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A variational approach to self-similar solutions for semilinear heat equations

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Abstract.

We study the existence of self-similar solutions for semilinear heat equations by making use of the methods for semilinear elliptic equations. In particular, via the variational approach, we show the existence of the second solution, which implies the non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data.

§1. Introduction

In this paper we review some recent results [19, 20] for the existence of self-similar solutions to semilinear heat equations by making use of the methods for semilinear elliptic equations. Let us consider the Cauchy problem for semilinear heat equations with singular initial data:

$$w_t = \Delta w + w^p \qquad \text{in } \mathbf{R}^N \times (0, \infty), \tag{1}$$

$$w(x, 0) = \lambda a\,(x/|x|)\,|x|^{-2/(p-1)} \qquad \text{in } \mathbf{R}^N \setminus \{0\}, \tag*{$(2)_\lambda$}$$

where $N > 2$, $p > 1$, $a : S^{N-1} \to \mathbf{R}$, and $\lambda > 0$ is a parameter. We assume that $a \in L^\infty(S^{N-1})$ and $a \geq 0$, $a \not\equiv 0$. A typical case is $a \equiv 1$.

The equation (1) is invariant under the similarity transformation

$$w(x,t) \mapsto w_\mu(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

In particular, a solution w is said to be *self-similar*, when $w = w_\mu$ for all $\mu > 0$, that is,

$$w(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0. \tag{3}$$

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Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1), see, e.g., [15, 16, 5, 24].

If w is a self-similar solution of (1) and has an initial value $A(x)$, then we easily see that A has the form $A(x) = A(x/|x|)|x|^{-2/(p-1)}$. Then the problem of existence of self-similar solutions is essentially depend on the solvability of the Cauchy problem (1)-$(2)_\lambda$. In this paper we consider the existence of self-similar solutions for the problem (1)-$(2)_\lambda$. In particular, we study the multiplicity of self-similar solutions to the problem in the subcritical and critical cases. The idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data goes back to the study by Giga and Miyakawa [12] for the Navier-Stokes equation in vorticity form.

It is well known by [9, 14, 17] that if $1 < p \le (N+2)/N$ then (1) has no time global solution w such that $w \ge 0$ and $w \not\equiv 0$. (See also [28, 15].) Then the condition $p > (N+2)/N$ is necessary for the existence of positive self-similar solutions of (1).

We briefly review some results concerning the Cauchy problem for (1) with initial data in $L^q(\mathbf{R}^N)$. Weissler [26, 27] showed that the IVP (1) with $w(x,0) = A \in L^q(\mathbf{R}^N)$ admits a unique time-local solution if $q \ge N(p-1)/2$. He also showed in [28] that the solution exists time-globally if $q = N(p-1)/2$ and if $\|A\|_{L^q(\mathbf{R}^N)}$ is sufficiently small. Giga [11] has constructed a unique local regular solution in $L^\alpha(0,T : L^\beta)$, where α and β are chosen so that the norm of $L^\alpha(0,T : L^\beta)$ is invariant under scaling. On the other hand, for $1 \le q < N(p-1)/2$, Haraux and Weissler [13] constructed a solution $w_0 \in C([0,\infty); L^q(\mathbf{R}^N))$ of (1) satisfying $w_0(x,t) > 0$ for $t > 0$ and $\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} \to 0$ as $t \to 0$ when $(N+2)/N < p < (N+2)/(N-2)$ by seeking solutions of self-similar form. Therefore, the Cauchy problem

$$
(4) \qquad \begin{cases} w_t = \Delta w + w^p & \text{in } \mathbf{R}^N \times (0,\infty) \\ w|_{t=0} = 0 & \text{in } \mathbf{R}^N \end{cases}
$$

admits a non-unique solution in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \le q < N(p-1)/2$ when $(N+2)/N < p < (N+2)/(N-2)$.

Kozono and Yamazaki [18] constructed Besov-type function spaces based on the Morrey spaces, and then obtained global existence results for the equation (1) and the Navier-Stokes system with small initial data in these spaces. Cazenave and Weissler [5] proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equations and the equation (1) with small initial data by using the weighted norms. By [18, 5] the problem (1)-$(2)_\lambda$ admits a time-global solution for sufficiently small $\lambda > 0$.

We note here that the equation (1) with $p > N/(N-2)$ has a positive singular stationary solution $W(x) = L|x|^{-2/(p-1)}$, where

$$L = \left[\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right]^{1/(p-1)}.$$

Galaktionov and Vazquez [10] investigated the uniqueness of solutions to the problem (1)-$(2)_\lambda$ in the case where $a \equiv 1$ and $\lambda = L$, and showed that the problem has a classical self-similar solution for $t > 0$ with certain values of p. In [10, p. 41] they also conjectured that the problem (1)-$(2)_\lambda$ has exactly two solutions (the minimal and maximal) when $N/(N-2) < p \leq (N+2)/(N-2)$. Recently, some improvements of the results [10] was shown by [25].

Letting $\mu = t^{-1/2}$ in (3), we see that the self-similar solution w of (1) has the form

$$w(x,t) = t^{-1/(p-1)}u(x/\sqrt{t}), \tag{5}$$

where u satisfies the elliptic equation

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + u^p = 0 \qquad \text{in } \mathbf{R}^N. \tag{6}$$

In addition, if w satisfies $(2)_\lambda$ in the sense of $L^1_{\text{loc}}(\mathbf{R}^N)$, then u satisfies

$$\lim_{r\to\infty} r^{2/(p-1)}u(r\omega) = \lambda a(\omega) \qquad \text{for a.e. } \omega \in S^{N-1}. \tag{7_λ}$$

Conversely, if $u \in C^2(\mathbf{R}^N)$ is a solution of (6) satisfying $(7)_\lambda$, then the function w defined by (5) satisfies (1)-$(2)_\lambda$ in the sense of $L^1_{\text{loc}}(\mathbf{R}^N)$. (See Lemma B.1 in [19].)

In this paper we investigate the problem (6)-$(7)_\lambda$ by making use of the methods for semilinear elliptic equations to derive the results for the Cauchy problem (1)-$(2)_\lambda$. First, we show the existence of the minimal solution by employing the comparison results based on the maximum principle. Next, by applying the variational method due to [1, 6, 4], we show the existence of the second solution of the problem (6)-$(7)_\lambda$ in the subcritical and critical cases, which implies the non-uniqueness of solutions to the problem (1)-$(2)_\lambda$.

This paper is organized as follows: In Section 2, we show the existence of the minimal solutions, and in Section 3, we introduce the weighted Sobolev space and recall some related results. In Sections 4 and 5, we show the existence of second solutions by employing variational arguments in the subcritical and critical cases, respectively.

§2. Existence of the minimal solution

In this section, we show the existence of positive minimal solutions of the problem (6)-$(7)_\lambda$. For simplicity, we define $\mathcal{L}u$ by

$$\mathcal{L}u = \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u$$

for $u \in C^2(\mathbf{R}^N)$. First we obtain the maximum principle for the operator $\mathcal{L}u$.

Lemma 1. *Let $p > (N+2)/N$. Assume that $-\mathcal{L}u \geq 0$ in $\mathbf{R}^N$, and that*

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)}u(x) \geq 0.$$

Then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$. In particular, if $-\mathcal{L}u \geq 0$ and $u \geq 0$ in $\mathbf{R}^N$ then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$.

Proof. From Lemma 2.1 in [19], there exists a positive function ϕ satisfying $\mathcal{L}\phi = 0$ in $\mathbf{R}^N$ and

$$\lim_{r\to\infty} r^{2/(p-1)}\phi(r\omega) = 1 \quad \text{for a.e. } \omega \in S^{N-1}.$$

Let $v(x) = u(x)/\phi(x)$. Then v satisfies $\liminf_{|x|\to\infty} v(x) \geq 0$ and

$$-\Delta v - \left(\frac{2}{\phi}\nabla\phi + \frac{1}{2}x\right) \cdot \nabla v \geq 0 \quad \text{in } \mathbf{R}^N.$$

By the maximum principle [22] we have $v > 0$ or $v \equiv 0$ in Ω, which implies that $u > 0$ or $u \equiv 0$ in Ω. Q.E.D.

We obtain the comparison result for the operator $\mathcal{L}u$ by employing Lemma 1. For the proof, see [19, Proposition 2.2].

Lemma 2. *Assume that $p > (N+2)/N$, and that $\alpha, \beta \in L^\infty(S^{N-1})$ satisfy $0 \leq \alpha(\omega) \leq \beta(\omega)$ for a.e. $\omega \in S^{N-1}$. Suppose that there exists a positive function v satisfying*

$$\text{(8)} \qquad \begin{cases} -\mathcal{L}v \geq v^p & \text{in } \mathbf{R}^N \text{ and} \\ \lim\limits_{r\to\infty} r^{2/(p-1)}v(r\omega) = \beta(\omega), & \text{a.e. } \omega \in S^{N-1}. \end{cases}$$

Then there exists a positive solution u of the problem

$$\begin{cases} -\mathcal{L}u = u^p & \text{in } \mathbf{R}^N \text{ and} \\ \lim\limits_{r\to\infty} r^{2/(p-1)}u(r\omega) = \alpha(\omega), & \text{a.e. } \omega \in S^{N-1}. \end{cases}$$

Furthermore, for any positive function v satisfying (8), *one has $u \le v$ in $\mathbf{R}^N$.*

By making use of Lemma 2, we obtain the existence of positive minimal solutions.

Theorem 1. *Assume that $p > (N+2)/N$. Then there exists a constant $\overline{\lambda} > 0$ such that*

(i) *for $0 < \lambda < \overline{\lambda}$,* (6)-(7)$_\lambda$ *has a positive minimal solution $\underline{u}_\lambda \in C^2(\mathbf{R}^N)$; the solution $\underline{u}_\lambda$ is increasing with respect to λ and satisfies $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $\lambda \to 0$;*

(ii) *for $\lambda > \overline{\lambda}$, there are no positive solutions $u \in C^2(\mathbf{R}^N)$ of* (6)-(7)$_\lambda$.

Sketch of Proof. For each $\lambda > 0$ we introduce the solution set

$$S_\lambda = \{u \in C^2(\mathbf{R}^N) : u \text{ is a positive solution of (6)-(7)}_\lambda\}.$$

Let $v = v(r)$ with $r = |x|$ be a positive solution of (6) satisfying

$$\lim_{r\to\infty} r^{2/(p-1)} v(r) = \ell$$

for some $\ell > 0$. The existence of such v is obtained by [13, Theorem 5]. Take $\lambda > 0$ so small that $\lambda \le \ell/\|a\|_{L^\infty(S^{N-1})}$. By applying Lemma 2 with $\alpha(\omega) = \lambda a(\omega)$ and $\beta(\omega) \equiv \ell$, we obtain a positive solution u_λ of (6)-(7)$_\lambda$, that is, $S_\lambda \ne \emptyset$. Applying again Lemma 2 with $v = u_\lambda$ and $\alpha(\omega) = \beta(\omega) = \lambda a(\omega)$, we obtain a positive solution $\underline{u}_\lambda$ of (6)-(7)$_\lambda$ such that $\underline{u}_\lambda \le u_\lambda$. Furthermore, we have $\underline{u}_\lambda \le u$ for all $u \in S_\lambda$. This implies that $\underline{u}_\lambda$ is the minimal solution of S_λ.

Assume that $S_{\lambda_0} \ne \emptyset$ for some $\lambda_0 > 0$. Let $\lambda \in (0, \lambda_0)$. Then, by applying Lemma 2 with $\alpha(\omega) = \lambda a(\omega)$ and $\beta(\omega) = \lambda_0 a(\omega)$, we have a positive solution u_λ of (6)-(7)$_\lambda$. Therefore, $S_\lambda \ne \emptyset$ for all $\lambda \in (0, \lambda_0)$.

Let $\overline{\lambda} = \sup\{\lambda > 0 : S_\lambda \ne \emptyset\}$. As a consequence, we obtain $\overline{\lambda} > 0$ and, for $\lambda \in (0, \overline{\lambda})$, $S_\lambda \ne \emptyset$ and there exists a minimal solution $\underline{u}_\lambda \in S_\lambda$. For the monotonicity properties of $\underline{u}_\lambda$, see (i) and (ii) of Lemma 4.2 in [19]. By (iii) of Lemma 4.2 in [19], we obtain $\overline{\lambda} < \infty$. By the definition of $\overline{\lambda}$, we can conclude that (6)-(7)$_\lambda$ has no positive solution for $\lambda > \overline{\lambda}$. Q.E.D.

Furthermore, we obtain the following result for the linearized eigenvalue problem, which plays a crucial role in the proofs of the existence of the second solutions.

Lemma 3. *Let $\underline{u}_\lambda$ be the minimal solution obtained in Theorem* 1 *for $\lambda \in (0, \overline{\lambda})$. Then the linearized eigenvalue problem*

$$\begin{cases} -\Delta w - \dfrac{1}{2} x \cdot \nabla w - \dfrac{1}{p-1} w = \mu p (\underline{u}_\lambda)^{p-1} w \quad \text{in } \mathbf{R}^N, \\ w \in H^1_\rho(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue $\mu = \mu(\lambda) > 1$. Moreover, $\mu(\lambda)$ is strictly decreasing in $\lambda \in (0, \overline{\lambda})$.

For the proof, see [19, Lemma 5.2].

§3. Weighted Sobolev space

Put $\rho(x) = e^{|x|^2/4}$. Then the equation (6) can be written as

$$\nabla \cdot (\rho \nabla u) + \rho \left(\frac{1}{p-1} u + u^p \right) = 0.$$

Escobedo-Kavian [8] investigated the corresponding functional

$$I_0(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} \rho dx$$

on the weighted functional spaces

$$L^q_\rho(\mathbf{R}^N) = \left\{ u \in L^q(\mathbf{R}^N) : \int_{\mathbf{R}^N} u^q \rho dx < \infty \right\} \quad \text{for } 1 \le q < \infty$$

and

$$H^1_\rho(\mathbf{R}^N) = \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2) \rho dx < \infty \right\}.$$

We recall here some results about the weighted Sobolev space $H^1_\rho(\mathbf{R}^N)$. For the proof, we refer to [8, 15],

Lemma 4. (i) *For every $u \in H^1_\rho(\mathbf{R}^N)$, there holds*

$$\frac{N}{2} \int_{\mathbf{R}^N} u^2 \rho dx \le \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx.$$

(ii) *The embedding $H^1_\rho(\mathbf{R}^N) \subset L^{p+1}_\rho(\mathbf{R}^N)$ is continuous for $1 \le p \le (N+2)/(N-2)$, and is compact for $1 \le p < (N+2)/(N-2)$.*

It was shown by [8, 29] that there exists a solution u_0 of the problem

$$(9)\qquad \begin{cases} \Delta u + \dfrac{1}{2}x \cdot \nabla u + \dfrac{1}{p-1}u + u^p = 0 & \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N, \end{cases}$$

with $(N+2)/N < p < (N+2)/(N-2)$. Moreover, it was shown in [8] that the solution u_0 satisfies $u_0 \in C^2(\mathbf{R}^N)$ and $u_0(x) = O(e^{-|x|^2/8})$ as $|x| \to \infty$. The uniqueness of the solution to the problem (9) was obtained in [21, Corollary 2] by combining the results [7, 30] and [21].

Now put

$$(10)\qquad w_0(x,t) = t^{-1/(p-1)}u_0(x/\sqrt{t}),$$

where u_0 is the unique solution of the problem (9). We note that $u_0 \in L^q(\mathbf{R}^N)$ for all $q \geq 1$ and

$$\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} = t^{-1/(p-1)+N/2q}\|u_0\|_{L^q(\mathbf{R}^N)}.$$

Then w_0 solves the the Cauchy problem (4) in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \leq q < N(p-1)/2$. By the uniqueness of the problem (9), we find that w_0 defined by (10) coincides with the non-unique solution of (4) constructed by [13].

§4. Existence of the second solution: subcritical case

Let $\underline{u}_\lambda$ be the positive minimal solution of (6)-$(7)_\lambda$ obtained in Theorem 1. In order to find a second solution of (6)-$(7)_\lambda$ we introduce the following problem:

$$(11)_\lambda\qquad \begin{cases} \Delta u + \dfrac{1}{2}x \cdot \nabla u + \dfrac{1}{p-1}u + g(u, \underline{u}_\lambda) = 0 & \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N, \end{cases}$$

where $g(t,s) = (t+s)^p - s^p$. We easily see that, if $(11)_\lambda$ possesses a solution u_λ, then we can get another positive solution $\overline{u}_\lambda = \underline{u}_\lambda + u_\lambda$ of (6)-$(7)_\lambda$.

In this section we will show the existence of solutions of $(11)_\lambda$ in the subcritical case $(N+2)/N < p < (N+2)/(N-2)$ by using the variational method. To this end we define the corresponding functional of $(11)_\lambda$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^N}\left(|\nabla u|^2 - \frac{1}{p-1}u^2\right)\rho dx - \int_{\mathbf{R}^N} G(u, \underline{u}_\lambda)\rho dx$$

with $u \in H^1_\rho(\mathbf{R}^N)$, where

$$G(t,s) = \frac{1}{p+1}(t+s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t.$$

We see that a nontrivial critical point $u \in H^1_\rho(\mathbf{R}^N)$ of the functional I_λ is a weak solution of the equation in $(11)_\lambda$. Moreover, we have $u_\lambda \in C^2(\mathbf{R}^N)$ and $u_\lambda > 0$ in $\mathbf{R}^N$ by employing the bootstrap arguments and the maximum principle. (See [19].)

We will verify the existence of a nontrivial solution of $(11)_\lambda$ by means of the Mountain Pass lemma ([1, 23]). We show the following lemmas.

Lemma 5. *The functional I_λ satisfies the Palais-Smale condition, that is, any sequence $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ such that*

$$\{I_\lambda(u_k)\} \text{ is bounded} \quad \text{and} \quad I'_\lambda(u_k) \to 0 \text{ as } k \to \infty$$

contains a convergent subsequence.

Lemma 6. *For $\lambda \in (0, \overline{\lambda})$ there exist some constants $\delta = \delta(\lambda) > 0$ and $\eta = \eta(\lambda) > 0$ such that $I_\lambda(u) \geq \eta$ for all $u \in H^1_\rho(\mathbf{R}^N)$ with $\|\nabla u\|_{L^2_\rho} = \delta$.*

Lemma 7. *Let u_0 be the solution of the problem* (9), *and let $0 < \lambda < \overline{\lambda}$. Then* (i) *$I_\lambda(tu_0) < 0$ for sufficient large t;* (ii) *$\sup_{t>0} I_\lambda(tu_0) \leq I_0(u_0)$.*

For the proofs of Lemmas 5-7, See [19, Lemmas 5.4-5.6]. In the proofs, the results in Lemma 3 play an important role. By making use of the results in Lemmas 5-7, we obtain the following:

Theorem 2. *Assume that $(N+2)/N < p < (N+2)/(N-2)$. Then, for $0 < \lambda < \overline{\lambda}$, there exists a positive solution $\overline{u}_\lambda$ of* (6)-$(7)_\lambda$ *satisfying $\overline{u}_\lambda > \underline{u}_\lambda$, $\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N)$, and*

$$\overline{u}_\lambda(x) - \underline{u}_\lambda(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \to \infty. \tag{12}$$

Furthermore,

$$\overline{u}_\lambda - \underline{u}_\lambda \to u_0 \quad \text{in } H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \quad \text{as } \lambda \to 0, \tag{13}$$

where u_0 is the unique solution of the problem (9). *In particular, $\overline{u}_\lambda \to u_0$ in $L^\infty(\mathbf{R}^N)$ as $\lambda \to 0$.*

Sketch of Proof. From (i) of Lemma 7, there exists a constant $T > 0$ such that $e = Tu_0$ satisfies $\|\nabla e\|_{L^2_\rho} > \delta$ and $I_\lambda(e) \le 0$, where δ is the constant appearing in Lemma 6. Denote

$$c = \inf_{v \in \Gamma} \max_{s \in [0,1]} I_\lambda(v(s)), \tag{14}$$

where $\Gamma = \{v \in C([0,1]; H^1_\rho(\mathbf{R}^N)) : v(0) = 0,\ v(1) = e\}$. Then, from Lemma 6 and (ii) of Lemma 7, it follows that

$$0 < \eta \le c \le I_0(u_0). \tag{15}$$

The Mountain Pass Lemma [1, 23] enables us to find a critical point $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ of $I_\lambda(u)$. Hence, u_λ is a weak solution of the equation in $(11)_\lambda$ and satisfies $I_\lambda(u_\lambda) \le I_0(u_0)$. By [19, Proposition A.1], we have $u_\lambda \in C^2(\mathbf{R}^N)$ and $u_\lambda(x) \to 0$ as $|x| \to \infty$. For the decay property (12), see the proof of Proposition 5.1 in [19].

We will show the outline of the proof of (13). We observe that $\{u_\lambda\}$ is bounded in $H^1_\rho(\mathbf{R}^N)$ as $\lambda \to 0$. Then there exists a sequence $\lambda_k \to 0$ and $v_0 \in H^1(\mathbf{R}^N)$ with $v_0 \ge 0$ such that

$$u_{\lambda_k} \rightharpoonup v_0 \quad \text{weakly in } H^1_\rho(\mathbf{R}^N) \quad \text{as } k \to \infty.$$

Moreover, we can show that u_{λ_k} converges to v_0 in $H^1_\rho(\mathbf{R}^N)$, and that v_0 is a solution of the problem (9). We note here that $\liminf_{\lambda \to 0} \eta(\lambda) > 0$ in Lemma 6. (See [20, Lemma 3.2].) Then, from (14) and (15), we have $v_0 \not\equiv 0$. By the uniqueness of the solution to the problem (9), we conclude that $v_0 \equiv u_0$. Thus, we obtain $u_\lambda \to u_0$ in $H^1_\rho(\mathbf{R}^N)$ as $\lambda \to 0$. For the detail, see [19, Proof of Proposition 5.2.]. Q.E.D.

Now we consider the Cauchy problem (1)-$(2)_\lambda$. Recall that, if u is a solution of (6)-$(7)_\lambda$, then the function w defined by (5) is a solution of (1)-$(2)_\lambda$ in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$, and that w_0 defined by (9) coincides with the non-unique solution of (4) constructed by [13]. As a consequence of Theorems 1 and 2, we obtain the following results.

Corollary 1. *Assume that $p > (N+2)/N$. Then there exists a constant $\overline{\lambda} > 0$ such that*

(i) *for $0 < \lambda < \overline{\lambda}$, (1)-$(2)_\lambda$ has a positive self-similar solution $\underline{w}_\lambda$; the solution $\underline{w}_\lambda(\cdot, t)$ satisfies, for each fixed $t > 0$,*

$$\|\underline{w}_\lambda(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } \lambda \to 0;$$

(ii) *for $\lambda > \overline{\lambda}$, (1)-$(2)_\lambda$ has no positive self-similar solutions.*

Assume, furthermore, that $p < (N+2)/(N-2)$. *Then* (1)-(2)$_\lambda$ *has a positive self-similar solution* $\overline{w}_\lambda$ *satisfying* $\overline{w}_\lambda > \underline{w}_\lambda$ *in* $\mathbf{R}^N \times (0,\infty)$ *for* $0 < \lambda < \overline{\lambda}$. *The solution* $\overline{w}_\lambda$ *satisfies, for each fixed* $t > 0$,

$$\|\overline{w}_\lambda(\cdot,t) - w_0(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } \lambda \to 0,$$

where w_0 *is the non-unique solution of* (4) *in* $C([0,\infty); L^q(\mathbf{R}^N))$ *for* $1 \le q < N(p-1)/2$, *which is constructed by* [13].

§5. Existence and nonexistence of the second solutions: critical case

In this section we consider the existence and nonexistence of second solutions of the problem (6)-(7)$_\lambda$ in the critical case $p = (N+2)/(N-2)$. For the detail of the proofs in this section, we refer to [20].

For the critical growth case, there are serious difficulties in obtaining solutions by using variational methods because the Sobolev embedding $H^1 \subset L^{p+1}$ is not compact. It is well known that this lack of compactness exhibits many interesting existence and nonexistence phenomena. See, e.g., [4, 2].

We show the existence of second solutions of the problem (6)-(7)$_\lambda$ by following the argument due to Brezis-Nirenberg [4]. Let us denote by S the best Sobolev constant of the embedding $H^1(\mathbf{R}^N) \subset L^{2N/(N-2)}(\mathbf{R}^N)$, which is given by

$$S = \inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbf{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbf{R}^N} |u|^{2N/(N-2)} dx\right)^{(N-2)/N}}.$$

In the critical case, the functional I_λ satisfies the following local Palais-Smale condition.

Lemma 8. *Let* $p = (N+2)/(N-2)$. *Then* I_λ *satisfies the* $(PS)_c$ *condition for* $c \in (0, S^{N/2}/N)$, *that is, any sequence* $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ *such that*

$$I_\lambda(u_k) \to c \in \left(0, \frac{1}{N} S^{N/2}\right) \quad \text{and} \quad I'_\lambda(u_k) \to 0 \quad \text{as } k \to \infty$$

contains a convergent subsequence.

By the Mountain Pass lemma with Lemma 8, we obtain the following existence result.

Lemma 9. *Let $p = (N+2)/(N-2)$. Assume that there exists $v_0 \in H^1_\rho(\mathbf{R}^N)$ with $v_0 \geq 0$, $v_0 \not\equiv 0$ such that*

$$\sup_{t>0} I_\lambda(tv_0) < \frac{1}{N} S^{N/2}. \tag{16}$$

Then there exists a positive solution $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ of $(11)_\lambda$.

Remark. We obtain $u_\lambda \in C^2(\mathbf{R}^N)$ by employing the estimate due to Brezis-Kato [3], based on the Moser's iteration technique.

In order to find a positive function $v_0 \in H^1_\rho(\mathbf{R}^N)$ satisfying (16), we set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}} \rho^{-1/2} \quad \text{and} \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{L^{p+1}_\rho}}$$

for $\varepsilon > 0$, where $\phi \in C_0^\infty(\mathbf{R}^N)$ is a cut off function. We remark that the functional I_λ can be written as

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1}u^2\right)\rho dx - \frac{1}{p+1}\int_{\mathbf{R}^N} u^{p+1}\rho dx \\ &\quad - \int_{\mathbf{R}^N} H(u, \underline{u}_\lambda)\rho dx \\ &\equiv I_0(u) - \int_{\mathbf{R}^N} H(u, \underline{u}_\lambda)\rho dx, \end{aligned}$$

where

$$H(t,s) = G(t,s) - \frac{1}{p+1} t^{p+1}.$$

Lemma 10. *For sufficient small $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that* $\sup_{t>0} I_\lambda(tv_\varepsilon) = I_\lambda(t_\varepsilon v_\varepsilon)$. *Moreover, as $\varepsilon \to 0$ we have*

$$I_0(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} S^{N/2} + \begin{cases} O(\varepsilon), & N \geq 5 \\ O(\varepsilon|\log\varepsilon|), & N = 4 \\ O(\varepsilon^{1/2}), & N = 3 \end{cases}$$

$$\int_{\mathbf{R}^N} H(t_\varepsilon v_\varepsilon, \underline{u}_\lambda)\rho dx \geq \begin{cases} C\varepsilon^{3/4}, & N = 5 \\ C\varepsilon^{1/2}, & N = 4 \\ C\varepsilon^{1/4}, & N = 3 \end{cases}$$

with some constant $C > 0$.

Remark. For instance, in the case $N = 6$, we obtain

$$\int_{\mathbf{R}^N} H(t_\varepsilon v_\varepsilon, \underline{u}_\lambda)\rho dx \geq C\varepsilon \quad \text{as } \varepsilon \to 0.$$

Then we can not ensure the condition (16) generally in this case.

Combining the results in Lemmas 9 and 10, we obtain the following:

Theorem 3. *Let $p = (N+2)/(N-2)$ and $N = 3, 4, 5$. Then, for $0 < \lambda < \overline{\lambda}$, the problem* (6)-(7)$_\lambda$ *has a positive solution $\overline{u}_\lambda \in C^2(\mathbf{R}^N)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$ and $\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N)$.*

On the other hand, for the case $N \geq 6$ we obtain the uniqueness result in the radial class by employing the Pohozaev type identity.

Theorem 4. *Let $p = (N+2)/(N-2)$ and $N \geq 6$. Assume that $a \equiv 1$ in* (7)$_\lambda$. *Then there exists a constant $\lambda_0 \in (0, \overline{\lambda})$ such that* (6)-(7)$_\lambda$ *has no positive radial solutions $u \in C^2(\mathbf{R}^N)$ with $u \not\equiv \underline{u}_\lambda$ for $\lambda \in (0, \lambda_0)$, that is,* (6)-(7)$_\lambda$ *has a unique positive radial solution $\underline{u}_\lambda$ for $0 < \lambda < \lambda_0$.*

Let us consider the Cauchy problem (1)-(2)$_\lambda$ with $p = (N+2)/(N-2)$. As a consequence of Theorems 3 and 4, we obtain the following results.

Corollary 2. *Assume that $p = (N+2)/(N-2)$ in* (1)-(2)$_\lambda$.

(i) *Let $N = 3, 4, 5$. For $0 < \lambda < \overline{\lambda}$, the problem* (1)-(2)$_\lambda$ *has a positive self-similar solution $\overline{w}_\lambda$ satisfying $\overline{w}_\lambda(x,t) > \underline{w}_\lambda(x,t)$ for $(x,t) \in (\mathbf{R}^N \times (0,\infty))$.*

(ii) *Let $N \geq 6$ and $a \equiv 1$ in* (2)$_\lambda$. *Then there exists a constant $\lambda_0 \in (0, \overline{\lambda})$ such that* (1)-(2)$_\lambda$ *has a unique positive radially symmetric self-similar solution $\underline{w}_\lambda$ for $0 < \lambda < \lambda_0$.*

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