

Logarithmic stable maps

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Abstract.

We introduce the notion of a logarithmic stable map from a minimal log prestable curve to a log twisted semi-stable variety of form $xy = 0$. We study the compactification of the moduli spaces of such maps and provide a perfect obstruction theory, applicable to the moduli spaces of (un)ramified stable maps and stable relative maps. As an application, we obtain a modular desingularization of the main component of Kontsevich's moduli space of elliptic stable maps to a projective space.

§1. Introduction

1.1.

In papers [11, 21], the admissibility, or equivalently, the predeformability of J. Li was introduced. It is a condition on maps from curves to semi-stable varieties which are étale locally of form $xy = 0$. The condition is natural. It is a necessary and sufficient condition to deform, étale locally at the domain, such a map to a map from a smooth domain curve to a smooth target. It is, however, not a condition friendly to moduli problems ([21, 22, 10]). If suitable log structures are given both on the domain curve and the target space, the admissibility amounts to the requirement that the map is a log morphism which is simple at the inverse image of the singular locus. For the separatedness of moduli spaces of such maps, it will be imposed that the log structures on the domain prestable curves are minimal (see 3.7); the log structures on the targets are extended log twisted ones (see 4.3); and the automorphism groups of the log morphisms are finite. Those maps, satisfying the above conditions, will be called log stable maps. Once suitable log structures

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are introduced, due to the construction of log cotangent complexes by Olsson in [29] among other things, it is straightforward to show that the moduli spaces of log stable maps carry the relative perfect obstruction theory identical to that of usual stable maps case if the tangent sheaves of the targets are replaced by the log tangent sheaves (see 7.1). In log sense, both sources and targets of the maps we consider are smooth, which is one of reasons why log geometry works well in the study of moduli spaces of such stable maps. The advantage of the usage of log geometry is manifest in 7.2: The log admissibility suffices to deform, étale locally at the target, such a map to a map from a smooth domain curve to a smooth target.

We explain the results of this paper, more precisely. Let \mathbf{k} be a fixed algebraically closed field of characteristic zero. Let \mathcal{B} be an algebraic stack whose objects form a collection of $(W/S, W \rightarrow X)$, where S, X are schemes over a base algebraic \mathbf{k} scheme Λ ; W is an algebraic space over S ; W/S is a proper flat family of semi-stable varieties of form $xy = 0$; and $W \rightarrow X$ is a map. Assume that \mathcal{B} is smooth over Λ and has a universal family \mathcal{U} over \mathcal{B} . Main examples we have in mind are these: the stack \mathfrak{X} of FM spaces of a smooth projective variety X ([15]); the stack of expanded degeneration spaces of a smooth projective variety with respect to a smooth divisor ([21, 19]); and the stack of expanded degeneration spaces of a semi-stable, projective, degeneration space of form $xy = 0$ ([21]).

Main Theorem A. *The moduli stack $\overline{M}_{g,n}^{\log}(\mathcal{U}/\mathcal{B})$ of (g, n) log stable maps to $W/S \in \mathcal{B}$ is an algebraic stack with a perfect obstruction theory over Λ . Furthermore if the moduli stack $\overline{M}_{g,n}(\mathcal{U}/\mathcal{B})$ of its underlying stable maps is a proper DM stack over Λ , so is $\overline{M}_{g,n}^{\log}(\mathcal{U}/\mathcal{B})$ over Λ .*

We refer to subsection 6.3 for the precise meaning of the statement of Main Theorem A. The theorem yields an explicit description of a perfect obstruction theory for the log versions of the stack of stable (un)ramified maps ([15]) and the stack of stable relative maps of J. Li ([22]). In fact, the moduli stack $\overline{M}_{g,n}^{\log}(\mathcal{U}/\mathcal{B})$ is equipped with a log structure, and hence it is a log algebraic stack. When \mathcal{B} is the stack \mathfrak{X} of FM spaces of a projective nonsingular variety X over \mathbf{k} ([15]), after (un)ramified condition and the strong stability being imposed, the stack $\overline{M}_{g,n}^{\log}(\mathfrak{X}^+/\mathfrak{X})$ is a proper DM log stack over \mathbf{k} . Here \mathfrak{X}^+ is the universal family of \mathfrak{X} .

When the genus is 1, we will consider a variant of stable (un)ramified maps, namely, elliptic log stable maps to chain type FM spaces of X . The key condition on these, possibly non-finite, maps is that either the

genus 1 components or the loops of the rational components are non-constantly mapped under the maps. See 8.1 for the precise definition.

Main Theorem B. *The moduli stack $\overline{M}_1^{\log, \text{ch}}(\mathfrak{X}^+/\mathfrak{X})$ of elliptic log stable maps to chain type FM spaces of X is a proper DM-stack over \mathbf{k} , carrying a perfect obstruction theory. When X is a projective space $\mathbb{P}_{\mathbf{k}}^r$, the stack is smooth over \mathbf{k} .*

Here the smoothness means the usual smoothness as a DM stack. Hence it provides a moduli-theoretic desingularization of the main component of stable maps. It would be interesting to find an explicit relationship with Vakil and Zinger’s desingularization in [34]. The space $\overline{M}_1^{\log, \text{ch}}(\mathfrak{X}^+/\mathfrak{X})$ can be perhaps used to algebro-geometrically establish quantum Lefschetz hyperplane section principle for elliptic case and to prove elliptic Mirror Conjecture for Calabi–Yau hypersurfaces in a projective space. The hyperplane section principle for reduced genus 1 Gromov–Witten invariants and the comparisons between reduced and standard genus 1 invariants are accomplished in [24, 36, 37].

1.2. Remarks

The minimality condition on log prestable curves, defined here, is a condition stronger than the minimality introduced by Wewers in [35]. The idea that log structures should be useful in (relative) Gromov–Witten theory has been around many years ([22, 33]). After finishing the paper, the author learned of Siebert’s long unfinished project [33] on log GW invariants that includes a construction of the virtual fundamental class without using log cotangent complexes. After posting the paper, the author was informed that there is another approach by Abramovich and Fantechi using twisted stacky structures along the singular loci ([2]).

1.3. Conventions

Throughout the paper, all schemes are locally noetherian schemes over Λ unless otherwise stated. The readers who are familiar with log geometry can skip Section 2, keeping in mind the following conventions. Log structures considered are always fine. Log schemes will be denoted by (X, M) (or X^\dagger) when X is an underlying scheme and M is a sheaf of monoid on the étale site $X_{\text{ét}}$ of X , equipped with a homomorphism $\alpha : M \rightarrow \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$ under α . Sometimes X alone will denote the log scheme. For a monoid P , P_X or simply P will mean a constant sheaf of the monoid on $X_{\text{ét}}$. The relative log differential sheaf of $X^\dagger \rightarrow Y^\dagger$ will be written by $\Omega_{X/Y}^\dagger$. In order to avoid confusion, we often use separated notations f and \underline{f} for a log morphism and its underlying morphism, respectively. The symbol e_i is the i -th element

of the standard basis of \mathbb{N}^m . For maps $X \rightarrow Y$ and $Z \rightarrow Y$, the fiber product $X \times_Z Y$ will be written by $X|_Z$; and by X_z if Z is a point z . We will consider log structures also on algebraic spaces, extending the definition obviously.

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§2. Basics on Logarithmic Geometry

Closely following [14, 26], we recall the basic terminology on log geometry which we use in this paper.

2.1. Monoids

A *monoid* is a set P with an associative commutative binary operation $+$ with a unity, zero. We assume that a homomorphism between monoids preserves the unities. An equivalence relation on a monoid P is called a *congruence relation* if $a \sim b$ iff $a + p \sim b + p$ for all $p \in P$. Note that when \sim is a congruence relation there is a unique monoid structure on P/\sim such that the projection $P \rightarrow P/\sim$ is a monoid homomorphism. A monoid P is called *integral* if $p + p_1 = p + p_2$ implies that $p_1 = p_2$. A homomorphism $h : P \rightarrow Q$ between integral monoids is called an *integral homomorphism* if for any $h(p_1) + q_1 = h(p_2) + q_2$, there are $p'_1, p'_2 \in P, q \in Q$ such that $q_1 = h(p'_1) + q, q_2 = h(p'_2) + q$, and $p_1 + p'_1 = p_2 + p'_2$. The *cokernel* of a homomorphism $h : P \rightarrow Q$ is defined to be the induced monoid on Q/P where the coset is given as: $q \sim q'$ iff $q + h(p) = q' + h(p')$ for some $p, p' \in P$. The *group* P^{gp} associated to a monoid P is defined to be the induced monoid on $P \times P/\sim$, where $(p, q) \sim (p', q')$ iff $p + q' + r = p' + q + r$ for some $r \in P$. A monoid is called *sharp* if there is no nonzero unit (here units are, by definition, invertible elements). A nonzero element u in a sharp monoid is called *irreducible* if $u = p + q$ implies that p or q is zero.

2.2. Log structures

A *pre-logarithmic* structure on a scheme X is a pair (M, α) where M is a sheaf of unital commutative monoids on the étale site $X_{\text{ét}}$ of X

and α is a monoidal sheaf homomorphism from M to \mathcal{O}_X . Here \mathcal{O}_X is taken to be a sheaf of monoid with respect to the multiplications. When $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$ via α , it is called *logarithmic structure*. For a given pre-log structure (M, α) , the *associated log structure* M^a is defined to be the amalgamated sum (i.e., the pushout)

$$M \oplus_{\alpha^{-1}(\mathcal{O}_X)} \mathcal{O}_X^\times := M \oplus \mathcal{O}_X^\times / \sim,$$

where $(m, u) \sim (m', u')$ if there are $a, a' \in \alpha^{-1}(\mathcal{O}_X^\times)$ for which $a + m = a' + m'$ and $\alpha(a)u = \alpha(a')u'$.

If $f : X \rightarrow Y$ is a map between schemes and N is a log structure on Y , then define the *pullback* f^*N to be the associated log structure of the prelog structure of $f^{-1}N \rightarrow f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$.

A *log morphism* $f : (X, M) \rightarrow (Y, N)$ between log schemes consists of a scheme morphism $f : X \rightarrow Y$ and a monoidal sheaf homomorphism $f^b : f^*N \rightarrow M$ making the diagram

$$\begin{array}{ccc} f^*N & \longrightarrow & M \\ & \searrow & \downarrow \\ & & \mathcal{O}_X \end{array}$$

commute. Later, for simplicity, we often say that $f^b : f^*N \rightarrow M$ is a homomorphism *over log structure maps* if the diagram commutes.

A *chart* P_X of a log structure M on X is $\theta : P_X \rightarrow M$, where P_X is the constant sheaf of a finitely generated integral monoid P making its associated log structure isomorphic to the log structure M under θ . When a chart exists étale locally everywhere on X , we say that the log structure is *fine*. From now on, we consider only fine log structures. We will denote the separable closure of $p \in X$ by \bar{p} . The quotient sheaf $M/\alpha^{-1}(\mathcal{O}_X^\times)$, denoted by \bar{M} , is called the *characteristic*. It is useful to note that $f^*\bar{N} = \bar{f^*N}$. A fine log structure is called *locally free* if for every point $p \in X$, $\bar{M}_{\bar{p}}$ is finitely generated and free, i.e. $\bar{M}_{\bar{p}} \cong \mathbb{N}^r$ for some integer r which possibly depends on p . If M is locally free, then for every $p \in X$, there is a chart $\theta : \mathbb{N}_X^r \rightarrow M$ by which $\mathbb{N}^r \rightarrow \bar{M}_{\bar{p}}$ is an isomorphism and $M_{\bar{p}} \cong \bar{M}_{\bar{p}} \oplus \mathcal{O}_{\bar{p}}^\times$ (Lemma 3.3.1).

Let M and M' be log structures on X with $\bar{M}_{\bar{p}}$ and $\bar{M}'_{\bar{p}'}$ sharp, where $p \in X$. A homomorphism h from M to M' will be called *simple* at $p \in X$ if: $\bar{h} : \bar{M}_{\bar{p}} \rightarrow \bar{M}'_{\bar{p}'}$ is injective, and for any irreducible element $b \in \bar{M}'_{\bar{p}'}$, there is an irreducible element a in $\bar{M}_{\bar{p}}$ such that $\bar{h}(a) = mb$ for some positive integer m , where \bar{h} is the induced map from h . When M and M' are locally free, it means that $\bar{M}_{\bar{p}} \rightarrow \bar{M}'_{\bar{p}'}$ is form of a diagonal

map $d = (d_1, \dots, d_r) : \mathbb{N}^r \rightarrow \mathbb{N}^r$, where the standard basis e_i maps to $d_i e_i$ and $d_i \neq 0$ for all i .

A *chart of a log morphism* f is a triple $(P_X \rightarrow M, Q_Y \rightarrow N, Q \rightarrow P)$ of the chart P for M , the chart Q for N , and a homomorphism $Q \rightarrow P$ making the diagram

$$\begin{array}{ccc} P_X & \longrightarrow & M \\ \uparrow & & \uparrow \\ Q_X & \longrightarrow & f^*N \end{array}$$

commutative. For any log morphism $f : (X, M) \rightarrow (Y, N)$ and any étale local chart $Q \rightarrow N$, there is a chart $(P_X \rightarrow M, Q_Y \rightarrow N, Q \rightarrow P)$ of the map f étale locally by Lemma 2.10 in [14].

A log morphism $f : (X, M) \rightarrow (Y, N)$ is called:

- *log smooth* if étale locally, there is a chart $(P_X \rightarrow M, Q_Y \rightarrow N, Q \rightarrow P)$ of f such that:
 - $\text{Ker}(Q^{gp} \rightarrow P^{gp})$ and the torsion part of $\text{Coker}(Q^{gp} \rightarrow P^{gp})$ are finite groups.
 - The induced map $X \rightarrow Y \times_{\text{Spec}(\mathbb{Z}[Q])} \text{Spec} \mathbb{Z}[P]$ is smooth in the usual sense.
- *integral* if for every $p \in X$, the induced map $\overline{N}_{f(\overline{p})} \rightarrow \overline{M}_{\overline{p}}$ is integral.
- *vertical* if M/f^*N is a sheaf of groups under the induced monoidal operation.
- *strict* if $f^b : f^*N \rightarrow M$ is an isomorphism.

All above notions are preserved under base changes.

The amalgamation of integral monoids

$$\begin{array}{ccc} & & P_1 \\ & & \uparrow \theta \\ P_2 & \longleftarrow & Q \end{array}$$

is not longer integral in general. However, when θ is integral, the amalgamation is always integral. Hence in that case, the base change, in other words, the fiber product of an integral log morphism f_1 is defined to be

$$(X_1, M_1) \times_{(Y, N)} (X_2, M_2) := (X_1 \times_Y X_2, (P_1 \oplus_Q P_2)^\alpha)$$

where P_1, P_2, Q are charts of the log morphisms $f_i : (X_i, M_i) \rightarrow (Y, N)$. See Proposition 4.1 in [14] for various equivalent definitions of integral

morphisms. For instance, we see that the underlying map of a smooth and integral morphism is flat (Corollary 4.5 [14]).

§3. Log Curves

This section deals with the definition and some properties of the allowed domains of log stable maps. The domains will be minimal log prestable curves defined in Subsection 3.7. We start with fixing a notion of log prestable curves basically following [25, 35, 13, 27] for our purpose.

3.1. 1st definition

A log morphism $\pi : (C, M) \rightarrow (S, N)$ is called a *log prestable curve* over a fine log scheme (S, N) if every geometric fiber of π is a prestable curve and π is a proper, log smooth, integral, and vertical morphism.

According to [13] and [27], this definition is equivalent to the following second definition.

3.2. 2nd definition

A log morphism $\pi : (C, M) \rightarrow (S, N)$ is called a *log prestable curve* over (S, N) if:

- (1) The underlying map $\underline{\pi} : C \rightarrow S$ is a S -family of prestable curves.
- (2)
 - The restriction of $\pi^*N \rightarrow M$ to the $\underline{\pi}$ -smooth locus $(C/S)^{\text{sm}}$ is an isomorphism.
 - If p is a singular point with respect to $\underline{\pi}$, then there are an étale neighborhood U of \bar{p} , an affine étale neighborhood $\text{Spec} A$ of $\pi(\bar{p})$ in S , and a chart $((\mathbb{N}^2 \oplus_{\mathbb{N}} Q)_C \rightarrow M, Q_S \rightarrow N, Q \rightarrow \mathbb{N}^2 \oplus_{\mathbb{N}} Q)$ of π such that the induced map

$$U \rightarrow \text{Spec}(A \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[\mathbb{N}^2 \oplus_{\mathbb{N}} Q])$$

is étale, where $\mathbb{N} \rightarrow \mathbb{N}^2$ is the diagonal Δ , sending $e_1 \mapsto e_1 + e_2$.

- (3) The log structure of N is fine.

3.3. Remarks

Here are some remarks on the definitions above.

3.3.1. Since the homomorphism Δ and the monoid Q are integral, the monoid $\mathbb{N}^2 \oplus_{\mathbb{N}} Q$ and the homomorphism $\mathbb{N}^2 \oplus_{\mathbb{N}} Q \rightarrow Q$ are also integral by Proposition 4.1 in [14].

3.3.2. Let $Q \rightarrow N$ be a chart at $s := \pi(\bar{p})$, which induces an isomorphism $Q \rightarrow \overline{N_s}$. Such a chart is called good at s in the past literature. Then at any node \bar{p} of $C_{\bar{s}}$, étale locally f has a chart $((\mathbb{N}^2 \oplus_{\mathbb{N}} Q)_C \rightarrow \mathcal{O}_C, Q_S \rightarrow \mathcal{O}_S, Q \rightarrow \mathbb{N}^2 \oplus_{\mathbb{N}} Q)$ satisfying: the induced map $\mathcal{O}_S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[\mathbb{N}^2 \oplus_{\mathbb{N}} Q] \rightarrow \mathcal{O}_C$ is étale, and the induced map $\mathbb{N}^2 \oplus_{\mathbb{N}} Q \rightarrow \overline{M_{\bar{p}}}$ is an isomorphism. We can see this by working on the characteristics $Q = \overline{N_s}$ and $P = \overline{M_{\bar{p}}}$, as following. By the second definition of log prestable curves and the statement 1 in Lemma 3.3.1 below, we know that P is the amalgamation sum of

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{e_1 \mapsto e_1 + e_2} & \mathbb{N}^2 \\ \downarrow & & \\ Q & & \end{array}$$

Now we lift P and Q to make charts for M and N , using 2 and 3 in Lemma 3.3.1.

Lemma 3.3.1. *Let (M, α) be a fine log structure on a scheme X and let $x \in X$.*

1. *A homomorphism $\theta : P \rightarrow M_{\bar{x}}$ induces a chart étale locally at \bar{x} if and only if the induced homomorphism $P/(\alpha \circ \theta)^{-1}(\mathcal{O}_{\bar{x}}^{\times}) \rightarrow \overline{M_{\bar{x}}}$ is an isomorphism.*

2. *Suppose that $\theta : \overline{M_{\bar{x}}} \rightarrow M_{\bar{x}}$ is a lift of $M_{\bar{x}} \rightarrow \overline{M_{\bar{x}}}$, then θ provides a local chart of M at \bar{x} .*

3. (Proposition 2.1 [28]) *Let (X, M) be a fine log scheme, and let $x \in X$ such that the characteristic of the residue field of x is zero. There is a lift of $M_{\bar{x}} \rightarrow \overline{M_{\bar{x}}}$.*

Proof. 1. This is alluded in [14]. The “only if” part is clear. We prove the “if” part. It amounts to showing the natural homomorphism

$$P \oplus_{(\alpha \circ \theta)^{-1}(\mathcal{O}_{\bar{x}}^{\times})} \mathcal{O}_{\bar{x}}^{\times} \rightarrow M_{\bar{x}}$$

is an isomorphism. It is clearly surjective. For 1-1, let $\theta(p) + u = \theta(p') + u'$, where $(p, u), (p', u') \in M_{\bar{x}} \times \mathcal{O}_{\bar{x}}^{\times}$. Then $\overline{\theta(p)} = \overline{\theta(p')}$ in $\overline{M_{\bar{x}}}$ and so by assumption

$$(3.3.1) \quad p + v = p' + v'$$

for some $v, v' \in (\alpha \circ \theta)^{-1}(\mathcal{O}_{\bar{x}}^{\times})$. Here we abuse notation. Thus, $\theta(p) + v = \theta(p') + v'$. This together with $\theta(p) + u = \theta(p') + u'$ implies that $\theta(p) + v + u' = \theta(p') + v' + u' = \theta(p) + u + v'$, which in turn shows that

$$(3.3.2) \quad u' + v = u + v'.$$

By (3.3.1) and (3.3.2), $(p, u) \sim (p', u')$.

2. This follows from 1 since $(\alpha \circ \theta)^{-1}(\mathcal{O}_x^\times) = \{0\}$. Q.E.D.

3.4.

For a prestable curve C/S , there is a canonical log structure $M^{C/S}$ on C (resp. $N^{C/S}$ on S) induced from the log structure on the stacks $\mathfrak{M}_{g,1}$ (resp. \mathfrak{M}_g) of genus g , 1-pointed (resp. no-pointed) prestable curves ([25, 13]). Here the log structures on a smooth cover is given by the boundary divisors of singular curves and the log structures are defined on smooth topology ([28]). Thus, the natural log morphism $(C, M^{C/S}) \rightarrow (S, N^{C/S})$ is a log prestable curve. More generally, for a given homomorphism $N^{C/S} \rightarrow N$ from the canonical log structure $N^{C/S}$ to a fine log structure N on S , the amalgamated sum M defined to be the log structure associated to the prelog structure

$$M^{C/S} \oplus_{\pi^{-1}N^{C/S}} \pi^{-1}N$$

yields a log prestable curve over (S, N) . All log prestable curves are obtained in this manner according to Proposition 2.3 (and Remark 2.4) in [13] and Theorem 2.7 in [27].

3.5. 3rd definition

A pair $(C/S, N^{C/S} \rightarrow N)$ is called a *log prestable curve* over (S, N) if C/S is a prestable curve over S , and $N^{C/S} \rightarrow N$ is a homomorphism from the canonical log structure $N^{C/S}$ on S , induced from the family C/S , to a fine log structure N on S .

3.6. Special coordinates

Let C/S be a prestable curve; let p be a nodal point; $A := \mathcal{O}_{\pi(\bar{p})}$; and $R := \mathcal{O}_{\bar{p}}$ be the strict henselianization of $A[u, v]/(uv - t)$ at the ideal (u, v, \mathfrak{m}_A) , where $t \in \mathfrak{m}_A$.

Then we call (u, v) a *special coordinate pair* at the node p defined by the ideal (u, v, \mathfrak{m}_A) . Based on the coordinates, in fact, the canonical log structure is constructed. Locally define prelog structures $\mathbb{N}^2 \rightarrow \mathcal{O}_C$ by sending $e_1 \mapsto u$, $e_2 \mapsto v$ and define $\mathbb{N} \rightarrow \mathcal{O}_S$ by sending $e_1 \mapsto uv$. Since by Lemma 3.6.1 special coordinates are unique up to multiplications by unique elements in \mathcal{O}_C^\times whose product is in \mathcal{O}_S^\times , such prelog structures are isomorphic up to unique isomorphism. They can be glued together. See [13] for the detail of the construction of the canonical log structure on a (pre)stable curve C/S .

Lemma 3.6.1. ([13]) *Provided with the notation as in the beginning of 3.6, we have:*

1. Let u', v' be elements in R . Suppose that the ideals (u', v', \mathfrak{m}_A) and (u, v, \mathfrak{m}_A) coincide, and the product $u'v'$ is an element in A . Then there are $a, b \in R^\times$ such that $u' = au, v' = bv$ (or $u' = av, v' = bu$) and $ab \in A^\times$.

2. Suppose that $u^l = au^l$ and $v^l = bv^l$, where $l \in \mathbb{N}_{\geq 1}, a, b \in R^\times$. If $ab \in A^\times$, then $a = b = 1$.

Proof. The first statement and the second statement with $l = 1$ are proven in [13]. The proof of Lemma 2.2 in [13] for $l = 1$ works also for the general l since $R \subset \hat{R}$, where \hat{R} is the t -adic completion of R . Q.E.D.

For example, in the localization of $\Lambda[u, v]/(uv)$ at the ideal (u, v) , let $u' = (1 + u)u, v' = (1 + v)v$. Then we have $u' = (1 + u - \frac{v}{1+v})u$ and $v' = (1 + v - \frac{u}{1+u})v$, where the product $(1 + u - \frac{v}{1+v})(1 + v - \frac{u}{1+u}) = 1$.

Corollary 3.6.2. *Let l be a positive integer and let $\pi : (C, M) \rightarrow (S, N)$ be a log prestable curve. Then with the notation as in the beginning of Subsection 3.6, in $M_{\bar{p}}$ there is a unique pair γ_u, γ_v —which will be denoted by $l \log u$ and $l \log v$, respectively—such that $\gamma_u + \gamma_v \in N_{\pi(\bar{p})}$ and $\alpha(\gamma_u) = u^l, \alpha(\gamma_v) = v^l$, where α is the structure map $M_{\bar{p}} \rightarrow \mathcal{O}_{\bar{p}}$.*

Proof. The existence follows from the second definition of log prestable curves. If (γ'_u, γ'_v) is another such pair. Then it is clear that $\gamma'_u = \gamma_u$ and $\gamma'_v = \gamma_v$ in $\overline{M}_{\bar{p}}$. Hence, $\gamma'_u = \gamma_u + c_u, \gamma'_v = \gamma_v + c_v$, where $c_u, c_v \in \mathcal{O}_{\bar{p}}^\times$. Since $\gamma'_u + \gamma'_v, \gamma_u + \gamma_v \in N_{\pi(\bar{p})}$ and $\gamma'_u + \gamma'_v = \gamma_u + \gamma_v$ in $\overline{N}_{\pi(\bar{p})}$, we see that $c_u c_v \in \mathcal{O}_{\pi(\bar{p})}^\times$. Now we apply Lemma 3.6.1, to conclude the proof. Q.E.D.

3.7. A minimal log prestable curve

We call a log prestable curve

$$(C/S, N^{C/S} \rightarrow N)$$

minimal in weak sense ([35]) when N is locally free and there is no proper *locally free* submonoid of N containing the image of $N^{C/S}$. This amounts that for all $s \in S, \overline{N}_{\bar{s}}$ is a finitely generated free monoid and there is no proper free submonoid of $\overline{N}_{\bar{s}}$ containing

$$\text{Im}(\overline{N}_{\bar{s}}^{C/S} \rightarrow \overline{N}_{\bar{s}}).$$

It is called *minimal* in strong sense furthermore if for every irreducible $b \in \overline{N}_{\bar{s}}$, there is an irreducible element $a \in \overline{N}_{\bar{s}}^{C/S}$ such that $a = lb$ for some positive integer l . Since we will use only ‘minimal in strong sense’

in this paper, by ‘minimal’, we will mean ‘minimal in strong sense’ unless otherwise stated.

3.8. Remark

It is straightforward to check that the fibered categories $\mathfrak{M}_g^{\text{log}}$ of log prestable curves in either definitions, 1st one and 3rd one, are equivalent as following. Let us choose pullbacks once for all and then identify them, for example $(f \circ g)^* M = g^*(f^* M)$, if there exist canonical isomorphisms. A morphism, i.e., arrow, $(C', M')/(S', N') \rightarrow (C, M)/(S, N)$ is a pair (h, ϕ) , where $h : (C', M') \rightarrow (C, M)$ and $\phi : (S', N') \rightarrow (S, N)$ are morphisms between log schemes such that:

- The underlying scheme morphisms provide a cartesian square diagram

$$(3.8.1) \quad \begin{array}{ccc} C' & \xrightarrow{\quad} & C \\ & \searrow \underline{h} & \downarrow \pi \\ & & S \\ & \swarrow \underline{\phi} & \downarrow \pi \\ S' & \xrightarrow{\quad} & S. \end{array}$$

- $h^b : h^* M \rightarrow M'$ and $\phi^b : \phi^* N \rightarrow N'$ are isomorphisms making the diagram

$$\begin{array}{ccc} M' & \longleftarrow & \underline{h}^* M \\ \uparrow & & \uparrow \\ \underline{\pi}^* N' & \longleftarrow & \underline{h}^* \underline{\pi}^* N = (\underline{\pi}')^* \underline{\phi}^* N \end{array}$$

commute over log structure maps.

In view point of the third definition, a morphism $(C'/S', N^{C'/S'} \rightarrow N') \rightarrow (C/S, N^{C/S} \rightarrow N)$ is a pair $(\underline{h} : C' \rightarrow C, \underline{\phi} : (S', N') \rightarrow (S, N))$ such that \underline{h} and $\underline{\phi}$ give rise to a cartesian square diagram of underlying schemes as in (3.8.1) and the diagram

$$\begin{array}{ccc} \underline{\phi}^* N & \xrightarrow{\quad} & N' \\ & \searrow \underline{\phi}^b & \downarrow \\ & & N^{C'/S'} \\ \uparrow & & \uparrow \\ \underline{\phi}^* N^{C/S} & \xrightarrow{\quad} & N^{C/S} \end{array}$$

which commutes over log structure maps. Here the equality means the second part in the pair of the canonical isomorphisms $(\underline{h}^* M^{C/S},$

$\phi^* N^{C/S} \rightarrow (M^{C'/S'}, N^{C'/S'})$ between two pairs of canonical log structures on C'/S' . We will see later that the fibered category of log prestable curves is an algebraic stack over the category (Sch/Λ) of schemes over Λ .

§4. Log twisted FM type spaces

This section deals with the allowed targets of log stable maps, called log twisted FM type spaces.

4.1. (log) FM type spaces

An algebraic space W over a scheme S is called a *FM type space* if $W \rightarrow S$ is a projective, flat map whose geometric fibers are semi-stable varieties of form $xy = 0$, i.e., for every $s \in S$, étale locally, there is an étale map $W_{\bar{s}} \rightarrow \text{Spec}k(\bar{s})[x, y, z_1, \dots, z_{r-1}]/(xy)$ where x, y, z_i are independent variables with only one relation $xy = 0$.

Furthermore, if W/S allows a special log morphism, then we call it a *log FM type space*.

Here we say that for a FM type space W/S , a log smooth morphism

$$\pi : (W, M^{W/S}) \rightarrow (S, N^{W/S})$$

is *special* ([27]) if:

- $M^{W/S}$ and $N^{W/S}$ are locally free.
- For any $w \in W$, the induced map $\pi^* \overline{N}_{\bar{w}}^{W/S} \rightarrow \overline{M}_{\bar{w}}^{W/S}$ is either an isomorphism or the part of the cocatesian diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\Delta} & \mathbb{N}^2 \\ \downarrow h & \scriptstyle e_1 \mapsto e_1 + e_2 & \downarrow \\ \overline{N}_{\pi(\bar{w})}^{W/S} & \longrightarrow & \overline{M}_{\bar{w}}^{W/S} \end{array}$$

where $h(e_1)$ is an irreducible element.

- The natural map $W_{\bar{s}}^{\text{sing}} \rightarrow \text{Irr} \overline{N}_{\bar{s}}$ induced by the above diagram gives rise to a bijection

$$\text{Irr} W_{\bar{s}}^{\text{sing}} \rightarrow \text{Irr} \overline{N}_{\bar{s}},$$

where $\text{Irr} W_{\bar{s}}^{\text{sing}}$ is the set of irreducible components of the singular locus $W_{\bar{s}}^{\text{sing}}$ of $W_{\bar{s}}$ and $\text{Irr} \overline{N}_{\bar{s}}$ is the set of irreducible elements of $\overline{N}_{\bar{s}}$.

We will also call the special log structure the *canonical* one (since it is unique up to a unique isomorphism by Theorem 2.7 in [27]) and denote it by $M^{W/S}$ and $N^{W/S}$, respectively as in the definition above.

4.2.

As a generalization of log twisted curves ([31]) to a FM type space W/S we define the following.

A *log twisted FM type space* is a pair $(W/S, N^{W/S} \rightarrow N)$, where W/S is a log FM type space and $N^{W/S} \rightarrow N$ is a *simple* map from the canonical log structure $N^{W/S}$ to a locally free log structure N of S . By Lemma 3.3.1, this amounts that étale locally at $s \in S$ there is a commuting diagram of charts:

$$(4.2.1) \quad \begin{array}{ccc} N^{W/S} & \longrightarrow & N \\ \uparrow \theta^{W/S} & & \uparrow \theta \\ \mathbb{N}^m & \longrightarrow & \mathbb{N}^m, \end{array}$$

where the bottom map is a diagonal map between the constant sheaves \mathbb{N}^m and $\theta^{W/S}$ (resp. θ) induces an isomorphism from \mathbb{N}^m to $\overline{N}_s^{W/S}$ (resp. \overline{N}_s).

4.3.

An *extended log twisted FM type space* is a pair $(W/S, N^{W/S} \rightarrow N)$, where W/S is a log FM type space and $N^{W/S} \rightarrow N$ is an extended simple map from the canonical log structure $N^{W/S}$ to a log structure N of S . This means, as the definition, that étale locally there is a commuting diagram of charts:

$$(4.3.1) \quad \begin{array}{ccccc} N^{W/S} & \longrightarrow & & \longrightarrow & N \\ \uparrow & & & & \uparrow \\ \mathbb{N}^m & \longrightarrow & \mathbb{N}^m & \longrightarrow & \mathbb{N}^m \oplus \mathbb{N}^{m'}, \end{array}$$

where the first bottom map is a diagonal map, the second bottom map is the natural monomorphism $(\text{id}_{\mathbb{N}^m}, 0)$, and vertical maps induce isomorphisms between \mathbb{N}^m (resp. $\mathbb{N}^m \oplus \mathbb{N}^{m'}$) and $\overline{N}_s^{W/S}$ (resp. \overline{N}_s). In particular, N is a locally free log structure on S . We often write simply $(d, 0) : N^{W/S} \rightarrow N$ for (4.3.1) with the diagonal map being $d = (d_1, \dots, d_m)$.

We endow a log structure M on W by the amalgamated sum

$$(M^{W/S} \oplus_{\pi^{-1}N^{W/S}} \pi^{-1}N)^a.$$

Conversely, from Theorem 2.7 of [27] we arrive at another equivalent definition.

4.4.

An *extended log twisted FM type space* is a proper, log smooth, integral, vertical morphism $\pi : (W, M) \rightarrow (S, N)$ such that the underlying map $\pi : W \rightarrow S$ is a FM space; the log structure N is locally free; π^b is an isomorphism on the smooth locus of W/S ; and at a singular point of W/S , étale locally π has a chart:

$$\begin{array}{ccc}
 \pi^{-1}N & \longrightarrow & M \\
 \uparrow & & \uparrow \\
 Q := \mathbb{N}^{r-1} \oplus \mathbb{N} & \xrightarrow{(\text{id}, \phi)} & P := \mathbb{N}^{r-1} \oplus B
 \end{array}$$

where the monoid B is the amalgamated sum in

$$\begin{array}{ccc}
 \mathbb{N} & \xrightarrow[e_1+e_2]{\Delta} & \mathbb{N}^2 \\
 \times d \downarrow & & \downarrow \\
 \mathbb{N} & \xrightarrow{\phi} & B
 \end{array}$$

the induced map $\mathcal{O}_S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \rightarrow \mathcal{O}_W$ is smooth; and for every $s \in S$, the natural map $W_{\bar{s}}^{\text{sing}} \rightarrow \text{Irr}\bar{N}_{\bar{s}}$ induced by the above diagram gives rise to an injection

$$\text{Irr}W_{\bar{s}}^{\text{sing}} \rightarrow \text{Irr}\bar{N}_{\bar{s}}.$$

4.5. Log twisted FM spaces

For a nonsingular projective variety X over \mathbf{k} , let $X[n]$ be the Fulton–MacPherson configuration space of n labeled distinct points in X and let $X[n]^+$ be its universal space ([8]).

In [15], a Fulton–MacPherson degeneration space or in short, a *FM space* W of X over S is defined to be a pair $(W \rightarrow S, \pi_X : W \rightarrow X)$ of maps, where: W is an algebraic space over a scheme S ; and for every $s \in S$, there are an étale neighborhood T of \bar{s} , and a cartesian diagram

$$\begin{array}{ccc}
 W|_T & \longrightarrow & X[n]^+ \\
 \downarrow & & \downarrow \\
 T & \longrightarrow & X[n]
 \end{array}$$

for some n such that, through the diagram, π_X is compatible with the composite

$$X[n]^+ \longrightarrow X^{n+1} \xrightarrow{\text{pr}_{n+1}} X.$$

By the blowup construction of the universal family $X[n]^+$ from $X[n] \times X$, it is clear that $X[n]^+ \rightarrow X[n]$ is a log FM type space, and thus FM spaces are log FM type spaces.

There are two more examples constructed in [19]: $(X_D^{[n]})^+/X_D^{[n]}$ (the configuration space of n labeled points away from a smooth closed subvariety D in X) and $X_D[n]^+/X_D[n]$ (the configuration space of n labeled distinct points away from a smooth closed subvariety D in X). The constructions in [19] are valid over any closed field \mathbf{k} instead of \mathbb{C} , without any changes.

§5. Log stable maps

In this section, we define log stable maps and study their properties. We first need to explain what the underlying maps of the log stable maps are.

5.1. Admissible maps and stable unramified maps

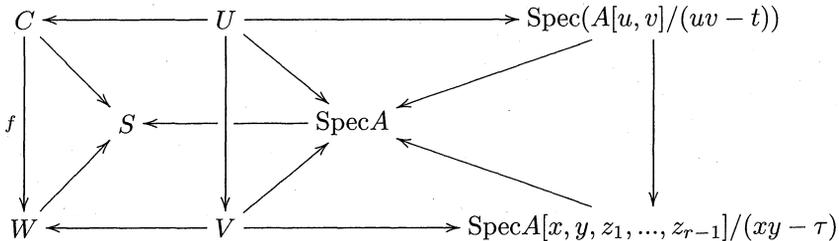
We recall some definitions in [21, 15].

5.1.1. A triple

$$(\star) \quad ((C/S, \mathbf{p}), W/S, f : C \rightarrow W)$$

is called a n -pointed, genus g , *admissible map* to a FM type space W/S if:

- (1) $(C/S, \mathbf{p} = (p_1, \dots, p_n))$ is a n -pointed, genus g , prestable curve over S .
- (2) W/S is a FM type space.
- (3) $f : C \rightarrow W$ is a map over S .
- (4) (*Admissibility*) If a point $p \in C$ is mapped into the relatively singular locus $(W/S)^{\text{sing}}$ of W/S , then étale locally at \bar{p} , f is factorized as



where all 5 horizontal maps are formally étale; u, v, x, y, z_i are indeterminates; $x = u^l, y = v^l$ under the far right vertical map for some positive integer l ; t, τ are elements in the maximal ideal \mathfrak{m}_A of the local ring A ; and \bar{p} is mapped to the point defined by the ideal (u, v, \mathfrak{m}_A) .

We call a node \bar{p} of $C_{\bar{s}}$ *distinguished* (resp. *nondistinguished*) if $f(\bar{p}) \in W_{\pi(\bar{p})}^{\text{sing}}$ (resp. $f(\bar{p}) \in W_{\pi(\bar{p})}^{\text{sm}}$), where π is the map $C \rightarrow S$.

5.1.2. *Remark* The admissibility can be stated as below, too. If $p \in f^{-1}((W/S)^{\text{sing}})$, then under $f, x = c_1 u^l, y = c_2 v^l$ for some $c_i \in R^\times$ such that $c_1 c_2 \in A^\times$ where $A := \mathcal{O}_{\pi(\bar{p})}$; $R := \mathcal{O}_{\bar{p}}$ is form of the strict henselianization of $A[u, v]/(uv - t)$ at the ideal (u, v, \mathfrak{m}_A) ; $\mathcal{O}_{f(\bar{p})}$ is form of the strict henselianization of $A[x, y, z_1, \dots, z_{r-1}]/(xy - \tau)$ at the ideal $(x, y, z_1, \dots, z_{r-1}, \mathfrak{m}_A)$; and t, τ are in the maximal ideal \mathfrak{m}_A of A .

Two definitions are equivalent: One direction is clear. We prove the other direction. Consider polynomials $w^l - c_i$ in $R[w]$ where w is an indeterminate. Each polynomial has a linear coprime factorization over R/\mathfrak{m}_R . Since R is a henselian ring, the linear factorization can be lifted over R , providing a solution $c_i^{1/l} \in R^\times$ to $w^l = c_i$. Now we have $x = (u')^l$ and $y = (v')^l$, where $u' = c_1^{1/l} u, v' = c_2^{1/l} v$. On the other hand, note that $u'v' \in A$ since $(u'v')^l = xy = \tau \in A$. Therefore, we can apply Lemma 3.6.1 to replace $c_i^{1/l}$ by elements $a_i \in R^\times$ such that $a_1 a_2 \in A^\times$. Finally we conclude that $u'v' \in \mathfrak{m}_A$.

5.1.3. Let a FM type space W/S be equipped with a map $\pi_X : W \rightarrow X$ from W to a scheme X . An admissible map (\star) is called a *stable admissible map* to a FM type space W/S of X if the *stability with respect to π_X* holds, namely: The automorphism group $\text{Aut}_X(f_{\bar{s}})$ is finite for all $s \in S$, where $\text{Aut}_X(f_{\bar{s}})$ consists of all pairs (h, φ) of automorphisms h of $C_{\bar{s}}$ preserving n -labeled points and automorphisms φ of $W_{\bar{s}}$ with respect X such that they are compatible with $f_{\bar{s}}$, i.e., $h(\mathbf{p}_{\bar{s}}) = \mathbf{p}_{\bar{s}}, \pi_X \circ \varphi = \pi_X$, and $f_{\bar{s}} \circ h = \varphi \circ f_{\bar{s}}$.

5.1.4. Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}_{\geq 1}^n$. A stable admissible map (\star) is called a *stable μ -ramified map* to a FM space of a smooth projective scheme X over \mathbf{k} if:

- W/S is a FM space of X .
- For all $s \in S, f(p_i)_{\bar{s}}$ are pairwise distinct for all i .
- (*Strong Stability*) For all $s \in S, \text{Aut}_X(f_{\bar{s}})$ is a finite group. Furthermore, every end component of $W_{\bar{s}}$ contains either a non-line image of an irreducible component of $C_{\bar{s}}$ or the images of at least two labeled points.

- $f((C/S)^{\text{sing}}) \subset (W/S)^{\text{sing}}$ and f is unramified everywhere on the relatively smooth locus $(C/S)^{\text{sm}}$, possibly except at the labeled points.
- The ramification order at p_i is exactly μ_i .

Here the end components of $W_{\bar{s}}$ are the screen components which correspond the end nodes of the dual graph of $W_{\bar{s}}$. See [15] for the precise definition.

When $\mu = (1, \dots, 1)$, we call the map a *stable unramified map*. Here we follow the ramification order convention that ramification order 2 means the simple ramification.

5.2. Log stable maps

Combining all previous notions, we introduce a series of definitions.

5.2.1. A pair $((C, M)/(S, N), \mathbf{p})$ is called a n -pointed, genus g , (resp. minimal) log prestable curve over (S, N) if $(C, M)/(S, N)$ is a genus g (resp. minimal) log prestable curve and $(C/S, \mathbf{p})$ is a n -pointed prestable curve over S .

5.2.2. A log morphism

$$(\star\star) \quad (f : (C, M_C, \mathbf{p}) \rightarrow (W, M_W)) / (S, N)$$

is called a (g, n) log prestable map over (S, N) if:

- (1) $((C, M)/(S, N), \mathbf{p})$ is a n -pointed, genus g , minimal log prestable curve.
- (2) $(W, M_W)/(S, N)$ is an extended log twisted FM type space.
- (3) (*Corank = # Nondistinguished Nodes Condition*) For every $s \in S$, the rank of $\text{Coker}(N_{\bar{s}}^{W/S} \rightarrow N_{\bar{s}})$ coincides with the number of nondistinguished nodes on $C_{\bar{s}}$.
- (4) $f : (C, M_C) \rightarrow (W, M_W)$ is a log morphism over (S, N) .
- (5) (*Log Admissibility*) either of the following conditions, equivalent under the above four conditions, holds:
 - f is admissible.
 - $f^b : f^*M_W \rightarrow M_C$ is simple at every distinguished node.

5.2.3. *Log Admissibility* We want to see the explicit meaning of the last condition (5) in 5.2.2 under the rest conditions imposed. Provided with the notation in the admissible condition in 5.1.1, there exist $\log x$ and $\log y$ in $(M_W)_{f(\bar{p})}$ such that $\log x + \log y \in N_{\bar{s}}$, $\alpha_W(\log x) = x$, and $\alpha_W(\log y) = y$, where $\bar{p} \mapsto \bar{s}$ under $C \rightarrow S$, and $\alpha_W : M_W \rightarrow \mathcal{O}_W$ is the log structure map. Then $f^b(\log x)$ and $f^b(\log y)$ must be the $l \log u$ and the $l \log v$, respectively, since the pair $(f^b(\log x), f^b(\log y))$ satisfies the assumption in Corollary 3.6.2. Therefore, at a distinguished node p , the

log prestable map is described as: at chart levels there is a diagram

$$\left(\begin{array}{ccc} \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N} & \xrightarrow{\log x, \log y \mapsto l \log u, l \log v} & \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N} \\ & \swarrow e_1 \mapsto (0, e_1) & \nearrow e_1 \mapsto (0, e_1) \\ & \mathbb{N} & \end{array} \right) \oplus \mathbb{N}^{m+m'-1}$$

for some positive integer l , where LHS and RHS amalgamated sums are given by

$$\begin{array}{ccc} \mathbb{N} \ni e_1 & \longmapsto & de_1 \in \mathbb{N} \quad ; \quad \mathbb{N} \ni e_1 \longmapsto \Gamma e_1 \in \mathbb{N} \\ \downarrow & & \downarrow \\ \mathbb{N}^2 \ni e_1 + e_2 & & \mathbb{N}^2 \ni e_1 + e_2 \end{array}$$

for some positive integers d, Γ , respectively.

Note also that there is a natural map $\text{Irr} \overline{N}_{\bar{s}}^{C/S} \rightarrow \text{Irr} \overline{N}_{\bar{s}}$ by sending a to b if $\Gamma b = a$ under the map $\overline{N}_{\bar{s}}^{C/S} \rightarrow \overline{N}_{\bar{s}}$ for some positive integer Γ .

Summing these observations together, we claim that if f is a log prestable map then, étale locally at every geometric point $\bar{s} \rightarrow S$, there are commuting charts

$$(5.2.1) \quad \begin{array}{ccccc} \mathbb{N}^{m''} \oplus \mathbb{N}^{m'} & \xrightarrow{(\Gamma, \text{id})} & \mathbb{N}^m \oplus \mathbb{N}^{m'} & \xleftarrow{(d, 0)} & \mathbb{N}^m \\ \downarrow & & \downarrow & & \downarrow \\ N^{C/S} & \xrightarrow{\quad} & N & \xleftarrow{\quad} & N^{W/S} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{N}^{C/S} & \xrightarrow{\quad} & \overline{N} & \xleftarrow{\quad} & \overline{N}^{W/S} \end{array}$$

such that the vertical maps induce isomorphisms between chart monoids and the characteristics at \bar{s} , where $m'' =$ the number of distinguished nodes in $C_{\bar{s}}$; $m' =$ the number of nondistinguished nodes in $C_{\bar{s}}$; $m =$ the number of irreducible components of $W_{\bar{s}}^{\text{sing}}$; Γ is a ‘generalized diagonal’ matrix of size $m \times m''$

$$\Gamma = \left(\begin{array}{ccc|ccc|ccc} \Gamma_{1,1} \cdots \Gamma_{1,k_1} & & & 0 \cdots 0 & & \cdots & & 0 \cdots 0 \\ 0 \cdots 0 & & & \Gamma_{2,1} \cdots \Gamma_{2,k_2} & & 0 \cdots 0 & & 0 \cdots 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 \cdots 0 & & & \cdots & & \cdots 0 & & \Gamma_{m,1} \cdots \Gamma_{m,k_m} \end{array} \right) ;$$

and d is a diagonal (d_1, \dots, d_m) . For the existence of the diagram, first note that we have such maps (Γ, id) and d at the characteristic level due to: the simplicity of $\overline{N} \leftarrow \overline{N}^{W/S}$; the simplicity of f at distinguished nodes; and the minimality of the log prestable curve at nondistinguished nodes. Now to lift such maps at chart levels, start with a chart in the middle and then build the rest of the charts.

The minimality at distinguished nodes means that there is no non-trivial common divisors of positive integers $\Gamma_{i,k_1}, \dots, \Gamma_{i,k_i}$ for all $i = 1, \dots, m$. Let $\{p_{i,j}\}$ be the set of distinguished nodes on $C_{\overline{s}}$, which are mapped into the i -th component of $W_{\overline{s}}^{\text{sing}}$. Since $d_i = l_{i,j}\Gamma_{i,j}$, d_i must be the least common multiple of $l_{i,j}, \forall j$, where $l_{i,j} = \overline{f}_{p_{i,j}}^b$ comes from the positive multiplication map

$$\overline{f}_{p_{i,j}}^b : \mathbb{Z} \cong \overline{f^*(M_W/\pi^*N)}_{p_{i,j}} \rightarrow \mathbb{Z} \cong \overline{M_C/\pi^*N}_{p_{i,j}}$$

induced by f^b .

5.2.4. Let a FM type space W/S be equipped with a map $\pi_X : W \rightarrow X$ from W to a scheme X . A log prestable map $(\star\star)$ is called a *log stable map* if the stability condition holds, i.e., for each $s \in S$, the group of automorphisms (h, φ) is finite where:

- h is an automorphism of $((C, M_C)/(S, N))_{\overline{s}}$ preserving n -labeled points $\mathbf{p}_{\overline{s}}$.
- φ is an automorphism of $((W, M_W)/(S, N))_{\overline{s}}$ preserving $W_{\overline{s}} \rightarrow X$.
- $\varphi \circ f_{\overline{s}} = f_{\overline{s}} \circ h$.

We will see in 6.2.4 that the above stability holds if and only if the underlying automorphism stability holds. That is, \underline{f} is stable if and only if f is stable.

5.2.5. Let $\beta \in A_1(X)/\sim^{\text{alg}}$. A log prestable map $(\star\star)$ is called a (g, n, β) *log stable map to a FM space W/S of a smooth projective k variety X* if:

- W/S is a FM space of X over S .
- Stability Condition holds.
- $(\pi_X \circ f)_*[C_{\overline{s}}] = \beta$ for every $s \in S$.

5.2.6. Fix $\mu = (\mu_1, \dots, \mu_n)$. A log stable map $(\star\star)$ to a FM space W/S of X is called a *log stable μ -ramified map* if \underline{f} is stable μ -ramified map over S as in 5.1.4.

When $\mu = (1, \dots, 1)$, we call it a *log stable unramified map*. Note that in this case, indeed, f is *log unramified*, that is, $f^*\Omega_{W/S}^\dagger \rightarrow \Omega_{C/S}^\dagger$ is surjective at every $p \in C$.

5.3. Remarks

5.3.1. Every usual stable map to a smooth projective \mathbf{k} variety X from a prestable curve can be an underlying stable map of a unique log stable map.

5.3.2. Suppose that a log prestable curve $(C, M_C)/(S, N)$; an extended twisted FM space $(W, M_W)/(S, N)$; and an admissible map $f : C/S \rightarrow W/S$ are given. Then there is a unique locally closed subscheme S' of S where f is a log prestable map over $(S', N_{|_{S'}})$ satisfying the universal property: If $Z \rightarrow S$ is a scheme morphism such that $f_{|_Z} : (C_{|_Z}, (M_C)_{|_Z}) \rightarrow (W_{|_Z}, (M_W)_{|_Z})$ is a log prestable map over $(Z, N_{|_Z})$ if and only if $Z \rightarrow S$ uniquely factors as $Z \rightarrow S' \rightarrow S$. Here, for example, $N_{|_Z}$ is the log structure of the pullback of N under $Z \rightarrow S$.

First of all the above statement makes sense because of 5.2.3 where we have seen that f^b is determined by \underline{f} and log structures on C/S and W/S . The minimality and the corank = # nondistinguished node condition are open conditions. Let V be the open subscheme of S where these conditions are satisfied. Then we need to consider only the condition that f becomes a log morphism. Since $f^b(\log x) = l \log u$ and $f^b(\log y) = l \log v$, the condition becomes the equality $\log x + \log y = l \log u + l \log v$ in N , which defines a locally closed subscheme Z in V by the following reason.

Let M be a fine log structure on a scheme Y_2 and φ be a scheme morphism from a scheme Y_1 to Y_2 . Then $\varphi^*M = \varphi^{-1}(\overline{M})$. This together with $M_{\overline{y}} \cong \overline{M}_{\overline{y}} \oplus \mathcal{O}_{\overline{y}}^\times$, $y \in Y_2$, implies that the requirement of the ‘equality’ of two sections of M defines a closed subscheme of an open subscheme in Y_2 .

§6. Stacks

In this section, we show that the moduli stacks of log stable maps are log algebraic (resp. proper DM) stacks over Λ whenever the corresponding stacks of underlying stable maps are algebraic (resp. proper DM) over Λ . Since some stacks considered here are not separated, we use terminology algebraic stacks.

6.1. Log stacks

Following [13], we introduce log stacks and examples.

6.1.1. Let (Sch/Λ) be the category of locally noetherian schemes over Λ . A pair (\mathcal{F}, L) is called a *log stack* if: \mathcal{F} is a stack over (Sch/Λ) , and L is a functor from \mathcal{F} to the category LOG of fine log schemes over

Λ , whose morphisms are strict log morphisms, making the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & (\text{Sch}/\Lambda) \\ L \downarrow & \nearrow \text{forgetful} & \\ \text{LOG} & & \end{array}$$

commute. Furthermore when \mathcal{F} is an algebraic stack, it is called a log algebraic stack. Here a stack \mathcal{F} over (Sch/Λ) is said to be *algebraic* if the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times_{(\text{Sch}/\Lambda)} \mathcal{F}$ is representable and of finite presentation (see §4 in [7] for the definition), and it allows a smooth cover by a scheme.

A log scheme (X, M) can be considered as a log stack (h_X, L_M) , where $h_X(Y) = \text{Hom}_{(\text{Sch}/\Lambda)}(Y, X)$ and $L_M(f : Y \rightarrow X) = (Y, f^*M)$.

6.1.2. For a fine log scheme Y , \mathcal{L}_Y will denote the stack of fine log schemes over Y : The objects are fine log schemes over the log scheme Y , and morphisms from Z/Y to Z'/Y are strict log morphisms $h : Z \rightarrow Z'$ over Y . The log stack \mathcal{L}_Y was denoted by Log_Y in [28]. To make notation easy, we use the symbol \mathcal{L}_Y .

For a log stack (\mathcal{F}, L) , $\mathcal{L}_{\mathcal{F}}$ is defined as: An object is a pair $(x \in \mathcal{F}(X), (X, M) \leftarrow L(x))$, where $\mathcal{F}(X)$ is the collection of objects of \mathcal{F} over X , and the arrow is a (possibly non-strict) log morphism. A morphism from $(x \in \mathcal{F}(X), (X, M) \leftarrow L(x))$ to $(y \in \mathcal{F}(Y), (Y, N) \leftarrow L(y))$ is a pair $(x \rightarrow y, (X, M) \rightarrow (Y, N))$, where the first arrow is a morphism in the stack \mathcal{F} , and the second arrow is a strict log morphism for which the diagram

$$\begin{array}{ccc} (X, M) & \longrightarrow & (Y, N) \\ \uparrow & & \uparrow \\ L(x) & \xrightarrow{L(x \rightarrow y)} & L(y) \end{array}$$

commutes.

Note that for a log scheme (X, M) , $\mathcal{L}_{(h_X, L_M)}$ is equivalent to $\mathcal{L}_{(X, M)}$ as log stacks.

Lemma 6.1.1. *If a log stack \mathcal{F} is algebraic, then $\mathcal{L}_{\mathcal{F}}$ is algebraic.*

Proof. The following argument is due to Olsson. Take a smooth cover of the algebraic stack \mathcal{F} by a scheme Z . Then we obtain the diagram

$$\begin{array}{ccc} \mathcal{L}_{\mathcal{F}} & \longrightarrow & \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{L}_{\mathcal{F}} \times_{\mathcal{F}} Z & \longrightarrow & Z. \end{array}$$

Note that the fibered category $\mathcal{L}_{\mathcal{F}} \times_{\mathcal{F}} Z$ is equivalent to \mathcal{L}_Z , where Z is endowed with the log structure induced from the cover. Since \mathcal{L}_Z is an algebraic stack due to [28], we can apply Lemma 6.1.2 to the diagram to conclude that $\mathcal{L}_{\mathcal{F}}$ is an algebraic stack. Q.E.D.

Lemma 6.1.2. ([3]) *Let \mathcal{F} be a stack. Suppose that there are: a morphism from \mathcal{F} to an algebraic stack \mathcal{G} and a smooth surjective map from a scheme U to \mathcal{G} , for which $\mathcal{F} \times_{\mathcal{G}} U$ is algebraic. Then \mathcal{F} is algebraic.*

6.2. Stacks of log prestable curves and extended log twisting FM type spaces

6.2.1. As usual, we can define the category $\mathfrak{M}_g^{\text{log}}$ of genus g , log prestable curves over (Sch/Λ) . Then using the inverse image of log structures, we see that the category is fibered in groupoids over (Sch/Λ) . Furthermore, using the gluing of sheaves, it is a stack. In fact, the stack $\mathfrak{M}_g^{\text{log}}$ is equivalent to $\mathcal{L}_{\mathfrak{M}_g}$. Hence it is a log algebraic stack.

Let $\mathfrak{M}_{g,n}^{\text{log}}$ be the algebraic stack $\mathfrak{M}_g^{\text{log}} \times_{\mathfrak{M}_g} \mathfrak{M}_{g,n}$ so that there is no interesting log structures on markings.

6.2.2. We consider a stack \mathcal{B} of certain $(W/S, W \rightarrow X)$, where W/S are log FM type spaces, and $W \rightarrow X$ are maps from W to a fixed X . A morphism between them is a cartesian diagram preserving X . By definition in 4.1, any object W/S in \mathcal{B} can be realized as an underlying space of an extended log twisted FM type space. Suppose that \mathcal{B} is an algebraic stack.

Since \mathcal{B} is a log stack by the canonical log structures, we can consider $\mathcal{L}_{\mathcal{B}}$. Let \mathcal{B}^{tw} (resp. \mathcal{B}^{etw}) be the full substack of $\mathcal{L}_{\mathcal{B}}$ whose objects are (resp. extended) log twisted FM type spaces whose underlying spaces are in \mathcal{B} . By Lemma 3.3.1, they are open substacks of the algebraic stack $\mathcal{L}_{\mathcal{B}}$, and hence they are also algebraic stacks.

Assume that there is a smooth scheme B over Λ and a smooth morphism $B \rightarrow \mathcal{B}$, defined by a ‘universal’ family U/B . Then, \mathcal{B}^{tw} and \mathcal{B}^{etw} are smooth over Λ . Indeed, we can formulate a smooth versal space as following. Let $W/k(\bar{p})$ be the pullback of U at a point $p \in B$. Then, a formal versal space of \mathcal{B}^{etw} at $W/k(\bar{p})$ with an extended log twisting (4.3.1) is

$$\text{Spf } \hat{\mathcal{O}}_{\bar{p}}[[x_1, \dots, x_m, y_1, \dots, y_{m'}]] / (x_1^{d_1} - \tau_1, \dots, x_m^{d_m} - \tau_m),$$

where x_i, y_j are indeterminates;

$$(d_1, \dots, d_m, \underbrace{0, \dots, 0}_{m'}) : \overline{N}_{\bar{p}}^{U/B} \rightarrow \overline{N}_{\bar{p}};$$

$\hat{\mathcal{O}}_{\bar{p}} \ni \tau_i$ are $\alpha^{U/B}(e_i)$ for a chart $\mathbb{N}^m \rightarrow N^{U/B}$ at \bar{p} ; and $m =$ the number of $\text{Irr}U_{\bar{p}}^{\text{sing}}$.

6.2.3. Note that $\mathfrak{M}_{g,n}^{\log} \times_{\text{LOG}} \mathcal{B}^{\text{etw}}$ is equivalent to the stack whose objects are pairs of log prestable curves $(C, M_C)/(S, N)$ and extended log twisting FM type spaces $(W, M_W)/(S, N)$. Since $\mathfrak{M}_{g,n}^{\log} \times_{\text{LOG}} \mathcal{B}^{\text{etw}}$ is a fiber product of algebraic stacks over an algebraic stack, it is algebraic. It is formally smooth over Λ , because the projection $\mathfrak{M}_{g,n}^{\log} \times_{\text{LOG}} \mathcal{B}^{\text{etw}} \rightarrow \mathcal{B}^{\text{etw}}$ is smooth by Proposition 3.14 in [14], and \mathcal{B}^{etw} is formally smooth over Λ .

6.2.4. Consider a log twisting (4.2.1); assume that the charts are global charts over S . Then the automorphism functor

$$\text{Aut}_{W/S}(W/S, N \xleftarrow{d} N^{W/S})$$

over W/S , i.e., fixing W/S , is representable by

$$\text{Spec} \mathcal{O}_S[z_1^{\pm 1}, \dots, z_m^{\pm 1}] / (z_i^{d_i} - 1, z_i \alpha(e_i) - \alpha(e_i))$$

over S , where $\alpha : N \rightarrow \mathcal{O}_S$ is the structure map of the log structure. Similarly,

$$\text{Aut}_{(C/S, W/S)}(C/S, W/S, N^{C/S} \xrightarrow{(\Gamma, \text{id})} N \xleftarrow{(d, 0)} N^{W/S}),$$

fixing all underlying scheme structures, is representable by

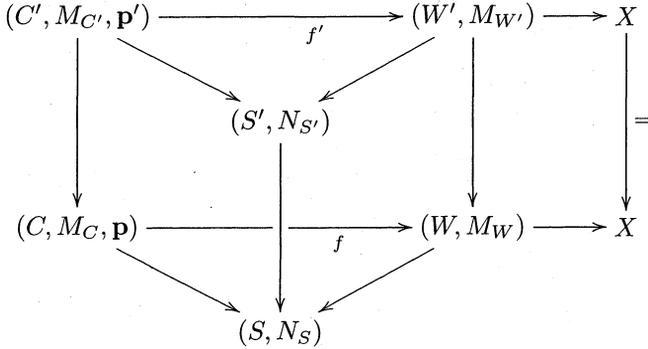
$$\text{Spec} \mathcal{O}_S[z_1^{\pm 1}, \dots, z_m^{\pm 1}] / (z_i^{G_i} - 1, z_i \alpha(e_i) - \alpha(e_i))$$

over S , where log twistings are given as in (5.2.1) with the global charts over S , and $G_i = \text{GCD}(\Gamma_{i,j}, \forall j)$.

6.3. The stack of log stable maps

Denote by \mathcal{U} be the stack of all pairs $(W/S, \sigma)$, where W/S is an object in \mathcal{B} and σ is a section of $W \rightarrow S$. The arrows are morphisms in \mathcal{B} preserving sections. Define the category $\overline{M}_{g,n}^{\log}(\mathcal{U}/\mathcal{B})$ of (g, n) log stable maps to FM type spaces in the stack \mathcal{B} . A morphism from $((f : (C', M_{C'}, \mathbf{p}') \rightarrow (W', M_{W'})) / (S', N_{S'}))$ to $((f : (C, M_C, \mathbf{p}) \rightarrow$

$(W, M_W)/(S, N_S)$ is a commutative diagram



where two side squares with the edge $(S', N_{S'}) \rightarrow (S, N_S)$ are fiber products in log sense preserving labeled points, and the log structure $N_{S'}$ of S' is naturally isomorphic to the pullback of the log structure N_S of S . Then it is a log stack over (Sch/Λ) ; there is a natural map

$$\overline{M}_{g,n}^{\text{log}}(\mathcal{U}/\mathcal{B}) \rightarrow \mathfrak{M}_{g,n}^{\text{log}} \times_{\text{LOG}} \mathcal{B}^{\text{etw}}.$$

Similarly define the stack $\overline{M}_{g,n}(\mathcal{U}/\mathcal{B})$ by taking off all log structures. There is a natural map

$$\overline{M}_{g,n}(\mathcal{U}/\mathcal{B}) \rightarrow \mathfrak{M}_{g,n} \times_{(\text{Sch}/\Lambda)} \mathcal{B}.$$

Note that since f^b is uniquely determined by f and log structures on its source and target, as seen in 5.2.3, $\overline{M}_{g,n}^{\text{log}}(\mathcal{U}/\mathcal{B})$ is a full substack of

$$\overline{M}_{g,n}(\mathcal{U}/\mathcal{B}) \times_{(\mathfrak{M}_{g,n} \times \mathcal{B})} (\mathfrak{M}_{g,n}^{\text{log}} \times_{\text{LOG}} \mathcal{B}^{\text{etw}}).$$

In what follows, a DM stack over Λ means an algebraic stack over Λ for which the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times_{(\text{Sch}/\Lambda)} \mathcal{F}$ is separated and unramified.

Theorem 6.3.1. *If $\overline{M}_{g,n}(\mathcal{U}/\mathcal{B})$ is an algebraic stack (resp. a proper DM stack) over Λ , then so is $\overline{M}_{g,n}^{\text{log}}(\mathcal{U}/\mathcal{B})$.*

Proof. Since $\overline{M}_{g,n}(\mathcal{U}/\mathcal{B})$ is a full substack of an algebraic stack, the isomorphism functors are representable and of finite presentation. Now, the algebraic stack part of the statement follows from Remark 5.3.2. The properness follows from that of $\overline{M}_{g,n}(\mathcal{U}/\mathcal{B})$ and Lemma 6.3.2 since $\overline{M}_{g,n}^{\text{log}}(\mathcal{U}/\mathcal{B})$ is of finite presentation over Λ . Due to the finiteness of automorphisms, the diagonal is unramified. Q.E.D.

Lemma 6.3.2. *Let R be a DVR with an algebraically closed residue field; let $(W/S, W \rightarrow X)$ be a log FM type space, where $S = \text{Spec}R$; and let C/S be a prestable curve over S . Suppose that a log stable map $f_\xi : (C_\xi, M_{C_\xi}) \rightarrow (W_\xi, M_{W_\xi})$ over (ξ, N_ξ) is given, where ξ is the generic point of S . Assume that there is a stable admissible map $\underline{f} : C \rightarrow W$ over S , extending the underlying map f_ξ . Then there exists a unique pair of a minimal log prestable curve $(\bar{C}, M_{\bar{C}})/(S, N)$ and an extended log twisted FM type space $(W, M_W)/(S, N)$, extending $(C_\xi, M_{C_\xi})/(\xi, N_\xi)$ and $(W_\xi, M_{W_\xi})/(\xi, N_\xi)$, respectively, such that the stable admissible map \underline{f} becomes a log morphism over (S, N) .*

$$\begin{array}{ccc}
 (C_\xi, M_{C_\xi}) & \xrightarrow{f_\xi} & (W_\xi, M_{W_\xi}) \\
 & \searrow & \swarrow \\
 & (\xi, N_\xi) &
 \end{array}
 , \quad
 \begin{array}{ccc}
 C & \xrightarrow{\underline{f}} & W \\
 & \searrow & \swarrow \\
 & S &
 \end{array}$$

Proof. Let p be the closed point of S . We use the following index sets:

- I_1 (resp. $I_1 \amalg I_2$) = an index set for the components of the singular locus of W at $\bar{\xi}$ (resp. p).
- I'_1 (resp. $I'_1 \amalg I'_2$) = an index set for the nondistinguished nodes of C at $\bar{\xi}$ (resp. p).
- I''_1 (resp. $I''_1 \amalg I''_2$) = an index set for the distinguished nodes of C at $\bar{\xi}$ (resp. p).

Then by 5.2.3 we may assume that the following compatible charts are given:

$$\begin{array}{ccccc}
 N_{\bar{\xi}}^{C/S} & \longrightarrow & N_{\bar{\xi}} & \longleftarrow & N_{\bar{\xi}}^{W/S} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{N}^{I''_1+I'_1} & \xrightarrow{(\Gamma, \text{id})} & \mathbb{N}^{I_1+I_2} & \xleftarrow{(d^{(1)}, 0)} & \mathbb{N}^{I_1}
 \end{array}$$

where $d^{(1)}$ is a monoid homomorphism from $\mathbb{N}^{I''_1}$ to \mathbb{N}^{I_1} , and the sums of index sets is used for the disjoint unions of them, for simplicity.

For $i \in I''_2$, let (u_i, v_i) be a special coordinate pair at the node p_i corresponding to i , and let (x_i, y_i) be the part of coordinates at $f(p_i)$ as in the admissible condition so that $x_i = u_i^{l_i}$ and $y_i = v_i^{l_i}$ under f . For $j \in I_2$, define $d_j^{(2)} = \text{LCM}(l_i, \forall i \mapsto j)$, where $i \mapsto j$ means that $f(p_i)$ is in the component of W_p^{sing} corresponding to $j \in I_2$.

First define a prelog structure

$$\alpha : P := \mathbb{N}^{I_1+I'_1+I_2+I'_2} \rightarrow \mathcal{O}_S,$$

where α is determined by

$$\alpha(e_j) = \begin{cases} 0 & j \in I_1 + I'_1 \\ (\alpha^{W/S}(e_j))^{1/d_j^{(2)}} & j \in I_2 \\ \alpha^{C/S}(e_j) & j \in I'_2 \end{cases}$$

where $\alpha^{W/S}$ (resp. $\alpha^{C/S}$) is the structure map from $N^{W/S}$ (resp. $N^{C/S}$) to \mathcal{O}_S , and $(\alpha^{W/S}(e_j))^{1/d_j^{(2)}}$ is a root (we choose a choice of the roots) whose $d_j^{(2)}$ -th power is $\alpha^{W/S}(e_j)$. Set $N = P^a$, the log structure associated to P . Now we construct a log prestable curve structure on C/S and an extended log twisting on W/S . It amounts to establishing certain log morphisms $N^{C/S} \rightarrow N \leftarrow N^{W/S}$. We define them by homomorphisms between their charts as in diagram

$$\begin{array}{ccccc} N^{C/S} & \xrightarrow{\quad\quad\quad} & N & \xleftarrow{\quad\quad\quad} & N^{W/S} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{N}^{I''_1+I'_2+I'_1+I'_2} & \xrightarrow{(\Gamma_1, \Gamma_2, \text{id}, \text{id})} & \mathbb{N}^{I_1+I_2+I'_1+I'_2} & \xleftarrow{(d^{(1)}, d^{(2)}, 0, 0)} & \mathbb{N}^{I_1+I_2} \end{array}$$

where $\Gamma_2(e_i) = \frac{d_i^{(2)}}{i} e_j$ for $i \in I'_2$, $i \mapsto j$, and $d^{(2)} = (d_j^{(2)})_{j \in I_2}$. The existence part of Lemma is verified.

When N is defined, the ambiguity occurs only in the choices of roots $\alpha^{W/S}(e_j)^{1/d_j^{(2)}}$. Let N' be a constructed one, using another choice. Then there is a unique isomorphism $N \rightarrow N'$ commuting with maps from $N^{C/S}$ and $N^{W/S}$: The isomorphism is determined by

$$\begin{aligned} N &\rightarrow N' \\ e_j &\mapsto e_j + \alpha(e_j)/\alpha'(e_j), \end{aligned}$$

where α' is the log structure map of N' , and $j \in I_2$ so that $\alpha(e_j)/\alpha'(e_j) \in \mathcal{O}_S^\times$. Q.E.D.

6.3.1. *Remarks* 1. Assume that the stability condition of admissible maps to U/B , with respect to π_X (i.e., fixing X), is an open condition. Then we claim that $\overline{M}_{g,n}(U/B)$ is a DM stack over Λ by the following argument. Consider the stack $\overline{M}_{g,n}(U/B; X)$ of admissible maps to the rigid target U/B , stable with respect to π_X , which is a full substack of the

Kontsevich moduli space $\overline{M}_{g,n}(U)$ of (g, n) stable maps to U . Then the stack is a DM stack over Λ by Theorem 2.11 in [21]. This in turn shows that $\overline{M}_{g,n}(\mathcal{U}/\mathcal{B})$ is a DM stack over Λ . Here the quasi-separatedness can be directly shown.

2. The explicit description in 5.2.3 shows that when S is a geometric point Speck , then for an underlying admissible map f , the number of isomorphism classes of log prestable maps realizing f is finite. The numerical data l, d, Γ are uniquely determined. What remain undetermined are maps from $N^{W/S}$ and $N^{C/S}$ to N . The choices, however, are finite since f is a log morphism. We illustrate the reason by an example, for simplicity. Suppose that the ranks of $N^{W/S}$ and N are 1 so that we may write a map from $N^{W/S} = \mathbb{N} \oplus \mathbf{k}^\times$ to $N = \mathbb{N} \oplus \mathbf{k}^\times$ by $e_1 \mapsto de_1$. We express a map $N^{C/S} = \mathbb{N}^m \oplus \mathbf{k}^\times$ to N by $e_i \mapsto \Gamma_i e_i + \rho_i$. Note that ρ_i must satisfy constraint $\rho_i^{l_i} = 1$, since f is a log morphism. This shows the finiteness. Some of them could be isomorphic under maps $N \rightarrow N$ sending $e_1 \mapsto e_1 + \rho$, where $\rho \in \mathbf{k}^\times$ must satisfy the condition $\rho^d = 1$. Hence, in this example, there are $|\prod_i \mathbb{Z}/l_i \mathbb{Z}| / |\mathbb{Z}/d\mathbb{Z}|$ many realizations of log prestable maps up to isomorphisms, where the action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathbb{Z}/l_i \mathbb{Z}$ is given by $a \cdot (a_1, \dots, a_m) = (\Gamma_1 a + a_1, \dots, \Gamma_m a + a_m)$, $a \in \mathbb{Z}/d\mathbb{Z}$ and $a_i \in \mathbb{Z}/l_i \mathbb{Z}$.

3. Using the deformation and obstruction theory at the long exact sequence in Section 7, one may try directly to prove the first part of Theorem 6.3.1, using Artin's theorem in [4]. We do not pursue this approach here.

§7. Perfect Obstruction Theory

7.1.

In this subsection, we show that there is a natural perfect obstruction theory on the log algebraic stack $\overline{M}_{g,n}^{\text{log}}(\mathcal{U}/\mathcal{B})$. The method parallel to [6, 5] will work if the cotangent complexes are replaced by the logarithmic cotangent complexes, as following. We first consider the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{U}^{\text{etw}} \\ & & \downarrow \pi \\ & & \mathcal{M} \end{array}$$

where \mathcal{C} (resp. \mathcal{U}^{etw}) is the universal family of the moduli log stack $\mathcal{M} := \overline{M}_{g,n}^{\text{log}}(\mathcal{U}/\mathcal{B})$ (resp. \mathcal{B}^{etw}) and f is the evaluation map. We regard f and π as log morphisms between log stacks. Let \mathcal{C} be the universal

family of $\mathfrak{M}_{g,n}^{\log}$ and let $\mathfrak{M}\mathfrak{B}$ be $\mathfrak{M}_{g,n}^{\log} \times_{LOG} \mathcal{B}^{\text{etw}}$. There is the composite of natural maps

$$f^*L_{\mathcal{U}^{\text{etw}}/\mathcal{B}^{\text{etw}}}^\bullet \rightarrow L_{\mathcal{C}/\mathcal{B}^{\text{etw}}}^\bullet \rightarrow L_{\mathcal{C}/\mathcal{C} \times_{LOG} \mathcal{B}^{\text{etw}}}^\bullet \cong \pi^*L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^\bullet$$

between logarithmic cotangent complexes. See [29] for the definition of logarithmic complexes and functorial properties. Taking the tensor product of the composite and the relative dualizing sheaf ω_π of π , we obtain an element in

$$\text{Hom}_{D^b(\mathcal{C})}(f^*L_{\mathcal{U}^{\text{etw}}/\mathcal{B}^{\text{etw}}}^\bullet \otimes \omega_\pi[1], \pi^*L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^\bullet \otimes \omega_\pi[1]).$$

This in turn yields an element in

$$\text{Hom}_{D^b(\mathcal{M})}(R\pi_*f^*L_{\mathcal{U}^{\text{etw}}/\mathcal{B}^{\text{etw}}}^\bullet \otimes \omega_\pi[1], L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^\bullet)$$

since

$$\pi^*L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^\bullet \otimes \omega_\pi[1] \cong \pi^!L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^\bullet$$

and $R\pi_!$ is left adjoint to $\pi^!$. Finally by Grothendieck–Verdier duality we have a natural homomorphism

$$(7.1.1) \quad E^\bullet \rightarrow L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^{\bullet \geq -1}$$

where $E^\bullet := (R\pi_*f^*T_{\mathcal{U}^{\text{etw}}/\mathcal{B}^{\text{etw}}}^\dagger)^\vee$ and $L_{\mathcal{M}/\mathfrak{M}\mathfrak{B}}^{\bullet \geq -1}$ is two-term $[-1, 0]$ truncation of the logarithmic relative cotangent complex. Note that the latter complex is isomorphic to the usual relative cotangent complex since the map $\mathcal{M} \rightarrow \mathfrak{M}\mathfrak{B}$ is strict. The homomorphism (7.1.1) of the complexes is a perfect obstruction theory since the relative deformation/obstruction theory for log morphisms is as expected as in Theorem 5.9 of [29], and E^\bullet can be realized as a two-term complex of locally free coherent sheaves. This defines a virtual fundamental class of $\overline{M}_{g,n}^{\log}(\mathcal{U}/\mathcal{B})$ by [23, 6, 5].

The absolute obstruction theory $F^\bullet \rightarrow L_{\mathcal{M}}^\bullet$ can be obtained as in [9, 16] so that there is a distinguished triangle:

$$(7.1.2) \quad \tau^*L_{\mathfrak{M}\mathfrak{B}}^\bullet \rightarrow F^\bullet \rightarrow E^\bullet,$$

where τ is the natural map $\mathcal{M} \rightarrow \mathfrak{M}\mathfrak{B}$. Consider a deformation situation of the extensions over $\text{Spec}A[I]$ of a given $(f, C^\dagger, W^\dagger, S^\dagger)$ over $S := \text{Spec}A$, where A is a reduced noetherian Λ -algebra; I is a finite A -module; and $A[I]$ is the trivial ring extension of A by I . Let $\text{Aut}(C^\dagger \times_{S^\dagger} W^\dagger)$ be the set of automorphisms of the trivial extension of $C^\dagger \times_{S^\dagger} W^\dagger$ over $\text{Spec}A[I]^\dagger$, whose restriction to S^\dagger is the identity. Here

the log structure on $\text{Spec}A[I]^\dagger$ is given by the pullback of $\text{Spec}A[I] \rightarrow \text{Spec}A$. Also, let $\text{Def}_I(C^\dagger \times_{S^\dagger} W^\dagger)$ be the set of isomorphism classes of the extensions over S^\dagger . Then combining Proposition 3.14 in [14] and Theorem 5.9 in [29], we obtain an A -module exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Aut}_I(C^\dagger \times_{S^\dagger} W^\dagger) \rightarrow \text{RelDef}(f) = R^0\pi_* f^* T_{W^\dagger/S^\dagger} \otimes_{\mathcal{O}_S} I \\ &\rightarrow \text{Def}(f) \rightarrow \text{Def}_I(C^\dagger \times_{S^\dagger} W^\dagger) \xrightarrow{\varphi} \text{RelOb}(f) = R^1\pi_* f^* T_{W^\dagger/S^\dagger} \otimes_{\mathcal{O}_S} I \\ &\rightarrow \text{Ob}(f) \rightarrow 0 \end{aligned}$$

where $\text{Ob}(f)$ is defined to be the cokernel of φ . When $I \cong A$, we may also use (7.1.2) and Proposition III, 12.2 in [12] to derive the above long exact sequence.

7.2. Log admissible covers

Let A be an Artinian local ring over \mathbf{k} , with the maximal ideal \mathfrak{m}_A . Consider a small extension R of A by I and an admissible map f , locally at a distinguished node described by

$$\begin{array}{ccc} (A[x, y, z_1, \dots, z_{r-1}]/(xy - \tau))^{\text{sh}} & \xrightarrow{f^*} & (A[u, v]/(uv - t))^{\text{sh}} \\ x, y & \mapsto & u^l, v^l, \end{array}$$

where superscript sh means the strict henselianization at the ‘origin’. We want to compute a local obstruction of extending f to an admissible map over a given extended domain and a given extended target near the node. The obstruction is $\tilde{\tau} - \tilde{t}^l \in I$ as explained in [10] if the extensions are given by $xy = \tilde{\tau} \in \mathfrak{m}_R$ and $uv = \tilde{t} \in \mathfrak{m}_R$ (locally at singular points), respectively. Here $\tilde{\tau}$ and \tilde{t} are extensions of τ and t , respectively. This vanishes if f is a log morphism and the extended source and target are logarithmic ones. Indeed, the equation $[\log \tilde{\tau}] = [l \log \tilde{t}]$ in \overline{N}_R implies that $\tilde{\tau} = (1 + c)\tilde{t}^l$ where $c \in I$. Since $\tilde{t} \in \mathfrak{m}_R$, we see that $\tilde{\tau} = \tilde{t}^l$.

Provided that f is a log morphism, for a log extension on the target with $xy = \tilde{\tau}$, there is a unique log extension with $uv = \tilde{t}$ such that $\tilde{\tau} = \tilde{t}^l$. This is due to the existence of element $\tilde{s} \in \mathfrak{m}_R$ such that $\tilde{s}^d = \tilde{\tau}$ and $s^\Gamma = t$, where \tilde{s} is an extension of s . Hence, an infinitesimal deformation of $(W, M_W)/(S, N)$ uniquely determines an infinitesimal deformation of $(C, M_C)/(S, N)$ at distinguished nodes. In particular, this shows that when the target is a projective smooth curve X , the natural map

$$(f : (C, M_C, \mathfrak{p}) \rightarrow (W, M_W)/(S, N)) \xrightarrow{\Psi} (X[n]^{\text{tw}} \leftarrow (W/S, N \leftarrow N^{W/S}, f(\mathfrak{p})))$$

is étale, where $\overline{M}_{g,n,\mu}^{\text{log}}(\mathfrak{X}^+/\mathfrak{X}, \beta)$ is the moduli stack of (g, n, β) log stable μ -ramified maps (5.2.6), and $X[n]^{\text{tw}}$ is the stack of n -pointed log twisted

stable FM spaces of X . Here ‘stable’ means that there are at least two special points on screens of FM spaces. By the following Lemma $X[n]^{\text{tw}}$ is an open stack of $\mathcal{L}_{X[n]}$. (Note that $X[n]^{\text{tw}}$ is not separated over \mathbf{k} if $n \geq 2$.) Essentially, it is already proven in [35] that the map Ψ is étale.

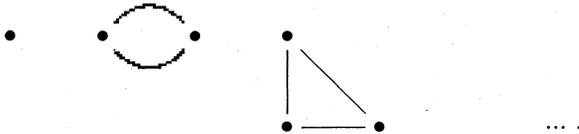
Lemma 7.2.1. ([15]) *Let X be a smooth variety over \mathbf{k} . The stack of FM spaces with n distinct, smooth, sections is equivalent to the scheme $X[n]$ with the universal space $X[n]^+$.*

According to 6.2.2, $X[n]^{\text{tw}}$ is smooth over \mathbf{k} . Therefore, the finite map from the smooth stack $\overline{M}_{g,n,\mu}^{\text{log}}(\mathfrak{X}^+/\mathfrak{X}, \beta)$ to $\overline{M}_{g,n,\mu}(\mathfrak{X}^+/\mathfrak{X}, \beta)$ is the normalization map. The normalization is also constructed by the stack of balanced twisted covers in [1].

§8. Chain Type

8.1.

In this section, we provide a modular desingularization of the main component of the moduli space of elliptic stable maps to a projective space. The main component is, by definition, the irreducible component of the moduli space, containing all the elliptic stable maps with smooth domains. This makes sense due to the connectedness result in [18]. First, note that any prestable curve C over \mathbf{k} of genus 1 has a unique subcurve C_0 which is either arithmetic $g = 1$ irreducible component or a loop of rational curves. We call C_0 the essential part of the curve C . Its dual graph looks like:



Note that the dualizing sheaf ω_{C_0} is trivial. Let $\overline{M}_1^{\text{log, ch}}(\mathfrak{X}^+/\mathfrak{X}, \beta)$ be the moduli stack of $(g = 1, n = 0, \beta \neq 0)$ log stable maps (f, C, W) satisfying the following conditions additional to those in 5.2.5. For every $s \in S$:

- Every end component of $W_{\bar{s}}$ contains the entire image of the essential part of $C_{\bar{s}}$ under $f_{\bar{s}}$.
- The image of the essential part of $C_{\bar{s}}$ is nonconstant.

Here, it is possible that some of irreducible components in the essential part are mapped to points. Note that the dual graph of the target $W_{\bar{s}}$ must be a chain. Such a log stable map is called an elliptic log stable map to a chain type FM space W of the smooth projective variety X . Set $\overline{M}_1^{\text{log, ch}}(\mathfrak{X}^+/\mathfrak{X}) := \coprod_{\beta} \overline{M}_1^{\text{log, ch}}(\mathfrak{X}^+/\mathfrak{X}, \beta)$.

8.2. Proof of Main Theorem B

By Theorem 6.3.1, it is enough to prove the properties for the moduli stack of the non-log version of elliptic log stable maps to chain type FM spaces of X . It is a DM stack of finite type over \mathbf{k} by Remark 1 in 6.3.1.

Properness. We use the valuative criterion of properness. We show that the criterion holds, by adapting the properness argument in [15]. Let $f_\xi : C_\xi \rightarrow W_\xi$ be an elliptic stable map to a chain type W over a quotient field of the DVR $R = \mathbf{k}[[t]]$, and let ξ and p be the generic point and the closed point of R , respectively. For simplicity assume that $W_\xi = X \times \xi$.

— Uniqueness: Suppose that there are two extensions $f_i : C^{(i)} \rightarrow W^{(i)}$ of f over R . To show that f_1 and f_2 are equivalent over f , it is enough to verify that $W^{(1)}$ and $W^{(2)}$ are isomorphic over X and R . To prove it, we construct sections of $W^{(i)} \rightarrow \text{Spec}R$. Consider two sections passing through the end component of $W_p^{(i)}$. Also, consider, for each ruled component, a section passing through the ruled component. Let N_i be two plus the number of ruled component of $W_p^{(i)}$ unless $W_p^{(i)} = X$. Let $N_i = 0$ if $W_p^{(i)} = X$. We may assume that those N_i sections are pairwise distinct at p . Let W be the FM type space of X over R , separating each set of N_i sections: W is defined to be $g^*X[N_1, N_2]^+$, where $g : \text{Spec}R \rightarrow X[N_1, N_2]$ is the map associated to two sets of sections. Here $X[N_1, N_2]$ denotes the compactification of configurations of two sets of N_i distinct labeled points, $i = 1, 2$. The precise construction of $X[N_1, N_2]$ can found in [15]. The space $X[N_1, N_2]$ has the universal space $X[N_1, N_2]^+$ with two sets of N_i distinct sections. We claim that W is isomorphic to W_i over X and R . To see it, we first remove two sections associated to the end component of $W^{(2)}$. Then still, W with the remained sections is stable since the domain of the stable map limit to the new target W has genus 1 and the stable map limit agrees with f_i after the contraction of suitable screens of the target W and then stable contraction of the domain curve. Now remove the rest of sections associated ruled components of $W^{(2)}$. W with N_1 sections remains stable otherwise there will be a component of curve $C_p^{(1)}$ which is mapped into a singular locus of $W^{(1)}$. Thus, $W \cong W^{(1)}$ over X and R . The same method shows that $W \cong W^{(2)}$ over X and R , and hence $W^{(1)} \cong W^{(2)}$ over X and R .

— Existence: Consider each irreducible component C_γ of the essential subcurve of C_ξ where $f_\xi(C_\gamma)$ is nonconstant. The f_ξ restricted to C_γ is μ -ramified, for some $\mu = (\mu_1, \dots, \mu_n)$. Consider the screens created by taking the limit of f_ξ restricted to C_γ as a μ -ramified stable map. See [15] for the construction. There is a natural partial order on the set

of all the screens for all γ . Take the first one (which needs the smallest magnifying power to be visible). It exits since $g = 1$. Now consider two general sections passing through the first screen, and then the stable map limit with the new target separating the sections. If necessary, take a further expansion along the singular locus of the target at the special fiber so that there are no domain components which are mapped into singular locus of the new target. This process ends in a finite step as in [15] and yields a stable admissible map to a chain type FM space of X over R , which satisfies the desired requirements.

Smoothness. This is simply because the relative obstruction space in Section 7 vanishes as following. Let $f : C^\dagger \rightarrow W^\dagger$ over \mathbf{k}^\dagger be an elliptic log stable map to a chain type FM space W of $X := \mathbb{P}^r_{\mathbf{k}}$. In what follows, T^\dagger denotes the log tangent sheaf $T_{W^\dagger/\mathbf{k}^\dagger}$.

Decompose C to be the union $\bigcup C_i$ of the essential part C_0 and the irreducible rational components $C_i, i \geq 1$; let f_i be the map f restricted to C_i . To prove $H^1(C, f^*T^\dagger) = 0$, we claim that $H^1(C_i, f_i^*T^\dagger) = 0$ for all i and the evaluation map

$$\bigoplus_i H^0(C_i, f_i^*T^\dagger) \rightarrow \bigoplus_{\{i,j:i \neq j\}} f_i^*T^\dagger|_{C_i \cap C_j}$$

is surjective. For the latter, by the induction argument on the number of irreducible rational components $C_i, i \geq 1$, it suffices that $H^1(C_i, f_i^*T^\dagger \otimes \mathcal{O}_{C_i}(-p)) = 0$ if p is a point in C_i and $i \geq 1$. When $W = X$ case, it follows from the Euler sequence, since $H^1(C_0, f_0^*\mathcal{O}_X(1)) = H^0(C_0, f_0^*\mathcal{O}_X(-1) \otimes \omega_{C_0})^\vee = 0$ and $H^1(C_i, f_i^*\mathcal{O}_X(1) \otimes \mathcal{O}_{C_i}(-p)) = 0$ for all $i \geq 1$. Similarly, when W is singular, the claim follows from the generalized Euler sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_Y &\rightarrow (\bigoplus \mathcal{O}_Y(1)) \oplus \mathcal{O}_Y \rightarrow T_{|Y}^\dagger \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_Y(-E) &\rightarrow (\bigoplus \pi^*\mathcal{O}_{\mathbb{P}^r}(1))(-E) \rightarrow T_{|Y}^\dagger \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_Y(-E) &\rightarrow (\bigoplus \pi^*\mathcal{O}_{\mathbb{P}^r}(1)(-E)) \oplus \mathcal{O}_Y(-E) \rightarrow T_{|Y}^\dagger \rightarrow 0 \end{aligned}$$

for the logarithmic tangent sheaf restricted to the end, the root, and a ruled component Y of W , respectively. The explanation of the sequences is in order. The first sequence is standard. The second one can be obtained from an isomorphism

$$\pi^*T_X(-E) \rightarrow T_{|Y}^\dagger.$$

This isomorphism can be seen by a local description of blowup $\pi : Y \rightarrow X$ at a point, with the exceptional divisor E (for example, see the proof

of Lemma 1 in [17]). The third one results from the gluing of the previous two sequences. Q.E.D.

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