_w$  be a path from  $u^0$  to  $w^{-1}(u^0)$  in  $U'$ . For elements  $w_1, w_2 \in \widetilde{W}$ , we define the composition  $\gamma_{w_2} \circ \gamma_{w_1}$  of  $\gamma_{w_1}$  and  $\gamma_{w_2}$  as the path composed of  $\gamma_{w_1}$  and the path  $\gamma_{w_2}$  mapped by  $w_1^{-1}$  (see Fig5). The class of  $\gamma$  will be denoted by  $\bar{\gamma}$ . The map  $\bar{\gamma}_w \rightarrow w$  is a homomorphism onto  $\widetilde{W}$ .

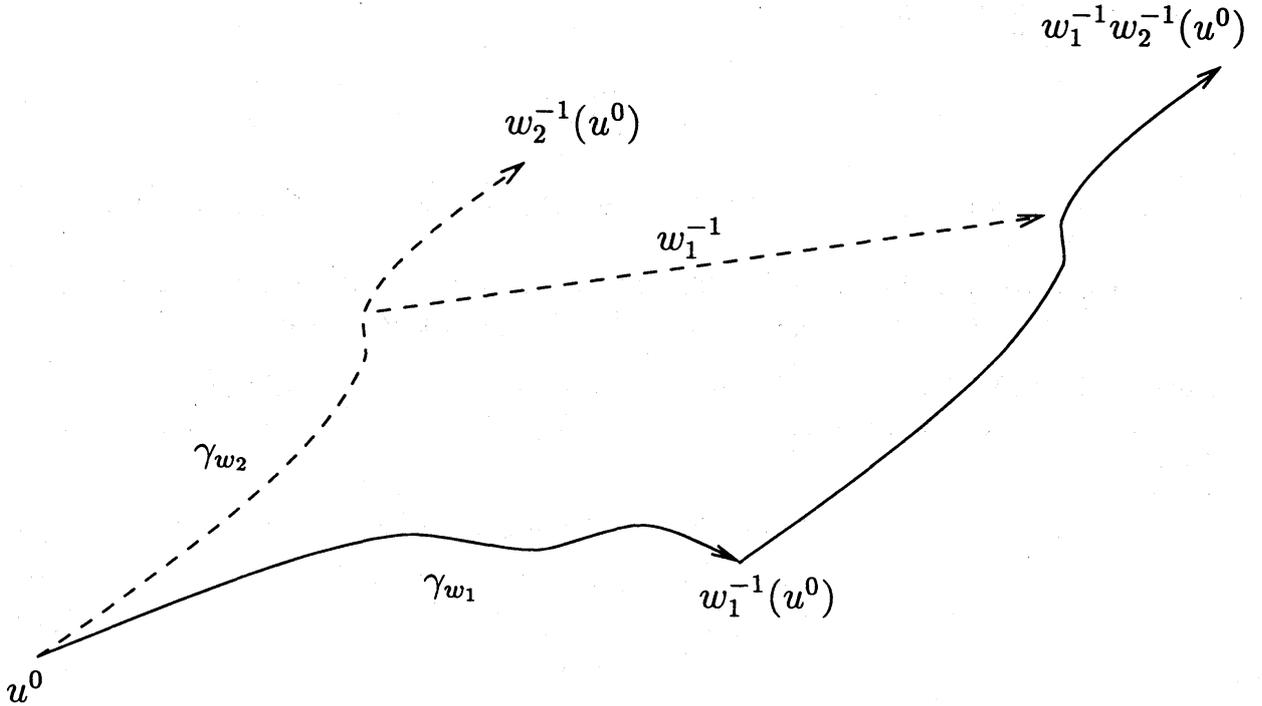


FIGURE 5. Composition of paths

It is convenient to choose  $u^0$  and the generators of  $\pi_1$  as follows. Set  $\Re = \text{Re}$ ,  $\Im = \text{Im}$ ,

$$C = (\sqrt{-1}\mathbb{R})^n \setminus \{u \in (\sqrt{-1}\mathbb{R})^n \mid 0 < \Im u_\alpha < 2\pi \text{ for every } \alpha \in \Sigma_+\}.$$

Then  $\mathbb{C}^n \setminus C$  is a simply connected open subset of  $U'$ . Let us take  $u^0 \in \mathbb{C}^n$  such that  $\Re u_\alpha^0 \gg 0$  for  $\alpha \in \Sigma_+$ . For any element  $w \in \widetilde{W}$ , we denote a path from  $u^0$  to  $w^{-1}(u^0)$  in  $\mathbb{C}^n \setminus C$  by  $\gamma_w$ . This condition simply means that whenever  $u_\alpha \in i\mathbb{R}$  intersects the imaginary axis it must go through the 'window'  $0 < \Im u_\alpha < 2\pi$ .

For any element  $w \in \widetilde{W}$  we define an element  $\bar{\gamma}_w$  of  $\pi_1$  to be the image of  $\gamma_w$ . Since  $\mathbb{C}^n \setminus C$  is simply connected,  $\bar{\gamma}_w$  depends only on  $w$ . We set  $\tau_i = \gamma_{s_i}$  and choose  $\chi_i$  to be a path from  $u^0$  to the point  $u'$  with the same coordinates  $u'_j = u_j^0$

for  $j \neq i$  and  $u'_i = u_i^0 + 2\pi\sqrt{-1}$ . The structure of  $\pi_1$  is described in the following theorem from [51].

**Theorem 3.3.**

$$\pi_1 = \langle \bar{\tau}_1, \dots, \bar{\tau}_n, \bar{\chi}_1, \dots, \bar{\chi}_n \rangle, \quad (3.14)$$

$$\bar{\tau}_i \text{ satisfy the Coxeter relations,} \quad (3.15)$$

$$[\bar{\chi}_i, \bar{\chi}_j] = [\bar{\tau}_i, \bar{\chi}_j] = 0 \quad (i \neq j), \quad (3.16)$$

$$\bar{\tau}_i^{-1} \bar{\chi}_i \bar{\tau}_i^{-1} = \bar{\chi}_{s_i(b_i)}. \quad (3.17)$$

Here for  $b = \sum_{i=1}^n k_i b_i$  we put

$$\bar{\chi}_b = \prod_{i=1}^n \bar{\chi}_i^{k_i}. \quad (3.18)$$

Fig6 proves the relation (3.17). It shows the  $u_i$ -coordinate only, which is sufficient for this relation.

Let us introduce the affine Hecke algebra  $\mathcal{H}_\Sigma^t$  associated with a root system  $\Sigma$  as a quotient of the group algebra of  $\pi_1$  by the quadratic relations.

**Definition 3.3.** The affine Hecke algebra associated with a root system  $\Sigma$  is an associative  $\mathbb{C}$ -algebra generated by  $1, T_1, \dots, T_n, X_1, \dots, X_n$  with the following relations:

$$T_i \text{ satisfy the Coxeter relations,} \quad (3.19)$$

$$[X_i, X_j] = [T_i, X_j] = 0 \quad i \neq j, \quad (3.20)$$

$$T_i^{-1} X_i T_i^{-1} = X_{s_i(b_i)}, \quad (3.21)$$

$$(T_i - t)(T_i + t^{-1}) = 0. \quad (3.22)$$

The monomials  $X_b$  are defined as in (3.18),  $t \in \mathbb{C}^*$ . Here and above we mean the homogeneous Coxeter relations:  $T_i T_j T_i \dots = T_j T_i T_j \dots$ ,  $m_{ij}$  factors on each side, where  $m_{ij} = 2, 3, 4$  whenever the corresponding vertices in the Dynkin diagram are connected by 0,1,2 laces.

Let  $\Phi$  be an invertible solution of the AKZ equation associated with  $\Sigma$ , defined in a neighborhood of  $u^0$ . Then, for  $w \in \widetilde{W}$ ,  $w^{-1}(\Phi)$  is defined near  $w^{-1}(u^0)$  (see (2.30)). Let  $\gamma$  be a path in  $U'$  from  $u^0$  to  $w^{-1}(u^0)$ . Denote by  $(w^{-1}(\Phi))_{\bar{\gamma}}$  the analytic continuation of  $w^{-1}(\Phi)$  back to  $u^0$  along the path  $\gamma$ , where  $\bar{\gamma}$  denotes the class of  $\gamma$  in the fundamental group  $\pi_1$ . We will also use the projection homomorphism  $\widetilde{W} \rightarrow W$  sending  $w = \bar{w}b$  to  $\bar{w}$  for  $b \in B, \bar{w} \in W$ . Using this homomorphism we can extend the action of  $W$  from (2.30) to  $\widetilde{W}$ , multiplying  $\Phi$  on the left by  $\bar{w}$ .

Let us define the monodromy  $T_{\bar{\gamma}}$  to be the ratio

$$T_{\bar{\gamma}} = (w^{-1}(\Phi))_{\bar{\gamma}}^{-1} \Phi = (\Phi(w(u)))_{\bar{\gamma}}^{-1} \cdot \bar{w} \cdot \Phi. \quad (3.23)$$

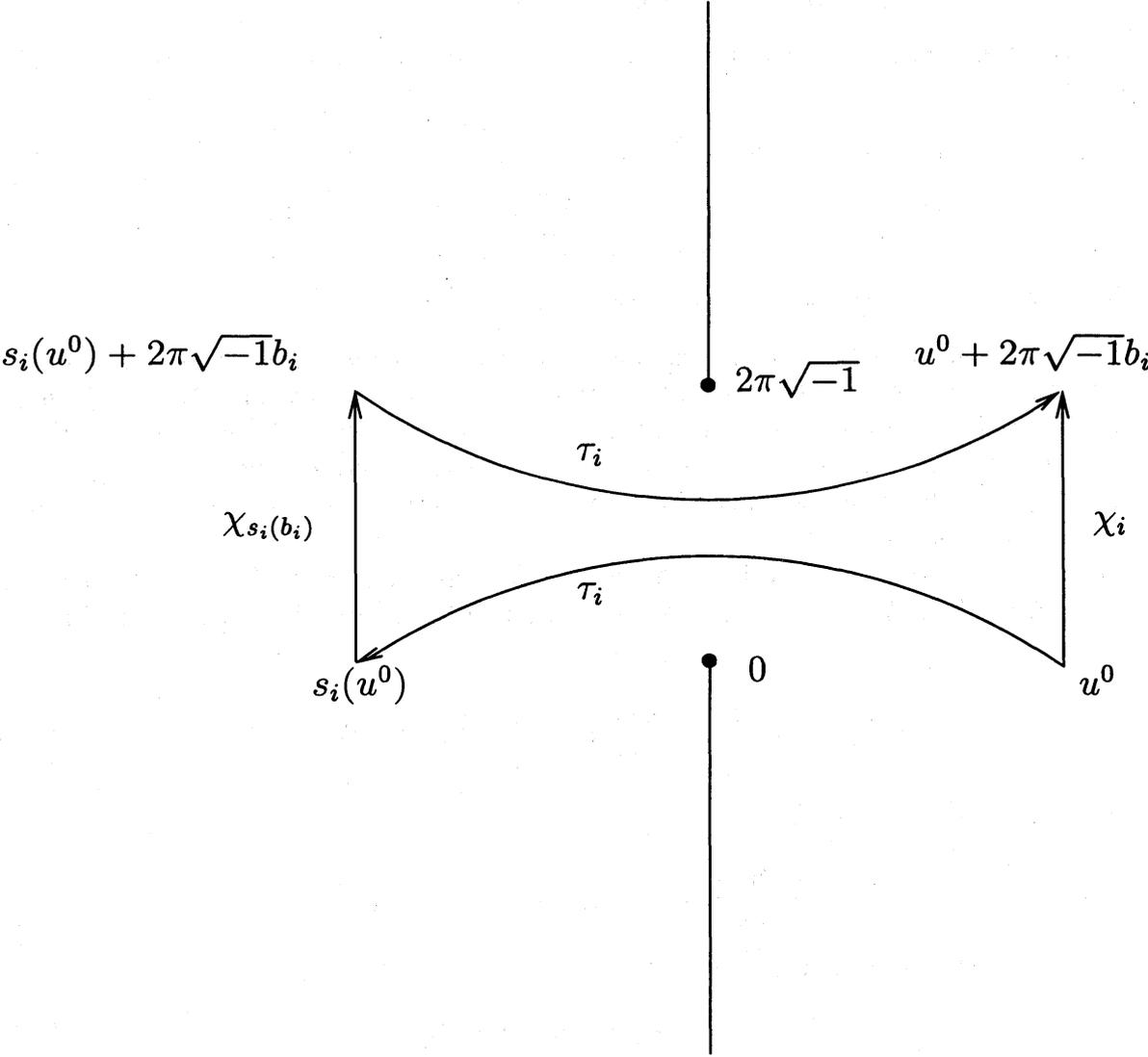


FIGURE 6. Proof of the relation (3.17)

Here the dot means the product in  $\mathcal{H}'_\Sigma$ . Since  $\Phi$  and  $w^{-1}(\Phi)$  both satisfy the same AKZ equation,  $T_\gamma$  does not depend on  $u$ . So it is an invariant of the homotopy class of  $\gamma$  and is always invertible. If we choose  $u^0$  and the paths  $\gamma_w$  in  $\mathbb{C}^n \setminus C$  as above, then  $T_w$  for  $w \in \widetilde{W}$  are well-defined. The monodromy is a homomorphism from  $\pi_1$  (but not from  $\widetilde{W}$ ), which readily results from the definition.

As a preparation for an explicit computation of  $\{T_w\}$  in the next section, we shall introduce a special class of solutions  $\Phi$ .

**Proposition 3.4.** *For generic  $\lambda$ , there exists a unique solution  $\Phi_{as}(u)$  of the AKZ equation such that*

$$\Phi_{as}(u) = \widehat{\Phi}(u) e^{\sum_{i=1}^n u_i x_i} \quad \text{for} \quad (3.24)$$

$$\widehat{\Phi}(u) = 1 + \sum_{m=(m_1, \dots, m_n), m_i \geq 0, m \neq 0} \Phi_m e^{-\sum_{i=1}^n m_i u_i}, \quad (3.25)$$

where  $\Re u \rightarrow \infty$ , and  $\Phi_m$  are independent of  $u$ .

We call the solution in the proposition the *asymptotically free solution*. To be more exact, we need either to complete  $\mathcal{H}'_\Sigma$ , or restrict ourselves with finite-dimensional representations of this algebra. Then establishing the (local) convergence is easy. In these notes we will follow the second way. We give general formulas, which are quite rigorous in finite-dimensional representations (say, in the induced representations).

Let us examine the condition necessary for the existence of the asymptotically free solutions in the case of  $A_1$ . A general consideration follows the same lines. In this case,

$$\Phi_{as}(u) = \left( 1 + \sum_{m>0} \Phi_m e^{-mu} \right) e^{ux} = \widehat{\Phi}(u) e^{ux}. \quad (3.26)$$

The equation (2.39) leads to

$$\frac{\partial \widehat{\Phi}(u)}{\partial u} = k \frac{s}{e^u - 1} \widehat{\Phi}(u) + [x, \widehat{\Phi}(u)]. \quad (3.27)$$

Comparing the coefficients of  $e^{-mu}$ :

$$-m\Phi_m = [x, \Phi_m] + (\text{terms with } \Phi_j, j < m). \quad (3.28)$$

Given a representation of  $\mathcal{H}'_\Sigma$ , we find  $\Phi_m$  assuming that  $m + \text{ad}(x)$  is invertible for any  $m > 0$  in this representation. Therefore, setting  $\text{Spec}(x) = \{\mu_j\}$ , the conditions  $m + \mu_i - \mu_j \neq 0$ ,  $m = 1, 2, \dots$ , ensure the existence of the asymptotically free solutions. The convergence estimates are straightforward. These conditions are fulfilled in generic induced representations.

**3.3. Lusztig's isomorphisms via the monodromy.** In this section we establish an isomorphism between  $\mathcal{H}'_\Sigma$  and  $\mathcal{H}^t_\Sigma$  using the monodromy of the AKZ equation.

Let us fix an invertible solution  $\Phi(u)$  of the AKZ system in a neighborhood of  $u^0 \in U^* = \mathbb{C}^n \setminus C \subset U'$ . The functions  $\Phi(w(u))$  will be extended to  $u^0$  through  $U^*$ . Since  $\mathcal{H}_\Sigma$  is infinite-dimensional, we have to consider all formulas in finite dimensional representations. Once we get the final expressions it is not difficult to find a proper completion of the degenerate Hecke algebra for them.

**Theorem 3.5** ([13]). *There exists a homomorphism from  $\mathcal{H}^t_\Sigma$  to  $\mathcal{H}'_\Sigma$  given by*

$$T_j \longmapsto T'_j, \quad X_j \longmapsto X'_j,$$

where

$$T'_j = \Phi(s_j(u))^{-1} s_j \Phi(u), \quad X'_j = \Phi(u - 2\pi\sqrt{-1}b_j)^{-1} \Phi(u).$$

If  $t = \exp(\pi\sqrt{-1}k)$  is sufficiently general (say, not a root of unity), then it is an isomorphism at the level of finite dimensional representations or after a proper completion.

Under the notation (3.23),  $T'_j = T_{\bar{\tau}_j}$  and  $X'_j = T_{\bar{x}_j}$ . Hence the relations (3.19)–(3.21) result from Theorem 3.3, and only the quadratic relations (3.22) need to be proved. We omit a simple direct proof since these relations follow from the exact formulas below.

Let us find the formulas for  $T'_j$  and  $X'_j$  for the asymptotically free solution  $\Phi_{as}(u)$ . Given  $b \in B$ , we set

$$X_b = \prod_{j=1}^n X_j^{k_j} \quad \text{for } b = \sum_{j=1}^n k_j b_j,$$

and define  $X'_b$  analogously.

**Theorem 3.6** ([12]). *Let us choose the asymptotically free solution  $\Phi_{as}(u)$  as  $\Phi(u)$ . Then*

$$(a) \quad X'_j = \exp(2\pi\sqrt{-1}x_j),$$

$$(b) \quad s_i - \frac{k}{x_{a_i}} = g(x_{a_i}) \left( T'_i + \frac{t - t^{-1}}{X'_{a_i} - 1} \right),$$

where the function  $g(v)$  is defined by

$$g(v) = \frac{\Gamma(1+v)^2}{\Gamma(1+k+v)\Gamma(1-k+v)},$$

and (b) is in fact a formula for  $T'_i$  in terms of  $\{s, x\}$ .

We will give a sketch of the proof of Theorem 3.6. The statement (a) is immediate, since

$$\Phi_{as}(u - 2\pi\sqrt{-1}b_j) = \hat{\Phi}(u)e^{\sum u_i x_i - 2\pi\sqrt{-1}x_j} = \Phi_{as}(u)e^{-2\pi\sqrt{-1}x_j}.$$

To prove the statement (b), we reduce the problem to the  $A_1$  case. Let us fix the index  $i$  ( $1 \leq i \leq n$ ). Set  $E(u) = e^{\sum_{i=1}^n u_i x_i}$ , so that  $\Phi_{as}(u) = \hat{\Phi}(u)E(u)$ . Let us define  $\Phi^{(i)}(u)$  as follows:

$$\Phi^{(i)}(u) = \Phi^{\infty(i)}(u_i)E(u),$$

where  $\Phi^{\infty(i)}(u_i) = \lim_{\Re u_j \rightarrow +\infty (j \neq i)} \hat{\Phi}(u)$ . The AKZ system for  $\Phi^{(i)}(u)$  reads:

$$\frac{\partial \Phi^{(i)}}{\partial u_i} = \left( k \frac{s_i}{e^{u_i} - 1} + x_i \right) \Phi^{(i)}, \quad (3.29)$$

$$\frac{\partial \Phi^{(i)}}{\partial u_j} = x_j \Phi^{(i)} \quad (j \neq i). \quad (3.30)$$

**Reduction procedure.** Since the monodromy  $T'_i$  does not depend on  $u$ , the point  $u^0$  and the path connecting  $u^0$  and  $s_i(u^0)$  may be replaced by any deformations in  $U'$  or their limits. Provided the existence, the resulting monodromy coincides with  $T'_i$ . For instance,  $T'_i$  equals

$$T'_i = (\Phi^{(i)}(s_i(u)))^{-1} s_i \Phi^{(i)}(u).$$

Indeed, the latter is the limiting monodromy for a path with  $\Re u_j (j \neq i)$  approaching the infinity. We note that  $\Re u_j (s_i(u^0)) \rightarrow +\infty$  if  $\Re u_j (u^0)$  does.

In the reduced equations (3.29) and (3.30), we may diminish the values, considering the subalgebra of  $\mathcal{H}'_\Sigma$  generated by  $x_j$  ( $1 \leq j \leq n$ ), and  $s_i$ . In this algebra, the following elements are central:

$$x_j (j \neq i), \quad x_i - \frac{1}{2}x_{a_i}.$$

Hence, if we define  $E^{(i)}(u)$  by

$$E^{(i)}(u) = e^{\sum_{j=1}^n u_j x_j - u_i x_{a_i}/2},$$

it enjoys the following properties:

- (i)  $E^{(i)}(u)$  commutes with  $s_i$ ,
- (ii)  $E^{(i)}(s_i(u)) = E^{(i)}(u)$ .

The second property can be verified directly:

$$\sum_j (s_i(u))_j x_j - \frac{1}{2} (s_i(u))_i x_{a_i} = \sum_j (u_j - (a_i, \alpha_j) u_i) x_j + \frac{1}{2} u_i x_{a_i} = \sum_j u_j x_j - \frac{1}{2} u_i x_{a_i}.$$

We have used that  $(s_i(u))_j = (\alpha_j, s_i(u))$  and  $\sum_j (a_i, \alpha_j) x_j = x_{a_i}$ . Setting

$$\tilde{\Phi}^{(i)}(u) = \Phi^{(i)}(u)E^{(i)}(u)^{-1},$$

the system of equations (3.29),(3.30) becomes precisely the AKZ equation for  $\tilde{\Phi}^{(i)}(u)$  in the  $A_1$  case:

$$\frac{\partial \tilde{\Phi}^{(i)}(u)}{\partial u_i} = \left( k \frac{s_i}{e^{u_i} - 1} + \frac{1}{2} x_{a_i} \right) \tilde{\Phi}^{(i)}(u), \quad (3.31)$$

$$\frac{\partial \tilde{\Phi}^{(i)}(u)}{\partial u_j} = 0 \quad (j \neq i). \quad (3.32)$$

Because of the above properties of  $E^{(i)}(u)$ , the monodromy of  $\tilde{\Phi}^{(i)}(u)$  coincides with  $T'_i$ . However  $\tilde{\Phi}^{(i)}(u)$  can be expressed in terms of the hypergeometric function, which conclude the proof up to a straightforward calculation.

To explain the structure of the formula for  $T'$ , let us involve the intertwiners of  $\mathcal{H}_\Sigma^t$ . They are defined similar to those in the degenerate case:

$$f_i = s_i - \frac{k}{x_{a_i}} \text{ for } \mathcal{H}'_\Sigma, \quad F_i = T_i + \frac{t - t^{-1}}{X_{a_i}^{-1} - 1} \text{ for } \mathcal{H}_\Sigma^t.$$

**Lemma 3.7.**  $F_i X_b = X_{s_i(b)} F_i$ .

It readily results from the definition of  $\mathcal{H}_\Sigma^t$  (cf. 3.10).

The image  $F'_i$  of  $F_i$  in  $\mathcal{H}'_\Sigma$  with respect to the homomorphism constructed in Theorem 3.5 can be represented as  $F'_i = g_i(x) f_i$  for a function  $g_i$  of  $x$ . Indeed,  $f_i X'_b = X'_{s_i(b)} f_i$ , which gives the proportionality. Recall that  $X'_b = \exp(2\pi\sqrt{-1}x_b)$ . Here  $g_i(x)$  must be of the form  $g(x_{a_i})$  for a function  $g$  in one variable, and can be calculated using the hypergeometric equation (3.31). We omit the details (see [12]).

We note that the quadratic relations for  $T'_i$  can be made quite obvious using the same reduction (the exact formulas above are not necessary). Set  $i = 1$  to simplify the indices. We switch from (3.31) to (2.18) with two variables  $z_1, z_2$  and a parameter  $z_0$ :

$$\frac{\partial \Phi'}{\partial z_j} = \left[ k \left( \frac{s_1}{z_j - z_k} \right) + \frac{\Omega_j}{z_j - z_0} \right] \Phi' \quad (j = 1, 2, k = 3 - j). \quad (3.33)$$

When  $z_0 = 0$  the substitutions are as follows

$$2x_1 = \Omega_1 - \Omega_2 + ks_1, \quad u_1 = \log(z_1/z_2), \quad \Phi' = \Phi^{(1)}(u_1)(z_1 z_2)^{-1/2(\Omega_1 + \Omega_2 + ks_1)}.$$

The monodromy corresponding to the transposition of  $z_1$  and  $z_2$  for  $z_0 = 0$  coincides with  $T'_1$ . It does not depend on  $z_0$  up to a conjugation (the same reduction argument applied to the KZ-equation with three variables). Sending  $z_0$  to infinity we eliminate the  $\Omega$ -terms. The monodromy of the resulting equation can be calculated immediately. Since it is conjugated to  $T'_1$  we get the desired quadratic relations.

Heckman in [39] used a similar reduction approach when calculating the monodromy of the quantum many-body problem (also called the Heckman-Opdam system). Our next aim is to establish an isomorphism of AKZ and the latter.

Combining Heckman's formulas and mine for the AKZ, which coincide since the representation of  $\mathcal{H}^t$  is the same, we readily conclude that these equations are isomorphic for generic  $\lambda$ . This will be made much more constructive below. We will also consider any  $\lambda$ .

*Remark 3.2.* Let us apply Theorem 3.6 to the standard rational KZ equation in the  $GL_n$  case. We calculated the monodromy of

$$\frac{\partial \Phi}{\partial v_i} = \left( k \sum_{j>i} \frac{s_{ij}}{e^{v_i-v_j} - 1} - k \sum_{j<i} \frac{s_{ij}}{e^{v_j-v_i} - 1} + y_i \right) \Phi \quad (1 \leq i \leq n).$$

Taking special  $y_i = k \sum_{j=i+1}^n s_{ij}$  and substituting  $z_i = e^{v_i}$ , we come to

$$\frac{\partial \Phi}{\partial z_i} = k \sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} \Phi \quad (1 \leq i \leq n).$$

It corresponds to the simplest  $\Omega_{ij} = 0$  in (2.18). By the way, these  $\{y\}$  induce a homomorphism from  $\mathcal{H}'_n$  to  $\mathbb{C}\mathbb{S}_{n+1}$  due to Drinfeld. Diagonalizing the commuting elements  $\sum_{j>i} s_{ij}$  we recover the monodromy computed by Tsuchiya-Kanie [68]. It also gives an explicit example of the general results on the monodromy of the rational KZ over Lie algebras due to Drinfeld and Kohno (see [49]).

*Remark 3.3.* In Theorems 3.5 and 3.6, we established the isomorphism

$$\mathcal{H}'_{\Sigma} \simeq \mathcal{H}'_{\Sigma}, \quad X_j \longmapsto t^{2x_j},$$

where  $t = e^{\pi\sqrt{-1}k}$  and represented it as a relation between the intertwiners of the degenerate and non-degenerate affine Hecke algebras:

$$F_j = T_j + \frac{t - t^{-1}}{X_{a_j}^{-1} - 1} \longmapsto g(x_{a_j}) \left( s_j - \frac{k}{x_{a_j}} \right).$$

This construction can be naturally generalized. In fact we need only a very mild restriction on  $g(x)$  to get such a homomorphism. Normalizing the intertwiners to make them 'unitary' ( $f^2 = 1 = F^2$ ), we come to the simplest possible map:

$$X_j \longmapsto t^{2x_j}, \quad \frac{F_j}{t + \frac{t-t^{-1}}{X_{a_j}^{-1} - 1}} \longmapsto \frac{s_j - \frac{k}{x_{a_j}}}{1 - \frac{k}{x_{a_j}}}.$$

Actually here we have four formulas in one since we can put the denominators on the right and on the left. One of them was found by Lusztig in [52].

**3.4. The isomorphism of AKZ and QMBP.** Here we present the isomorphism between the AKZ equation and the quantum many-body problem (QMBP). The latter will appear as a ‘trace’ of the first.

We will need a variant of the general notion of monodromy by A. Grothendieck. Let us fix the notations:

$${}^w\Phi(u) = \Phi(w^{-1}(u)), \quad w = \bar{w}b \in \widetilde{W} = W \ltimes B, \quad u \in \mathbb{C}^n.$$

Given a finite union  $C$  of affine real closed half-hyperplanes, we set  $U = \mathbb{C}^n \setminus C$  assuming that

- (i)  $U$  does not contain ‘bad hyperplanes’  $\prod_{\alpha \in \Sigma_+} (e^{u_\alpha} - 1) = 0$ ,
- (ii)  $U$  is simply connected,
- (iii)  $(\mathbb{C}^n \setminus \cup_{w \in \widetilde{W}} w(C)) / \widetilde{W}$  is connected.

We shall refer to such  $C$  as a *system of cutoffs*. In §3.2, a special system of cutoffs ( $U^*$ ) has been already used in order to compute the monodromy.

Let us fix a system of cutoffs  $C$  and  $U$ . Then for each  $w \in \widetilde{W}$  there is a path  $\gamma_w$  (unique up to homotopy) joining  $u^0$  and  $w^{-1}(u^0)$ . So the choice of  $C$  implies a choice of representatives  $\bar{\gamma}_w$  in the fundamental group  $\pi_1(U'/\widetilde{W}, u^0)$ . Here  $U'$  is the complement of the union of ‘bad hyperplanes’ (3.13).

We pick a solution  $\Phi$  of the AKZ equation in  $U$  and define the monodromy function  $\mathcal{T}_w$  ( $w \in \widetilde{W}$ )

$$\bar{w}\Phi = w^{-1}\Phi \cdot \mathcal{T}_w \quad w = \bar{w}b \in \widetilde{W}. \quad (3.34)$$

Here  $\Phi$  is invertible at least at one point and is extended analytically to the whole  $U$ . The values are in the endomorphisms of any finite-dimensional representation of  $\mathcal{H}'_\Sigma$  (we will apply the construction to the induced representations).

The monodromy  $\{\mathcal{T}_w\}_{w \in \widetilde{W}}$  satisfies the following:

- (a) (1-cocycle condition)  $v^{-1}(\mathcal{T}_w)\mathcal{T}_v = \mathcal{T}_{wv} \quad \forall w, v \in \widetilde{W}$ ,
- (b)  $\frac{\partial}{\partial u_i}\mathcal{T}_w = 0$ , and hence  $\mathcal{T}_w$  is locally constant.

The property (b) holds since both  $\Phi$  and  $w(\Phi) = \bar{w}^w\Phi$  satisfy the same differential equation of the first order (the AKZ equation). It readily results in the invertibility of  $\mathcal{T}_w$  on  $\mathbb{C}^n - \cup_{w \in \widetilde{W}} w(C)$ . The latter set is not connected, so  $\mathcal{T}$  is not just a constant.

Next let us introduce the operators  $\sigma_w, \sigma'_w$  ( $w \in \widetilde{W}$ ), acting on functions  $F$  on  $U$ :

$$\begin{aligned} (\sigma_w F)(u) &= ({}^{w^{-1}}F)(u) = F(w(u)), \quad \sigma_i = \sigma_{s_i}, \\ (\sigma'_w F)(u) &= ({}^{w^{-1}}F)(u)\mathcal{T}_w, \quad \sigma'_i = \sigma'_{s_i}. \end{aligned}$$

The relations for the operators  $\sigma'_w$  are the same as for the permutations  $\sigma_w$ :

$$\begin{aligned}
\text{a)} \quad & \sigma'_w \sigma'_v = \sigma'_{vw}, \\
\text{b)} \quad & \sigma'_w u_b = u_{w^{-1}(b)} \sigma'_w, \\
\text{c)} \quad & \sigma'_w \partial_b = \partial_{w^{-1}(b)} \sigma'_w, \quad \partial_b(u_\alpha) = (b, \alpha).
\end{aligned} \tag{3.35}$$

Note that the property a) follows from the 1-cocycle condition for  $\{\mathcal{T}_w\}_{w \in \tilde{W}}$ . Indeed,

$$\begin{aligned}
(\sigma'_w \sigma'_v)(F) &= \sigma'_w(\sigma'_v(F)) \\
&= \sigma'_w(v^{-1} F \mathcal{T}_v) \\
&= w^{-1}(v^{-1} F \mathcal{T}_v) \mathcal{T}_w \\
&= w^{-1} v^{-1} F (w^{-1} \mathcal{T}_v) \mathcal{T}_w \\
&= (vw)^{-1} F \mathcal{T}_{vw} \\
&= \sigma'_{vw}(F).
\end{aligned}$$

Let  $Sol_{AKZ}$  be the space of solutions of the AKZ equation with values in  $\mathcal{H}'_\Sigma$ . When we consider the AKZ equation on a finite-dimensional  $\mathcal{H}'_\Sigma$ -module  $V$ , we will denote the space of its solutions by  $Sol_{AKZ}(V)$ . Starting with AKZ let us go to QMBP. In what follows,  $\Phi \in Sol_{AKZ}$  or  $\Phi \in Sol_{AKZ}(\text{End}(V))$ . In the latter case all operators act on  $\text{End}(V)$ -valued functions.

(1) Using  $s_\alpha \Phi = \sigma'_{s_\alpha} \Phi$ , we rewrite the AKZ equation:

$$\begin{aligned}
x_i \Phi &= \left( \frac{\partial}{\partial u_i} - k \sum_{\alpha \in \Sigma_+} \nu_\alpha^i \frac{s_\alpha}{e^{u_\alpha} - 1} \right) \Phi \\
&= \left( \frac{\partial}{\partial u_i} - k \sum_{\alpha \in \Sigma_+} \nu_\alpha^i (e^{u_\alpha} - 1)^{-1} \sigma'_{s_\alpha} \right) \Phi \quad (1 \leq i \leq n).
\end{aligned}$$

Let us denote:

$$\mathcal{D}'_i = \frac{\partial}{\partial u_i} - k \sum_{\alpha \in \Sigma_+} \nu_\alpha^i (e^{u_\alpha} - 1)^{-1} \sigma'_{s_\alpha}.$$

The local invertibility of  $\Phi$  and the relations  $\mathcal{D}'_i \Phi = x_i \Phi$  result in the commutativity

$$[\mathcal{D}'_i, \mathcal{D}'_j] = 0 \quad \forall i, j.$$

Here one can use that the commutators do not contain the derivatives, which readily results from the relations for  $\sigma'$ . Moreover, the commutativity follows from these relations algebraically. It was proved in [16] (see [19] for a more conceptual proof based on the induced representations). It also

follows from the corresponding difference theory, where this and similar statements are much simpler (and completely conceptual).

- (2) Since the multiplication by  $x_i$  commutes with  $\mathcal{D}'_j$ , we get

$$p(x_1, \dots, x_n)\Phi = p(\mathcal{D}'_1, \dots, \mathcal{D}'_n)\Phi$$

for any polynomial  $p \in \mathbb{C}[x_1, \dots, x_n]$ .

- (3) For  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  let us take an  $\mathcal{H}'_\Sigma$ -module  $V_\lambda$  with the following properties:

(i)  $p(x_1, \dots, x_n) = p(\lambda_1, \dots, \lambda_n)$  on  $V_\lambda$  for any  $p \in \mathbb{C}[x_1, \dots, x_n]^W$ ,

$$(3.36)$$

- (ii) there exists a linear map  $\text{tr} : V_\lambda \rightarrow \mathbb{C}$  satisfying

$$\text{tr}(wa) = \text{tr}(a) \quad \forall w \in W, a \in V_\lambda. \quad (3.37)$$

Let  $p(x_1, \dots, x_n)$  be a polynomial. Using the commutation relations (3.35), we can write

$$p(\mathcal{D}'_1, \dots, \mathcal{D}'_n) = \sum_{w \in W} \mathcal{D}'_w^{(p)} \sigma'_w,$$

where  $\mathcal{D}'_w^{(p)}$  are differential operators (they do not contain  $\sigma'$ ). They are scalar and commute with  $\mathcal{H}'_\Sigma$ . Thus

$$p(x_1, \dots, x_n)\Phi = \sum_{w \in W} \mathcal{D}'_w^{(p)} \sigma'_w \Phi = \sum_{w \in W} \mathcal{D}'_w^{(p)} w\Phi.$$

Now, we assume that  $p$  is  $W$ -invariant. Applying  $\text{tr}$  (see (3.36) and (3.37)), we come to

$$p(\lambda_1, \dots, \lambda_n)\psi = L'_p \psi \text{ for } L'_p = \sum_{w \in W} \mathcal{D}'_w^{(p)},$$

where

$$\psi(u) = \text{tr}(\Phi(u))$$

is a  $\mathbb{C}$ -valued function. The differential operators  $L'_p$  are  $W$ -invariant, which follows from the same construction (we will reprove this algebraically below).

Let us introduce the *trigonometric Dunkl operators*  $\mathcal{D}_i$  ( $1 \leq i \leq n$ ) replacing  $\sigma'$  by  $\sigma$ :

$$\mathcal{D}_i = \frac{\partial}{\partial u_i} - k \sum_{\alpha \in \Sigma_+} \nu_\alpha^i (e^{u_\alpha} - 1)^{-1} \sigma_{s_\alpha}.$$

Repeating the above construction, define  $\mathcal{D}^{(p)}$  for a  $W$ -invariant polynomial  $p$  by

$$p(\mathcal{D}_1, \dots, \mathcal{D}_n) = \sum_{w \in W} \mathcal{D}_w^{(p)} \sigma_w.$$

Since in the construction of  $L'_p$  and  $L_p$  we use only the commutation relations (3.35) for  $\sigma'_w$  and  $\sigma_i$  these operators just coincide. The trigonometric Dunkl operators are from [14]. Dunkl introduced their rational counterparts (see also [19] and references

therein). When defining my operators I also used [40]. Heckman's 'global Dunkl operators' are sufficient to introduce QMBP, but do not commute.

We are now in a position to introduce the QMBP with the eigenvalue  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ . It is the following system of differential equations for a  $\mathbb{C}$ -valued function  $\psi$ .

$$L_p \psi = p(\lambda_1, \dots, \lambda_n) \psi \quad (p \in \mathbb{C}[x_1, \dots, x_n]^W).$$

It is known [38] (and easy to see by looking at the leading terms of  $L_p$ ) that the dimension of the space of solutions  $\psi$  is  $|W|$ .

Summarizing, we come to the theorem.

**Theorem 3.8.** *Applying  $\text{tr}$  we get a homomorphism*

$$\text{tr} : \text{Sol}_{AKZ}(V_\lambda) \longrightarrow \text{Sol}_{QMBP}(\lambda).$$

Here  $\text{Sol}_{QMBP}(\lambda)$  denotes the space of solutions to QMBP with the eigenvalue  $\lambda$ .

We can say more for concrete representations, especially for the induced representations  $J_\lambda^\circ$  (see (3.4)). We define the 'trace'

$$\text{tr} : J_\lambda^\circ \longrightarrow \mathbb{C}$$

as the map dual to the embedding

$$\mathbb{C} \longrightarrow J_\lambda = \mathbb{C}[x_1, \dots, x_n]/L_\lambda$$

sending 1 to  $1 \in \mathbb{C}[x_1, \dots, x_n]$ . Here  $L_\lambda$  denotes the ideal generated by  $p(x) - p(\lambda)$ ,  $p \in \mathbb{C}[x_1, \dots, x_n]^W$ . One easily checks that  $\text{tr}$  satisfies the conditions (3.37).

**Theorem 3.9** ([13]). *For any  $\lambda \in \mathbb{C}^n$ ,  $\text{tr}$  gives an isomorphism*

$$\text{tr} : \text{Sol}_{AKZ}(J_\lambda^\circ) \xrightarrow{\sim} \text{Sol}_{QMBP}(\lambda).$$

*Proof.* The key observation:

$$\text{for any } \mathcal{H}'_\Sigma\text{-submodule } 0 \neq M \subset J_\lambda^\circ, \text{ we have } \text{tr}|_M \neq 0. \quad (3.38)$$

Indeed, if  $0 \neq f \in M$ , then there exists a polynomial  $p(x) \in \mathbb{C}[x_1, \dots, x_n]$  such that  $f(p) \neq 0$ . However  $f(p) = \text{tr}(p(f)) \in \text{tr}(M)$ .

To prove Theorem 3.9, it is enough to show the injectivity of  $\text{tr}$ , since the surjectivity will then follow by comparing the dimensions of the solution spaces (both of them are  $|W|$ ). So let us suppose that for  $\varphi(u) \in \text{Sol}_{AKZ}(J_\lambda^\circ)$  identically

$$\text{tr}(\varphi) = 0. \quad (3.39)$$

We will show that

$$\text{tr}(\mathcal{H}'_\Sigma \varphi) = 0. \quad (3.40)$$

Differentiating (3.39),

$$0 = \operatorname{tr}\left(\frac{\partial\varphi}{\partial u_i}\right) = k \sum_{\alpha \in \Sigma_+} \nu_\alpha^i \operatorname{tr}\left(\frac{s_\alpha}{e^{u_\alpha} - 1}\varphi\right) + \operatorname{tr}(x_i\varphi).$$

By the  $W$ -invariance of  $\operatorname{tr}$ ,  $\operatorname{tr}(s_\alpha\varphi) = \operatorname{tr}(\varphi) = 0$ . Hence

$$\operatorname{tr}(x_i\varphi) = 0. \quad (3.41)$$

Differentiating this equation by  $u_j$  we have

$$0 = k \sum_{\alpha \in \Sigma_+} \nu_\alpha^j \operatorname{tr}\left(x_i \frac{s_\alpha}{e^{u_\alpha} - 1}\varphi\right) + \operatorname{tr}(x_i x_j \varphi).$$

Using the commutation relations of  $x_j$  and  $s_\alpha$ , we deduce from (3.39), (3.41) that

$$\operatorname{tr}(x_i x_j \varphi) = 0.$$

Proceeding in the same way, we establish that

$$\operatorname{tr}(x_{i_1} \cdots x_{i_l} \varphi) = 0$$

for any  $i_1, \dots, i_l$ . Combining this with the  $W$ -invariance of  $\operatorname{tr}$ , we get (3.40).

For each  $u^0$ , consider the submodule  $M = \mathcal{H}'_\Sigma \varphi(u^0) \subset J_\lambda^\circ$ . Then  $\operatorname{tr}|_M = 0$ , and from the key observation above, we deduce that  $M = 0$ . This completes the proof of Theorem 3.9.  $\square$

The map from Theorem 3.9 was found by Matsuo [56] for induced representations  $I_\lambda$ . He proved his theorem algebraically (without the passage through the trigonometric Dunkl operators discussed above) using an explicit presentation for AKZ in  $I_\lambda$ . The isomorphism for  $J_\lambda^\circ$  (or for  $I_\lambda$  with properly ordered  $\lambda$  - (3.6)) was established independently and simultaneously by Matsuo and the author in [19]. He proved that a certain determinant is non-zero for properly ordered  $\lambda$ . I used the modules  $J$ . Matsuo was the first to conjecture that the QMBP (the Heckman-Opdam system) and a certain specialization of the trigonometric KZ from [10] are isomorphic. The affine KZ were defined in full generality a bit later (in [12]).

Let us give the formula for the simplest  $L_p$ .

**Example 3.1.** Let  $p_2(x_1, \dots, x_n) = \sum_{i=1}^n x_{\alpha_i} x_i$ . Then we have

$$L_2 = L_{p_2} = \sum_{i=1}^n \partial_{\alpha_i} \partial_i + \sum_{\alpha \in \Sigma_+} (\alpha, \alpha) \frac{k(1-k)}{(e^{u_\alpha/2} - e^{-u_\alpha/2})^2}.$$

It was studied in [59].

*Remark 3.4.* More generally, let  $A$  be a  $\mathbb{C}[W]$ -module and

$$V_{A,\lambda} = \left( \text{Ind}_{\mathbb{C}[W]}^{\mathcal{H}'_{\Sigma}}(A) / L_{\lambda} \right)^{\circ}.$$

As before,  $L_{\lambda}$  is the ideal generated by  $p(x) - p(\lambda)$ ,  $p \in \mathbb{C}[x_1, \dots, x_n]^W$ . Then the following holds

$$\text{Sol}_{AKZ}(V_{A,\lambda}) \xrightarrow{\sim} \text{Sol}_{QMBP_A}(\lambda)$$

where now the right hand side means a matrix version of QMBP (sometimes it is called spin-QMBP). It was introduced in [19] for the first time. It is a certain unification of the Haldane-Shastry model and that by Calogero-Sutherland.

For example, the  $L$ -operator corresponding to  $p_2$  above reads

$$L_2 = \sum_{i=1}^n \partial_{\alpha_i} \partial_i + \sum_{\alpha \in \Sigma_+} (\alpha, \alpha) \frac{k(s_{\alpha}^* - k)}{(e^{u_{\alpha}/2} - e^{-u_{\alpha}/2})^2}.$$

where by  $s_{\alpha}^*$  we mean the image of  $s_{\alpha}$  in  $\text{Aut}(A)$ .

**3.5. The  $GL_n$  case.** Let us describe AKZ and QMBP in the  $GL_n$  case.

In §2.3, we introduced the degenerate affine Hecke algebra of type  $GL_n$ . It is the algebra

$$\mathcal{H}'_n = \langle \mathbb{C}\mathbb{S}_n, y_1, \dots, y_n \rangle$$

subject to the following relations:

$$\begin{aligned} s_i y_i - y_{i+1} s_i &= k, & s_i y_j &= y_j s_i \quad (i \neq j, j+1), \\ y_i y_j &= y_j y_i \quad (1 \leq i, j \leq n). \end{aligned}$$

As in §2.3, we will use the coordinates  $v_i$ .

To prepare the passage to the difference case, we conjugate the AKZ for  $GL_n$  by the function  $\Delta^k$  for  $\Delta = \prod_{i < j} (e^{v_i} - e^{v_j})$ . The equation becomes as follows:

$$\frac{\partial \Phi}{\partial v_i} = \left( k \left( \sum_{j(>i)} \frac{s_{ij} - 1}{e^{v_i - v_j} - 1} - \sum_{j(<i)} \frac{s_{ij} - 1}{e^{v_j - v_i} - 1} \right) + y_i + k \left( i - \frac{n+1}{2} \right) \right) \Phi. \quad (3.42)$$

Only in this form it can be quantized (see §4.2). The system is consistent and  $\mathbb{S}_n$ -invariant.

The corresponding Dunkl operators are given by the formula

$$\begin{aligned} \mathcal{D}_i &= \frac{\partial}{\partial v_i} \\ &- k \left( \sum_{j(>i)} (e^{v_i - v_j} - 1)^{-1} (\sigma_{ij} - 1) - \sum_{j(<i)} (e^{v_j - v_i} - 1)^{-1} (\sigma_{ij} - 1) + i - \frac{n+1}{2} \right). \end{aligned}$$

Here  $\sigma_{ij}$  stands for the transpositions of the coordinates:

$$\sigma_{ij} v_i = v_j \sigma_{ij}.$$

Similarly,  $\sigma_w$  means the permutation of the coordinates corresponding to  $w^{-1}$ .

The main point of the theory is that they satisfy the relations from the degenerate Hecke algebra:

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 = [\mathcal{D}_i, y_j], \quad i \neq j, \quad \sigma_{ii+1} \mathcal{D}_i - \mathcal{D}_{i+1} \sigma_{ii+1} = k.$$

It holds for any root systems. This statement is from [19]. In these notes we will deduce these relations from the difference theory (where they are almost obvious). These relations readily give that  $p(\mathcal{D}_1, \dots, \mathcal{D}_n)$  and the corresponding  $L_p$  are  $W$ -invariant for the  $W$ -invariant polynomials. Use the description of the center of  $\mathcal{H}'$  to see this.

In the case of  $GL_n$ , given symmetric  $p \in \mathbb{C}[x_1, \dots, x_n]^{\mathbb{S}_n}$ ,

$$p(\mathcal{D}_1, \dots, \mathcal{D}_n) = \sum_{w \in \mathbb{S}_n} D_w^{(p)} \sigma_w,$$

where  $D_w^{(p)}$  are scalar differential operators,

$$L_p = p(\mathcal{D}_1, \dots, \mathcal{D}_n)|_{\text{symm.poly.}} = \sum_{w \in \mathbb{S}_n} D_w^{(p)}.$$

Let us take the elementary symmetric polynomials:

$$e_m(x) = \sum_{i_1 < \dots < i_m} x_{i_1} \cdots x_{i_m},$$

as  $p$ , setting  $L_m = L_{e_m}$ . Clearly

$$L_1 = \sum_{i=1}^n \frac{\partial}{\partial v_i}.$$

The next operator is:

$$L_2 = \sum_{i < j} \frac{\partial^2}{\partial v_i \partial v_j} - \frac{k}{2} \sum_{i < j} \coth \left( \frac{v_i - v_j}{2} \right) \left( \frac{\partial}{\partial v_i} - \frac{\partial}{\partial v_j} \right) - \frac{k^2}{4} \binom{n+1}{3}.$$

When we replace  $e_2$  by  $p = \sum_i x_i^2$ , the corresponding  $L$ -operator is conjugated (by  $\Delta^k$ ) to the original Sutherland operator up to a constant term [67]. For special values of the parameter  $k$ , these operators are the radial parts of the Laplace operators on the symmetric spaces. A particular case was considered by Koornwinder. The rational counterpart is due to Calogero. It is equivalent to a rational variant of the AKZ (an extension of the rational  $W$ -valued KZ from [10] by the  $x$ -s). Here the  $J_\lambda$ -modules cannot be represented as  $I_\lambda$ , the theorem holds in terms of  $J$  only (see [19]). See also [38] and [61].

## 4. ISOMORPHISM THEOREMS FOR THE QAKZ EQUATION

Let us now turn to the  $q$ -deformations. We introduce the quantum affine Knizhnik–Zamolodchikov (QAKZ) equation, and show that there is an isomorphism between solutions of the QAKZ equation and solutions of the generalized Macdonald eigenvalue problem.

**4.1. Affine Hecke algebras and intertwiners.** In this section we recall the definition of the affine Hecke algebra  $\mathcal{H}_n^t$  in the case of  $GL_n$ .

Let  $t \in \mathbb{C}^*$  be a parameter. Then  $\mathcal{H}_n^t$  is the algebra defined over  $\mathbb{C}$  by the following set of generators and relations:

$$\begin{aligned} \text{generators} & : T_1, \dots, T_{n-1}, Y_1, \dots, Y_n, \\ \text{relations} & : (T_i - t)(T_i + t^{-1}) = 0, \quad (1 \leq i \leq n-1) \end{aligned} \quad (4.1)$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad (1 \leq i \leq n-2) \quad (4.2)$$

$$T_i T_j = T_j T_i, \quad (|i - j| > 1) \quad (4.3)$$

$$Y_i Y_j = Y_j Y_i, \quad (1 \leq i, j \leq n) \quad (4.4)$$

$$Y_i T_j = T_j Y_i, \quad (j \neq i, i-1), \quad (4.5)$$

$$T_i^{-1} Y_i T_i^{-1} = Y_{i+1}. \quad (1 \leq i \leq n-1) \quad (4.6)$$

The relations (4.1) are called the quadratic relations, (4.2)-(4.3) the Coxeter relations, (4.4) the commutativity, and (4.5),(4.6) the cross relations.

Set

$$P = T_1 \cdots T_{i-1} Y_i T_i^{-1} \cdots T_{n-1}^{-1}.$$

It follows from the defining relations (4.1)-(4.6) that the right hand side is independent of  $i$  ( $1 \leq i \leq n$ ) and therefore equals to

$$P = T_1 \cdots T_{n-1} Y_n = Y_1 T_1^{-1} \cdots T_{n-1}^{-1}. \quad (4.7)$$

**Lemma 4.1.** *The algebra  $\mathcal{H}_n^t$  can be presented as*

$$\mathcal{H}_n^t = \langle T_1, \dots, T_{n-1}, P \rangle / \sim, \quad (4.8)$$

where the quotient is by the quadratic relations (4.1), the Coxeter relations (4.2)-(4.3) and the following:

- (a)  $PT_{i-1} = T_i P$  ( $1 < i < n$ ),
- (b)  $P^n$  is central.

*Proof.* Notice that in terms of  $Y_i$ 's we have  $P^n = Y_1 \cdots Y_n$ . The relations (a) and (b) readily follow from (4.7) and the defining relations (4.1)-(4.6). For instance,

$$PT_1 P^{-1} = Y_1 T_1^{-1} (T_2^{-1} T_1 T_2) T_1 Y_1^{-1} = Y_1 T_1^{-1} (T_1 T_2^{-1} T_1^{-1}) T_1 Y_1^{-1} = T_2.$$

To establish (4.8), we start with  $T_1, \dots, T_{n-1}, P$  and introduce the elements  $Y_1, \dots, Y_n$  by

$$Y_1 = PT_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \dots$$

We must check the commutativity  $Y_1 Y_2 = Y_2 Y_1$ ,  $T_j Y_1 = Y_1 T_j$  ( $j > 1$ ), etc. using (a), (b). The first reads

$$Y_1 T_1^{-1} Y_1 T_1^{-1} = T_1^{-1} Y_1 T_1^{-1} Y_1.$$

We plug in the above formula for  $Y_1$  and move  $P$  to the left. The commutativity with 'distant'  $T$  is obvious. The other relations formally follow from them. We leave the verifications to the reader as an exercise.  $\square$

**4.2. The QAKZ equation.** In this section, we introduce the QAKZ equation.

**Definition 4.1.** For  $u \in \mathbb{C}$ , we define the intertwiners by

$$F_i(u) = \frac{T_i + \frac{t - t^{-1}}{e^u - 1}}{t + \frac{t - t^{-1}}{e^u - 1}}. \quad (4.9)$$

They satisfy

$$F_i(u) F_i(-u) = 1, \quad (4.10)$$

$$F_i(u) F_{i+1}(u+v) F_i(v) = F_{i+1}(v) F_i(u+v) F_{i+1}(u). \quad (4.11)$$

The second relation can be deduced from Lemma 3.7 as we did for the degenerate Hecke algebra.

The *quantum affine Knizhnik-Zamolodchikov* (QAKZ) equation is the following system of difference equations for a function  $\Phi(v)$  that takes values in  $\mathcal{H}_n^t$  (or any  $\mathcal{H}_n^t$ -module).

$$\begin{aligned} & \Phi(v_1, \dots, v_i + h, \dots, v_n) \\ &= F_{i-1}(v_i - v_{i+1} + h) \dots F_1(v_i - v_1 + h) T_1 \dots T_{i-1} Y_i \\ & \times T_i^{-1} \dots T_{n-1}^{-1} F_{n-1}(v_i - v_n) \dots F_i(v_i - v_{i+1}) \\ & \times \Phi(v_1, \dots, v_i, \dots, v_n) \quad (i = 1, \dots, n). \end{aligned} \quad (4.12)$$

Here  $h$  is a new parameter.

**Theorem 4.2.** *The QAKZ system (4.12) is self-consistent. It is invariant in the following sense: if  $\Phi(v)$  is a solution, then so is*

$$F_i(v_{i+1} - v_i)^{s_i} \Phi(v) = {}^{s_i} (F_i(v_i - v_{i+1}) \Phi(v)).$$

This follows from (4.10), (4.11). Later we will make it quite obvious.

Let us discuss the quasi-classical limit of the QAKZ system. Setting

$$t = e^{kh/2} = q^k, \quad q = e^h,$$

let  $h \rightarrow 0$ . The generators  $T_i, Y_i$  are supposed to have the form

$$\begin{aligned} T_i &= s_i + \frac{kh}{2} + \dots \quad (s_i^2 = 1), \\ Y_i &= 1 + hy_i + \dots, \end{aligned}$$

where by  $\dots$  we mean terms of order  $h^2$ . The relations of the degenerate affine Hecke algebra for  $s_i, y_i$  can be readily verified. Using the formula

$$tT_i^{-1}F_i(u) = 1 + \frac{kh}{e^u - 1}(s_i - 1) + \dots,$$

we find that

$$h^{-1}(\Phi(\dots, v_i + h, \dots) - \Phi(\dots, v_i, \dots)) = \left\{ y_i + k \left( \sum_{j(>i)} \frac{s_{ij} - 1}{e^{v_i - v_j} - 1} - \sum_{j(<i)} \frac{s_{ij} - 1}{e^{v_j - v_i} - 1} + i - \frac{n+1}{2} \right) \right\} \Phi(\dots, v_i, \dots) + \dots.$$

Hence the AKZ equation (3.42) is a semi-classical limit ( $h \rightarrow 0$ ) of the QAKZ equation.

To make the QAKZ equations more transparent, let us discuss the action of the affine Weyl group. The affine Weyl group of type  $GL_n$  is the semi-direct product

$$\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \mathbb{Z}^n,$$

where

$$\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}\gamma_i$$

is a free abelian group of rank  $n$ . Define the action of  $\tilde{\mathfrak{S}}_n$  on a vector  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$  by

$$\begin{aligned} s_{ij}v &= (v_1, \dots, v_j, \dots, v_i, \dots, v_n) = s_{ji}v, & i < j, \\ \gamma_i v &= (v_1, \dots, v_i + h, \dots, v_n), & \gamma_i(v_j) = v_j - h\delta_{ij}. \end{aligned}$$

We also introduce

$$\pi = \gamma_1 s_1 \cdots s_{n-1} = s_1 \cdots s_{n-1} \gamma_n.$$

Its action on  $\mathbb{R}^n$  and the coordinates reads as

$$\pi v = (v_n + h, v_1, \dots, v_{n-1}), \quad \pi v_n = v_1 - h, \quad \pi v_1 = v_2, \dots.$$

**Lemma 4.3.**  $\tilde{\mathfrak{S}}_n$  can be presented as

$$\tilde{\mathfrak{S}}_n = \langle s_1, \dots, s_{n-1}, \pi \rangle / \sim,$$

where the relations are

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad (|i - j| > 1), \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

and

- (a)  $\pi s_{i-1} \pi^{-1} = s_i$  ( $1 < i < n$ ),
- (b)  $\pi^n$  is central.

It is convenient to represent the elements  $\gamma_i, \pi$  graphically.

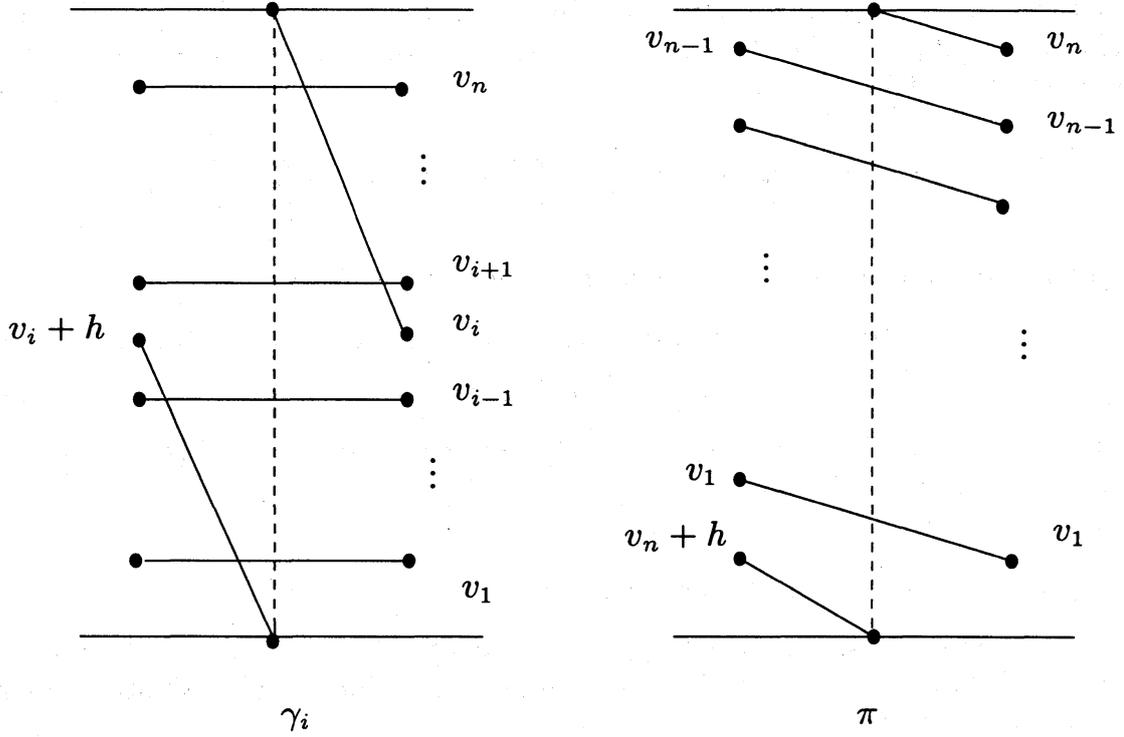

 FIGURE 7. Graphs for  $\gamma$  and  $\pi$ 

Fig7 shows a reduced decomposition of  $\gamma_i$ :

$$\gamma_i = s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i.$$

For a function  $\Psi(v)$  with values in  $\mathcal{H}_n^t$ , let

$$\tilde{s}_i(\Psi) = F_i(v_{i+1} - v_i)^{s_i} \Psi, \quad (4.13)$$

$$\tilde{\pi}(\Psi) = P^\pi \Psi. \quad (4.14)$$

**Theorem 4.4** ([15]). *The formulas (4.13), (4.14) can be extended to an action of  $\tilde{\mathfrak{S}}_n$ .*

We denote this action by  $\tilde{\mathfrak{S}}_n \ni w : \Psi \mapsto \tilde{w}(\Psi)$ . For instance,

$$\begin{aligned} \tilde{\gamma}_i(\Psi)(v_1, \dots, v_n) &= F_{i-1}(v_{i-1} - v_i)^{-1} \cdots F_1(v_1 - v_i)^{-1} P \\ &\quad \times F_{n-1}(v_i - v_n - h) \cdots F_i(v_i - v_{i+1} - h) \Psi(v_1, \dots, v_i - h, \dots, v_n). \end{aligned}$$

Hence the QAKZ equation simply means the invariance of  $\Phi(v)$  with respect to the pairwise commuting elements  $\gamma_i$ :

$$\text{QAKZ} \iff \tilde{\gamma}_i(\Phi) = \Phi \quad (i = 1, \dots, n). \quad (4.15)$$

Let us connect QAKZ with the q-KZ introduced by Smirnov and Frenkel-Reshetikhin [66, 37]. We fix an  $N$ -dimensional complex vector space  $V$  and introduce  $T \in \text{End}(V \otimes V)$  by

$$T = (t - t^{-1}) \sum_{i < j} E_{ii} \otimes E_{jj} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + t \sum_{i=1}^N E_{ii} \otimes E_{ii}$$

due to Baxter and Jimbo. The algebra  $\mathcal{H}_n^t$  acts on  $V^{\otimes n}$  by

$$T_i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes T(a_i \otimes a_{i+1}) \otimes \cdots \otimes a_n, \quad (4.16)$$

$$P(a_1 \otimes \cdots \otimes a_n) = C a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}, \quad (4.17)$$

where  $a_i \in V$  and  $C = \text{diag}(\lambda_1, \dots, \lambda_n)$ . One can check that this action is well-defined by a direct calculation.

For  $N = n$ , let

$$(V^{\otimes n})_0 = \text{span}\{e_{w(1)} \otimes \cdots \otimes e_{w(n)} \mid w \in \mathbb{S}_n\}$$

be the 0-weight subspace. Here  $e_1, \dots, e_n$  denote the standard basis of  $V$ . It is easy to see that this subspace is closed under the action of  $\mathcal{H}_n^t$ . We state the next proposition without proof.

**Proposition 4.5.** *If  $N = n$  and  $\lambda = (\lambda_1, \dots, \lambda_n)$  is generic, then the 0-weight space  $(V^{\otimes n})_0$  is isomorphic to  $I_\lambda = \text{Ind}_{\mathbb{C}[Y_1, \dots, Y_n]}^{\mathcal{H}_n^t}(\lambda)$ .*

Writing down AQKZ in  $(V^{\otimes n})_0$  we get the q-KZ (for  $GL_n$  and in the fundamental representation). Combining this observation with the isomorphism with the Macdonald eigenvalue problem (our next aim) we can explain why the Macdonald polynomials appear in many calculations involving the vertex operators.

**4.3. The monodromy cocycle.** Let  $\Phi$  be a solution of the QAKZ equation. Thanks to (4.15),  $\tilde{w}(\Phi)$  is also a solution of the QAKZ equation for any  $w \in \tilde{\mathbb{S}}_n$ . We define  $\mathcal{T}_w \in \mathcal{H}_n^t$  by

$${}^w \mathcal{T}_w = \Phi^{-1} \tilde{w}(\Phi) \quad \text{for } w \in \tilde{\mathbb{S}}_n$$

and call it the *monodromy cocycle*. It follows from (4.13) and (4.14) that

$$F_i(v_i - v_{i+1})\Phi = {}^{s_i} \Phi \mathcal{T}_i \quad (4.18)$$

and

$$P\Phi = \pi^{-1} \Phi \mathcal{T}_\pi. \quad (4.19)$$

Here  $\mathcal{T}_i$  stands for  $\mathcal{T}_{s_i}$ .

**Lemma 4.6.**

$$w_2^{-1}(\mathcal{T}_{w_1})\mathcal{T}_{w_2} = \mathcal{T}_{w_1 w_2} \quad \text{for } w_1, w_2 \in \tilde{\mathbb{S}}_n.$$

Indeed,

$$\Phi^{w_1 w_2} \mathcal{T}_{w_1 w_2} = \widetilde{w_1 w_2}(\Phi) = \tilde{w}_1(\tilde{w}_2 \Phi) = \tilde{w}_1(\Phi^{w_2} \mathcal{T}_{w_2}) = \Phi^{w_1} \mathcal{T}_{w_1}^{w_1 w_2} \mathcal{T}_{w_2}.$$

The QAKZ equation implies that  $\mathcal{T}_{\gamma_i} = 1$ . Hence  $\mathcal{T}_w$  depends only on the image  $\bar{w}$  of  $w$  in  $\mathbb{S}_n$ .

Let  $\mathcal{F}(\mathbb{C}^n, \mathcal{H}_n^t)$  be the set of  $\mathcal{H}_n^t$ -valued function on  $\mathbb{C}^n$ . Next we define two anti-actions of  $\tilde{\mathbb{S}}_n$ :

$$\sigma_w(\Psi) = w^{-1} \Psi, \quad (4.20)$$

$$\sigma'_w(\Psi) = w^{-1} \Psi \mathcal{T}_w, \quad (4.21)$$

where  $w \in \tilde{\mathbb{S}}_n$  and  $\Psi \in \mathcal{F}(\mathbb{C}^n, \mathcal{H}_n^t)$ . Lemma 4.6 means exactly that  $\sigma'$  is an anti-action (i.e.  $\sigma'_{w_1 w_2} = \sigma'_{w_2} \sigma'_{w_1}$ ). For instance,  $\sigma_{\gamma_i}(v_i) = v_i + h = \sigma'_{\gamma_i}(v_i)$ .

We note that in the difference theory the monodromy can be always made trivial. Indeed, the 1-cocycle  $\{\mathcal{T}_w, w \in W\}$  is always a co-boundary because of the Hilbert 90 theorem. Hence conjugating solutions of AQKZ we can always get rid of the monodromy. So the above actions  $\sigma, \sigma'$  are not too much different in contrast to the differential theory.

This argument can be applied to the AQKZ itself, although the group  $\mathbb{Z}^n$  is infinite. We can formally solve the QAKZ equation as follows. Let  $\Psi \in \mathcal{F}(\mathbb{C}^n, \mathcal{H}_n^t)$ . Then the infinite sum

$$\sum_{b \in B} \tilde{b}(\Psi), \quad (4.22)$$

where  $B = \bigoplus_{i=1}^n \mathbb{Z} \gamma_i \subset \tilde{\mathbb{S}}_n$ , satisfies the AQKZ, provided the convergence. For example, if  $\Psi$  is rapidly decreasing, then one can check that  $\sum_{b \in B} \tilde{b}(\Psi)$  is convergent.

We see that constructing  $\text{End}(V)$ -valued solution  $\Phi$  to QAKZ for finite dimensional  $\mathcal{H}_n^t$ -modules  $V$  poses no problem. What is more difficult is to ensure a proper asymptotical behavior.

**4.4. Isomorphism of QAKZ and the Macdonald eigenvalue problem.** In this subsection, we introduce the Macdonald eigenvalue problem and prove its equivalence to the QAKZ equation. This is a  $q$ -analogue of the relation between AKZ and QMBP discussed in §3.4.

Let  $\Phi$  be a solution of the QAKZ equation with values in  $\text{End}(V)$  for a  $\mathcal{H}_n^t$ -module  $V$ . We assume that it is invertible for sufficiently general  $v$ . Setting  $\sigma'_i = \sigma'_{s_i}$ , we get from (4.18) and (4.9):

$$F_i(v_i - v_{i+1}) \Phi = \sigma'_i(\Phi),$$

$$T_i \Phi = \left( t \sigma'_i + \frac{t - t^{-1}}{e^{v_i - v_{i+1}} - 1} (\sigma'_i - 1) \right) \Phi.$$

Let us introduce the operator  $\hat{T}'_i$  ( $1 \leq i \leq n$ ) by

$$\hat{T}'_i = t \sigma'_i + \frac{t - t^{-1}}{e^{v_i - v_{i+1}} - 1} (\sigma'_i - 1). \quad (4.23)$$

Then  $\hat{T}'_i \Phi = T_i \Phi$  and  $\sigma'_\pi \Phi = P \Phi$  (see (4.19)). The operators  $\hat{T}'_i$  and  $\sigma'_\pi$  commute with the left multiplication by  $T_j$ ,  $P$  and any elements from  $\mathcal{H}_n^t$ . Using all these:

$$\begin{aligned} Y_i \Phi &= T_{i-1}^{-1} \cdots T_1^{-1} P T_{n-1} \cdots T_{i+1} T_i \Phi \\ &= T_{i-1}^{-1} \cdots T_1^{-1} P T_{n-1} \cdots T_{i+1} \hat{T}'_i \Phi \\ &= \hat{T}'_i T_{i-1}^{-1} \cdots T_1^{-1} P T_{n-1} \cdots T_{i+1} \Phi \\ &\dots \\ &= \hat{T}'_i \cdots \hat{T}'_{n-1} \sigma'_\pi (\hat{T}'_1)^{-1} \cdots (\hat{T}'_{i-1})^{-1} \Phi. \end{aligned}$$

We come to the following definition:

$$\Delta'_i = \hat{T}'_i \cdots \hat{T}'_{n-1} \sigma'_\pi (\hat{T}'_1)^{-1} \cdots (\hat{T}'_{i-1})^{-1}, \quad 1 \leq i \leq n. \quad (4.24)$$

Since  $Y_i \Phi = \Delta'_i \Phi$  and  $Y_i$  commute with each other,

$$[\Delta'_i, \Delta'_j] = 0.$$

By the construction, the operators  $\Delta'_i$  act in  $\text{End}(V)$ -valued functions. However if we understand them formally the commutativity can be deduced from the relations

$$\sigma'_i v_i = v_{i+1} \sigma'_i, \quad (4.25)$$

$$\sigma'_i \sigma_{\gamma_i} = \sigma_{\gamma_{i+1}} \sigma'_i. \quad (4.26)$$

$$\sigma'_{\gamma_i} = \sigma_{\gamma_i} \quad (4.27)$$

The latter means that  $\mathcal{T}_{\gamma_i} = 1$ .

Let  $Q$  be a polynomial in  $n$  variables. Then

$$Q(Y_1, \dots, Y_n) \Phi = Q(\Delta'_1, \dots, \Delta'_n) \Phi$$

and we can represent

$$Q(\Delta'_1, \dots, \Delta'_n) = \sum_{w \in \mathbb{S}_n} D_w^{(Q)} \sigma'_w, \quad (4.28)$$

where  $D_w^{(Q)}$  are pure difference operators, which do not contain  $\sigma'_w$  ( $w \in \mathbb{S}_n$ ).

For symmetric  $Q$ , we introduce a difference operator of Macdonald type  $M_Q$  by

$$M_Q = \sum_{w \in \mathbb{S}_n} D_w^{(Q)}.$$

Let  $\varphi$  be a  $\mathbb{C}$ -valued function on  $\mathbb{C}^n$ . The system

$$M_Q \varphi = Q(\lambda_1, \dots, \lambda_n) \varphi. \quad (4.29)$$

will be called the Macdonald eigenvalue problem. The operators  $M_Q$  can be calculated for  $\sigma$  instead of  $\sigma'$ . As in the differential case, the result will be the same.

Fix  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ . We take a left  $\mathcal{H}_n^t$ -module  $V_\lambda$  with the following properties:

(1) for any symmetric polynomial  $Q$  in  $n$  variables and all  $a \in V_\lambda$ ,

$$Q(Y_1, \dots, Y_n)a = Q(\lambda_1, \dots, \lambda_n)a,$$

(2) there exists a  $\mathbb{C}$ -linear map  $\text{tr} : V_\lambda \rightarrow \mathbb{C}$  such that

$$\text{tr}((T_i - t)a) = 0$$

for all  $i$  and  $a \in V_\lambda$ .

As always, we fix a (local) invertible solution  $\Phi(v)$  of the QAKZ equation with values in  $\text{End}(V_\lambda)$ . Note that all  $V_\lambda$ -valued solutions of the QAKZ equation can be written in the form  $\varphi(v) = \Phi(v)a(v)$  for  $B$ -periodic  $V_\lambda$ -valued function  $a(v)$ :

$$a(\dots, v_i + h, \dots) = a(v) \text{ for } i = 1, \dots, n.$$

**Theorem 4.7.** *Let  $V_\lambda$  be an  $\mathcal{H}_n^t$ -module with the above properties,  $\text{Sol}_{\text{QAKZ}}(V_\lambda)$  be the space of solutions of the QAKZ equation with values in  $V_\lambda$ , and  $\text{Sol}_{\text{Mac}}(\lambda)$  the space of solutions of the Macdonald eigenvalue problem (4.29). Then*

$$\text{Sol}_{\text{QAKZ}}(V_\lambda) \xrightarrow{\text{tr}} \text{Sol}_{\text{Mac}}(\lambda).$$

*Proof.* Let  $\varphi(v) = \Phi(v)a \in \text{Sol}_{\text{QAKZ}}(V_\lambda)$ . Then

$$(\sigma'_i - 1)\Phi = \left( t + \frac{t - t^{-1}}{e^{v_i - v_{i+1}} - 1} \right)^{-1} (T_i - t)\Phi.$$

For a reduced decomposition  $w = s_{i_1} \cdots s_{i_l}$  of  $w \in \mathbb{S}_n$ ,

$$\begin{aligned} \sigma'_w - 1 &= \sigma'_{i_l} \cdots \sigma'_{i_1} - 1 \\ &= \sigma'_{s_{i_l}} \cdots \sigma'_{i_2} (\sigma'_{i_1} - 1) + \sigma'_{i_l} \cdots \sigma'_{i_2} - 1 \\ &\dots \\ &= \sum_{k=1}^l \sigma'_{i_l} \cdots \sigma'_{i_{k+1}} (\sigma'_{i_k} - 1). \end{aligned}$$

Since  $\sigma'_i$  commutes with the left action of  $\{T\}$ , we have

$$\begin{aligned} (\sigma'_w - 1)\Phi &= \sum_{k=1}^l \sigma'_{i_l} \cdots \sigma'_{i_{k+1}} (\sigma'_{i_k} - 1)\Phi \\ &= \sum_{k=1}^l \sigma'_{i_l} \cdots \sigma'_{i_{k+1}} (\text{a scalar function})(T_{i_k} - t)\Phi \\ &= \sum_{k=1}^l (\text{a scalar function})(T_{i_k} - t)\sigma'_{i_l} \cdots \sigma'_{i_{k+1}} \Phi. \end{aligned}$$

Using the commutativity of  $D_w^{(Q)}$  with  $T_i - t$  we represent  $D_w^{(Q)}(\sigma'_w - 1)\Phi$  as a sum  $\sum(T_i - t)\Psi_i$  for some  $\mathcal{H}_n^t$ -valued functions  $\Psi_i$ . Finally

$$\begin{aligned} Q(\lambda_1, \dots, \lambda_n)\Phi &= Q(\Delta'_1, \dots, \Delta'_n)\Phi \\ &= \sum_{w \in \mathbf{S}_n} D_w^{(Q)} \sigma'_w \Phi \\ &= \sum_{w \in \mathbf{S}_n} D_w^{(Q)} \Phi + \sum_{w \in \mathbf{S}_n} D_w^{(Q)}(\sigma'_w - 1)\Phi \\ &= M_Q \Phi + \sum (T_i - t)\Psi_i. \end{aligned}$$

Applying this relation to  $a \in V_\lambda$  and taking  $\text{tr}$ , we conclude:

$$Q(\lambda_1, \dots, \lambda_n) \text{tr}(\varphi) = M_Q \text{tr}(\varphi).$$

□

Let us now consider  $V_\lambda = J_\lambda^\circ$ . The definition is quite similar to the differential case. We start with

$$J_\lambda = \text{Ind}_{H_n^t}^{\mathcal{H}_n^t} (+) / L_\lambda.$$

Here  $H_n^t = \langle T_1, \dots, T_{n-1} \rangle \subset \mathcal{H}_n^t$ ,  $+ : H_n^t \rightarrow \mathbb{C}$  is the one-dimensional representation sending  $T_i$  to  $t$ , and  $L_\lambda$  is the ideal generated by  $p(Y_1, \dots, Y_n) - p(\lambda)$  ( $p \in \mathbb{C}[x_1, \dots, x_n]^{\mathbf{S}_n}$ ). As in §3.1,  $J_\lambda^\circ$  stands for the dual module defined via the anti-involution  $\circ$  of  $\mathcal{H}_n^t$ :

$$Y_i^\circ = Y_i, \quad T_i^\circ = T_i.$$

The main result of this subsection is the following theorem from [16, 18].

**Theorem 4.8.** *If  $V_\lambda = J_\lambda^\circ$ , then the map from  $\text{Sol}_{QAKZ}(V_\lambda)$  to  $\text{Sol}_{Mac}(\lambda)$  is injective.*

The theorem results from the following two lemmas.

**Lemma 4.9.** *Let  $K$  be a  $\mathcal{H}_n^t$ -submodule of  $J_\lambda^\circ$ . Then  $\text{tr}(K) = 0$  implies  $K = 0$ .*

The proof repeats that in the differential case (see ((3.38))).

**Lemma 4.10.** *Let  $\varphi$  be a  $V_\lambda$ -valued solution of the QAKZ equation. Assume  $\text{tr}(\varphi) = 0$ . Then  $\text{tr}(\mathcal{H}_n^t \varphi) = 0$ .*

*Proof.* First

$$\text{tr}(T_i \varphi) = t \text{tr}(\varphi) = 0$$

for all  $i$ . Then  $\pi = s_1 s_2 \cdots s_{n-1} \gamma_n$ ,

$$\begin{aligned} \sigma'_\pi - 1 &= \sigma_{\gamma_n} \sigma'_{n-1} \cdots \sigma'_1 - 1 \\ &= \sigma_{\gamma_n} \left( \sum_{k=1}^{n-1} \sigma'_{n-1} \cdots \sigma'_{k+1} (\sigma'_k - 1) \right) + \sigma_{\gamma_n} - 1, \end{aligned}$$

and  $\text{tr}(\sigma_{\gamma_n} \varphi) = \text{tr}(\varphi) = 0$ . Therefore, representing  $\varphi = \Phi a$  ( $a \in V_\lambda$ ), we have

$$\begin{aligned} \text{tr}(P\varphi) - \text{tr}(\varphi) &= \text{tr}((\sigma'_\pi - 1)\Phi a) \\ &= \text{tr}\left(\sigma_{\gamma_n} \left(\sum_{k=1}^{n-1} \sigma'_{n-1} \cdots \sigma'_{k+1} (\sigma'_k - 1)\Phi\right) a\right) \\ &= \sum_{k=1}^{n-1} \text{tr}(\sigma_{\gamma_n} \sigma'_{n-1} \cdots \sigma'_{k+1} f_i(v)(T_k - t)\Phi a) \\ &= \sum_{k=1}^{n-1} \text{tr}((T_k - t)\sigma_{\gamma_n} \sigma'_{n-1} \cdots \sigma'_{k+1} f_i(v)\Phi a) \\ &= 0, \end{aligned}$$

where  $f_i(v)$  are  $\mathbb{C}$ -valued function. Hence  $\text{tr}(P\varphi) = 0$  and

$$\begin{aligned} \text{tr}(Y_n \varphi) &= \text{tr}(T_{n-1}^{-1} \cdots T_1^{-1} P\varphi) \\ &= t^{1-n} \text{tr}(P\varphi) \\ &= 0. \end{aligned}$$

Now we shall prove that  $\text{tr}(Y_i \varphi) = 0$  for all  $i$  by induction. Assume that  $\text{tr}(Y_i \varphi) = 0$  for  $k+1 \leq i \leq n$ . Since  $Y_k = T_{k-1}^{-1} \cdots T_1^{-1} P T_{n-1} \cdots T_k$  it is enough to see that  $\text{tr}(P T_{n-1} \cdots T_k \varphi) = 0$ . Since  $\varphi$  is a solution of the QAKZ equation we have

$$\begin{aligned} \text{tr}(F_{k-1}^{-1} \cdots F_1^{-1} P F_{n-1} \cdots F_k \varphi) &= \text{tr}(\gamma_k^{-1} \varphi) \\ &= \gamma_k^{-1} \text{tr}(\varphi) \\ &= 0. \end{aligned}$$

On the other hand,

$$F_i(v) = c_i(v)(T_i + f_i(v))$$

where  $c_i(v)$  and  $f_i(v)$  are some scalar functions. Therefore

$$\begin{aligned} 0 &= \text{tr}(P F_{n-1} \cdots F_k \varphi) = \text{tr}(c_{n-1} \cdots c_k P (T_{n-1} + f_{n-1}(v)) \cdots (T_k + f_k(v)) \varphi) \\ &= \sum_{I=(i_1, \dots, i_l)} \text{tr}(c_I(v) P T_{i_l} \cdots T_{i_1} \varphi) \end{aligned}$$

where  $I = (i_1, \dots, i_l)$  is a sequence of integers such that  $k \leq i_1 < i_2 < \cdots < i_l \leq n-1$ , and  $c_I(v)$  is some scalar function. If  $I \neq I_0 = (k, k+1, \dots, n-1)$  then there are the following possibilities:

- (1)  $i_l \neq n-1$ ,
- (2)  $i_l = n-1$  and there exists an  $m$  ( $1 \leq m \leq l$ ) such that  $i_j - i_{j-1} = 1$  for any  $j = m+1, m+2, \dots, l$  and  $i_m - i_{m-1} > 1$ ,
- (3) otherwise.

case (1): As  $i_l < n - 1$ , we have

$$\begin{aligned} \operatorname{tr}(PT_{i_l} \cdots T_{i_1} \varphi) &= \operatorname{tr}(T_{i_l+1} \cdots T_{i_1+1} P \varphi) \\ &= t^l \operatorname{tr}(P \varphi) \\ &= 0. \end{aligned}$$

case (2): Since  $[T_i, T_j] = 0$  for  $|i - j| > 1$ ,

$$\begin{aligned} \operatorname{tr}(P(T_{i_l} \cdots T_{i_m})(T_{i_{m-1}} \cdots T_{i_1}) \varphi) &= \operatorname{tr}(P(T_{i_{m-1}} \cdots T_{i_1})(T_{i_l} \cdots T_{i_m}) \varphi) \\ &= \operatorname{tr}(T_{i_{m-1}+1} \cdots T_{i_1+1} PT_{i_l} \cdots T_{i_m} \varphi) \\ &= t^{m-1} \operatorname{tr}(PT_{i_l} \cdots T_{i_m} \varphi). \end{aligned}$$

By the induction hypothesis,  $\operatorname{tr}(PT_{i_l} \cdots T_{i_m} \varphi) = 0$ . Hence

$$\operatorname{tr}(PT_{i_l} \cdots T_{i_1} \varphi) = 0.$$

case (3): In this case  $I = (i_1, \dots, i_l)$  must be of the form  $i_l = n - 1$ ,  $i_{l-1} = n - 2, \dots, i_1 = n - l > k$ . By induction,  $\operatorname{tr}(PT_{i_l} \cdots T_{i_1} \varphi) = 0$ . So  $\operatorname{tr}(Y_i \varphi) = 0$  for all  $i$ .

Because of the relations between  $T$  and  $Y$ , it remains to check that

$$\operatorname{tr}(Y_{i_1} \cdots Y_{i_l} \varphi) = 0$$

for any  $l$ . One can show this by induction on  $l$ .  $\square$

**4.5. Macdonald operators.** We set

$$\hat{T}_i = t\sigma_i + \frac{t - t^{-1}}{e^{v_i - v_{i+1}} - 1}(\sigma_i - 1), \quad (1 \leq i \leq n - 1), \quad (4.30)$$

$$G_{ij} = t + \frac{t - t^{-1}}{e^{v_i - v_j} - 1}(1 - \sigma_{ij}), \quad (1 \leq i, j \leq n), \quad (4.31)$$

$$\Delta_i = \hat{T}_i \cdots \hat{T}_{n-1} \sigma_n \hat{T}_1^{-1} \cdots \hat{T}_{i-1}^{-1}. \quad (4.32)$$

Here  $\sigma_w$  are from (4.20),  $\sigma_{ij} = \sigma_{s_{ij}}$ .

Switching from  $\{T\}$  to  $\{G\}$ :

$$\begin{aligned} \hat{T}_i \sigma_i &= G_{ii+1}, \\ G_{ij}^{-1} &= t^{-1} - \frac{t - t^{-1}}{e^{v_i - v_j} - 1}(1 - \sigma_{ij}), \\ \Delta_i &= G_{ii+1} \cdots G_{in} \sigma_{\gamma_i} G_{1i}^{-1} \cdots G_{i-1i}^{-1}. \end{aligned}$$

Let  $e_m$  be the  $m$ -th elementary symmetric polynomial in  $n$  variables. We represent

$$e_m(\Delta_1, \dots, \Delta_n) = \sum_{w \in \mathfrak{S}_n} D_w^{(m)} \sigma_w, \quad (4.33)$$

for difference operators  $D_w^{(m)}$ , and define

$$M_m = M_{e_m} = \sum_{w \in \mathfrak{S}_n} D_w^{(m)}.$$

All these operators are  $W$ -invariant, which results from the following lemmas.

**Lemma 4.11.** *Consider the algebra  $\hat{\mathcal{H}}$  generated by  $\hat{T}_i$  ( $1 \leq i \leq n-1$ ),  $\Delta_j$  ( $1 \leq j \leq n$ ). Then  $T_i \mapsto \hat{T}_i$ ,  $Y_j \mapsto \Delta_j$  extends to an algebra isomorphism  $\mathcal{H}_n^t \xrightarrow{\sim} \hat{\mathcal{H}}$ . Moreover, if  $Q$  is a symmetric polynomial in  $n$  variables, then  $Q(\Delta_1, \dots, \Delta_n)$  is a central element in  $\hat{\mathcal{H}}$ .*

Actually this observation is the key point (it can be checked directly or with some representation theory). We note that the formulas for  $T$  generalize the so-called Demazure operations and the Bernstein-Gelfand-Gelfand operations. They were also studied by Lusztig and in a paper by Kostant-Kumar.

From now on we identify  $\hat{\mathcal{H}}$  with  $\mathcal{H}_n^t$ .

**Lemma 4.12.** *Let  $f(v_1, \dots, v_n)$  be a function on  $\mathbb{C}^n$ . Then  $f$  is symmetric if and only if  $(\hat{T}_i - t)f = 0$  for all  $i$ .*

**Lemma 4.13.** *Let  $Q$  be a symmetric polynomial in  $n$  variables. Then the operator  $Q(\Delta_1, \dots, \Delta_n)$  acts on the space of the symmetric polynomials in  $e^{v_i}$  ( $1 \leq i \leq n$ ).*

*Proof.* This follows immediately from Lemma 4.11 and 4.12.  $\square$

Let us calculate  $M_1$ . Since  $M_1$  is symmetric, it is enough to find the coefficient of  $\sigma_{\gamma_1}$ . Using the  $G$ -representation it is easy to see that  $\sigma_{\gamma_1}$  does not appear in  $\Delta_2, \dots, \Delta_n$ . The  $\sigma_{\gamma_1}$ -factor of  $\Delta_1$  is equal to  $\prod_{i=2}^n \frac{te^{v_1-v_i} - t^{-1}}{e^{v_1-v_i} - 1} \sigma_{\gamma_1}$ .

After the symmetrization we get the formula:

$$M_1 = \sum_{i=1}^n \prod_{j \neq i} \frac{te^{v_i} - t^{-1}e^{v_j}}{e^{v_i} - e^{v_j}} \sigma_{\gamma_i}.$$

Similarly,

$$M_m = \sum_{I=(i_1, \dots, i_m)} \prod_{i \in I} \prod_{j \notin I} \frac{te^{v_i} - t^{-1}e^{v_j}}{e^{v_i} - e^{v_j}} \sigma_{\gamma_{i_1}} \cdots \sigma_{\gamma_{i_m}}$$

where  $I = (i_1, \dots, i_m)$  is a sequence of integers such that  $1 \leq i_1 < \dots < i_m < n$ .

To recapitulate, let us consider the classical limit of the Macdonald operators. Setting  $q = e^h$  and  $t = q^{k/2}$ ,  $h \rightarrow 0$ , we have

$$\begin{aligned} \Delta_i &= 1 + h\mathcal{D}_i + O(h^2), \\ M_1 - n &= h \sum \frac{\partial}{\partial v_i} + O(h^2), \\ M_2 - (n-1)M_1 + \frac{n(n-1)}{2} &= h^2 L_2 + O(h^3). \end{aligned}$$

#### 4.6. Comments.

*Remark 4.1.* Take a solution  $\Phi = \Phi(v)$  of the QAKZ equation in an  $\mathcal{H}^t$ -module  $V$ , assuming that  $\Phi$  has the *trivial monodromy*. Then, for any polynomial  $p \in \mathbb{C}[x_1, \dots, x_n]$ , we have

$$p(Y_1, \dots, Y_n)\Phi = p(\Delta_1, \dots, \Delta_n)\Phi, \quad (4.34)$$

where  $\Delta_i$  are the difference Dunkl operators defined before. Note that  $\Delta'_i$  can be replaced by  $\Delta_i$  because the monodromy of  $\Phi$  is trivial. We also need a linear functional  $\text{pr} : V_\lambda \rightarrow \mathbb{C}$  for a vector  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  such that

$$\text{pr}(Y_i b) = \lambda_i \text{pr}(b) \quad (i = 1, \dots, n) \quad (4.35)$$

for any  $b \in V$ . Given any element  $a \in V$ , let us define a scalar-valued function  $\varphi = \varphi(v)$  setting

$$\varphi(v) = \text{pr}(\Phi(v)a) \in \mathbb{C}. \quad (4.36)$$

Then the formula (4.34) implies

$$p(\lambda_1, \dots, \lambda_n)\varphi = p(\Delta_1, \dots, \Delta_n)\varphi. \quad (4.37)$$

Thus, the scalar-valued function  $\varphi = \varphi(v)$  solves the *Dunkl eigenvalue problem*.

*Remark 4.2. Arbitrary root systems.* Let  $\Sigma = \{\alpha\} \in \mathbb{R}^n$  be any reduced root system of rank  $n$  (of type  $A, B, C, D, E, F$  or  $G$ ), and

$$\mathcal{H}^t = \langle T_1, \dots, T_n, X_1, \dots, X_n \rangle \quad (4.38)$$

the corresponding affine Hecke algebra. The *baxterization* (a parametric deformation satisfying the Yang-Baxter relations) of  $T_i$  will be given by

$$F_i = T_i + \frac{t - t^{-1}}{e^{u_i} - 1} \quad \text{with } u_i = (u, \alpha_i) \quad (4.39)$$

for each  $i = 1, \dots, n$ . We also have to use the element

$$T_0 = X_{\theta^\vee} T_\theta^{-1} \quad (4.40)$$

corresponding to the simple affine root  $\alpha_0 = \delta - \theta$  for  $\theta$  being the highest root. Its baxterization is quite similar:

$$F_0 = T_0 + \frac{t - t^{-1}}{e^{h-u_\theta} - 1}, \quad (4.41)$$

where  $u_\theta = (u, \theta)$ . The functions  $F_0, F_1, \dots, F_n$  satisfy the Yang-Baxter equations associated with the extended Dynkin diagram. For example, in the case of  $\mathfrak{o}^1 \Rightarrow \mathfrak{o}^2$ , we have

$$F_1(v)F_2(u+v)F_1(2u+v)F_2(u) = F_2(u)F_1(2u+v)F_2(u+v)F_1(v). \quad (4.42)$$

The arguments of  $F_i$  can also be determined graphically by means of the equivalent pictures of the reflection of two particles (see [15]).

Using  $T_0$ , the affine Hecke algebra  $\mathcal{H}^t$  has an alternative representation

$$\mathcal{H}^t = \langle T_0, T_1, \dots, T_n; \Pi \rangle, \quad (4.43)$$

where  $\Pi$  is a certain finite abelian group. The group  $\Pi$  is isomorphic to  $P^\vee/Q^\vee$ . It is the set of all elements of the extended affine Weyl group

$$\widetilde{W} = W \ltimes B, \quad B = \bigoplus_{i=1}^n \mathbb{Z} b_i, \quad (4.44)$$

preserving the set  $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$  of the simple affine roots. It gives the embedding of  $\Pi$  into the automorphism group of the extended Dynkin diagram. The action of  $\widetilde{W}$  on  $\mathbb{R}^n \oplus \mathbb{R}\delta$  is by the affine reflections and the corresponding shifts in the  $\delta$ -direction for  $B$ :

$$b(z + \zeta\delta) = z + (\zeta - (b, z))\delta.$$

**Lemma 4.14.**  $\widetilde{W} = \langle s_0, s_1, \dots, s_n; \Pi \rangle$  with  $s_0 = (\theta^\vee) \cdot s_\theta$ .

The group  $\Pi$  can be embedded into the affine Hecke algebra. The images  $P_\pi$  of the elements  $\pi \in \Pi$  permute  $\{T_i\}$  in the same way as  $\pi$  do in  $\widetilde{W}$  with  $\{s_i\}$ . The baxterization of the elements in  $\Pi$  is trivial:  $F_\pi = P_\pi$  for each  $\pi \in \Pi$ .

Keeping the notations of the previous sections, we have the following theorem.

**Theorem 4.15.** *Given any  $\mathcal{H}^t$ -valued function  $\Psi = \Psi(u)$ , the formulas*

$$\tilde{s}_i(\Psi) = s_i(F_i\Psi) \quad (4.45)$$

for all  $i = 0, 1, \dots, n$ , and

$$\tilde{\pi}(\Psi) = P_\pi \pi\Psi \quad (4.46)$$

for all  $\pi \in \Pi$  induce a representation of  $\widetilde{W}$ .

The QAKZ equation for  $\Sigma$  is the invariance condition  $\tilde{b}(\Phi) = \Phi$  for all  $b \in B$ . It can be shown that this equation is equivalent to the difference QMBP associated with the root system  $\Sigma$  defined via similar Dunkl operators. A conceptual proof of this isomorphism theorem is given by means of the intertwiners of double affine Hecke algebras (see [16, 18]).

## 5. DOUBLE AFFINE HECKE ALGEBRAS AND MACDONALD POLYNOMIALS

**5.1. Macdonald polynomials : the  $A_1$  case.** The subject of this section is to show how the Hecke algebra technique is applied to the Macdonald polynomials. We will concentrate on the duality and the recurrence relations. The key notion will be the double affine Hecke. Let us start with  $A_1$ .

The corresponding  $L$ -operator in the differential case reads as follows

$$L^{(k)} = \frac{\partial^2}{\partial u^2} + 2k \frac{e^u + e^{-u}}{e^u - e^{-u}} \frac{\partial}{\partial u} + k^2, \quad (5.1)$$

where  $k$  is a complex parameter. There are two special values of  $k$  when the operator  $L^{(k)}$  is very simple. For  $k = 0$  we have  $L^{(0)} = \partial^2 / \partial u^2$ . When  $k = 1$ ,

$$L^{(1)} = d^{-1} \frac{\partial^2}{\partial u^2} d, \quad \text{with } d = e^u - e^{-u}. \quad (5.2)$$

Similarly, we can conjugate by  $d^k$  for any  $k$ :

$$d^k L^{(k)} d^{-k} = \frac{\partial^2}{\partial u^2} - \frac{4k(k-1)}{(e^u - e^{-u})^2}. \quad (5.3)$$

Sometimes  $L$  is more convenient to deal with in this form.

Let us now consider the eigenvalue problem for the operator  $L^{(k)}$ :

$$L^{(k)} \varphi = \lambda^2 \varphi. \quad (5.4)$$

If  $k = 1$ , the solution of this equation is immediate:

$$\varphi(u; \lambda) = \frac{\sinh(u\lambda)}{\sinh(u) \sinh(\lambda)}. \quad (5.5)$$

In this normalization it is symmetric with respect to  $u$  and  $\lambda$ . Without  $\sinh(\lambda)$  it generalizes the characters of finite-dimensional representations of  $SL_2(\mathbb{C})$  ( $k=1$ ).

If  $k = 1/2$ , this operator is the *radial part* of the Casimir operator for the symmetric space  $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ . It is the restriction of the Casimir operator  $C$  on the double coset space  $SO_2 \backslash SL_2 / SO_2$  which is identified with a domain in  $\mathbb{R}^* / S_2$ . If  $k = 1, 2$ , then  $L^{(k)}$  corresponds in the same way to  $SL_2(\mathbb{C})/SU(2)$  and  $SL_2(\mathcal{K})/SU_2(\mathcal{K})$  for the quaternions  $\mathcal{K}$ .

For any  $k$ , one can find a family of even ( $u \rightarrow -u$ ) solutions of the form

$$p_n = e^{nu} + e^{-nu} + \text{lower integral exponents}, \quad (5.6)$$

such that

$$L^{(k)} p_n = (n+k)^2 p_n \quad (5.7)$$

for  $n = 0, 1, 2, \dots$ . This family of hyperbolic polynomials satisfy the orthogonality relations

$$\text{Constant Term } (p_n p_m d^{2k}) = c_n \delta_{nm}. \quad (5.8)$$

They are called the *ultraspherical polynomials*.

We can also consider the rational limit

$$\varrho^{(k)} = \frac{\partial^2}{\partial u^2} + \frac{2k}{u} \frac{\partial}{\partial u} \quad (5.9)$$

of the operator  $L^{(k)}$ , switching from the Sutherland model to the Calogero model. The solutions of the rational eigenvalue problem are expressed in terms of the Bessel function. In this case, the solutions can be normalized to ensure the symmetry between the variable and the eigenvalue. In the trigonometric case it is possible only for two special values  $k = 0, 1$ . It is one of the main demerits of the harmonic analysis on the symmetric spaces.

In the difference theory, this symmetry holds for any root systems. This discovery is expected to renew the Harish-Chandra theory. The so-called group case ( $k=1$ ) is an intersection point of the differential (classical) and difference (new) theories.

We now turn to the difference version. We set  $x = e^u$  and introduce the 'multiplicative difference'  $\Gamma_q$  acting as  $\Gamma_q(f(x)) = f(qx)$  and satisfying the commutation relation  $\Gamma_q x = qx\Gamma_q$ . The Macdonald operator  $L$  is expressed as follows:

$$L = \frac{tx - t^{-1}x^{-1}}{x - x^{-1}} \Gamma_q + \frac{tx^{-1} - t^{-1}x}{x^{-1} - x} \Gamma_q^{-1}. \quad (5.10)$$

The parameter  $k$  in the difference setup is determined from the relation  $t = q^k$ . When  $q = t$  (or  $k = 1$ ), the operator  $L$  is simple:

$$L = \frac{1}{x - x^{-1}} (\Gamma_q + \Gamma_q^{-1})(x - x^{-1}). \quad (5.11)$$

Compare this formula with (5.2) in the differential case and notice that (5.11) is easier to check than (5.2).

The eigenvalue problem

$$L\varphi = (\Lambda + \Lambda^{-1})\varphi \quad (5.12)$$

always has a self-dual family of solutions. When  $n = 0, 1, 2, \dots$ , there exists a unique family of the so-called  $q, t$ -ultraspherical (or Rogers-Askey-Ismail) Laurent polynomials

$$p_n = x^n + x^{-n} + \text{lower terms}, \quad (5.13)$$

which are symmetric with respect to the transformation  $x \rightarrow x^{-1}$ , and satisfy the equation

$$Lp_n = (tq^n + t^{-1}q^{-n})p_n. \quad (5.14)$$

The following duality theorem is proved in the next section by using the double affine Hecke algebra.

**Theorem 5.1 (Duality).**  $p_n(tq^m)p_m(t) = p_m(tq^n)p_n(t)$  for any  $m, n \geq 0$ .

If we set

$$\pi_n(x) = \frac{p_n(x)}{p_n(t)}, \quad (5.15)$$

the duality can be rewritten as follows:

$$\pi_n(tq^m) = \pi_m(tq^n) \quad (m, n = 0, 1, 2, \dots). \quad (5.16)$$

The Askey-Ismail polynomials are nothing but the Macdonald polynomials of type  $A_1$ . There are three main Macdonald's conjectures for the Macdonald polynomials associated with root systems (see [53, 54]):

- (1) the scalar product conjecture,
- (2) the evaluation conjecture,
- (3) the duality conjecture.

One may also add the Pieri rules to the list. These conjectures were justified recently using the double affine Hecke algebras in [17, 20]. The scalar product conjecture was established by Opdam in the differential setup (see [60]). He introduced the shift operators and deduced the norm-formula for any natural  $k$  from the trivial particular case  $k = 0$ . The passage to any  $k$  is based on the analytic continuation. I extended his approach to the  $q, t$ -case. The evaluation and the duality conjectures collapse in the differential case.

**5.2. A modern approach to  $q, t$ -ultraspherical polynomials.** The duality from Theorem 5.1 can be rephrased as the symmetry of a certain scalar product. Actually this product is a difference counterpart of the spherical Fourier transform. For any symmetric Laurent polynomials  $f, g \in \mathbb{C}[x + x^{-1}]$ , we set

$$\{f, g\} = (a(L)g)(t), \quad (5.17)$$

where  $a$  is a polynomial such that  $f(x) = a(x + x^{-1})$ . So we apply the operator  $a(L)$  to  $g$  and then evaluate the result at  $x = t$ .

**Theorem 5.2.**  $\{f, g\} = \{g, f\}$  for any  $f, g \in \mathbb{C}[x + x^{-1}]$ .

Theorem 5.1 follows from Theorem 5.2. Indeed, if  $f = \pi_m$  and  $g = \pi_n$ , we can compute the scalar product as follows:

$$\{\pi_m, \pi_n\} = (a(L)\pi_n)(t) = a(tq^n + t^{-1}q^{-n})\pi_n(t) = \pi_m(qt^n). \quad (5.18)$$

Use  $L\pi_n = (tq^n + t^{-1}q^{-n})\pi_n$  and the normalization  $\pi_n(t) = 1$ . Hence Theorem 5.2 implies  $\pi_m(qt^n) = \pi_n(qt^m)$ . Actually the theorems are equivalent, since  $p_n$  form a basis in the space of all symmetric Laurent polynomials.

**Definition 5.1.** The double affine Hecke algebra  $\mathcal{H}^{q,t}$  of type  $A_1$  is the quotient

$$\mathcal{H}^{q,t} = \langle X, Y, T \rangle / \sim, \quad (5.19)$$

by the relations for the generators  $X, Y, T$

$$TXT = X^{-1}, \quad T^{-1}YT^{-1} = Y^{-1}, \quad (5.20)$$

$$Y^{-1}X^{-1}YXT^2 = q^{-1}, \quad (T-t)(T+t^{-1}) = 0. \quad (5.21)$$

Here we consider  $q, t$  as numbers or parameters. The first point of the theory is the following statement of PBW type.

Any element of  $H \in \mathcal{H}$  can be uniquely expressed in the form

$$H = \sum_{\substack{i,j \in \mathbb{Z} \\ \epsilon=0,1}} c_{i\epsilon j} X^i T^\epsilon Y^j \quad (c_{i\epsilon j} \in \mathbb{C}). \quad (5.22)$$

The second important fact is the symmetry of  $\mathcal{H}^{q,t}$  with respect to  $X$  and  $Y$ .

**Theorem 5.3.** *There exists an anti-involution  $\phi : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\phi(X) = Y^{-1}$ ,  $\phi(Y) = X^{-1}$  and  $\phi(T) = T$ .*

Indeed,  $\phi$  transposes the first two relations and leaves the remaining invariant.

Next, we introduce the expectation value  $\{H\}_0 \in \mathbb{C}$  of an element  $H \in \mathcal{H}^{q,t}$  by

$$\{H\}_0 = \sum_{\substack{i,j \in \mathbb{Z} \\ \epsilon=0,1}} c_{i\epsilon j} t^{-i} t^\epsilon t^j, \quad (5.23)$$

using the expression (5.22). The definitions of  $\phi$  and  $\{ \}_0$  give that

$$\{\phi(H)\}_0 = \{H\}_0 \quad \text{for any } H \in \mathcal{H}^{q,t}. \quad (5.24)$$

Now we can introduce the operator counterpart of the pairing  $\{f, g\} = \{g, f\}$  on  $\mathcal{H}^{q,t} \times \mathcal{H}^{q,t}$ : setting

$$\{A, B\}_0 = \{\phi(A)B\}_0 \quad (5.25)$$

for any  $A, B \in \mathcal{H}^{q,t}$ .

The  $\phi$ -invariance of the expectation value (5.24) ensures that it is symmetric

$$\{A, B\}_0 = \{B, A\}_0.$$

We also remark that this pairing is non-degenerate for generic  $q, t$ .

Theorem 5.2 readily follows from

**Lemma 5.4.** *For any symmetric Laurent polynomials  $f(x), g(x) \in \mathbb{C}[x + x^{-1}]$ ,*

$$\{f(X), g(X)\}_0 = \{f, g\}.$$

To prove the lemma we need to introduce the *basic representation* of the double affine Hecke algebra  $\mathcal{H}^{q,t}$ . Consider the one-dimensional representation of the Hecke algebra  $\mathcal{H}_Y = \langle T, Y \rangle$  sending  $T \mapsto t$  and  $Y \mapsto t$ . We denote this representation simply by  $+$ . Then take the induced representation

$$V = \text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}}(+) = \mathcal{H} / \{\mathcal{H}(T - t) + \mathcal{H}(Y - t)\} \simeq \mathbb{C}[x, x^{-1}], \quad (5.26)$$

where the last isomorphism is  $x^n \leftrightarrow X^n \pmod{\mathcal{H}(T - t) + \mathcal{H}(Y - t)}$ . Under this identification of  $V$  with the ring  $\mathbb{C}[x, x^{-1}]$  of Laurent polynomials, the element  $X$  acts on  $\mathbb{C}[x, x^{-1}]$  as the multiplication by  $x$ , while  $T$  and  $Y$  act by the operators

$$\hat{T} = ts + \frac{t - t^{-1}}{x^2 - 1}(s - 1) \quad \text{and} \quad \hat{Y} = s\Gamma_q \hat{T}, \quad (5.27)$$

respectively. Here  $s(f)(x) = f(x^{-1})$ , the equality  $Hf(x) = g(x)$  in  $V$  means that  $Hf(X) - g(X) \in \mathcal{H}(T - t) + \mathcal{H}(Y - t)$ . The latter readily gives the desired formulas for  $\hat{T}, \hat{Y}$ .

The expectation value is the composition

$$\mathcal{H} \xrightarrow{\alpha} V \cong \mathbb{C}[x, x^{-1}] \xrightarrow{\beta} \mathbb{C}, \quad (5.28)$$

where  $\alpha$  is a residue mod  $\mathcal{H}(T - t) + \mathcal{H}(Y - t)$  and  $\beta(f) = f(t^{-1})$  is the evaluation map at  $t^{-1}$ . Take any  $f, g \in \mathbb{C}[X, X^{-1}]$ . Then

$$\{f(X), g(X)\}_0 = \{\phi(f(X))g(X)\}_0 = \{f(Y^{-1})g(X)\}_0 = f(\hat{Y}^{-1})(g)(t^{-1}). \quad (5.29)$$

The last equality follows from (5.28). If  $f$  and  $g$  are symmetric and  $f(X) = a(X + X^{-1})$ , then

$$\{f(X), g(X)\}_0 = a(L)(g)(t) = \{f, g\}, \quad (5.30)$$

since the operator  $\hat{Y} + \hat{Y}^{-1}$  acts on symmetric Laurent polynomials as  $L$ . It is straightforward. The duality is established.

This method of proving the duality theorem can be generalized to any root system.

We now discuss the application of the duality to the *Pieri rules*, the recurrence formulas for  $\pi_n$ 's with respect to the index  $n$ . First we will *discretize* functions and operators.

Recall that the renormalized  $q, t$ -ultraspherical polynomials  $\pi_n(x)$  are characterized by the conditions

$$L\pi_n = (tq^n + t^{-1}q^{-n})\pi_n, \quad \pi_n(t) = 1, \quad (5.31)$$

where

$$L = \frac{tx - t^{-1}x^{-1}}{x - x^{-1}}\Gamma + \frac{tx^{-1} - t^{-1}x}{x^{-1} - x}\Gamma^{-1}. \quad (5.32)$$

As always,  $\Gamma x = qx\Gamma$ . Denote the set of  $\mathbb{C}$ -valued functions on  $\mathbb{Z}$  by  $\text{Funct}(\mathbb{Z}, \mathbb{C})$ . For any Laurent polynomial  $f \in \mathbb{C}[x, x^{-1}]$  or more general rational function, we define  $\widehat{f} \in \text{Funct}(\mathbb{Z}, \mathbb{C})$  by setting

$$\widehat{f}(m) = f(tq^m) \quad \text{for all } m \in \mathbb{Z}. \quad (5.33)$$

Considering  $\mathcal{A} = \langle \mathbb{C}(x), \Gamma \rangle$  as an abstract algebra with the fundamental relation  $\Gamma x = qx\Gamma$ , the action of  $\mathcal{A}$  on  $\varphi \in \text{Funct}(\mathbb{Z}, \mathbb{C})$  is as follows:

$$\widehat{x}\varphi(m) = tq^m\varphi(m), \quad \widehat{\Gamma}\varphi(m) = \varphi(m+1). \quad (5.34)$$

The correspondence  $f \mapsto \widehat{f}$ , whenever it is well-defined (the functions  $f$  may have denominators), is an  $\mathcal{A}$ -homomorphism  $\mathbb{C}(x) \rightarrow \text{Funct}(\mathbb{Z}, \mathbb{C})$ .

Due to (5.31):

$$\widehat{L}\widehat{\pi}_n(m) = (tq^{-n} + t^{-1}q^n)\widehat{\pi}_n(m). \quad (5.35)$$

The Pieri rules result directly from this equality. Indeed, the duality  $\widehat{\pi}_n(m) = \widehat{\pi}_m(n)$  implies:

$$\mathcal{L}\widehat{\pi}_m(n) = (tq^n + t^{-1}q^{-n})\widehat{\pi}_m(n) = (\widehat{x} + \widehat{x}^{-1})\widehat{\pi}_m(n). \quad (5.36)$$

Here  $\mathcal{L}$  is  $\widehat{L}$  acting on the indices  $m$  instead of  $n$ . Explicitly,

$$\frac{t^2q^m - t^{-2}q^{-m}}{tq^m - t^{-1}q^{-m}}\widehat{\pi}_{m+1}(n) + \frac{q^{-m} - q^m}{t^{-1}q^{-m} - tq^m}\widehat{\pi}_{m-1}(n) = (\widehat{x} + \widehat{x}^{-1})\widehat{\pi}_m(n). \quad (5.37)$$

For generic  $q, t$ , the mapping  $f \rightarrow \widehat{f}$  is injective. Hence one can pull (5.37) back, removing the hats:

$$(x + x^{-1})\pi_m = \frac{t^2q^m - t^{-2}q^{-m}}{tq^m - t^{-1}q^{-m}}\pi_{m+1} + \frac{q^m - q^{-m}}{tq^m - t^{-1}q^{-m}}\pi_{m-1}. \quad (5.38)$$

This is the *Pieri formula* in the case of  $A_1$ . See [2]. We remark that this formula makes sense when  $m = 0$ , since the coefficient of  $\pi_{m-1}$  vanishes at  $m = 0$ . Generally speaking the ‘vanishing conditions’ are much less obvious.

The Pieri rules obtained above can be used to prove the so-called *evaluation conjecture* describing the values of  $p_n$  at  $x = t$ . Applying (5.38) repeatedly, we get the formula

$$(x + x^{-1})^\ell \pi_m = c_{\ell, m} \pi_{m+\ell} + \text{lower terms} \quad (5.39)$$

for each  $\ell = 0, 1, 2, \dots$ . The leading coefficient  $c_{\ell, m}$  can be readily calculated:

$$c_{\ell, m} = \prod_{i=0}^{\ell-1} \frac{t^2q^{m+i} - t^{-2}q^{-m-i}}{tq^{m+i} - t^{-1}q^{-m-i}}. \quad (5.40)$$

Let us look at (5.39) for  $m = 0$ :

$$(x + x^{-1})^\ell = c_{\ell,0} p_\ell + \text{lower terms.} \quad (5.41)$$

Comparing the coefficients of  $x^\ell + x^{-\ell}$ , we have  $1 = c_{\ell,0}/p_\ell(t)$ , since  $p_\ell = x^\ell + x^{-\ell} + \dots$ . Hence

$$p_\ell(t) = c_{\ell,0} = \prod_{i=0}^{\ell-1} \frac{t^2 q^i - t^{-2} q^{-i}}{t q^i - t^{-1} q^{-i}}. \quad (5.42)$$

This value is easy to calculate directly (the formulas for  $p_n$  are known). However the method described in this section is applicable to arbitrary root systems. We need only the duality, which is the main advantage of the difference harmonic analysis in contrast to the classical Harish-Chandra theory.

**5.3. The  $GL_n$  case.** In this last subsection, we will discuss the double affine Hecke algebra and applications for  $GL_n$ . Since we have already clarified the  $A_1$  case in full detail, we will try to get concentrated on the main points only.

In the  $GL_n$  case, the Macdonald operators  $M_0 = 1, M_1, \dots, M_n$  are as follows:

$$M_m = \sum_{I=\{i_1 < \dots < i_m\}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - t^{-1} x_j}{x_i - x_j} \Gamma_{i_1} \dots \Gamma_{i_m}. \quad (5.43)$$

In this normalization,  $t = q^{k/2}$  (cf. the differential case). For instance, the so-called group case is for  $k = 1/2$  (in contrast to  $SL_2$  considered above).

The Macdonald polynomials  $p_\lambda$  for  $GL_n$  satisfy the Macdonald eigenvalue problem:

$$M_m p_\lambda = e_m(t^{n-1} q^{\lambda_1}, \dots, t^{-n+1} q^{\lambda_n}) p_\lambda \quad (m = 0, 1, \dots, n), \quad (5.44)$$

where  $\lambda = (\lambda_1, \dots, \lambda_n)$  are partitions, i.e., sequences of integers  $\lambda_i \in \mathbb{Z}$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ . Here  $e_m$  is the elementary symmetric function of degree  $m$ . Given  $\lambda$ ,  $p_\lambda = p_\lambda(x)$  is a symmetric polynomial in  $x = (x_1, \dots, x_n)$  of degree  $|\lambda| = \sum_{i=1}^n \lambda_i$  in the form

$$p_\lambda(x) = x_1^{\lambda_1} \dots x_n^{\lambda_n} + \text{lower order terms.} \quad (5.45)$$

The lower order terms are understood in the sense of the dominance ordering. Namely, a partition obtained from  $\lambda$  by subtracting simple roots is lower than  $\lambda$ . For instance,

$$(\lambda_1, \lambda_2, \dots) > (\lambda_1 - 1, \lambda_2 + 1, \dots) > (\lambda_1 - 1, \lambda_2, \lambda_3 + 1, \dots) > \dots \quad (5.46)$$

We will use the abbreviation

$$t^{2\rho} q^\lambda = (t^{n-1} q^{\lambda_1}, \dots, t^{-n+1} q^{\lambda_n}), \quad (5.47)$$

where  $2\rho = (n-1, n-3, \dots, -n+1)$ . So  $M_m p_\lambda = e_m(t^{2\rho} q^\lambda) p_\lambda$ . Using  $k: t = q^{k/2}$ , and  $t^{2\rho} q^\lambda = q^{k\rho + \lambda}$ .

Given a partition  $\lambda$ , we set

$$\pi_\lambda(x) = \frac{p_\lambda(x)}{p_\lambda(t^{2\rho})} = \frac{p_\lambda(x_1, \dots, x_n)}{p_\lambda(t^{n-1}, t^{n-3}, \dots, t^{-n+1})}. \quad (5.48)$$

**Theorem 5.5 (Duality).** *For any partitions  $\lambda$  and  $\mu$ , we have*

$$\pi_\lambda(t^{2\rho} q^\mu) = \pi_\mu(t^{2\rho} q^\lambda). \quad (5.49)$$

This duality theorem implies the following *Pieri formula*.

**Theorem 5.6.**

$$e_m(x)\pi_\lambda(x) = \sum_{|I|=m} \prod_{\substack{i \in I \\ j \notin I}} \frac{t^{2(j-i)+1} q^{\lambda_i - \lambda_j} - t^{-1}}{t^{2(j-i)} q^{\lambda_i - \lambda_j} - 1} \pi_{\lambda + \epsilon_I}(x), \quad (5.50)$$

where  $\epsilon_I = \sum_{i \in I} \epsilon_i$  (sum of unit vectors).

Here the summation is taken only over subsets  $I \subset \{1, 2, \dots, n\}$  ( $|I| = m$ ) such that  $\lambda + \epsilon_I$  remain partitions (generally speaking, dominant). It happens automatically, since the coefficient of  $\pi_{\lambda + \epsilon_I}(x)$  on the right vanishes unless  $\lambda + \epsilon_I$  is dominant.

We can also determine the value of the Macdonald polynomial  $p_\lambda(x)$  at  $x = t^{2\rho}$  exactly by the method used in the  $A_1$  case. The formula was conjectured by Macdonald and proved by Koornwinder. The above theorems (for  $GL_n$ ) are also due to Macdonald and Koornwinder. See also [31]. For arbitrary roots they were established in my recent papers.

Our approach is based on the double Hecke algebras. The operators  $M_m$  appear naturally using the operators  $\Delta_i$  from (4.32). The latter describe the action of the generators  $Y_i$  in the induced representation  $\text{Ind}_{\mathcal{H}_Y}^{\mathcal{H}}(+)$  isomorphic to the algebra of Laurent polynomials  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . So the analogy with (5.26) is complete.

The *double affine Hecke algebra* (DAHA)  $\mathcal{H} = \mathcal{H}^{q,t}$  for  $GL_n$  is the algebra generated by the following two commutative algebras of Laurent polynomials in  $n$  variables:

$$\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \quad \text{and} \quad \mathbb{C}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}], \quad (5.51)$$

and the Hecke algebra of type  $A_{n-1}$ :

$$\mathcal{H} = \langle T_1, \dots, T_{n-1} \rangle \quad (5.52)$$

with the standard braid and quadratic relations. The remaining relations are as follows:

$$T_i X_i T_i = X_{i+1} \quad (i = 1, \dots, n-1), \quad T_i X_j = X_j T_i \quad (j \neq i, i+1), \quad (5.53)$$

$$T_i^{-1} Y_i T_i^{-1} = Y_{i+1} \quad (i = 1, \dots, n-1), \quad T_i Y_j = Y_j T_i \quad (j \neq i, i+1), \quad (5.54)$$

$$Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2, \quad (5.55)$$

$$\tilde{Y} X_j = q X_j \tilde{Y} \quad \text{and} \quad \tilde{X} Y_j = q^{-1} Y_j \tilde{X}. \quad (5.56)$$

Here  $\tilde{X} = \prod_{i=1}^n X_i$  and  $\tilde{Y} = \prod_{i=1}^n Y_i$ . They commute with  $\{T_1, \dots, T_{n-1}\}$  thanks to 5.53 and 5.55.

When  $q = 1, t = 1$  we come to the *elliptic* or 2-extended *Weyl group* of type  $GL_n$  due to Saito [63]. If  $q = 1$  and there are no quadratic relations, the corresponding group is the *elliptic braid group* ( $\pi_1$  of the product of  $n$  elliptic curves without the diagonals divided by  $\mathbb{S}_n$ ). It was calculated by Birman [6] and Scott.

Establishing the connection with (4.32),  $X_i = e^{v_i}$ ,  $T_i = \hat{T}_i$  and  $Y_i = \Delta_i$  give the so-called polynomial (or basic) representation of  $\mathcal{H}$ .

There is another version of this definition, using the element  $\pi$ . It is introduced from the formula

$$Y_1 = T_1 \cdots T_{n-1} \pi^{-1} \quad (5.57)$$

and has the following commutation relations with  $X_i$  and  $T_i$ :

$$\pi X_i = X_{i+1} \pi \quad (i = 1, \dots, n-1), \quad \pi X_n = q^{-1} X_1 \pi \quad (5.58)$$

and

$$\pi T_i = T_{i+1} \pi \quad (i = 1, \dots, n-2). \quad (5.59)$$

In the polynomial representation this element coincides with  $\pi$  from Lemma 4.3. Note that it acts on the functions  $X_i = e^{v_i}$  through the action of  $\pi^{-1}$  on vectors  $v$ . Considered formally,  $\pi$  is the image of the element  $P$  from Lemma 4.1 with respect to the Kazhdan-Lusztig automorphism, sending  $T \rightarrow T^{-1}$ ,  $Y \rightarrow Y^{-1}$ ,  $t \rightarrow t^{-1}$ .

Since

$$T_{i-1} \cdots T_1 X_1 T_1 \cdots T_{i-1} = X_i, \quad T_1 \cdots T_{i-1} Y_i T_{i-1} \cdots T_1 = Y_1, \quad (5.60)$$

we can reduce the list of generators. Namely,

$$\mathcal{H} = \langle X_1, Y_1, T_1, \dots, T_{n-1} \rangle \quad (5.61)$$

or

$$\mathcal{H} = \langle X_1, \pi, T_1, \dots, T_{n-1} \rangle. \quad (5.62)$$

In terms of  $\{T, \pi, X\}$ , the list of defining relations of  $\mathcal{H}$  is as follows:

- (a)  $X_i X_j = X_j X_i$  ( $1 \leq i, j \leq n$ ),
- (b) the braid relations and quadratic relations for  $T_1, \dots, T_{n-1}$ ,
- (c)  $\pi X_i = X_{i+1} \pi$  ( $i = 1, \dots, n-1$ ) and  $\pi^n X_i = q^{-1} X_i \pi^n$  ( $i = 1, \dots, n$ ),
- (d)  $\pi T_i = T_{i+1} \pi$  ( $i = 1, \dots, n-2$ ) and  $\pi^n T_i = T_i \pi^n$  ( $i = 1, \dots, n-1$ ).

For instance, let us deduce (5.55) from these formulas. Substituting, the left hand side equals:

$$\begin{aligned} & (T_1 \pi T_{n-1}^{-1} \cdots T_2^{-1}) X_1 (T_2 \cdots T_{n-1} \pi^{-1} T_1^{-1}) X_1^{-1} \\ &= T_1 \pi T_{n-1}^{-1} \cdots T_2^{-1} (T_2 \cdots T_{n-1}) X_1 \pi^{-1} (T_1^{-1} X_1^{-1} T_1^{-1}) T_1 \\ &= T_1 (\pi X_1 \pi^{-1} X_2^{-1}) T_1 = T_1^2. \end{aligned}$$

This representation, however, is not convenient from the viewpoint of the symmetry between  $X_i$  and  $Y_i$ , which will be discussed next. It is better to use  $\{Y\}$  instead of  $\pi$ .

The algebra  $\mathcal{H}$  contains the following two affine Hecke algebras:

$$\mathcal{H}_X^t = \langle X_1, \dots, X_n, T_1, \dots, T_{n-1} \rangle,$$

$$\mathcal{H}_Y^t = \langle Y_1, \dots, Y_n, T_1, \dots, T_{n-1} \rangle. \quad (5.63)$$

They are isomorphic to each other by the correspondence  $X_i \leftrightarrow Y_i^{-1}$ . This map can be extended to an anti-involution of  $\mathcal{H}$ . It is a general statement which holds for any root systems.

**Theorem 5.7.** *There exists an anti-involution  $\phi : \mathcal{H}^{q,t} \rightarrow \mathcal{H}^{q,t}$  such that  $\phi(X_i) = Y_i^{-1}$ ,  $\phi(Y_i) = X_i^{-1}$  for  $i = 1, \dots, n$ , and  $\phi(T_i) = T_i$  ( $i = 1, \dots, n-1$ ). It preserves  $q, t$ .*

*Proof.* We need to check that the relation (5.55) is self-dual with respect to  $\phi$ . The other relations are obviously  $\phi$ -invariant. One has:

$$\begin{aligned} 1 &= T_1^{-2} Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^{-1} \{Y_1^{-1} (T_1 X_1 T_1^{-1}) Y_1 T_1^{-1} X_1^{-1}\} \\ &= Y_1^{-1} X_2 T_1^{-2} Y_1 (T_1^{-1} X_1^{-1} T_1^{-1}) = Y_1^{-1} X_2 T_1^{-2} Y_1 X_2^{-1}. \end{aligned}$$

The latter can be rewritten as  $Y_1 X_2^{-1} Y_1^{-1} X_2 = T_1^2$ , which is the  $\phi$ -image of (5.55).  $\square$

Using this involution we can establish the duality theorem for the  $GL_n$  case in the same way as we did in the  $A_1$  case. Generalizing the theory to the case of arbitrary roots we can prove the Macdonald conjectures and much more. It gives a very convincing example of the power of the modern difference-operator methods.

## 6. ABSTRACT KZ AND ELLIPTIC QMBP

We will introduce general  $r$ -matrix KZ and use them to prove the self-consistency of the affine Knizhnik–Zamolodchikov equations. Another application will be the elliptic quantum many-body problem for arbitrary root systems, generalizing that due to Olshanetsky–Perelomov, and the double affine KZ.

**6.1. Abstract  $r$ -matrices.** Recall the notations for root systems. Let  $\Sigma = \{\alpha\} \subset \mathbb{R}^n$  be a finite root system of rank  $n$  of type  $A_n, B_n, \dots, G_2$ . We normalize the inner product of  $\mathbb{R}^n$  setting  $(\theta, \theta) = 2$ , where  $\theta \in \Sigma$  is the maximal root. Let  $\{\alpha_1, \dots, \alpha_n\}$  be the simple roots,  $\Sigma_+ = \{\alpha = \sum_{i=1}^n c_i \alpha_i \in \Sigma \mid \forall i, c_i \geq 0\}$  the set of positive roots,  $\{b_1, \dots, b_n\} \subset \mathbb{R}^n$  the dual fundamental weights (i.e.  $(b_i, \alpha_j) = \delta_{ij}$ ),  $P^\vee$  the lattice spanned by  $\{b_1, \dots, b_n\}$ ,  $W$  the Weyl group generated by the reflections

$$s_\alpha(u) = u - \frac{2(\alpha, u)}{(\alpha, \alpha)} \alpha, \quad u \in \mathbb{R}^n, \alpha \in \Sigma, \quad s_i = s_{\alpha_i}.$$

For  $u \in \mathbb{R}^n$  and  $\alpha \in \Sigma$ ,  $u_\alpha = (\alpha, u) \in \mathbb{R}$  and  $u_i = (\alpha_i, u)$ . So if  $\alpha = \sum_{i=1}^n c_i \alpha_i$  then  $u_\alpha = \sum_{i=1}^n c_i u_i$ . We will use the derivations:

$$\partial_v(u_\alpha) = (v, \alpha), \quad \partial_i(u_\alpha) = (b_i, \alpha), \quad i = 1, \dots, n, \quad v \in \mathbb{R}^n. \quad (6.1)$$

Let us begin with the formal theory of the following partial differential equations:

$$\partial_i \Phi(u) = \left( \sum_{\alpha \in \Sigma_+} \nu_\alpha^i r_\alpha \right) \Phi(u). \quad (6.2)$$

Here the values of  $\Phi(u)$  are in a vector space  $V$  over  $\mathbb{C}$ ,  $r_\alpha = r_\alpha(u_\alpha)$  is a function of one variable  $u_\alpha$  with the values in  $\text{End}(V)$ ,  $\nu_\alpha^i = \text{mult}_{\alpha_i}(\alpha) = (b_i, \alpha)$ .

More generally, we set

$$\partial_v \Phi(u) = \left( \sum_{\alpha \in \Sigma_+} (v, \alpha) r_\alpha \right) \Phi(u), \quad (6.3)$$

for  $v \in \mathbb{R}^n$  ( $v = b_i$  in the above formula). Let

$$X_i = \sum_{\alpha \in R_+} \nu_\alpha^i r_\alpha, \quad X_v = \sum_{\alpha \in R_+} (v, \alpha) r_\alpha. \quad (6.4)$$

The compatibility of system (6.2) (the cross-derivative integrability conditions) is equivalent to a purely algebraic commutativity of  $\{X_i\}$  ( $\Leftrightarrow [X_v, X_{v'}] = 0 \forall v, v' \in \mathbb{R}^n$ ). Indeed,  $\partial_i(\nu_\alpha^j r_\alpha) = \partial_j(\nu_\alpha^i r_\alpha)$  for all  $\alpha$ . We are going to establish the necessary conditions for the compatibility.

For  $\alpha, \beta \in \Sigma$  ( $\beta \neq \pm\alpha$ ), let  $\langle \alpha, \beta \rangle$  be the subspace of  $\mathbb{R}^n$  spanned by  $\alpha$  and  $\beta$ . Then  $\Sigma \cap \langle \alpha, \beta \rangle$  is a root system of rank 2.

**Proposition 6.1.** *If the operators*

$$X_v^{(\alpha, \beta)} = \sum_{\substack{\gamma \in \Sigma_+ \\ \gamma \in \Sigma \cap \langle \alpha, \beta \rangle}} (v, \gamma) r_\gamma \quad (v \in \langle \alpha, \beta \rangle) \quad (6.5)$$

*commute for  $v \in \langle \alpha, \beta \rangle$  (say, for  $v = \alpha$  and  $v' = \beta$ ) and any arbitrary given pair  $\{\alpha, \beta\} \subset \Sigma_+$ , then  $[X_i, X_j] = 0$  ( $\forall i, j = 1, \dots, n$ ).*

By Proposition 6.1, it is sufficient to consider the rank 2 cases (Fig8.). One may assume that  $\alpha, \beta$  are the *standard simple roots* from the figure, switching to proper linear combinations if necessary. The commutativity conditions are directly connected with the quantum Yang–Baxter equations from [9]. The definition is as follows.

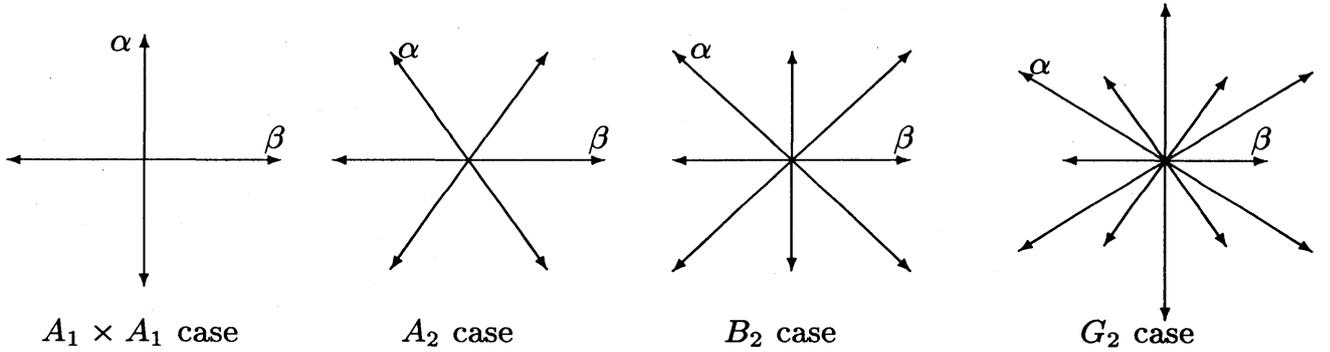


FIGURE 8. The root systems of rank two

We set formally

$$R_\alpha = 1 + hr_\alpha + O(h^2). \quad (6.6)$$

The classical Yang–Baxter equations are the coefficient of  $h^2$  in the quantum Yang–Baxter equations for  $R$ :

0)  $A_1 \times A_1$  case:  $R_\alpha R_\beta = R_\beta R_\alpha \Rightarrow$

$$[r_\alpha, r_\beta] = 0. \quad (6.7)$$

1)  $A_2$  case:  $R_\alpha R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_\alpha \Rightarrow$

$$[r_\alpha, r_{\alpha+\beta} + r_\beta] + [r_{\alpha+\beta}, r_\beta] = 0. \quad (6.8)$$

2)  $B_2$  case:  $R_\alpha R_{\alpha+\beta} R_{\alpha+2\beta} R_\beta = R_\beta R_{\alpha+2\beta} R_{\alpha+\beta} R_\alpha \Rightarrow$

$$[r_\alpha, r_{\alpha+\beta} + r_{\alpha+2\beta} + r_\beta] + [r_{\alpha+\beta}, r_{\alpha+2\beta} + r_\beta] + [r_{\alpha+2\beta}, r_\beta] = 0. \quad (6.9)$$

$$\begin{aligned}
3) \ G_2 \text{ case: } R_\alpha R_{\alpha+\beta} R_{2\alpha+3\beta} R_{\alpha+2\beta} R_{\alpha+3\beta} R_\beta &= R_\beta R_{\alpha+3\beta} R_{\alpha+2\beta} R_{2\alpha+3\beta} R_{\alpha+\beta} R_\alpha \Rightarrow \\
& [r_\alpha, r_{\alpha+\beta} + r_{2\alpha+3\beta} + r_{\alpha+2\beta} + r_{\alpha+3\beta} + r_\beta] + \\
& + [r_{\alpha+\beta}, r_{2\alpha+3\beta} + r_{\alpha+2\beta} + r_{\alpha+3\beta} + r_\beta] + \\
& + [r_{2\alpha+3\beta}, r_{\alpha+2\beta} + r_{\alpha+3\beta} + r_\beta] + [r_{\alpha+2\beta}, r_{\alpha+3\beta} + r_\beta] + [r_{\alpha+3\beta}, r_\beta] = 0.
\end{aligned} \tag{6.10}$$

The structure of the quantum Yang–Baxter equations is very simple: the product of all  $R_\alpha$  for positive  $\alpha$  clockwise coincides with the product taken counterclockwise. Respectively, the left hand sides of the classical equations are the sums of the commutators  $[r_{\alpha'}, r_{\beta'}]$  where  $\alpha', \beta'$  are ordered clockwise. All roots are positive in  $\Sigma$ .

However the classical Yang–Baxter equations are not exactly what we need for the commutativity. Indeed, the coefficients of all  $[r, r]$ –commutators in these equations are 1, but they are not always 1 in the commutators of  $X_v^{(a,b)}$ . The simplest example is  $B_2$ . This discrepancy can be clarified only in the theory of quantum KZ (see [15]). However it is not difficult to find the desired conditions without any reference to QKZ, as I did in my first paper devoted to the Knizhnik–Zamolodchikov equations [10].

**Proposition 6.2.** *For any pair  $\{\alpha, \beta\} \in \Sigma_+$  such that the entire intersection of  $\Sigma$  and  $\langle \alpha, \beta \rangle$  is represented by one of the pictures above, we impose the conditions:*

0)  $A_1 \times A_1$  case: (6.7),

1)  $A_2$  case: (6.8),

2)  $B_2$  case: (6.9) and  $([r_\alpha, r_{\alpha+2\beta}] = 0)$ ,

3)  $G_2$  case: (6.10),  $(\alpha' \perp \beta' \Rightarrow [r_{\alpha'}, r_{\beta'}] = 0)$ , and ( the  $A_2$  relation (6.8) for the long roots), where  $\alpha', \beta' \in \langle \alpha, \beta \rangle \cap \Sigma_+$ .

*Then the operators  $X$  are pairwise commutative.*

Let us consider a bit more special situation. For the same root system  $\Sigma$ , we set

$$x_i = \sum_{\alpha \in \Sigma_+} k_\alpha \nu_\alpha^i r_\alpha, \tag{6.11}$$

where  $k_\alpha = k_\beta$  if there exists  $w \in W$  such that  $w(\alpha) = \beta$  (in other words,  $k_\alpha = k_\beta$  if  $(\alpha, \alpha) = (\beta, \beta)$ ). We call such  $k$  invariant.

**Proposition 6.3.** *Let  $\Sigma$  be a root system of rank  $n$ . If a pair of positive roots  $\alpha, \beta$  is standard (i.e. corresponds to one of the pictures above) in a  $2D$ –subsystem of  $\Sigma$ , which may be less than the complete intersection with  $\langle \alpha, \beta \rangle$ , then the conditions*

0)  $A_1 \times A_1$  case: (6.7), 1)  $A_2$  case: (6.8),

2)  $B_2$  case: (6.9), 3)  $G_2$  case: (6.10)

*are sufficient for the commutativity of the operators  $x_i$  ( $i = 1, \dots, n$ ) for arbitrary invariant  $k$ .*

**6.2. Degenerate Hecke algebras.** One may use the same definitions for  $\{r_\alpha\}$  defined on all roots (positive and negative). Let us assume that  $r$  satisfies the conditions of Proposition 6.3 for positive roots. Then there are two natural extensions of  $r$  to all roots, where the relations (0-3) hold for any standard  $\alpha, \beta$  (positive or not in  $\Sigma$ ). The first is the extension by 0:  $r_{-\alpha} = 0$  ( $\alpha \in \Sigma_+$ ). The second is given by  $r_{-\alpha} = r_\alpha$  ( $\alpha \in \Sigma_+$ ). One more extension exists for the  $W$ -invariant  $r$ .

We need to suppose that the Weyl group  $W$  acts on an algebra  $\mathcal{R}$  containing  $\{r_\alpha\}$  provided the relations  $w(r_\alpha) = r_{w(\alpha)}$  ( $\alpha, w(\alpha) \in \Sigma_+, w \in W$ ). Such  $r$  are called  $W$ -invariant. Given a root  $\beta \in \Sigma$ , we set  $r_\beta = w(r_\alpha)$  for any  $w \in W$  and  $\alpha \in \Sigma_+$  satisfying  $w(\alpha) = \beta$ . The definition does not depend on the particular choice of  $w$  and  $\alpha$ . The group  $W$  acts on all  $\{r_\alpha | \alpha \in \Sigma\}$  by the same formulas as for positive roots. We will call  $W$ -invariant  $r$  satisfying the assumption of Proposition 6.3 for all roots *invariant  $r$ -matrices*.

**Proposition 6.4.** *For an invariant  $r$ -matrix, we set*

$$x_b = \sum_{\alpha \in \Sigma_+} k_\alpha(b, \alpha) r_{\alpha(u)}, \quad (6.12)$$

where  $u \in \mathbb{R}^n$  and  $k_\alpha = k_\beta$  if  $\exists w \in W$  s.t.  $w(\alpha) = \beta$ . Then

$$[x_b, x_{b'}] = 0 \quad (\forall b, b' \in \mathbb{R}^n), \quad (6.13)$$

$$s_i(x_b) - x_{s_i(b)} = k_{\alpha_i}(b, \alpha_i)(s_i(r_{\alpha_i}) + r_{\alpha_i}). \quad (6.14)$$

*Proof.* The relation (6.13) results readily from Proposition 6.3. As to (6.14), it follows from the relation  $s_i(\Sigma_+) = (\Sigma_+ \setminus \{\alpha_i\}) \cup \{-\alpha_i\}$ .  $\square$

If  $W \subset \mathcal{R}$ ,  $w(a) = waw^{-1}$  for  $w \in W$  for  $a \in \mathcal{R}$ , and  $s(r_{\alpha_i}) + r_{\alpha_i} = s_i$ , we get

$$s_i x_b s_i - x_{s_i(b)} = k_{\alpha_i}(b, \alpha_i) s_i. \quad (6.15)$$

So we come back to the definition the degenerate Hecke algebra  $\mathcal{H}'_\Sigma$ :

**Definition 6.1.** The *degenerate Hecke algebra*  $\mathcal{H}'_\Sigma$  for  $\Sigma$  is generated by

$$\{s_\alpha(\alpha \in R), y_b(b \in P^\vee)\}$$

with the defining relations

$$[x_b, x_{b'}] = 0 \quad (\forall b, b' \in P^\vee), \quad (6.16)$$

$$s_i x_b - x_{s_i(b)} s_i = k_{\alpha_i}(b, \alpha_i), \quad (6.17)$$

and those from the Weyl group.

Now we are able to construct quite a few interesting representations of  $\mathcal{H}'_\Sigma$  using various  $r$ -matrices. For instance, let us apply this machinery to establish the compatibility and  $W$ -invariance of AKZ. Let us double the set of variables adding

pairwise commutative  $\{v_a (a \in P^\vee)\}$  provided that  $v_{a+b} = v_a + v_b$  ( $a, b \in P^\vee$ ). However, in contrast to  $\{u_a\}$ , they will not commute with  $W$ :

$$wv_\alpha = v_{w(\alpha)}w \quad (\alpha \in P^\vee, w \in W).$$

We set

$$r_\alpha^\circ = -(e^{v_\alpha} - 1)^{-1}s_\alpha. \quad (6.18)$$

This  $r$ -matrix is derived from the  $R$ -matrix

$$R_\alpha = t + (t - t^{-1})(e^{v_\alpha} - 1)^{-1}(1 - s_\alpha), \quad (6.19)$$

when  $h \rightarrow 0$  ( $t = 1 + h$ ). It is easy to check that the latter does satisfy relations (0-3) for  $R$  above. The claim is due to Lusztig. There is a conceptual proof based on the explicit formulas for the generators  $\{T\}$  in the polynomial representation of the affine Hecke algebra induced from the character  $\{T_i \mapsto t, 0 \leq i \leq n\}$ . Anyhow  $r^\circ$  satisfies Proposition 6.3.

**Proposition 6.5.** *The elements*

$$x_b^\circ = \sum_{\alpha > 0} k_\alpha(b, \alpha)r_\alpha^\circ \quad (b \in P^\vee) \quad (6.20)$$

satisfy the relations of the degenerate Hecke algebra  $\mathcal{H}'_\Sigma$ . The representation  $\mathcal{H}'_\Sigma \rightarrow \text{End}(\mathbb{C}[e^{v_a}, a \in P^\vee])$  sending  $w \mapsto w, x_b \mapsto x_b^\circ$  is faithful.

Finally,

$$\hat{r}_\alpha = \hat{r}_\alpha(u_\alpha) = \frac{s_\alpha}{e^{u_\alpha} - 1} + r_\alpha^\circ. \quad (6.21)$$

It is an invariant  $r$ -matrix and moreover unitary:

$$\hat{r}_\alpha(u_\alpha) + s_\alpha \hat{r}_\alpha(-u_\alpha) s_\alpha = 0. \quad (6.22)$$

This  $r$  is nothing else but the  $W$ -extension of the intertwining operators of the degenerate Hecke algebra, where the generators  $\{T\}$  are taken in the “polynomial” representation  $\mathbb{C}[e^{v_a}]$ .

**Proposition 6.6.** 1) *The following system of partial differential equations*

$$\partial_b \Phi(u) = \sum_{\alpha > 0} k_\alpha(b, \alpha) \hat{r}_\alpha \Phi(u) \quad (6.23)$$

for  $\hat{r}_\alpha$  from (6.21),  $\Phi(u) \in \mathbb{C}[e^{v_a}]$  and invariant  $\{k_\alpha\}$  is self-consistent and  $W$ -invariant.

2) *The AKZ with the values in  $\mathcal{H}'_\Sigma$*

$$\frac{\partial \Phi}{\partial u_i} = \left( \sum_{\alpha \in \Sigma_+} k_\alpha \nu_\alpha^i \frac{s_\alpha}{e^{u_\alpha} - 1} + x_i \right) \Phi \quad (1 \leq i \leq n) \quad (6.24)$$

is self-consistent and  $W$ -invariant.

Here the  $W$ -invariance of (6.23) follows from (6.22). Recall that the system  $\partial_b \Phi(u) = A_b \Phi(u)$  ( $b \in P^\vee$ ) is said to be self-consistent if

$$[\partial_b - A_b, \partial_{b'} - A_{b'}] = 0 \quad (b, b' \in P^\vee). \quad (6.25)$$

The invariance means that if  $\Phi(u)$  is a solution so are  $s_i \Phi(s_i(u))$  for all  $i$ . The second claim readily results from the first. Collecting all  $r^\circ$  together and replacing them by  $\{x\}$ , we get the AKZ in the representation  $\mathbb{C}[e^{v_a}]$ , which is faithful.

**6.3. Elliptic QMBP.** Another application of the general theory of the classical  $r$ -matrices will be a generalization of the elliptic quantum many-body problem from [59] to arbitrary root systems. Olshanetsky and Perelomov introduced it for  $GL_n$ . Ochiai, Oshima and Sekiguchi in [58] generalized their construction to arbitrary classical root systems. More exactly, they considered the quantum Hamiltonian in the form

$$H = \sum_{i=1}^n \partial_{\alpha_i} \partial_i + \sum_{\alpha > 0} V(u_\alpha) \quad (6.26)$$

and deduced from the existence of the higher conservation laws (differential operators commuting with  $H$ ) that the potential  $V$  has to be the Weierstrass  $\wp$ -function or its degenerations. Arbitrary root systems were covered in [22]. We will reproduce here the simplest variant of the construction from this paper.

Let us first recall the construction of the Calogero-Sutherland operators. We keep the notation from the previous section. The root system  $\Sigma(\in \mathbb{R}^n)$  is arbitrary. We will use the same  $u_\alpha = (u, \alpha)$  imposing the relation  $s_\alpha u_b s_\alpha^{-1} = u_{s_\alpha(b)}$ , where  $s_\alpha \in W$ . So in this section they do not commute with  $W$ .

**Proposition 6.7.** *The operators*

$$\mathcal{D}_b = \partial_b - \sum_{\alpha \in \Sigma_+} k_\alpha(b, \alpha) (e^{u_\alpha} - 1)^{-1} s_\alpha \quad (6.27)$$

are pairwise commutative for  $b \in P^\vee$  and satisfy the relations of the degenerate affine Hecke algebra:

$$s_i \mathcal{D}_b - \mathcal{D}_{s_i(b)} s_i = k_i(b, \alpha_i). \quad (6.28)$$

Here  $k$  is invariant:  $k_\alpha = k_\beta$  if  $(\exists w \in W \text{ s.t. } w(\alpha) = \beta)$ . The statement is obvious since  $r_\alpha^\circ = -(e^{u_\alpha} - 1)^{-1} s_\alpha$  obey Proposition 6.3 and the relation

$$r_\alpha^\circ + s_\alpha r_\alpha^\circ s_\alpha = s_\alpha. \quad (6.29)$$

Then we proceeded in Section 3.4 as follows. Let  $\mathbb{C}[x_1, \dots, x_n]^W$  be the algebra of  $W$ -invariant polynomials. We set

$$\mathcal{L}_p = p(\mathcal{D}_{b_1}, \dots, \mathcal{D}_{b_n}), \quad (6.30)$$

where  $p \in \mathbb{C}[x_1, \dots, x_n]^W$ . Since  $w\mathcal{L}_p w^{-1} = \mathcal{L}_p$  the operators  $L_p = \mathcal{L}_p|_{Sym}$  (restriction to the symmetric function) are pairwise commutative and  $W$ -invariant. If  $p_2 = \sum_{i=1}^n x_{\alpha_i} x_{b_i}$ , then

$$L_{p_2} = \sum_{i=1}^n \partial_{\alpha_i} \partial_{b_i} + \sum_{\alpha \in \Sigma_+} \frac{k_\alpha (1 - k_\alpha)(\alpha, \alpha)}{(e^{\frac{u\alpha}{2}} - e^{-\frac{u\alpha}{2}})^2}. \quad (6.31)$$

This operator is the Hamiltonian of the Sutherland model.

Proposition 6.3 can be readily extended to affine root systems provided the convergence of the products of  $D_i$ . The affine root system is the set

$$\Sigma^a = \{[\alpha, k] \in \mathbb{R}^{n+1} \mid \alpha \in \Sigma, k \in \mathbb{Z}\}, \quad (6.32)$$

$$\Sigma_+^a = \{[\alpha, k] \mid \alpha \in \Sigma, k \in \mathbb{Z}_{>0}\} \cup \{\alpha = [\alpha, 0] \mid \alpha \in \Sigma_+\}. \quad (6.33)$$

The *affine Weyl group*  $W^a$  is generated by the reflections  $s_{\tilde{\alpha}}$  ( $\tilde{\alpha} = [\alpha, k] \in R^a$ ) acting in  $\mathbb{R}^{n+1}$ :

$$s_{\tilde{\alpha}}([v, \xi]) = [v, \xi] + 2 \frac{(\alpha, v)}{(\alpha, \alpha)} \tilde{\alpha}, \quad (6.34)$$

where  $v \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}$ .

The *extended affine Weyl group*  $\widetilde{W}$  is generated by  $W$  and the “translations” corresponding to  $b \in P^\vee = \bigoplus_{i=1}^n \mathbb{Z}b_i$ :

$$b([v, \xi]) = [v, -(b, v) + \xi], \quad (6.35)$$

where  $v \times \xi \in \mathbb{R}^{n+1}$ . On the space  $\mathbb{R}^{n+1}$ ,  $W$  acts preserving  $\xi$ . The group  $\widetilde{W}$  contains  $W^a$  generated by  $W$  and  $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}a_i$  for  $a_i = \alpha_i^\vee = 2\alpha_i/(\alpha, \alpha)$ . One has:

$$W^b = W \ltimes P^\vee, \quad W^b = \Pi \ltimes W^a, \quad (6.36)$$

for the group  $\Pi$  isomorphic to  $P^\vee/Q^\vee$ .

Concerning the abstract theory, one starts with an  $W$ -invariant nonaffine  $r$ -matrix  $r_\alpha$  and assume that  $\widetilde{W}$  acts on the algebra  $\mathcal{R}$  which contains  $\{r_\alpha\}$ , provided the relations  $b(r_\alpha) = r_\alpha$  whenever  $(b, \alpha) = 0$ . This is sufficient to extend  $r$  to a  $\widetilde{W}$ -invariant affine  $r$ -matrix:  $r_{[b, \xi]} = b(r_\alpha)$  for any  $b \in P^\vee$  such that  $(b, \alpha) = -\xi$ . The natural setup here is that from Proposition 6.1. The additional conditions which appear in this proposition to ensure the commutativity of  $\{X\}$  (the commutativity of  $r$  for orthogonal long roots in the case of  $B_2$ , etc.) are exactly those which are necessary to ensure the existence of an affine extension. This coincidence is not by chance and can be explained in full in the difference (quantum) theory.

Once an affine  $r$ -matrix is given, we can introduce the operators  $X$  and  $x$  and get a formal proof of their commutativity. The sums for these operators are infinite so we need to provide the convergence to make it rigorous. Let us demonstrate how it works for the elliptic QMB.

We will define the *double affine Dunkl operators*, which are differential operators with the coefficients from  $\text{Funct}(\mathbb{C}^n, \mathbb{C}_\infty \widetilde{W})$ , where  $\mathbb{C}_\infty \widetilde{W} = \{\sum_{\tilde{w} \in \widetilde{W}} c_{\tilde{w}} \tilde{w}, c_{\tilde{w}} \in \mathbb{C}\}$  consists of infinite sums in contrast to the standard group algebra  $\mathbb{C}\widetilde{W}$ . For each  $b \in P^\vee$ ,

$$\mathcal{D}_b = \partial_b - \sum_{\tilde{\alpha} \in \Sigma_+^a} k_\alpha (e^{u_{\tilde{\alpha}}} - 1)^{-1} s_{\tilde{\alpha}}, \quad (6.37)$$

where  $\tilde{\alpha} = [\alpha, m]$ ,  $k_\alpha$  is an arbitrary  $W$ -invariant function on  $\Sigma$ , and  $e^{u_{\tilde{\alpha}}} = e^{u_\alpha + m\eta}$  for a fixed  $\eta \in \mathbb{C}$  (by definition).

**Lemma 6.8.** *If  $\text{Re}\eta > 0$  then the products of operators  $\mathcal{D}_c$  are well-defined and can be represented as infinite sums*

$$\mathcal{D}_{c_1} \dots \mathcal{D}_{c_k} = \sum_{\tilde{w} \in W^b} \Psi_{\tilde{w}}(u, \eta) \tilde{w}, \quad (6.38)$$

where  $\Psi_{\tilde{w}}(u, \eta)$  are differential operators with the coefficients meromorphic in  $\mathbb{C}^n \ni v$ . Moreover, the absolute values of the coefficients of  $\Psi_{\tilde{w}}(u, \eta)$  are bounded point-wise (apart from the singularities) by a function  $C(u, \eta) \epsilon(u, \eta)^{l(\tilde{w})}$ , where  $0 \leq \epsilon(u, \eta) < 1$ ,  $C > 0$ , and  $l(\tilde{w})$  is the length of  $\tilde{w}$  in  $\widetilde{W}$  with respect to the generators  $\{s_i, 0 \leq i \leq n\}$ .

See [22] for the proof and the exact definition of the length. The condition  $\text{Re}\eta > 0$  is replaced by a less exact inequality in [22], but the claim holds for all positive  $\text{Re}\eta$ .

**Theorem 6.9.** *If  $\text{Re}\eta > 0$  then we have the following relations:*

$$[\mathcal{D}_b, \mathcal{D}_{b'}] = 0, \quad (6.39)$$

$$s_i \mathcal{D}_b - \mathcal{D}_{s_i(b)} s_i = k_i(b, \alpha_i), \quad (6.40)$$

$$s_0 \mathcal{D}_b - \mathcal{D}_{s_\theta(b)} s_0 = -k_\theta(b, \theta), \quad (6.41)$$

where  $\theta$  is the maximal root of  $\Sigma$  and  $s_0 = s_{[-\theta, 1]} \in W^a \subset \widetilde{W}$ .

For  $p \in \mathbb{C}[x_1, \dots, x_n]^W$ , we set

$$\mathcal{L}_p = p(\mathcal{D}_{b_1}, \dots, \mathcal{D}_{b_n}). \quad (6.42)$$

Then  $w\mathcal{L}_p w^{-1} = \mathcal{L}_p$  ( $w \in W^b$ ) and the operators  $L_p = \mathcal{L}_p|_{\text{Sym}}$  are pairwise commutative, i.e.  $[L_p, L_{p'}] = 0$ . However now  $|_{\text{Sym}}$  is the restriction to the  $\widetilde{W}$ -invariant functions in contrast to the non-affine theory. More explicitly, if we have  $p(\mathcal{D}_{b_1}, \dots, \mathcal{D}_{b_n}) = \sum_{\tilde{w}} \Psi_{\tilde{w}}(u, \eta) \tilde{w}$  where  $\Psi_{\tilde{w}}(u, \eta)$  is the differential operator and does not contain the elements from  $\widetilde{W}$ , then we have

$$L_p = \sum_{\tilde{w}} \Psi_{\tilde{w}}(u, \eta). \quad (6.43)$$

The convergence readily follows from the lemma. The coefficients of  $L_p$  are  $\tilde{W}$ -invariant functions.

For instance, if we put  $p_2 = \sum_{i=1}^n x_{\alpha_i} x_{b_i}$ , then

$$L_{p_2} = \sum_{i=1}^n \partial_{\alpha_i} \partial_{b_i} + \sum_{\alpha \in \Sigma_+} (\alpha, \alpha) k_\alpha (k_\alpha - 1) \tilde{\zeta}'(u_\alpha), \quad (6.44)$$

where

$$\tilde{\zeta}(t) = \sum_{m=0}^{\infty} \frac{1}{e^{m\eta+t} - 1} - \sum_{m=1}^{\infty} \frac{1}{e^{m\eta-t} - 1}, \quad \tilde{\zeta}'(t) = \frac{d\tilde{\zeta}(t)}{dt}. \quad (6.45)$$

Note that

$$\tilde{\zeta}(t + \eta) = \tilde{\zeta}(t) + 1, \quad \tilde{\zeta}(t) + \tilde{\zeta}(-t) = -1, \quad (6.46)$$

$$-\tilde{\zeta}'(t) + c = \wp(t; \Omega) = \frac{1}{t^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left\{ \frac{1}{(t - \omega)^2} - \frac{1}{\omega^2} \right\} \quad (6.47)$$

for some constant  $c$ . Here  $\Omega = \{2\pi\sqrt{-1}\mathbb{Z} + \eta\mathbb{Z}\}$  is a lattice in  $\mathbb{C}$ .

The above construction is more interesting when we consider the Dunkl operators with  $(s_{\tilde{\alpha}} - 1)$  instead of  $s_{\tilde{\alpha}}$ . Still they are pairwise commutative. However relation (6.41) is more complicated and the reduction to the  $L$ -operators is somewhat different. The latter is governed by the degenerate double affine Hecke algebra with zero central charge. The resulting operators preserve the Looijenga space of (formal) theta-functions of level  $c = -kg$  for the dual Coxeter number  $g$  (it is the dot product  $k \cdot h$  if  $k$  has more than one component). See [22]. When  $k = 1$  (the group case), we come to the relation  $c + g = 0$  coinciding with the critical level condition for the Kac-Moody algebras ( $c$  is the central charge). It obviously indicates that double Hecke algebras are on the right track.

**6.4. Double affine KZ.** Continuing the main line of this course let us establish the relations of the elliptic QMBP to KZ-equations. The commutativity of  $u_\alpha$  (the coordinates) with  $\tilde{W}$  and  $\mathcal{H}'_0$  is resumed. We follow [22].

**Definition 6.1.** The 0-level *degenerate double affine Hecke*  $\mathcal{H}'_0$  is generated by pairwise commutative  $\{x_b\}$  and  $W^a$  satisfying the relations

$$s_i x_b - x_{s_i(b)} s_i = k_i(b, \alpha_i), \quad (6.48)$$

$$s_0 x_b - x_{s_0(b)} s_0 = -k_\theta(b, \theta), \quad (6.49)$$

where  $s_0 = s_{[-\theta, 1]} \in W^a$ .

**Theorem 6.10.** *The double affine KZ (DAKZ)*

$$\partial_b(\Phi) = \left( \sum_{\tilde{\alpha} \in \Sigma_+^a} k_\alpha (e^{u_{\tilde{\alpha}}} - 1)^{-1} s_{\tilde{\alpha}} + x_b \right) \Phi, \quad (6.50)$$

is  $\widetilde{W}$ -invariant and self-consistent. Here  $\tilde{\alpha} = [\alpha, m]$ ,  $k_\alpha$  is an arbitrary  $W$ -invariant function on  $\Sigma$ , and  $e^{u_{\tilde{\alpha}}} = e^{u_\alpha + m\eta}$  for a fixed  $\eta$ .

Let us consider (6.50) in the representation of  $\mathcal{H}'_0$  induced from the character  $\{s_i \mapsto 1\}$  of  $W^a$ . It is isomorphic to  $\mathbb{C}[x_1, \dots, x_n]$ . We can reduce it furthermore since the symmetric polynomials  $p(x_1, \dots, x_n) (\in \mathbb{C}[x_1, \dots, x_n]^W)$  are central in  $\mathcal{H}'_0$ . We fix  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  and define  $\text{AKZ}_\lambda$  as  $\text{AKZ}$  in the representation  $J_\lambda^o$  dual to  $J_\lambda = \mathbb{C}[x_1, \dots, x_n]/((p(x) - p(\lambda)) \forall p)$ . Here  $p$  are symmetric. By dual we mean  $\text{Hom}(J, \mathbb{C})$  with the action of  $\mathcal{H}'_0$  via the anti-involution preserving the generators  $s_i, x_b$ . It is similar to the considerations of Section 3.4. Because of the definition of  $J_\lambda^o$ , there is a natural  $W^a$ -invariant map  $\text{tr} : J_\lambda^o \rightarrow \mathbb{C}$  dual to the embedding  $\mathbb{C} \rightarrow \mathbb{C}[x_1, \dots, x_n]$ .

**Theorem 6.11.** *The map  $\Phi \mapsto \phi = \text{tr}(\Phi)$  is an isomorphism of the space of all (local) solutions of  $\text{DAKZ}_\lambda$  and the space of solutions of the eigenvalue problem  $L_p \phi = p(\lambda)\phi$  for  $p \in \mathbb{C}[x_1, \dots, x_n]^W$ .*

The proof follows the same lines as in the non-affine case. We note that the double affine KZ can be defined for the central extension  $\mathcal{H}'$  (non-zero level) of  $\mathcal{H}'_0$  as well as the double affine Dunkl operators. The algebra  $\mathcal{H}'$  is much more interesting. However the  $\widetilde{W}$ -invariant  $L$ -operators can be constructed only for  $\mathcal{H}'_0$ , which eventually leads to the relation generalizing the critical level condition in the Kac-Moody theory (see the Introduction). We consider the KZ via the Kac-Moody algebras in the next sections.

7. FACTORIZATION AND  $r$ -MATRICES

We will introduce the  $r$ -matrix KZ in this section closely connected with the  $r$ -matrix Kac–Moody algebras defined in [7]. They are directly connected with the interpretation of  $r$  matrices via the factorization of the Kac–Moody algebras. The basic trigonometric  $r$  will be considered.

**7.1. Basic trigonometric  $r$ -matrix.** Let us start with the following example.

**Example 7.1.** We set  $V = \mathbb{C}_N^{\otimes n} = \overbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}^n$  and

$$r(x) = \frac{1}{2} \coth\left(\frac{x}{2}\right) P + \frac{1}{2} \sum_{1 \leq l < m \leq N} (e_{lm} \otimes e_{ml} - e_{ml} \otimes e_{lm}) + D. \quad (7.1)$$

Here  $\coth(x) = (e^x + e^{-x})/(e^x - e^{-x})$ ,  $P = \sum_{1 \leq l, m \leq N} e_{lm} \otimes e_{ml}$  for the standard generators  $\{e_{ab}\}$  of  $End(\mathbb{C}^N)$  with the entries  $\delta_{li}\delta_{mj}$ ,  $D$  is any diagonal matrix ( $D = \sum c_{lm} e_{ll} \otimes e_{mm}$ ). So the values of  $r$  belong to  $End(\mathbb{C}^N \otimes \mathbb{C}^N)$ . Note that  $P$  is the permutation matrix:  $P(v_1 \otimes v_2) = v_2 \otimes v_1$ .

We put  $r^{ij} = r(u_i - u_j)^{(i,j)}$  where

$$C^{(i,j)} = \sum 1 \otimes \cdots \otimes \overset{i}{a} \otimes \cdots \otimes \overset{j}{b} \otimes \cdots \otimes 1 \text{ if } C = \sum a \otimes b \quad (7.2)$$

for any matrices  $\{a, b\}$ . Note that  $r^{21} = (r(u_2 - u_1))^{(2,1)}$ . The following relations can be verified directly:

$$[r^{ij}, r^{ik} + r^{jk}] - [r^{ik}, r^{kj}] = 0, \quad (7.3)$$

$$[r^{ij}, r^{kl}] = 0, \quad r^{ij} + r^{ji} = D^{(i,j)} + D^{(j,i)}, \quad (7.4)$$

for pairwise distinct indices  $i, j, k, l$  ( $r^{kj}$  is not a misprint!).

When  $D^{(i,j)} + D^{(j,i)} = 0$  we get the relations

$$[r^{ij}, r^{ik} + r^{jk}] + [r^{ik}, r^{jk}] = 0, \quad (7.5)$$

which are nothing else but the assumptions of Proposition 6.3 for the root system  $A_{n-1}$  (the pairs  $\{ij\}$  can be identified with the roots). This  $r$ -matrix is invariant and unitary. So we can introduce the corresponding KZ-equation.

Identifying  $P^{(i,j)}$  with the transpositions  $s_{ij}$  we can represent it in the form 2.23 from Section 2.2 (here we use  $u$  instead of  $v$ ). The formulas for the operators  $x_i$  are straightforward. They involve  $D$ . So we arrive at a representation of the degenerate affine Hecke algebra  $\mathcal{H}'_n$  of type  $GL_n$  in  $V$ . Any weight subspaces of  $V$  with respect to the standard action of  $\mathfrak{gl}_N$  are  $\mathcal{H}'_n$ -submodules. Actually it is not necessary to impose the relation  $D^{(i,j)} + D^{(j,i)} = 0$ . One can define KZ for any  $D$ . We are coming to this.

Let us generalize this example to an arbitrary simple finite dimensional Lie algebra  $\mathfrak{g}$  (rank =  $l$ ) over  $\mathbb{C}$ . Let  $\Sigma = \{\alpha\} \subset \mathbb{R}^l$  be the root system associated with  $\mathfrak{g}$ ,  $(, )$  the  $W$ -invariant inner product on  $\mathbb{R}^l$  normalized by the condition  $(\theta, \theta) = 2$  for the maximal root  $\theta$  with respect to the simple roots  $\alpha_1, \dots, \alpha_l$ .

We choose nonzero

$$e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, \alpha \in \Sigma_+, \quad (7.6)$$

provided the relations

$$[h_\alpha, e_\beta] = (\alpha, \beta)e_\beta, [h_\alpha, f_\beta] = -(\alpha, \beta)f_\beta, \quad (7.7)$$

$$\text{for } h_\alpha = \frac{(\alpha, \alpha)}{2}[e_\alpha, f_\alpha]. \quad (7.8)$$

The elements  $\{e_\alpha\}, \{f_\alpha\}$  are linearly independent and generate the Borel subalgebras  $\mathfrak{b}_\pm$  respectively. The elements  $h_m = h_{\alpha_m}$  ( $m = 1, \dots, l$ ) form a basis in the Cartan subalgebra  $\mathfrak{h}$ ;  $h_\alpha = \sum_{m=1}^l (\alpha, b_m)h_m$  for the fundamental coweights  $b_m$  ( $(b_m, \alpha_n) = \delta_{mn}$ ).

Let us connect the above form on  $\mathbb{R}^l$  with the the *standard invariant form*  $(f, f')$  on  $f \in \mathfrak{g} \ni f'$  (by invariant we mean that  $([f, g], h) = (f, [g, h])$ ). All invariant forms are proportional. The standard normalization is as follows:

$$(h_\alpha, h_\beta) = (\alpha, \beta), \quad (7.9)$$

$$(e_\alpha, f_\beta) = \frac{2}{(\alpha, \alpha)}\delta_{\alpha\beta}, \quad (7.10)$$

$$(e_\alpha, e_\beta) = 0 = (f_\alpha, f_\beta). \quad (7.11)$$

See [42]. In terms of this form, the definition of  $\{h_\alpha\}$  does not depend on the particular choice of  $e_\alpha, f_\alpha$  in the corresponding weight spaces:

$$h_\alpha = \frac{[e_\alpha, f_\alpha]}{(e_\alpha, f_\alpha)}. \quad (7.12)$$

Let us connect the standard form with the Killing form  $(f, f')_K = \text{Tr}(\text{ad} f \text{ad} f')$  for  $f, f' \in \mathfrak{g}$ :

$$(f, f')_K = (2g)(f, f'), \quad g = 1 + (\rho, \theta) \text{ (the dual Coxeter number)}, \quad (7.13)$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha$ .

We set

$$\Omega = \sum_a I_a \otimes I_a^* \in \mathfrak{g} \otimes \mathfrak{g}, \quad (7.14)$$

for a basis  $\{I_a \text{ (} a = 1, \dots, \dim \mathfrak{g})\}$  of  $\mathfrak{g}$  and its dual  $\{I_a^*\}$ :  $(I_a, I_{a'}^*) = \delta_{a, a'}$ . The definition of  $\Omega$  does not depend on the choice of the basis  $\{I_a\}$ . Using the canonical

generators:

$$\Omega = \sum_{m=1}^n h_m \otimes h_m^* + \sum_{\alpha \in R^+} \frac{1}{(e_\alpha, f_\alpha)} (e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha), \quad (7.15)$$

where  $\{h_m^* = h_{b_m}\}$  are dual to  $\{h_m\}$ .

Let  $A : \mathfrak{h} \rightarrow \mathfrak{h}$  be a linear map. The *basic trigonometric  $r$ -matrix* is given by the formula:

$$r(x) = \frac{1}{2} \coth\left(\frac{x}{2}\right) \Omega + \frac{1}{2} \sum_{\alpha \in R^+} \frac{f_\alpha \otimes e_\alpha - e_\alpha \otimes f_\alpha}{(e_\alpha, f_\alpha)} + \frac{1}{2} \sum_{m=1}^l A(h_m) \otimes h_m^*. \quad (7.16)$$

It takes values in  $\mathfrak{g} \otimes \mathfrak{g}$ . We use the same notation  $r^{ij} = r(u_i - u_j)^{(i,j)}$ ,  $(a \otimes b)(i, j) = 1 \otimes \cdots \otimes \overset{m}{a} \otimes \cdots \otimes \overset{j}{b} \otimes \cdots \otimes 1$  for  $U(\mathfrak{g})^{\otimes n}$ ,  $U(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$ . Generalizing Example 7.1 we arrive at

**Proposition 7.1.**

$$[r^{ij}, r^{ik} + r^{jk}] - [r^{ik}, r^{kj}] = 0, [r^{ij}, r^{lm}] = 0, \quad (7.17)$$

$$r^{ij} + r^{ji} = \Theta^{ij}, \quad \Theta = (1/2) \sum_{m=1}^l (A(h_m) \otimes h_m^* + h_m^* \otimes A(h_m)) \quad (7.18)$$

for pairwise distinct indices ( $l$  is not the rank).

**7.2. Factorization and  $r$ -matrices.** We will establish the equivalence of  $r$ -matrices and Lie subalgebras complementary to the standard holomorphic subalgebra in a Kac–Moody algebra.

In this section we fix a simple Lie algebra  $\mathfrak{g}$ , an open neighbourhood  $U_0 \subset \mathbb{C}$  of 0, and the set of pairwise distinct points  $u_1, \dots, u_n \in U_0$  such that  $u_i - u_j \in U_0$  ( $1 \leq i \neq j \leq n$ ).

**Definition 7.1.** Let  $r(x)$  be a holomorphic  $\mathfrak{g} \otimes \mathfrak{g}$ -valued function of  $x \in U_0 \setminus \{0\}$ . We call  $r(x)$  a *quasi-unitary  $r$ -matrix* if:

( $\alpha$ )  $r(x) - \frac{\Omega}{x}$  is holomorphic at 0,

( $\beta$ )  $[r^{12}, r^{13} + r^{23}] = [r^{12}, r^{32}]$ ,

( $\gamma$ )  $r^{12} + r^{21} = \Theta$ ,  $d(\Theta)/dx = 0$ .

It is called *unitary* if  $r^{12} + r^{21} = 0$ .

Here  $r^{ij} = r(x_i - x_j)^{(i,j)}$  for three independent variables  $x_1, x_2, x_3$  ( $x = x_1 - x_2$ ). For instance,  $r^{21} = (r(-x))^{(2,1)}$ . The relations ( $\alpha, \beta, \gamma$ ) must be fulfilled whenever  $x_i - x_j \in U_0 \setminus \{0\}$ . The element  $\Omega$  is from (7.14).

From now on we set  $x_i = x - u_i$ ,

$$\tilde{\mathfrak{g}}^i = \mathfrak{g}((x_i)) = \left\{ \sum_{k \geq p} g_k x_i^k \mid p \in \mathbb{Z}, g_k \in \mathfrak{g} \right\}, \quad (7.19)$$

$$\tilde{\mathfrak{g}}_0^i = \mathfrak{g}[[x_i]] = \left\{ \sum_{k \geq 0} g_k x_i^k \mid g_k \in \mathfrak{g} \right\}, \quad (7.20)$$

$$\tilde{\mathfrak{g}} = \prod_{i=1}^n \tilde{\mathfrak{g}}^i, \quad \tilde{\mathfrak{g}}_0 = \prod_{i=1}^n \tilde{\mathfrak{g}}_0^i. \quad (7.21)$$

All these are Lie algebras:

$$[f, g]_{\tilde{\mathfrak{g}}} = ([f^1, g^1], \dots, [f^n, g^n]) \text{ for } f = (f^1, \dots, f^n), g = (g^1, \dots, g^n) \in \tilde{\mathfrak{g}}, \quad (7.22)$$

where the components  $f^i, g^i$  of  $f, g$  are formal series in  $x_i$ ; their commutators are coefficient-wise. The Lie algebras  $\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0$  are counterparts of the groups of all and integral adèles in arithmetics.

The central extension  $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}c$  ( $c$  is the center element) of  $\tilde{\mathfrak{g}}$  is introduced as follows:

$$[f + \xi c, g + \zeta c]_{\hat{\mathfrak{g}}} = [f, g]_{\tilde{\mathfrak{g}}} + \text{Res} \left( \frac{df}{dx}, g \right) c, \quad (7.23)$$

$$\text{Res} \left( \frac{df}{dx}, g \right) = \sum_{i=1}^n \text{Res}_{x_i} \left( \frac{df^i}{dx_i}, g^i \right) dx_i, \quad (7.24)$$

where  $\text{Res}_{x_i} (\sum_k f_k x_i^k) dx_i = f_{-1}$ .

**Definition 7.2.** The Kac–Moody algebra  $\hat{\mathfrak{g}}$  is called *factorized* if it is endowed with a subspace, a *factorizing subalgebra*,  $\tilde{\mathfrak{g}}_r \subset \tilde{\mathfrak{g}}$  such that

- (a)  $\tilde{\mathfrak{g}}_r \oplus \tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{g}}$ .
- (b)  $\tilde{\mathfrak{g}}_r$  is a Lie subalgebra of  $\tilde{\mathfrak{g}}$ .
- (c)  $\tilde{\mathfrak{g}}_r$  is a Lie subalgebra of  $\hat{\mathfrak{g}}$ .

Given an  $r$ -matrix  $r(x)$  with values in  $\mathfrak{g} \otimes \mathfrak{g}$ , let us construct the corresponding factorizing subalgebra  $\tilde{\mathfrak{g}}_r$ . For  $f = (f^1, \dots, f^n) \in \tilde{\mathfrak{g}}$ , we define a function of  $x$  in a certain neighborhood of zero  $\tilde{U}_0 \subset U_0$ :

$$\bar{f}(x) = \text{Res}(r(x-y), 1 \otimes f) dy = \sum_{i=1}^n \text{Res}_{y_i} (r(x_i - y_i), 1 \otimes f^i(y_i)) dy_i, \quad (7.25)$$

where  $(a \otimes b, 1 \otimes c) = (b, c)a$  ( $a, b, c, \in \mathfrak{g}$ ). Let  $f_r^i = f_r^i(x_i)$  be the expansion of  $\bar{f}(x)$  at  $u_i$  ( $x_i = x - u_i$ ). We set

$$\bar{\mathfrak{g}} = \{\bar{f}\}, \quad \tilde{\mathfrak{g}}_r = \{f_r = (f_r^1(x_1), \dots, f_r^n(x_n)) \mid f \in \tilde{\mathfrak{g}}\}, \quad (7.26)$$

**Theorem 7.2.** *If  $r(x)$  is a quasi-unitary  $r$ -matrix, then  $\bar{\mathfrak{g}}$  is a Lie algebra and  $\tilde{\mathfrak{g}}_r$  is a factorizing subalgebra of  $\tilde{\mathfrak{g}}$ . Vice versa, every factorizing subalgebra  $\tilde{\mathfrak{g}}_r$  which is invariant with respect to the differentiation  $d/dx$  is associated to a quasi-unitary  $r$ -matrix defined on some neighborhood  $\tilde{U}_0$  of 0. The corresponding  $r$  is unique:*

$$r(x - u_i) = \sum_a \overline{\left(\frac{I_a}{x_i}\right)}(x) \otimes I_a^*, \quad (7.27)$$

where  $i$  can be arbitrary,  $\{I_a\}$  is a basis of  $\mathfrak{g}$ , and  $f = f_r + f_0$  is the factorization with respect to the decomposition  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_r \oplus \tilde{\mathfrak{g}}_0$ .

The proof is based on the relation

$$[f_r, g_r] + [f, g]_r = [f_r, g]_r + [f, g_r]_r, \quad f, g \in \tilde{\mathfrak{g}}, \quad (7.28)$$

following from (b) by considering the principal parts of all four terms. The coincidence of the principal parts is sufficient because of (a). We note that (7.28) results in the (equivalent) equality  $[f_0, g_0] + [f, g]_0 = [f_0, g]_0 + [f, g_0]_0$ . Expressing  $f_r, g_r$  in terms of  $r$  one comes to ( $\beta$ ). Here it is not necessary to assume that  $r$  depends on the difference (see [7, 8]). The interpretation of  $r$  as a projection was formalized by Manin (see [28]) in the definition of the so-called Manin triple, and by Semenov-Tjan-Shanskii [64]. They considered abstract (“constant”)  $r$ . In [7] it was done for  $r$  depending on the parameter, which resulted in a fruitful theory with many examples. This paper was stimulated by [4], where the unitary case was considered in detail. Note that the definition of the classical  $r$ -matrix appeared for the first time in paper [50].

Arbitrary  $\tilde{\mathfrak{g}}_r$  satisfying condition (a) of Definition 7.1 can be obtained by the construction from (7.25) and (7.26) for a certain series  $r(x, y)$  defined on the square of a formal neighborhood of the set  $\{u_i\}$ . If  $r(x, y)$  depends on the difference, i.e.  $r(x, y) = r(x - y)$ , then condition ( $\alpha$ ) is fulfilled. For such  $r$ , the relations ( $\beta$ ) and (b) are equivalent (see *ibid.*). Moreover, imposing ( $\alpha, \beta$ ) (or (a,b)), the remaining relations ( $\gamma$ ) and (c) are also equivalent. See [8, 13]. Let us consider some examples.

**Example 7.2.** The Yang  $r$ -matrix is given by the formula  $r(x) = \frac{\Omega}{x}$  for  $\Omega$  defined in (7.14). It is unitary:  $r^{12} + r^{21} = 0$ . The corresponding  $\tilde{\mathfrak{g}}_r$  consists of the sets  $(f_r^1(x_1), \dots, f_r^n(x_n)) \in \tilde{\mathfrak{g}}$  of expansions of  $\bar{f}$  at  $\{u_i\}$  from the Lie algebra  $\bar{\mathfrak{g}}$  of rational  $\mathfrak{g}$ -valued function on  $\mathbb{P}^1$  with poles at  $\{u_1 \dots u_n\}$  and the normalization condition  $f(\infty) = 0$ . We have

$$\overline{\left(\frac{g}{x_i}\right)} = \frac{g}{x - u_i} \quad (7.29)$$

for any  $i, g \in \mathfrak{g}$ , where  $\overline{(\cdot)}$  is defined in (7.25).

**Example 7.3.** Given a linear map  $A : \mathfrak{h} \rightarrow \mathfrak{h}$ , let us describe the factorizing subalgebra associated to  $r$  from (7.16):

$$r(x) = \frac{1}{2} \coth\left(\frac{x}{2}\right) \Omega + \frac{1}{2} \sum_{\alpha \in R^+} \frac{f_\alpha \otimes e_\alpha - e_\alpha \otimes f_\alpha}{(e_\alpha, f_\alpha)} + \sum_{n=1}^n A(h_i) \otimes h_i^*. \quad (7.30)$$

First of all,

$$\overline{\left(\frac{f_\alpha}{x_i}\right)} = \frac{e^{x_i/2}}{e^{x_i/2} - e^{-x_i/2}} f_\alpha, \quad \overline{\left(\frac{e_\alpha}{x_i}\right)} = \frac{e^{-x_i/2}}{e^{x_i/2} - e^{-x_i/2}} e_\alpha, \quad (7.31)$$

$$\overline{\left(\frac{h}{x_i}\right)} = \frac{1}{2} \coth\left(\frac{x_i}{2}\right) h + A(h) \quad (h \in \mathfrak{h}). \quad (7.32)$$

The Lie algebra  $\tilde{\mathfrak{g}}_r$  consists of  $(f_r^1(x_1), \dots, f_r^n(x_n)) \in \tilde{\mathfrak{g}}$ , where  $f_r^i(x_i) \in \tilde{\mathfrak{g}}^i$  are the expansions of  $\mathfrak{g}$ -valued functions  $\bar{f}(x)$  such that

- (1) they are rational in terms of  $v = e^x$  on  $v \in \mathbb{P}^1$ ,
- (2) have poles at  $v_1 = e^{u_1}, \dots, v_n = e^{u_n}$  only,
- (3)  $\bar{f}(v=0) \in \mathfrak{b}_+$ ,  $\bar{f}(v=\infty) \in \mathfrak{b}_-$ , and
- (4)  $(\bar{f}(0) + \bar{f}(\infty))|_{\mathfrak{h}} = A((\bar{f}(0) - \bar{f}(\infty))|_{\mathfrak{h}})$ ,

where  $\mathfrak{b}_+ = \langle e_\alpha, \alpha \in \Sigma_- \rangle$  and  $\mathfrak{b}_- = \langle f_\alpha, \alpha \in \Sigma_+ \rangle$ ,  $\mathfrak{h}$  is the Cartan subalgebra.

The simplest way to check that  $r$  is quasi-unitary is of course based on this interpretation. Indeed, the functions satisfying conditions (1–4) obviously form a Lie algebra. So we arrive at (7.28), which readily results in  $(\beta)$ .

Let us introduce the  $r$ -matrix KZ. Given a quasi-unitary  $r$ , let

$$\varrho = \sum_a \rho_a I_a \in U(\mathfrak{g}) \text{ for } \{\Omega/x - r(x)\}(x=0) = \sum_a \rho_a \otimes I_a, \quad (7.33)$$

where  $U(\mathfrak{g})$  is the enveloping algebra of  $\mathfrak{g}$ .

**Theorem 7.3.** *Setting*

$$R_i = \varrho^i - \sum_{j(\neq i)} r^{ji}, \quad 1 \leq j \leq n, \text{ for } 1 \leq i \leq n, \quad (7.34)$$

the following system of the differential equations for a  $U(\mathfrak{g})$ -valued function  $\Phi(u)$  is self-consistent:

$$\kappa \partial(\Phi) / \partial u_i = R_i \Phi, \quad 1 \leq i \leq n. \quad (7.35)$$

for all  $\kappa = k^{-1}$ .

The theorem can be checked by a straightforward calculation using the relation

$$[r^{12}, \varrho^1] + [\varrho^2, r^{21}] = [r^{12}, r^{21}]. \quad (7.36)$$

Recall that  $\varrho^i = \varrho^{(i)}$  is  $\varrho$  considered in  $i$ -th component of  $U(\mathfrak{g})^{\otimes n}$ . The calculation is more transparent for unitary  $r$ . The last relation becomes  $[r^{12}, \varrho^1 + \varrho^2] = 0$  in this case. Considering the basic trigonometric  $r$  from (7.16),

$$\varrho = h_\rho - \frac{1}{2} \sum_{m=1}^l A(h_m) h_m^*, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha. \quad (7.37)$$

So  $\varrho = (1/2) \sum_{\alpha > 0} h_\alpha \in \mathfrak{h}$  when  $A = 0$ .

**7.3. Comments.** If  $\tilde{\mathfrak{g}}_r$  satisfies conditions (a) and (b), then (c) is equivalent to the relation  $\text{Res}\left(\frac{df}{dx}, g\right) = 0$  for all  $f, g \in \tilde{\mathfrak{g}}_r$ . The suffix  $r$  stays for “ $r$ -matrix” or “rational”. The latter does not mean that this subalgebra (a counterpart of the group of principal adèles in arithmetics) must be associated with an algebraic curve. However in all known examples it is so. One can always find a Lie algebras of rational  $\mathfrak{g}$ -valued functions on  $\mathbb{P}^1$  or an elliptic curve (containing  $U_0$  as an open subset) such that  $\tilde{\mathfrak{g}}_r$  is formed by expansions of these functions at  $u_1, \dots, u_n$ . We conjecture that it is true for all  $r$ . More generally, let us suppose that the spaces

$$\tilde{\mathfrak{g}}/(\tilde{\mathfrak{g}}_r + \tilde{\mathfrak{g}}_0), \quad \tilde{\mathfrak{g}}_r \cap \tilde{\mathfrak{g}}_0 \quad (7.38)$$

are finite dimensional. Such  $\tilde{\mathfrak{g}}_r$  are  $\mathbb{C}$ -variants of discrete subgroups of adèle groups with quotients of finite volume in arithmetics.

**Conjecture 7.4.** *There exists a complete algebraic curve  $C$  over  $\mathbb{C}$  containing  $U_0$  such that for the Lie algebra  $\tilde{\mathfrak{g}}_C$  of the expansions at  $u_1, \dots, u_n$  of all  $\mathfrak{g}$ -valued rational functions on  $C$  with the poles apart from this set both spaces*

$$(\tilde{\mathfrak{g}}_r + \tilde{\mathfrak{g}}_C)/\tilde{\mathfrak{g}}_r, \quad (\tilde{\mathfrak{g}}_r + \tilde{\mathfrak{g}}_C)/\tilde{\mathfrak{g}}_C \quad (7.39)$$

*are finite dimensional.*

Its counterpart in arithmetics was proved by Margulis. It holds when either the number of points  $u$  or the rank of the arithmetic group is bigger than 1. It seems that we do not need this restriction in the Kac–Moody setup. As to the special ( $r$ -matrix) conjecture, see paper [4] (the unitary case) and [7, 8]. Almost nothing is known about the general conjecture (any curves).

We note that unitary  $r$ -matrices ( $r^{12} + r^{21} = 0$ ) have the following interpretation in terms of  $\tilde{\mathfrak{g}}_r$ :

$$\{f \in \tilde{\mathfrak{g}} \mid \text{Res}(f, \tilde{\mathfrak{g}}_r) dx = 0\} = \tilde{\mathfrak{g}}_r.$$

It looks similar to condition (c) but of course does not coincide with it. In (c),  $f$  must be replaced by  $df/dx$ .

An interesting problem concerns the independence of condition (c) in the definition of the factorizing subalgebra.

**Conjecture 7.5.** *The subspace  $\tilde{\mathfrak{g}}_r$  satisfying (a,b) is a Lie subalgebra of  $\hat{\mathfrak{g}}$  for a proper choice of the central extension.*

All central extensions are described by 2-cocycles on  $\tilde{\mathfrak{g}}$  (see [42]). I expect that the conjecture is true even in the setup of Conjecture 7.4. Let me give an outline of the proof in the  $r$ -matrix case. I need to reproduce the part of the paper [8] devoted to a generalization of the  $\tau$ -function introduced by Date, Jimbo, Kashiwara, and Miwa in paper [25], and a construction due to Kac, Peterson from [43].

The definition of the  $r$ -matrix  $\tau$ -function uses that the group  $\tilde{\mathcal{G}}_0 = \exp(\tilde{\mathfrak{g}}_0)$  has a natural structure of an infinite dimensional algebraic variety. Given  $f \in \tilde{\mathfrak{g}}$ , let us define the corresponding vector field  $D_f$  on this variety. It means a formula for  $\Phi^{-1}D_f(\Phi) \in \tilde{\mathfrak{g}}_0$  for the generic element  $\Phi \in \tilde{\mathcal{G}}_0$ . The coefficients of  $\Phi$  in the expansions with respect to  $x_1, \dots, x_n$  are the coordinates of  $\tilde{\mathcal{G}}_0$ . So equating both sides of the following relation we get the complete list of  $D_f$ -derivatives of these coefficients and a differentiation of  $\text{Funct}(\tilde{\mathcal{G}}_0)$ :

$$\Phi^{-1}D_f(\Phi) = (\Phi f \Phi^{-1})_0, \text{ where } \tilde{\mathfrak{g}} \ni f = f_0 + f_r, f_0 \in \tilde{\mathfrak{g}}_0, f_r \in \tilde{\mathfrak{g}}_r, \quad (7.40)$$

is the factorization with respect to  $\tilde{\mathfrak{g}}_r$ , by  $\Phi f \Phi^{-1}$  we mean the adjoint action. Note that  $\Phi^{-1}D_g(\Phi) = g$  for  $g \in \tilde{\mathfrak{g}}_0$ , so such  $D_g$  are left-invariant fields on the group  $\tilde{\mathfrak{g}}_0$ .

A simple straightforward calculation based on the very definition of the factorization (see (7.28)) gives that

$$D_{[f, f']} = [D_f, D_{f'}] \text{ on } \text{Funct}(\tilde{\mathcal{G}}_0). \quad (7.41)$$

Then we introduce the  $\tau$ -function as the infinite wedge product  $\tau = \wedge_g D_g$  of all vector fields  $D_g$  where  $g$  runs over the following natural basis of  $\tilde{\mathfrak{g}}_0$ :

$$\{I_A^K = ((I_{a_1} x_1^{k_1}), \dots, (I_{a_n} x_n^{k_n}))\}$$

Here  $\{I_a\}$  is a fixed basis of  $\mathfrak{g}$ ,  $A = (a_1, \dots, a_n)$ ,  $K = (k_1, \dots, k_n)$ ,  $k_i \geq 0$ . So  $\tau$  is a section of the  $\wedge^{\text{top}} \mathcal{T}$  for the tangent bundle  $\mathcal{T}$  of  $\tilde{\mathcal{G}}_0$ .

**Theorem 7.6.** *The commutators  $\hat{D}_f(\tau) = [D_f, \tau]$  are well-defined for  $f \in \tilde{\mathfrak{g}}$ ,*

$$\hat{D}_f(\tau) = (\text{Res}(\Phi^{-1}d\Phi, f)_K - \text{Res}(\Phi f \Phi^{-1}, \varrho)_K dx) \tau, \quad (7.42)$$

where  $\varrho \in \prod_{i=1}^n \mathfrak{g}$  for quasi-unitary  $r$  (see [8],  $\varrho = 0$  for quite a few  $r$ ), and

$$[\hat{D}_f, \hat{D}_{f'}] = \hat{D}_{[f, f']} + (2g) \text{Res} \left( \frac{df}{dx}, f' \right) \text{ on } \text{Funct}(\tilde{\mathcal{G}}_0)\tau \quad (7.43)$$

for the dual Coxeter number  $g$ .

Thus the central extension of  $\tilde{\mathfrak{g}}$  emerges naturally for the adjoint action of  $\{D\}$  in  $\wedge^{\text{top}} \mathcal{T}$ . Here  $2g$  appears because it is the ratio of the Killing form and the standard invariant form. Formula (7.42) is an analogue of that from [25]. I also establish in [8] that  $\tau$  coincides with the coinvariant (the next section) defined for the basic representation of  $\hat{\mathfrak{g}}$  with the central charge  $2g$ . The basic representations are irreducible quotients of the Weil modules for zero starting representations of  $\mathfrak{g}$ . A generalization of this relation and formula (7.43) to the coinvariants of arbitrary Verma and Weil modules does not seem difficult.

Returning to (c), this condition is not necessary in the theorem. Moreover, an arbitrary central extensions of  $\tilde{\mathfrak{g}}$  can be obtained for a proper choice of the basis in  $\tilde{\mathfrak{g}}_0$ . We have arrived at the standard cocycle because our basis  $\{I_A^K\}$  was invariant with respect to the differentiation  $d/dx$ . So, given  $\tilde{\mathfrak{g}}_r$ , it is sufficient to find a basis which is “compatible” with the adjoint action of this Lie subalgebra. Then the commutators of the differentiations  $D_f$  for the “rational”  $f$  will contain no central additions. I believe it is possible. As to the theory of KZ, it would give the definition of the  $r$ -matrix KZ for the most general class of  $r$ -matrices, which are not supposed to depend on the differences.

## 8. COINVARIANT AND INTEGRAL FORMULAS

We introduce the  $r$ -matrix coinvariant, establish that it satisfies the corresponding KZ when differentiated with respect to the Sugawara elements  $L_{-1}$ , and prove the integral formulas for basic KZ. Last we discuss the KZB. We keep the notation from the previous section.

**8.1. Coinvariant.** Let  $V = V_1 \otimes \cdots \otimes V_n$  for  $\mathfrak{g}$ -modules  $V_1, \dots, V_n$ . It has a natural structure of a  $\mathfrak{g}$ -module and a  $\tilde{\mathfrak{g}}_0$ -module:

$$\begin{aligned} g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) &= g^1(0)v_1 \otimes v_2 \otimes \cdots \otimes v_n + \\ &v_1 \otimes g^2(0)v_2 \otimes \cdots \otimes v_n + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes g^n(0)v_n, \end{aligned} \quad (8.1)$$

where  $g = (g^1, \dots, g^n)$ ,  $g^i = \sum_{m \geq 0} g_m^i x_i^m$ ,  $g^i(0) = g_0^i$ .

Setting  $\tilde{\mathfrak{g}}_0 \oplus \mathbb{C}c = \hat{\mathfrak{g}}_0$ , we define the induced module, the *Weil module*,

$$M_V^\sigma = \text{Ind}_{\hat{\mathfrak{g}}_0}^{\tilde{\mathfrak{g}}} V, \quad (8.2)$$

where the central element  $c$  acts as  $\sigma \in \mathbb{C}$ , i.e.  $c \cdot v = \sigma v \forall v$ . Because of the decomposition  $\tilde{\mathfrak{g}}_r \oplus \tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{g}}$ , given any  $m \in M_V^\sigma$ , there exists a unique element  $\pi(m) \in V$  such that  $m - \pi(m) \in \tilde{\mathfrak{g}}_r M_V^\sigma$ . The linear map  $\pi : M_V^\sigma \rightarrow V$  is called the *coinvariant*. Its defining property is

$$\pi(g_r m) = \pi(m) \quad (\forall g \in \tilde{\mathfrak{g}}_r, m \in M_V^\sigma). \quad (8.3)$$

Let  $\{I_a\}$  be a basis of  $\mathfrak{g}$  and  $\{I_a^*\}$  the dual basis. We put  $I_{a,k}^i = (I_a)^i x_i^k \in \tilde{\mathfrak{g}}^i$  and  $I_{a,k}^{*i} = (I_a^*)^i x_i^k$ . The *Sugawara element* of degree  $-1$  at  $u_i$  is given by the series

$$L_{-1}^i = \sum_{m \geq 0} \sum_a I_{a,-1-k}^i I_{a,m}^{*i}, \quad (8.4)$$

which belongs to a completion of  $U(\hat{\mathfrak{g}})$  (to a completion of  $U(\hat{\mathfrak{g}}^i)$  to be exact). The definition does not depend on the choice of the basis  $\{I_a\}$  and the action of these elements in  $M_V^\sigma$  is well-defined. They commute with each other,  $[L_{-1}^i, L_{-1}^j] = 0$ , because different components of  $U(\hat{\mathfrak{g}})$  are pairwise commutative.

We want to determine the dependence of the coinvariant on the positions of the points  $u_1, \dots, u_n$ . So we need to enlarge the algebras and modules under consideration assuming that the elements are functions on  $u$ . Let  $\mathcal{U}$  be the algebra of  $\mathbb{C}$ -functions of  $u_1, \dots, u_n \in U_0$ ,  $\hat{\mathfrak{g}}(U_0) = \mathcal{U} \otimes \hat{\mathfrak{g}}$ ,  $M_V^\sigma(U_0) = \mathcal{U} \otimes M_V^\sigma$ . Thus the values of  $\pi$  on  $M_V^\sigma(U_0)$  belong to  $\mathcal{U} \otimes V$ . The functions may have singularities along the diagonals  $\{u_i = u_j\}$ . We extend the derivatives  $\partial/\partial u_i$  from  $\mathcal{U}$  to  $g(u) \in \hat{\mathfrak{g}}(U_0)$  and  $m(u) \in M_V^\sigma(U_0)$ , setting

$$\frac{\partial}{\partial u_i} I_{a,k}^j = 0, \quad \frac{\partial}{\partial u_i} (I_{a,k}^j v) = 0, \quad (8.5)$$

for all "constant"  $v \in V$ .

We note that the latter definition leads to nontrivial formulas and very quickly if one combines it with the projection  $f \mapsto f_r$ . For instance, let  $m(u) = (g^i/x_i)_r v$  for  $v \in V$ ,  $g \in \mathfrak{g}$ ,  $g^i = g^{(i)} = 1 \otimes \cdots \otimes \overset{i}{g} \otimes \cdots \otimes 1$ . Then

$$\frac{\partial m(u)}{\partial u_k} = \frac{\partial}{\partial u_k} \left\{ \prod_{j=1}^n (r(x_j + u_j - u_i), g) \right\} v = \quad (8.6)$$

$$= \left\{ \prod_{j=1}^n \left( \delta_{jk} \frac{\partial}{\partial x_k} - \delta_{ik} \frac{\partial}{\partial x_j} \right) (r(x_j + u_j - u_i), g) \right\} v, \quad (8.7)$$

where  $(a \otimes b, c) = (b, c)a$  for the standard form on  $\mathfrak{g}$ . This examples demonstrates that the dependence of the coinvariant of  $u$  may be very complicated, since the calculation of  $\pi$  is based on the factorization. It also shows that one can express the  $u$ -derivatives in terms of  $x$ -derivatives, which is important in the next theorem.

We will use the notation from (7.35,7.33):

$$\varrho = \sum_a \rho_a I_a \in U(\mathfrak{g}) \text{ for} \\ (\Omega/x - r(x))(x=0) = \sum_a \rho_a \otimes I_a, R_i = \varrho^i - \sum_{j(\neq i)} r^{ji}. \quad (8.8)$$

Here  $\varrho^i = \varrho^{(i)}$  and the values of  $r^{ij} = r(u_i - u_j)^{(i,j)}$  are considered as endomorphisms of  $V$ . Note that  $-r^{ji} = r^{ij}$  for unitary  $r$ .

**Theorem 8.1.** *Let  $r(x)$  be a quasi-unitary  $r$ -matrix,  $\tilde{\mathfrak{g}}_r$  the factorizing subalgebra of  $\hat{\mathfrak{g}}$  corresponding to  $r(x)$ . For  $V$ ,  $M_V^\sigma$ ,  $\pi$  and  $L_{-1}^i$  defined as before,*

$$\pi \left( \kappa \frac{\partial}{\partial u_i} m(u) + L_{-1}^i m(u) \right) = \left( \kappa \frac{\partial}{\partial u_i} + \sum_{j \neq i} r^{i,j} (u_i - u_j) + \rho^i \right) \pi(m(u)), \quad (8.9)$$

where  $\kappa = \sigma + g$ ,  $g$  is the dual Coxeter number of  $\mathfrak{g}$ ,  $m(u) \in M_V^\sigma(U_0)$ .

*Proof.* We will consider here the case when  $\kappa = 0$ . Only this degeneration will be applied later to the integral formulas. See [13] for the general case. One has

$$L_{-1}^i v = \sum_a (I_a^i/x_i) I_a^{*i} v = \sum_a ((I_a^i/x_i)_r + \rho_a^i) I_a^{*i} v \quad (8.10)$$

for  $v \in V$ . The  $\tilde{\mathfrak{g}}_r$ -invariance of  $\pi$  (see (8.3)) gives that

$$\pi(L_{-1}^i v) = \sum_a \rho_a^i I_a^{*i} v - \sum_{a, j \neq i} (I_a^i/x_i)_r^j I_a^{*i} v, \quad (8.11)$$

where  $f_r^j$  here and further is the expansion of  $f_r$  ( $\in \tilde{\mathfrak{g}}_r$ ) at  $u_j$  with respect to  $x_j = x - u_j$ . However  $\sum_a (I_a^i/x_i)_r^j I_a^{*i} v = r^{ji} v = r(u_j - u_i)^{(i,j)}$  thanks to relation (7.27).

Any element of  $M_V^\sigma$  can be represented as  $v + \sum f_r m$  for proper  $f_r$  and  $m \in M_V^\sigma$ . Since the level is critical ( $\kappa = 0$ ) the Sugawara elements commute with any  $f = (f^1, \dots, f^n) \in \tilde{\mathfrak{g}}$ . Indeed, the general relation is

$$[L_{-1}^i, f] = -\kappa \partial f^i / \partial x_i. \quad (8.12)$$

See [42]. So  $\pi(L_{-1}^i(f_r m)) = \pi(f_r(L_{-1}^i m)) = 0$ , and (8.9) results from the above calculation with  $v$ .  $\square$

As a byproduct, we can establish the self-consistency of the  $r$ -matrix KZ from (7.35):

$$\kappa \partial(\Phi) / \partial u_i = R_i \Phi, \quad 1 \leq i \leq n. \quad (8.13)$$

Actually the theorem gives much more. We have a generic formula for solutions of this equation. Namely,  $\Phi = \exp\left(-\sum_{i=1}^n ((x_i/\kappa)L_{-1}^i)\right)m$  satisfies this equation for an arbitrary element of  $M_V^\sigma$  (constant, not from  $M_V^\sigma(U_0)$ ).

This is analogous to the claims that  $\tau$ -functions are universal solutions of soliton equations and cannot be used for constructing explicit solutions without special analytic or algebraic methods. In the soliton theory, the main constructive applications of  $\tau$ -functions are the Backlund–Darboux transformations and formulas in terms of  $\theta$ -functions and their degenerations. The integral formulas for KZ, which will be discussed next, have many common points with the BD-transforms as well as with  $\theta$ -solutions. It is not very surprising since the definition of the  $\tau$ -function ( $= \phi$ ) is literally the same. The difference is with the flows. In the soliton theory, we consider mainly the vector fields  $D_f$ ,  $f \in \tilde{\mathfrak{g}}$ , from (7.40). As to KZ, the flows correspond to the Sugawara elements.

**8.2. Integral formulas.** We keep the same notation. However now let us assume that every  $V_i$  ( $1 \leq i \leq n$ ) is a highest weight module relative to  $\mathfrak{b}_+$ . So  $V_i$  is generated by the vacuum vector  $\nu_i$  associated to a weight  $\lambda_i \in \mathbb{C}^n$  (the highest weight):  $h_\alpha(\nu_i) = (\alpha, \lambda_i)\nu_i$ . The above consideration will be applied to  $V' = V \otimes V_{n+1} \otimes \dots \otimes V_{n+m}$ , where  $V = V_1 \otimes \dots \otimes V_n$  for zero  $\hat{\mathfrak{g}}$ -modules  $V_i = \mathbb{C} = \mathbb{C}\nu_0$  ( $n+1 \leq i \leq n+m$ ). Respectively, we introduce the induced  $\hat{\mathfrak{g}}$  module  $M_{V'}^\sigma$ , where the central element  $c$  acts as the scalar  $\sigma$ , and the coinvariant  $\pi : M_{V'}^\sigma \rightarrow V'$ . Since the vacuum vectors  $\nu_0$  in  $V_{n+i}$  ( $i > 0$ ) are fixed, we can identify  $V$  and  $V'$ . Thus the values of the coinvariant will be actually in the same space  $V$ . Note that the Weil modules defined for zero  $\mathfrak{g}$ -modules are very nontrivial, and there are no obvious connections between  $M_V^\sigma$  and  $M_{V'}^\sigma$ . The points  $\{u_1, \dots, u_n\}$  will be called *old*,  $\{u_{n+1}, \dots, u_{n+m}\}$  *new*. The same names will be used for the corresponding indices.

We fix the sequence of numbers  $(n+1)', \dots, (n+m)' \in \{1, \dots, l(= \text{rank } \mathfrak{g})\}$ , which may coincide, and define

$$w = \pi(\hat{m}), \quad \hat{m} = \nu_1 \otimes \dots \otimes \nu_n \otimes \left(\frac{f_{(n+1)'}}{x_{n+1}}\right)\nu_0 \otimes \dots \otimes \left(\frac{f_{(n+m)'}}{x_{n+m}}\right)\nu_0, \quad (8.14)$$

$\hat{m} \in M_{V'}^\sigma$ ,  $w \in V = V'$ . The definition is applicable to any  $r$ -matrices. However  $w$  is simple enough to be used for the integral formulas when  $r$  are rather special.

From now on we will consider the basic trigonometric  $r$ -matrix from (7.16):

$$r(x) = \frac{1}{2} \coth\left(\frac{x}{2}\right) \Omega + \frac{1}{2} \sum_{\alpha \in R^+} \frac{f_\alpha \otimes e_\alpha - e_\alpha \otimes f_\alpha}{(e_\alpha, f_\alpha)} + \frac{1}{2} \sum_{m=1}^l A(h_i) \otimes h_i^*. \quad (8.15)$$

depending on an arbitrary endomorphism  $A : \mathfrak{h} \rightarrow \mathfrak{h}$ . Recall that

$$\varrho = h_\rho - \frac{1}{2} \sum_{k=1}^l A(h_k) h_k^*, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha. \quad (8.16)$$

We identify the elements from  $\mathfrak{h}$  with the corresponding weights from  $\mathbb{C}^l$ :

$$h_b = \sum_{k=1}^l (b, b_k) h_k, \quad h_k = h_{\alpha_k}, \quad (h_b, \lambda) = (b, \lambda) \text{ for } b, \lambda \in \mathbb{C}^l \quad (8.17)$$

and the invariant forms on  $\mathfrak{h}$  and the complexification of  $\mathbb{R}^l$ .

Let

$$\Lambda_i = \lambda_i \text{ for } 1 \leq i \leq n, \quad \Lambda_i = -\alpha_{i'} \text{ for } i > n, \quad \Lambda = \sum_{i=1}^{n+m} \Lambda_i. \quad (8.18)$$

The  $\Lambda_i$  for new  $i > n$  have nothing to do with the corresponding weights, which are zero. Such uniform notation is convenient in the following definitions:

$$\omega_i = \sum_{1 \leq k \leq n+m, k \neq i} (\Lambda_i, \Lambda_k) \frac{1}{2} \coth\left(\frac{u_i - u_k}{2}\right) + \left(\varrho - \frac{A^*(\Lambda)}{2}, \Lambda_i\right), \quad (8.19)$$

where  $(Aa, b) = (a, A^*b)$  on  $a, b \in \mathfrak{h}$ . These functions are logarithmic derivatives. Namely,

$$\begin{aligned} \omega_i &= \partial\omega / \partial u_i \text{ for } \omega = & (8.20) \\ &= \prod_{1 \leq i < j \leq n+m} (e^{(u_i - u_j)/2} - e^{(u_j - u_i)/2})^{(\Lambda_i, \Lambda_j)/\kappa} \prod_{1 \leq i \leq n+m} \exp\left(\left(\varrho - \frac{A^*(\Lambda)}{2}, \Lambda_i\right) \frac{u_i}{\kappa}\right). \end{aligned}$$

We will also use

$$\omega_J = \sum_{j \in J} \omega_j \quad (8.21)$$

for subsets of new points(indices)  $J \subset \{n+1, \dots, n+m\}$ .

**Definition 8.1.** 1) An ordered sequence  $c = (j_1, \dots, j_s; i)$  for pairwise distinct new indices  $j_1, \dots, j_s \in \{n+1, \dots, n+m\}$  and one old index  $1 \leq i \leq n$  is called a *chain*.

2) An ordered sequence  $d = (c_1, \dots, c_r)$  of chains is called a *diagram* if every new  $n+1 \leq j \leq n+m$  belongs to one and only one chain (no restrictions for old indices).

3) Given a linear ordering on new indices  $\succ$ , a diagram

$$d = \left( c_1 = (j_1^{(1)}, j_2^{(1)}, \dots), c_2 = (j_1^{(2)}, j_2^{(2)}, \dots), \dots, c_r = (j_1^{(r)}, j_2^{(r)}, \dots) \right)$$

is called *well-ordered* if  $j_1^{(1)} \succ j_1^{(2)} \succ \dots \succ j_1^{(r)}$ .

4) For a diagram  $d$  the component of an old  $i$ , denoted by  $comp_i(d)$ , is a union of all new indices connected with  $i$  by a chain from  $d$ .

**Definition 8.2.** 1) Given a chain  $c = (j_1, \dots, j_s; i)$ , we set

$$F_c = \frac{[\cdot \{f_{j'_1}, f_{j'_2}\}, f_{j'_3}, \dots, f_{j'_{s-1}}, f_{j'_s}]^i}{(e^{u_{j_1} - u_{j_2}} - 1)(e^{u_{j_2} - u_{j_3}} - 1) \dots (e^{u_{j_s} - u_i} - 1)}. \quad (8.22)$$

2) Given a diagram  $d = (d_1, \dots, d_r)$ , we set

$$F_d = F_r \dots F_2 F_1 \in U(\mathfrak{g})^{\otimes n}, \text{ where } F_j = F_{c_j}, F_{\{\emptyset; i\}} = 1. \quad (8.23)$$

**Theorem 8.2.** We fix an ordering  $\succ$ . Then  $w = \pi(\hat{m}) = \sum_d F_d(vac)$ , where  $vac = \nu_1 \otimes \dots \otimes \nu_n \otimes \nu_0 \otimes \dots \otimes \nu_0$  and the summation is over all well-ordered diagrams. In particular,  $w$  does not depend on the central charge  $\sigma$ . Vice versa,  $\sum_d F_d(vac)$  does not depend on the ordering. The following sums do not depend on the particular choice of the ordering either:

$$w_i[J] = \sum_d F_d(vac), \quad w_i(J) = \sum_{d'} F_{d'}(vac), \quad (8.24)$$

where the summation is over all well-ordered diagrams such that  $comp_i(d) = J$  and  $J \subset comp_i(d')$  (the second sum) for any set of new points  $J$ .

**Theorem 8.3.** For any old  $i$  ( $1 \leq i \leq n$ ) and  $R_i = \varrho^i - \sum_{j(\neq i)} r^{j_i}$ ,

$$R_i w = \omega_i w + \sum_{j=n+1}^{n+m} \omega_j w_i(j) = \omega_i w + \sum_J \omega_J w_i[J], \quad (8.25)$$

where  $J$  runs over all new subsets ( $\subset \{n+1, \dots, n+m\}$ ).

**Theorem 8.4.** For any old  $i$ ,

$$(R_i - \kappa \partial / \partial u_i) W = \kappa \sum_{j=n+1}^{n+m} \partial W_i(j) / \partial u_j = \kappa \sum_{\text{new } J} \partial W_i[J] / \partial u_j, \quad (8.26)$$

where  $W = w\omega$ ,  $W_i(j) = w_i(j)\omega$ ,  $W_i[J] = w_i[J]\omega$  for  $\omega$  from (8.20).

We will prove the first two theorems in the next section. Let us deduce the third one from them now. One has:

$$\kappa \partial w / \partial u_i = -\kappa \sum_{j=n+1}^{n+m} \partial w_i(j) / \partial u_j, \quad (8.27)$$

since  $\partial F_c / \partial u_i = \partial F_{j_1} / \partial u_{j_1} + \dots + \partial F_{j_s} / \partial u_{j_s}$  for any chain  $c = (j_1, \dots, j_s; i)$ . Indeed, the formula for  $F_c$  depends only on the differences  $u_{j_p} - u_{j_q}$  for  $1 \leq p, q \leq s+1$ ,  $j_{s+1} = i$ . Hence, the same holds for any  $F_d$  with  $\text{comp}_i(d) = J$ . The relations  $\sum_{j=n+1}^{n+m} \omega_j w_i(j) = \sum_{\{new\ J\}} \omega_J w_i[J]$  are obvious because  $w_i(j) = \sum_{J \ni j} w_i[J]$ .

To get the integral formulas for KZ, we set  $\Phi = \int W du_{n+1} \cdots du_{n+m}$ , where the integration contours are taken to ensure that  $\int (\partial W_i(j) / \partial u_j) du_{n+1} \cdots du_{n+m} = 0$  for all new  $j$ . Then  $(\kappa \partial / \partial u_i - R_i) \Phi = 0$ . The proper choice of the contours and the description of the spaces of corresponding solutions  $\Phi$  can be a difficult problem especially if  $\kappa$  are not assumed generic (see [69]). We discuss in this work the algebraic machinery only (the integrands but not the integrals).

**8.3. Proof.** For the sake of simplicity we will consider here the Yang  $r$ -matrix only. We refer the reader to [13] for the general case (somewhat more general than that considered above). Let us degenerate the main formulas replacing the trigonometric formulas (hyperbolic, to be exact) by the rational ones. From now on  $r = \Omega/x$  and

$$\begin{aligned} \omega_i &= \sum_{1 \leq k \leq n+m, k \neq i} \frac{(\Lambda_i, \Lambda_k)}{u_i - u_k}, \quad \omega_i = \partial w / \partial u_i, \\ \omega &= \prod_{1 \leq i < j \leq n+m} (u_i - u_j)^{(\Lambda_i, \Lambda_j) / \kappa}. \end{aligned} \quad (8.28)$$

Since  $r$  is unitary and  $\varrho = 0$  we can simplify  $R_i$ :

$$R_i = \sum_{j=1}^n r^{ij} = \sum_{j=1}^n \frac{\Omega^{ij}}{u_i - u_j}, \quad j \neq i, \quad 1 \leq i \leq n. \quad (8.29)$$

Given a chain  $c = (j_1, \dots, j_s; i)$ , we set

$$F_c = \frac{[\cdot \{f_{j'_1}, f_{j'_2}\}, f_{j'_3}], \dots, f_{j'_{s-1}}, f_{j'_s}]^i}{(u_{j_1} - u_{j_2})(u_{j_2} - u_{j_3}) \cdots (u_{j_s} - u_i)}. \quad (8.30)$$

The definition of  $F_d$  for a diagram  $d = (c_1, \dots, c_r)$  is the same:

$$F_d = F_r \cdots F_2 F_1 \in U(\mathfrak{g})^{\otimes n} \text{ for } F_j = F_{c_j}. \quad (8.31)$$

Let us fix the ordering and check that  $w = \pi(\hat{m}) = \sum_d F_d(vac)$ , where  $vac = \nu_1 \otimes \cdots \otimes \nu_n \otimes \nu_0 \otimes \cdots \otimes \nu_0$  and the summation is over all well-ordered diagrams. It is a straightforward calculation based directly on the defining property of the coinvariant, that is  $\pi(f_r \tilde{v}) = 0$  for  $f \in \mathfrak{g}_r$  and  $\tilde{v} \in M_{\tilde{V}}$ .

Recall that

$$\overline{\left(\frac{g}{x_i}\right)} = \frac{g}{x - u_i}, \quad \left(\frac{g}{x_i}\right)_r = \left(\frac{g}{x_1 + u_1 - u_i}, \dots, \frac{g}{x_{n+m} + u_{n+m} - u_i}\right), \quad (8.32)$$

where  $x_i = x - u_i$ ,  $g \in \mathfrak{g}$ ,  $1 \leq i \leq n + m$ . Here  $g/x_i$  is considered as an element of the  $i$ -th component of  $\tilde{\mathfrak{g}}$  (defined in a formal neighborhood of  $u_i$  only),  $g/(x - u_i)$  is a rational function of  $x$  (defined globally). So

$$\pi \left( \left( \frac{g}{x_i} \right)^i \tilde{v} \right) = \pi \left( \left( \frac{g}{x_i} \right)_r^i \tilde{v} \right) = -\pi \left( \sum_{1 \leq j \leq n+m, j \neq i} \left( \frac{g}{x_j + u_j - u_i} \right)^j \tilde{v} \right). \quad (8.33)$$

If  $\tilde{v} = \tilde{v}_1 \otimes \dots \otimes \tilde{v}_{n+m}$  and  $\tilde{v}_j \in V_j$  for some  $j \neq i$ , then further simplification is possible:

$$\left( \frac{g}{x_j + u_j - u_i} \right)^j \tilde{v} = \tilde{v}_1 \otimes \dots \otimes \left( \frac{g}{u_j - u_i} \right)^j \tilde{v}_j \otimes \dots \otimes \tilde{v}_{n+m}. \quad (8.34)$$

Moreover, if  $j$  is new, then  $(g/(u_j - u_i))^j \tilde{v}_j = 0$ .

If the point  $j$  is new ( $j > n$ ) and  $\tilde{v}_j = (f^j/x_j)\nu_0$  for  $f \in \mathfrak{g}$ , then we can proceed as follows:

$$\begin{aligned} \left( \frac{g}{x_j + u_j - u_i} \right)^j \tilde{v}_j &= \left( \frac{g}{x_j + u_j - u_i} \right)^j \frac{f^j}{x_j} \nu_0 = \\ &= \left[ \left( \frac{g}{x_j + u_j - u_i} \right)^j, \frac{f^j}{x_j} \right] \nu_0 = \frac{1}{u_i - u_j} \frac{[f, g]^j}{x_j} \nu_0. \end{aligned} \quad (8.35)$$

*Formula for the integrand.* Now let  $\hat{m}$  be from (8.14),  $w = \pi(\hat{m})$ ,

$$\hat{m} = \nu_1 \otimes \dots \otimes \nu_n \otimes \left( \frac{f_{(n+1)'}}{x_{n+1}} \right) \nu_0 \otimes \dots \otimes \left( \frac{f_{(n+m)'}}{x_{n+m}} \right) \nu_0. \quad (8.36)$$

We start with the new index  $i$  which is maximal with respect to the ordering  $\succ$  and use (8.33) in combination with (8.35) to clear up the  $i$ -th component, replacing  $(f_i^i/x_i)\nu_0$  by the sum over all remaining components. Applying (8.33) again and again we will eventually come to the element from  $V \otimes \nu_0 \otimes \dots \otimes \nu_0$  which contains pure  $\nu_0$  at all new points and coincides with its coinvariant. Note that we have to apply this procedure to *all* terms of the sum obtained after the previous step.

In this calculation, the terms are in one-to-one correspondence with the chains  $c = (j_1, \dots, j_s; i)$  from Definition 8.1. The indices  $j_1, \dots, j_s$  must be pairwise distinct because  $g\nu_j = 0$  for any  $g \in \mathfrak{g}$ . There can be only one old  $i$  in the chain due to (8.34). The old point is always the end of the simplification process. If all the chains have reached their endpoints, we start the next chain, picking the maximal new point among the untouched ones and follow the same procedure. The new points which have already been cleared of  $\{f\}$  will not participate. We come

to the definition of the diagram and establish the formula for  $w$ , which is the first claim of Theorem 8.2.

We can take any ordering  $\succ$  in this calculation. The result will be of course the same. If we did not have an interpretation of  $w$  in terms of the coinvariant, it would not be easy to establish that different orderings result in the same formula (cf. [65]).

Let us prove that

$$w_i[J] = \sum_d F_d(\text{vac}), \quad w_i(J) = \sum_{d'} F_{d'}(\text{vac}) \quad (8.37)$$

do not depend on the ordering as well. Here the summation is over all well-ordered diagrams such that  $\text{comp}_i(d) = J$  or  $\text{comp}_i(d) \subset J$ . It is sufficient to examine  $w_i[J]$  only. The quantities  $\{w_i(J)\}$  can be expressed in terms of  $\{w_i[J]\}$ . We will deform the positions of the points  $u_1, \dots, u_{n+m}$ . Let

$$\tilde{u}_j = u_j + \delta \text{ for } j \in \{J \cup i\}, \quad \tilde{u}_j = u_j \text{ for } j \notin \{J \cup i\}. \quad (8.38)$$

The terms  $F_d$  will remain unchanged iff they appear in the sum for  $w_i[J]$ . Moreover,

$$w_i[J] = \lim_{\delta \rightarrow \infty} \tilde{w}, \quad \text{where } \tilde{w} = w(\tilde{u}_1, \dots, \tilde{u}_{n+m}). \quad (8.39)$$

Indeed, all  $F_d$  with  $\text{comp}_i(d) \neq J$  will contain at least one  $\delta$  in the denominator. This representation does not depend on the ordering. The proof of Theorem 8.2 is completed. Note that  $w = \sum_J w_i[J]$  for any old  $i$ .

*Calculating  $R_i w$ .* Let us establish the formulas  $R_i w = \omega_i w + \sum_{new J} \omega_J w_i[J]$  (Theorem 8.3). To simplify the indices we set  $i = n$ . So we need to check that

$$\left( \sum_{1 \leq j < n} r^{nj} (u_n - u_j) \right) \pi(\hat{m}) = \sum_{J \subset \{n+1, \dots, n+m\}} \omega_J w_n[J] + \omega_n \pi(\hat{m}). \quad (8.40)$$

Since  $F_d$ ,  $w = \pi(\hat{m})$ , and  $w_i[J]$  do not depend on the central charge  $\sigma$ , one may put  $\kappa = \sigma + g = 0$  and apply Theorem 8.1:

$$\left( \sum_{1 \leq j < n} r^{ij} (u_n - u_j) \right) \pi(w) = \pi(L_{-1}^n w). \quad (8.41)$$

Explicitly,

$$\pi(L_{-1}^n w) = \pi \left( \left( \sum_{\alpha > 0} \left( \frac{e_\alpha}{x_n} \right) \frac{f_\alpha}{(e_\alpha, f_\alpha)} + \sum_{1 \leq p \leq l} \left( \frac{h_p}{x_n} \right) h_p^* \right)^n w \right). \quad (8.42)$$

Later on we will not show the upper right indices indicating the component if the confusion is impossible. Once there is  $x_j$ , it means that the corresponding element acts on the  $j$ -th component.

Making use of (8.33,8.34,8.35), we have

$$\pi \left( \left( \sum_{1 \leq p \leq l} \left( \frac{h_p}{x_n} \right) h_p^* \right) \hat{m} \right) = \omega_n \pi(\hat{m}). \quad (8.43)$$

Indeed, the left hand side is  $\pi(\hat{m})$  multiplied by

$$\sum_{j < n} \frac{1}{u_n - u_j} \sum_p (h_p, \Lambda_j)(h_p^*, \Lambda_n) = \sum_{j < n} \frac{(\Lambda_j, \Lambda_n)}{u_n - u_j}, \quad (8.44)$$

that is the contribution of the old points, plus that of the new points:

$$\sum_{j > n} \frac{1}{u_n - u_j} \sum_p (h_p, \Lambda_j)(h_p^*, \Lambda_n) = \sum_{j > n} \frac{(\Lambda_j, \Lambda_n)}{u_n - u_j}. \quad (8.45)$$

The formulas for  $j < n$  and  $j > n$  are the same but their meaning is different. In the second formula, we used that  $[h_p, f_{j'}] = -(h_p, \alpha_{j'}) = (h_p, \Lambda_j)$  due to the definition of  $\Lambda_j$  at the new points.

Let us now get rid of  $(e_\alpha/x_n)$ . The calculation is more involved because  $e_\alpha$  may "interact" with  $f_{j'}$  at the new points, but still not too complicated. Since  $\nu_j$  ( $1 \leq j < n$ ) are the highest weight vectors, we move  $e_\alpha$  only to the right (to new points):

$$\begin{aligned} & \pi \left( \left( \left( \frac{e_\alpha}{x_n} \right) \frac{f_\alpha}{(e_\alpha, f_\alpha)} \right)^n \hat{m} \right) = \\ & = \pi \left( \sum_{n+1 \leq j \leq n+m} \left( \frac{[e_\alpha, f_{j'}]}{(u_n - u_j)x_j} \right)^j \left( \frac{f_\alpha}{(e_\alpha, f_\alpha)} \right)^n \hat{m}\{j\} \right). \end{aligned} \quad (8.46)$$

Here  $\hat{m}\{j\}$  denotes  $\hat{m}$  without  $f_{j'}/x_j$ . Namely (see (8.36)):

$$\hat{m}\{j\} = \nu_1 \otimes \cdots \otimes \nu_n \otimes \cdots \left( \frac{f_{(j-1)'}}{x_{j-1}} \right) \nu_0 \otimes \nu_0 \otimes \left( \frac{f_{(j+1)'}}{x_{j+1}} \right) \nu_0 \otimes \cdots. \quad (8.47)$$

Next we move each  $([e_\alpha, f_{j'}]/x_j)$  to all components  $\tilde{j} \neq j$  to clear the  $j$ -th component of  $[e, f]$ . We will get triple commutators  $[[e, f], f]$  at new  $\tilde{j}$ . Next we will go to the components  $\hat{j} \neq j, \tilde{j}$  and will produce 4-term commutators, etc. The poles will be always simple. We stop the process when the  $s$ -fold commutator is 0 or when we come to the first old point (including  $u_n$ ).

Let us examine the successive commutators. Since  $f_{j'}$  are simple, any commutator given by this procedure is proportional to

$$(a) (e_\beta/x_k)^k, \quad (b) (h_\beta/x_k)^k, \quad \text{or} \quad (c) \text{ the central element } c \quad (8.48)$$

for a certain  $\beta > 0$  and new  $k$ .

In the first case we will continue and get another new point. In the case (b), there will be one more (final) step when we replace  $(h_\beta/x_k)^k$  by the corresponding

sum over all points (old and new). After this we will stop. As to (c), we just plug in  $c = \sigma = -g$ . Note that since one never gets  $f$ -s in this procedure, the only way to reach the old points is via (b), excluding  $u_n$ .

Thus we arrive again at the chains  $j_s = j, j_{s-1} = \tilde{j}, j_{s-2} = \hat{j}, \dots, j_1 = k$ . However now we read  $\{j\}$  in the opposite order and they describe the process of elimination of  $e_\alpha/x_n$ . Note that  $\beta = \alpha_{k'}$  for the last index  $k = j_1$ . Once we know all the indices it is not difficult to determine the exact formula up to the contribution of the central element (the case (c)) and the chains which go back to the starting point  $u_n$ . The calculation is based on the identities:

$$\begin{aligned} & \text{if } [\cdot\{f_{j'_1}, f_{j'_2}, \dots, f_{j'_s}\}] = cf_\alpha \text{ then} \\ & [\cdot\{e_\alpha, f_{j'_s}, f_{j'_{s-1}}, \dots, f_{j'_1}\}] = (-1)^{s-1} c(f_\alpha, e_\alpha) h_{j'_1}. \end{aligned} \quad (8.49)$$

Setting  $\phi = (u_n - u_{j_1})(u_{j_1} - u_{j_2}) \cdots (u_{j_{s-1}} - u_{j_s})$ , we get

$$\begin{aligned} & \pi \left( \left( \left( \frac{e_\alpha}{x_n} \right) \frac{f_\alpha}{(e_\alpha, f_\alpha)} \right) \hat{m} \right) = \\ & = \phi^{-1} \pi \left( ((f_\alpha)^n \otimes 1 \otimes \cdots (h_{\alpha_{k'}}/x_k) \cdots \otimes 1) \hat{m}\{J\} \right). \end{aligned} \quad (8.50)$$

Here  $k = j_1$ ,  $\hat{m}\{J\}$  denotes  $\hat{m}$  without  $(f_{j'}/x_j)$  for all  $j \in J$ . Namely,

$$\begin{aligned} \hat{m}\{J\} &= \nu_1 \otimes \cdots \otimes \nu_n \otimes \cdots \left( \frac{f_{1'}}{x_1} \right) \nu_0 \otimes \cdots \otimes \overbrace{\nu_0}^{j_1} \otimes \\ & \cdots \otimes \left( \frac{f_{(j)'}}{x_{(j)'}} \right) \nu_0 \cdots \overbrace{\nu_0}^{j_2} \cdots \left( \frac{f_{(j)'}}{x_{(j)'}} \right) \nu_0 \cdots \overbrace{\nu_0}^{j_s} \cdots \left( \frac{f_{(n+m)'}}{x_{n+m}} \right). \end{aligned} \quad (8.51)$$

Of course here the order of  $\{j\}$  may be arbitrary, say,  $j_2, \dots, j_1, \dots$ .

Compare the formula with the definition of  $F_c$ :

$$F_c = \phi^{-1} \pi \left( \left( (f_\alpha)^n \otimes 1 \otimes \cdots \overbrace{1}^k \cdots \otimes 1 \right) \hat{m}\{J\} \right) \quad (8.52)$$

for the chain  $c = (j_1, \dots, j_n; n)$ . We see that the expressions are different only because of  $h_{\alpha_{k'}}/x_k$ . The latter can be readily replaced by the sum over all old and new indices  $i \neq k$ . Note the cancelation of  $(-1)^{s-1}$  from the denominator of the first formula with  $(-1)^{s-1}$  from (8.49).

This procedure is just the first step. We still need to clear the new components  $i \notin J$  which contain  $\{f\}$ . We will follow exactly the process of calculating  $w$ , numerating the terms by the diagrams. The final formula will be the sum of  $F_d$  over all diagrams  $d$  multiplied by proper  $\omega$ -s coming from  $h_{\alpha_{k'}}/x_k$  at the first step. Here we need to use that  $w_n[J]$  do not depend on the ordering. Indeed, when eliminating  $e_\alpha$ , we cannot control the endpoint  $k = j_1$  of the (first) chain. So we change the ordering taking  $k$  as the maximal element.

We arrive at the formula

$$R_n w - \omega_n w - \sum_{\text{new } J} \omega_J w_n[J] = S, \quad (8.53)$$

where  $S$  is a sum of terms with the denominators which may not contain the “chain products” in the form  $(u_n - u_{j_1})(u_{j_1} - u_{j_2}) \cdots (u_{j_p} - u_i)$  for any new  $j$  and old  $i < n$ . It is the contribution of the terms of type (c) and the chains ending at the  $n$ -th component. The later can come from  $h$  (the case (b)) or from certain  $e$  if we go back and fuse them with  $(f_\alpha)\nu_n$ .

Let us check that  $S = 0$ . It can be represented as  $S = \sum_J S_n[J]$  exactly in the same manner as  $w = \sum_J w_n[J]$ . We will prove that  $S_n[J] = 0$  for any new set  $J$ . Consider the deformation  $\tilde{u}$  from (8.38) for  $i = n$  and tend  $\delta$  to  $\infty$ . Then  $w \rightarrow w_n[J]$ . In the sum  $\omega_n + \omega_J = \sum_{i,j} \frac{(\Lambda_i, \Lambda_n)}{u_i - u_j}$ , exactly one index from any pair  $\{i, j\}$  belongs to  $\{J \cup n\}$ . So it goes to 0 as  $\delta$  approaches  $\infty$ . Since all the terms of  $R_n$  contain the differences  $u_i - u_n$  for some  $i < n$  in the denominators, the left hand side of (8.53) identically equals zero. The right hand side tends to  $S_n[J]$ . Thus the letter is zero. The proof of the theorem is completed.

*Remark 8.1.* The direct (combinatorial) definition of the  $w$  and  $W$  in the rational case is due to Schechtman and Varchenko (Preprint MPI/89-51, 1989). They established that  $W$  satisfies KZ up to exact derivatives in terms of the new  $\{u_j, j > n\}$ . This theorem generalized the paper by Date, Jimbo, Matsuo, and Miwa [26] (the  $SL_2$ -case) and that by Matsuo for  $SL_n$ . There were also results of Dotsenko, Fateev, Aomoto, Christe, and Flume in this direction. An extended version of the MPI-preprint is [65].

The paper [13] (first published as preprint RIMS-699, 1990) contained the interpretation via the coinvariant, the trigonometric generalization, and Theorem 8.3 with the explicit formulas for the exact derivatives.

Theorem 8.3 has applications not only to KZ. Actually it is a pure algebraic statement and must have algebraic corollaries. Let us choose the parameters  $\{u_{n+1}, \dots, u_{n+m}\}$  to ensure the relations  $\omega_j = 0$  for  $j > n$ . Then  $w$  is an eigenvector of the pairwise commutative matrices  $R_i$  ( $1 \leq i \leq n$ ) with the eigenvalues  $\{\omega_i\}$ . This eigenvalue problem is called the Gaudin model. The first results on the diagonalization of  $\{R_i\}$  were obtained by Babujan and Flume in [3]. See also [32]. We will not discuss this direction here.

**8.4. Comment on KZB.** Concerning the elliptic examples, there is the so-called Baxter-Belavin  $r$ -matrix. It is unique among unitary elliptic ones. There are more examples of non-unitary type. A variant of Belavin’s  $r$  in infinite matrices was introduced by Shibukawa–Ueno. Theorem 8.1 holds for all such  $r$ . However there are problems with the integral formulas. Our method can be generalized, but we need to assume that the rational extensions  $\bar{f}$  of the elements  $f = f_\alpha/x_i$  and  $f = e_\alpha/x_i$  are proportional to  $f_\alpha$  and  $e_\alpha$ , respectively, for certain scalar meromorphic functions as coefficients of proportionality. As it was demonstrated in [35], the

KZB due to Bernard [5] leads to a Lie algebra of elliptic functions satisfying this very property (see also [33]). In this section, we will comment on it.

Let  $E$  be an algebraic elliptic curve,  $0$  its zero point,  $x$  a local parameter in a neighborhood  $U_0 \subset E$  of  $0$ . We fix pairwise distinct  $u_1, \dots, u_n \in U_0$ . Given  $\mu \in \mathbb{C}$ , a *Baker function*  $\varphi$  is a function on  $E$  regular apart from  $U_0$  such that  $e^{-\mu x^{-1}} \varphi(x)$  is meromorphic with poles at points  $u_1, \dots, u_n$ . So they have an “exponential” singularity at  $0$ . We denote the space of such functions by  $B_\mu$ .

Baker functions are determined by their principal parts, which may be arbitrary. More precisely, given  $p(x) = \sum_{i=1}^n \sum_{1 \leq j < \infty} c_{ij} (x - u_i)^{-j}$ , there exists a unique function  $\tilde{p}(x) \in B_\mu$  such that  $\tilde{p}(x) - p(x)$  is holomorphic on  $U_0$ . Here  $\mu$  is generic. Say, if  $\mu = 0$  and there is no essential singularity at  $0$ , one can define  $\tilde{p}$  only when the sum of residues of  $p$  equals zero.

Let  $\tilde{p}_0$  be a rational function on  $E$  with the principal part

$$q(x) = p(x) - \left( \sum_{i=1}^n \operatorname{Res}_{u_i} p(x) dx \right) x^{-1} \quad (8.54)$$

normalized by the condition  $\tilde{p}_0 - q(x) = x(\cdot)$  (no constant term at  $0$ ).

We keep the notation of the previous section:  $\mathfrak{g}$  is simple Lie algebra,  $\Sigma = \{\alpha\} \subset \mathbb{R}^l$  the corresponding root system. Let us fix a vector  $\lambda \in \mathbb{C}^n$  and set  $\lambda_\alpha = (\lambda, \alpha)$  for the standard invariant form on  $\mathbb{C}^l$ .

Given a principal part  $p(x) = \sum_{i=1}^k \sum_{j=1}^{m_i} \frac{c_{i,j}}{(x-u_i)^j}$ , let  $\tilde{p}_\alpha$  be the function  $\tilde{p}$  from  $B_{\lambda_\alpha}$  with the principal part  $p$ . Using this definition we set,

$$\overline{p(x)e_\alpha} = \tilde{p}_{-\alpha}(x)e_\alpha, \quad \overline{p(x)f_\alpha} = \tilde{p}_\alpha(x)f_\alpha, \quad (8.55)$$

$$\overline{p(x)h_\alpha} = \tilde{p}_0(x)h_\alpha - \left( \sum_{i=1}^k \operatorname{Res}_{u_i} p(x) dx \right) \partial_{\lambda_\alpha}, \quad (8.56)$$

where  $\partial_{\lambda_\alpha}$  is the differentiation  $\partial_{\lambda_\alpha}(\lambda_\beta) = (\alpha, \beta)$ .

Let us extend  $\mathbb{C}$  to the noncommutative algebra  $\mathcal{L}$  of differential operators of  $\lambda$  with meromorphic coefficients (we will not specify which and where). We set  $\mathfrak{G} = \mathfrak{G} \otimes \mathcal{L}$ .

**Proposition 8.5.** *The space  $\bar{\mathfrak{G}}$  linearly generated by  $\{\overline{p(x)e_\alpha}, \overline{p(x)f_\alpha}, \overline{p(x)h_\alpha}, h_\alpha\}$  for  $\alpha \in \Sigma_+$  and all principal parts  $p$  at  $\{u_1, \dots, u_n\}$  is a Lie algebra.*

*Proof.* The commutator  $[\overline{p(x)f_\alpha}, \overline{q(x)f_\beta}]$  is proportional to  $[f_\alpha, f_\beta]$  and the coefficient of proportionality is from  $B_{\lambda_\alpha + \lambda_\beta}$ . So it belongs to  $\bar{\mathfrak{G}}$ , as well as the commutators  $[f, e]$ ,  $[e, e]$ . The commutators  $[\overline{p(x)h_\alpha}, \overline{q(x)h_\beta}]$  are zero since  $\tilde{p}_0$  do not depend on  $\lambda$ . Let us calculate  $[\overline{q(x)h_\alpha}, \overline{p(x)f_\beta}]$ . The commutator  $[\tilde{q}_0 h_\alpha, \overline{p(x)f_\beta}]$  belongs to  $\bar{\mathfrak{g}}$ . As to the  $\partial_\lambda$ -term, we need to check, that

$$[c\partial_{\lambda_\alpha} + c\frac{h_\alpha}{x}, \overline{q(x)f_\beta}] \quad (8.57)$$

does not contain a pole at  $x = 0$ . It is easy, since

$$[\partial_{\lambda_\alpha} + \frac{h_\alpha}{x}, e^{\lambda_\beta x^{-1}} f_\beta] = \left( (\alpha, \beta) \frac{f_\beta}{x} - \frac{[h_\alpha, f_\beta]}{x} \right) e^{\lambda_\beta x^{-1}} = 0. \quad (8.58)$$

□

We note that if one of  $u_i$  is 0, say,  $u_1 = 0$ , then the definition still works well. For instance,  $\overline{x^{-1}h_\alpha} = -\partial_{\lambda_\alpha}$  at  $u_1 = 0$ .

One may introduce  $\tilde{\mathfrak{G}}$  and the Kac-Moody algebra  $\hat{\mathfrak{G}}$  using the same definitions. The coefficients of all series are taken from  $\mathcal{L}$ . Expanding, the elements from  $\tilde{\mathfrak{G}}$  we define  $\tilde{\mathfrak{G}}_r$ . We see that

$$\tilde{\mathfrak{G}} = \tilde{\mathfrak{G}}_r + \tilde{\mathfrak{G}}_0, \quad \tilde{\mathfrak{G}}_r \cap \tilde{\mathfrak{G}}_0 = \mathfrak{h} = \mathfrak{h} \otimes \mathcal{L}. \quad (8.59)$$

So  $\tilde{\mathfrak{G}}_r$  is a bit bigger than factorizing.

We can introduce the  $r$ -matrix describing the projection of  $\tilde{\mathfrak{G}}$  onto  $\tilde{\mathfrak{G}}_r$ . It will satisfy non-unitary Yang-Baxter equation “up to  $\mathfrak{h}$ ”, but will not depend on the difference.

The notion of the coinvariant must be properly changed. Let  $V_i$  be  $\mathfrak{g}$ -modules,  $V = \otimes_{i=1}^n V_i$ ,  $M_V^\sigma$  the corresponding Weil module. We set  $\mathfrak{M}_V^\sigma = M_V^\sigma \otimes \mathcal{L}$  and define the coinvariant

$$\pi : \mathfrak{M}_V^\sigma \rightarrow \mathfrak{B}/(\mathfrak{h}\mathfrak{B}) \text{ for } \mathfrak{B} = V \otimes \mathcal{L}. \quad (8.60)$$

The abstract relation between  $\pi$  and the Sugawara elements and the integral formulas can be extended to this setup. To get exactly KZB in the form [35], we need to add the derivative with respect to the  $\tau$ -parameter of the elliptic curve (the parabolic equation). Then we must switch to the  $r$ -matrix depending on the difference, which means that the rational continuation  $(\bar{\cdot})$  must be invariant with respect to the shifts in  $x$ . The Baker functions have to be replaced by multi-valued functions on  $E$  with multipliers. The definitions of  $\pi$ ,  $\hat{m}$  and  $w$  follow the same lines. Eventually we come to the integral formulas due to Felder-Varchenko [34]. The proof is similar to that from [13], excluding the parabolic equation (a special feature of KZB).

Note that the interpretation of KZB via Baker functions is equivalent to the Felder interpretation. It might be more convenient to try to extend the integral formulas to any algebraic curves (there are certain results in this direction). At least it clarifies why the integral formulas cannot be expected to be too “integrable” in the general theory.

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