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On the exterior problem for nonlinear wave equations with small initial data

Hideo Kubo

Abstract.

The aim of this note is to give an overview concerning the mixed problem for a system of nonlinear wave equations with small and smooth initial data. In particular, we are interested in the three and two space dimensional case.

§1. Introduction

The aim of this note is to give an overview concerning the mixed problem for nonlinear wave equations with small and smooth initial data. Let Ω be an unbounded domain in $\mathbf{R}^n$ $(n \geq 2)$ with compact and smooth boundary $\partial\Omega$. We put $\mathcal{O} := \mathbf{R}^n \setminus \Omega$, which is called an obstacle and is supposed to be non-empty. We consider the mixed problem for a system of nonlinear wave equations:

$$
\begin{aligned}
&(1) \qquad (\partial_t^2 - \Delta)u_i = F_i(\partial u, \nabla_x \partial u), && (t,x) \in (0,\infty) \times \Omega, \\
&(2) \qquad u(t,x) = 0, && (t,x) \in (0,\infty) \times \partial\Omega, \\
&(3) \qquad u(0,x) = \varepsilon\phi(x), \quad \partial_t u(0,x) = \varepsilon\psi(x), && x \in \Omega
\end{aligned}
$$

for $i = 1, \ldots, N$, where $\Delta = \sum_{j=1}^n \partial_j^2$, $\partial_t = \partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1, \ldots, n)$, and $\varepsilon > 0$. We assume ϕ, $\psi \in C_0^\infty(\overline{\Omega}\,; \mathbf{R}^N)$, namely they are smooth functions on $\overline{\Omega}$ vanishing outside some ball. We assume that each F_i is a smooth function in $\mathbf{R}^{(1+n)N}$ satisfying

$$
(4) \qquad F_i(\partial u, \nabla_x \partial u) = O(|\partial u|^q + |\nabla_x \partial u|^q), \quad 1 \leq i \leq N
$$

around $(\partial u, \nabla_x \partial u) = 0$ for some integer $q \geq 2$, together with the energy symmetric condition. In addition, we assume that (ϕ, ψ, F) satisfies the

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compatibility condition to infinite order for the mixed problem (1)–(3), that is, $(\partial_t^j u)(0,x)$, formally determined by (1) and (3), vanishes on $\partial\Omega$ for any non-negative integer j (notice that the values $(\partial_t^j u)(0,x)$ are determined by (ϕ,ψ,F) successively; for example we have $\partial_t^2 u(0) = \Delta_x\phi + F(\psi,\nabla_x\phi)$, and so on).

We may assume, without loss of generality, that $\mathcal{O} \subset B_1$ by the translation and scaling. Hence we always pose this assumption in the following.

It was shown by Shibata and Tsutsumi [21] that the mixed problem for (1)–(3) admits a unique global solution for sufficiently small initial data, when $n \geq 6$ and $\mathcal{O}$ is non-trapping. The argument is still effective to handle the problem for $3 \leq n \leq 5$, if $q \geq 3$ in (4) (see also [3]). To show the result, they deduced L^p-L^q estimates for the mixed problem from those for the Cauchy problem, based on the cut-off method. In this procedure, decay property for the local energy plays a crucial role.

We remark that if $\mathcal{O}$ is non-trapping and $n \geq 3$, then we have the following decay estimate of the local energy (for the proof, see for instance, [9, Appendix B]): Let $a > 1$, $\gamma \in (0,1]$, $m \in \mathbf{N}$, and set

$$\Omega_b = \Omega \cap \{x \in \mathbf{R}^n;\ |x| < b\}$$

for $b > 1$. Assume that $f \in C^\infty([0,T)\times\overline{\Omega};\mathbf{R})$ satisfies

$$f(t,x) \equiv 0 \text{ for } |x| \geq a \text{ and } t \in [0,T).$$

Then there is a positive number $C = C(\gamma,a,b,m)$ such that

$$(1+t)^\gamma \|v(t) : H^m(\Omega_b)\| \leq C \sup_{0\leq s\leq t} (1+s)^\gamma \|\partial^\alpha f(s) : H^{m-1}(\Omega)\|$$

for $t \in [0,T)$, where v is the solution to

$$(\partial_t^2 - \Delta_x)v = f, \qquad (t,x) \in (0,T)\times\Omega, \tag{5}$$

$$v(t,x) = 0, \qquad (t,x) \in (0,T)\times\partial\Omega, \tag{6}$$

$$v(0,x) = (\partial_t v)(0,x) = 0, \qquad x \in \Omega. \tag{7}$$

In treating the problem (1)–(3) for the case when $n = 3$ and $q = 2$, we need to exploit more precise information on the behavior of solutions, as was done in the study of the corresponding Cauchy problem. For instance, the following type of a-priori estimates for solutions would be useful:

$$|u(t,x)| \leq C(1+t+|x|)^{-1}\log\left(2 + \frac{1+t+|x|}{1+|t-|x||}\right), \tag{8}$$

$$|\partial_{t,x} u(t,x)| \leq C(1+|x|)^{-1}(1+|t-|x||)^{-1} \tag{9}$$

for $(t, x) \in [0, \infty) \times \overline{\Omega}$. But, due to the blow-up result for the corresponding Cauchy problem obtained by John [7], one can construct a blow-up solution to the mixed problem when $q = 2$ in view of the finite speed of propagation. This means that it is impossible to show the global solvability in time for our mixed problem in general. So, what we can expect is to get a lower bound of the lifspan such as

$$T_\varepsilon \geq \exp(C\varepsilon^{-1}) \tag{10}$$

for sufficiently small ε, where C is a constant and T_ε is the supremum of all positive number T such that there is a classical solution for the problem (1)–(3). In fact, in Keel, Smith and Sogge [11], [12] the estimate (10) was shown, provided either the nonlinearity depends only on the first derivatives of unknown, or the obstacle is star-shaped. These restrictions are removed in [14], by using a different approach.

By assuming in addition that the quadratic part of the nonlinearity has nice algebraic structure, called *null condition*, it was shown by Metcalfe, Nakamura, and Sogge [19] that $T_\varepsilon = +\infty$ for sufficiently small ε. An alternative proof for the result can be found in in Katayama and Kubo [9] (see also [2], [10], [17], [20], [18]). We remark that under the null condition, the quadratic part of the nonlinearity can be written as a linear combination of $Q_0(u_j, u_k)$ or $Q_{ab}(u_j, u_k)$, where Q_0 and Q_{ab} are the null forms defined by

$$Q_0(\xi, \eta) = (\partial_t \xi)(\partial_t \eta) - (\nabla_x \xi) \cdot (\nabla_x \eta), \tag{11}$$

$$Q_{ab}(\xi, \eta) = (\partial_a \xi)(\partial_b \eta) - (\partial_b \xi)(\partial_a \eta) \quad (0 \leq a < b \leq 3) \tag{12}$$

for real valued-functions $\xi = \xi(t, x)$ and $\eta = \eta(t, x)$.

Now, a natural question is if it is possible to obtain an analogous result in the case $n = 2$. Because decay property of solutions in two dimensional case is rather weak, compared with the case $n = 3$, we need more delicate treatment for establishing a-priori estimates.

Observe that one can construct a blow-up solution to the mixed problem also when $q = 3$ in (4), based on the blow-up result for the Cauchy problem given by Agemi [1]. For this, we assume that the nonlinearity $F = (F_1, F_2, \ldots, F_N)$ takes the following form

$$F_i(\partial u, \nabla_x \partial u) = \sum_{a=0}^{2} \sum_{j,k,l=1}^{N} g_i^{a,b,c} (\partial_a u_j)(\partial_b u_k)(\partial_c u_l), \quad 1 \leq i \leq N, \tag{13}$$

where $g_i^{a,b,c}$ $(a, b, c = 0, 1, 2)$ are real constants. When we consider the global solvability, we assume in addition that the nonlinearity satisfies the second null condition, that is,

$$\sum_{a=0}^{2}\sum_{j,k,l=1}^{N} g_i^{a,b,c}\lambda_j\lambda_k\lambda_l\,\omega_a\,\omega_b\,\omega_c = 0 \tag{14}$$

for any $i = 1, \ldots, N$, $(\lambda_1, \ldots, \lambda_N) \in \mathbf{R}^N$, $(\omega_1, \omega_2) \in S^1$ and $\omega_0 = -1$. This is a sufficient condition to assure the global existence result for the corresponding Cauchy problem. Under the condition, each term in the nonlinearity contains one of the null forms $Q_0(u_j, u_k)$ and $Q_{12}(u_j, u_k)$.

This note is organized as follows. In the next section precise statement of the global existence and almost global existence results for small initial data will be presented. In the section 3 we give an outline of the proof of these results.

§2. Statement of results

In order to state the results concerning the problem for (1)–(3), we introduce a couple of notations and a condition on the obstacle.

We denote by $X(T)$ the set of all

$$\Xi = (v_0, v_1, f) = (\vec{v}_0, f) \in C_0^\infty(\overline{\Omega}; \mathbf{R}^2) \times C_D^\infty([0,T) \times \overline{\Omega}; \mathbf{R})$$

satisfying the compatibility condition to infinite order for the mixed problem for (5), (6), and

$$v(0,x) = v_0(x),\ (\partial_t v)(0,x) = v_1(x), \quad x \in \Omega, \tag{15}$$

i.e., $(\partial_t^j v)(0,x)$, determined formally from (5) and (15), vanishes on $\partial\Omega$ for any non-negative integer j. Here $f \in C_D^\infty([0,T) \times \overline{\Omega}; \mathbf{R})$ means that $f \in C^\infty([0,T) \times \overline{\Omega}; \mathbf{R})$ and $f(t,\cdot) \in C_0^\infty(\overline{\Omega})$ for any fixed $t \in [0,T)$.

For $\Xi \in X(T)$ we denote by $S[\Xi]$ the solution to (5), (6), and (15).

For $a > 1$, $X_a(T)$ denotes the set of all $\Xi = (v_0, v_1, f) \in X(T)$ satisfying

$$v_0(x) = v_1(x) = f(t,x) \equiv 0 \text{ for } |x| \geq a \text{ and } t \in [0,T).$$

We put $\mathcal{H}^m(\Omega) = H^{m+1}(\Omega) \times H^m(\Omega)$ for every integer m.

We are ready to state the condition on the obstacle $\mathcal{O}$.

Definition 1. *Let $a > 1$, $\ell \in \mathbf{N}$, and $\gamma \in (0,1]$. We say that the obstacle $\mathcal{O}$ is* admissible, *if for any $\Xi = (v_0, v_1, f) \in X_a(T)$, $b > 1$, and*

$m \in \mathbf{N}$, *there exists a positive constant* $C = C(\gamma, a, b, m)$ *such that*

$$(16)\quad (1+t)^\gamma \|S[\Xi](t) : H^m(\Omega_b)\| \\ \leq C\Big(\|(v_0, v_1) : \mathcal{H}^{m+\ell-1}(\Omega)\| + \sup_{0\leq s\leq t}(1+s)^\gamma \|\partial^\alpha f(s) : H^{m+\ell-1}(\Omega)\|\Big)$$

holds for $t \in [0, T)$.

We remark that if the obstacle is non-trapping, then it is admissible with $\ell = 0$ in the above sense. On the other hand, when the obstacle is trapping, we would need some loss of derivatives, i.e., $\ell \geq 1$. Such a kind of estimates of the local energy was actually derived by Ikawa [5], [6].

Now we are in a position to state the results. The first one is a generalization of the almost global existence result by [14] for the admissible obstacle. Its proof can be done in the same line as in [9].

Theorem 1. *Let* $n = 3$ *and* ϕ, $\psi \in C_0^\infty(\overline{\Omega}; \mathbf{R}^N)$. *Assume that* (ϕ, ψ, F) *satisfies the compatibility condition to infinite order for the problem* (1)–(3) *and that* $\mathcal{O}$ *is admissible. Then there exist positive constants* ε_0 *and* C *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the mixed problem* (1)–(3) *admits a unique solution* $u \in C^\infty([0, T_\varepsilon) \times \overline{\Omega}; \mathbf{R}^N)$ *and* $T_\varepsilon \geq \exp(C\varepsilon^{-1})$ *holds.*

The second one is the global existence result due to [9].

Theorem 2. *Let all the assumptions in Theorem* 1 *be fulfilled. If* F *satisfies the null condition, then there exists a positive constant* ε_0 *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *we have* $T_\varepsilon = +\infty$, *and the weighted* L^∞-*estimates* (8) *and* (9) *holds.*

The third one is a counter part of Theorem 1 in the case $n = 2$ obtained by [15].

Theorem 3. *Let* $n = 2$ *and* ϕ, $\psi \in C_0^\infty(\overline{\Omega}; \mathbf{R}^N)$. *Assume that* (ϕ, ψ, F) *satisfies the compatibility condition to infinite order for the problem* (1)–(3) *and that* $\mathcal{O}$ *is star-shaped. If* F *takes the form of* (13), *then there exist positive constants* ε_0 *and* C *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the mixed problem* (1)–(3) *admits a unique solution* $u \in C^\infty([0, T_\varepsilon) \times \overline{\Omega}; \mathbf{R}^N)$ *and* $T_\varepsilon \geq \exp(C\varepsilon^{-2})$ *holds.*

The last one is a counter part of Theorem 2 in the case $n = 2$ (the detail will be given by [16]).

Theorem 4. *Let all the assumptions in Theorem* 3 *be fulfilled. If* F *satisfies the second null condition, then there exists a positive constant* ε_0 *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *we have* $T_\varepsilon = +\infty$, *and*

$$(17)\qquad |\partial_{t,x} u(t,x)| \leq C w_\nu(t,x)$$

holds for $(t,x) \in [0,\infty) \times \overline{\Omega}$ *and* $1/2 < \nu < 1$. *Here we put*

$$w_\nu(t,x) = \Big((1+|x|)^{-1/2}(1+|t-|x||)^{-\nu} \\ +(1+t+|x|)^{-1/2}(1+|t-|x||)^{-1/2}\Big). \tag{18}$$

We remark that these results can be extended to the case where the system of nonlinear wave equations has multiple speeds of propagation, because our proof does not require to use the boost $t\partial_j + x_j\partial_t$ $(1 \le j \le n)$ that commutes the d'Alembertian only if the propagation speed is 1. But, for simplicity of exposition, we restrict our attention to the case where the system has a common propagation speed.

§3. Outline of the proof

Main step of the proof is to derive weighted L^∞-estimates from the corresponding estimates for the Cauchy problem due to Yokoyama [22] and Hoshiga and Kubo [4], based on the cut-off argument developed in [21]. In this procedure, the decay property of the local energy given by (16) is crucial.

First of all, we introduce notations. We use

$$\partial_0 = \partial_t, \quad \partial_j \ (1 \le j \le n), \quad O_{jk} = x_j\partial_k - x_k\partial_j \ (1 \le j < k \le n) \tag{19}$$

and denote $\partial = (\partial_0, \partial_1, \ldots, \partial_n)$, $O = (O_{ij})_{1 \le j < k \le n}$. For a smooth function $\varphi(t,x)$, we set $|\varphi(t,x)|_k = \sum_{|\alpha|+|\beta| \le k} |\partial^\alpha O^\beta \varphi(t,x)|$.

For $\nu, \kappa \ge 0$, $c \ge 0$, and a non-negative integer k, we put

$$\|f(t) : N_k(\mathcal{W})\| = \sup_{(s,x) \in [0,t] \times \Omega} |x|^{(n-1)/2}\, \mathcal{W}(\lambda, s)\, |f(s,x)|_k, \tag{20}$$

$$\mathcal{A}_{\nu,k}[\vec{v}_0] = \sup_{x \in \Omega} (1+|x|)^\nu \big(|v_0(x)|_k + |\nabla_x v_0(x)|_k + |v_1(x)|_k\big), \tag{21}$$

where $\Xi = (\vec{v}_0, f) \in X(T)$ and $\mathcal{W}(\lambda, s)$ is a weight function. Typical choices of $\mathcal{W}(\lambda, s)$ are as follows:

$$z_{\nu,\kappa;c}(\lambda,s) = (1+s+\lambda)^\nu (1+|\lambda - cs|)^\kappa;$$
$$W_{\nu,\kappa}(\lambda,s) = (1+s+\lambda)^\nu \Big(\min\{(1+\lambda), (1+|\lambda - cs|)\}\Big)^\kappa.$$

For $\kappa \ge 0$, we define

$$\Psi_\kappa(t) = \begin{cases} \log(2+t) & \text{if } \kappa = 1, \\ 1 & \text{if } \kappa \ne 1, \end{cases}$$

Then we have the following basic estimates.

Propsoition 1. *Let $n = 3$ and let $\mathcal{O}$ be admisssible. Assume that $\Xi = (\vec{v}_0, f) \in X(T)$. If $1 \le \nu \le 2$ and $\kappa \ge 1$, then we have*

$$(1+|x|)(1+|t-|x||)^\rho |\partial S[\Xi](t,x)|_k \le C\left(\mathcal{A}_{\nu+2,k+\ell+4}[\vec{v}_0] + \Psi_\rho(t)\,\|f(t) : N_{k+\ell+4}(W_{\nu,\kappa})\|\right) \tag{22}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$. Here we put $\rho = \min(\nu, \kappa)$.

Propsoition 2. *Let $n = 2$ and let $\mathcal{O}$ be star-shaped. Assume that $\Xi = (\vec{v}_0, f) \in X(T)$. If $1/2 \le \nu < 1$, $\kappa \ge 1$, and $\mu > 0$, then we have*

$$(w_\rho(t,x))^{-1} |\partial S[\Xi](t,x)|_k \le C\left(\mathcal{A}_{2+\nu+\mu,k+4}[\vec{v}_0] + \Psi_\rho(t)\,\Psi_\kappa(t)\|f(t) : N_{k+4}(W_{\nu+(1/2),\kappa})\|\right) \tag{23}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$. Here we put $\rho = \min(\nu + (1/2), \kappa)$.

When one deal with the null form, it is convenient to use the vector fields $t\partial_j + x_j\partial_t$ ($j = 1, \dots, n$) and $t\partial_t + x \cdot \nabla_x$ (see [13]). But, the boundary condition (2) makes difficult to use them. In order to handle the null form by using the restricted vector fields (19), we make use of stronger decay property of a tangential derivative to the light cone $(\partial_t + \partial_r)u$. This idea was introduced by Katayama and Kubo [8].

Then one can establish needed estimates for solutions to prove the theorems.

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Graduate School of Information Sciences
Tohoku University, Sendai 980-8579, Japan

Current Address:
Department of Mathematics
Graduate School of Science
Hokkaido University, Sapporo 060-0810, Japan

E-mail address: `kubo@math.sci.hokudai.ac.jp`