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On the finiteness of abelian varieties with bounded modular height

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Abstract.

In this paper, we propose a definition of modular heights of abelian varieties defined over a field of finite type over $\mathbb{Q}$, and prove its bounding property, that is, the finiteness of abelian varieties with bounded modular height.

§ Introduction

The modular heights of abelian varieties and their bounding property played a crucial role in Faltings' first proof [2] of the Mordell conjecture. Although many important results concerning finiteness properties over number fields (conjectures of Tate, Shafarevich and Mordell among others) are now available over arbitrary fields of finite type over $\mathbb{Q}$, a similar generalization of the aforementioned theory of Faltings does not seem to have been explicitly formulated. In this paper, we propose a definition of the modular heights of abelian varieties and prove the finiteness of abelian varieties with bounded modular height over a general field of finite type over $\mathbb{Q}$.

Let K be a field of finite type over $\mathbb{Q}$. In order to properly define the height function over K, we have to fix a polarization of K (see [9]). A *polarization* of K is, by definition, a collection of data $(B; \overline{H}_1, \ldots, \overline{H}_d)$, where

- B is a normal and projective scheme over $\operatorname{Spec}(\mathbb{Z})$ such that its function field is isomorphic to K;
- $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and $\overline{H}_1, \ldots, \overline{H}_d$ are nef C^∞-hermitian line bundles on B.

Let A be an abelian variety over K. By use of the Néron model of A over B defined in codimension one (see Section 1.1), the *Hodge sheaf*

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$\lambda(A/K;B)$ attached to A is canonically defined as a reflexive sheaf of rank one on B. Moreover it carries a locally integrable singular hermitian metric $\|\cdot\|_{\mathrm{Fal}}$ induced by Faltings' metric on the good reduction part of the Néron model of A. The arithmetic first Chern class

$$\widehat{c}_1(\lambda(A/K;B), \|\cdot\|_{\mathrm{Fal}})$$

is represented by a pair of a Weil divisor and a locally integrable function. We define the *modular height* $h(A)$ of A as the arithmetic intersection number of $\widehat{c}_1(\lambda(A/K;B), \|\cdot\|_{\mathrm{Fal}})$ with $\overline{H}_1, \dots, \overline{H}_d$:

$$h(A) = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\lambda(A/K;B), \|\cdot\|_{\mathrm{Fal}})\right).$$

The main objective of the present paper is to show the following finiteness result:

Theorem A (cf. Theorem 6.1). *Assume that the arithmetic divisors $\overline{H}_1, \dots, \overline{H}_d$ are big. Then, for an arbitrary fixed real number c, the set of K-isomorphism classes of the abelian varieties over K with $h(A) \le c$ is finite.*

This theorem can be viewed as an Arakelov geometric analogue of a result of Moret-Bailly [8], where the ground field K is replaced by a function field over a finite field and the height is defined by means of the ordinary intersection theory.

In our proof, we have to look at the compactified moduli space of abelian varieties and the local behavior of Faltings' metric around the boundary. We do not need, however, strong assertions due to Faltings-Chai [4]; basic facts stated in [12] together with a lemma of Gabber (Lemma 1.2.2) are sufficient for our purpose.

The present paper is organized as follows. Basic notions and facts are prepared in Section 1. In Section 2, we study height functions of singular hermitian line bundles with logarithmic singularities. In Section 3, we observe some properties of the Faltings modular height. The proof of the bounding property is done from Section 4 through Section 6.

Finally we would like to express hearty thanks to the referee for a lot of comments to improve the paper.

§1. Preliminaries

1.1. Néron model

Let B be a noetherian normal integral scheme and K its function field. Let A be an abelian variety over K. A smooth group scheme $\mathcal{A} \to$

B is called a *Néron model of A over B* if the following two conditions are satisfied:

(a) The generic fiber $\mathcal{A} \times_B \operatorname{Spec}(K)$ of $\mathcal{A} \to B$ is isomorphic to A over K;

(b) (Universal property) If $\mathcal{X} \to B$ is a smooth B-scheme and X its generic fiber, then any K-morphism $X \to A$ uniquely extends to a B-morphism $\mathcal{X} \to \mathcal{A}$.

If B is a Dedekind scheme, then there exists a Néron model of A over B (cf. [1]). When B has higher dimension, we still have a partial Néron mode $\mathcal{A}$ of A defined over a big open subset $U \subseteq B$ (*i.e.* $B \setminus U$ is of codimension ≥ 2), and we call $\mathcal{A}$ a *Néron model over B in codimension one*:

Proposition 1.1.1. *There exists a Néron model of A over a big open set U of B.*

Proof. Let us begin with the following lemma:

Lemma 1.1.2. *Let S be a noetherian normal integral scheme and K its function field. Let A be an abelian variety over K, and let $\mathcal{A} \to S$ be a smooth group scheme over S such that, for each point x of codimension one in S, the restriction of $\mathcal{A} \to S$ to $\mathcal{A} \times_S \operatorname{Spec}(\mathcal{O}_{S,x})$ is a Néron model of A over $\operatorname{Spec}(\mathcal{O}_{S,x})$. If $\mathcal{X} \to S$ is a smooth S-scheme and X its generic fiber, then any K-morphism $X \to A$ uniquely extends to an S-morphism $\mathcal{X} \to \mathcal{A}$.*

Proof. This follows from the universal property of Néron models and Weil's extension theorem (cf. [1, Theorem 1 in 4.4]). □

Let us go back to the proof of Proposition 1.1.1. First of all, we choose a non-empty Zariski open set U_0 of B and an abelian scheme $\mathcal{A}_0 \to U_0$ whose generic fiber is A. Let $x_1, \ldots, x_l$ be points of codimension one in $B \setminus U_0$. Then there are open neighborhoods $U_1, \ldots, U_l$ of $x_1, \ldots, x_l$ respectively, and smooth group schemes $\mathcal{A}_i$ over U_i of finite type with the following properties:

(i) $x_j \notin U_i$ for all $i \neq j$.

(ii) The restriction of $\mathcal{A}_i \to U_i$ to $\mathcal{A} \times_{U_i} \operatorname{Spec}(\mathcal{O}_{B,x_i})$ is a Néron model of A over $\operatorname{Spec}(\mathcal{O}_{B,x_i})$ for all i.

(iii) $\mathcal{A}_i \to U_i$ is an abelian scheme over $U_i \setminus \overline{\{x_i\}}$ for all i.

For each $i = 0, \ldots, l$, let A_i be the generic fiber of $\mathcal{A}_i \to U_i$ and $\phi_i : A \to A_i$ an isomorphism over K. Note that $x_1, \ldots, x_l \notin U_i \cap U_j$ for $i \neq j$. Thus, by Lemma 1.1.2, the isomorphism $\phi_j \circ \phi_i^{-1} : A_i \to A_j$ over K extends uniquely to an isomorphism $\psi_{ji} : \mathcal{A}_i|_{U_i \cap U_j} \to \mathcal{A}_j|_{U_i \cap U_j}$ over $U_i \cap U_j$. Clearly, $\psi_{kj} \circ \psi_{ji} = \psi_{ki}$. Thus, if we set $U = U_0 \cup U_1 \cup \cdots \cup U_l$, then we can construct a smooth group scheme $\mathcal{A}$ over U of finite type such that $\mathcal{A}|_{U_i}$ is isomorphic to $\mathcal{A}_i$ over U_i. The universal property of $\mathcal{A} \to U$ is obvious by Lemma 1.1.2. □

1.2. Semiabelian reduction

Let B be a noetherian normal integral scheme and K the function field of B. Let A be an abelian variety over K. We say A has *semiabelian reduction over B in codimension one* if there are a big open set U of B (*i.e.* $\operatorname{codim}(B \setminus U) \geq 2$) and a semiabelian scheme $\mathcal{A} \to U$ such that the generic fiber of $\mathcal{A} \to U$ is isomorphic to A.

Proposition 1.2.1. *Let B, K and A be same as above. Let m be a positive integer which has a factorization $m = m_1 m_2$ with $m_1, m_2 \geq 3$ and m_1 and m_2 relatively prime (for example $m = 12 = 3 \cdot 4$). If $A[m](\overline{K}) \subseteq A(K)$, then A has semiabelian reduction in codimension one over B.*

Proof. Let x be a point of codimension one in B. Then there is m_i which is not divisible by the characteristic of the residue field of $\mathcal{O}_{B,x}$. Moreover, $A[m_i](\overline{K}) \subseteq A(K)$. Thus, by [11, exposé 1, Corollaire 5.18], A has semiabelian reduction at x.

Let U_0 be a non-empty Zariski open subset of B over which we can take an abelian scheme $\mathcal{A}_0 \to U_0$ whose generic fiber is A. Let $x_1, \ldots, x_l$ be points of codimension one in $B \setminus U_0$. Then there are open neighborhoods $U_1, \ldots, U_l$ of $x_1, \ldots, x_l$, and semiabelian schemes $\mathcal{A}_i$ over U_i with the following properties:

(i) $x_j \notin U_i$ for all $i \neq j$.

(ii) $\mathcal{A}_i \to U_i$ is an abelian scheme over $U_i \setminus \overline{\{x_i\}}$.

Thus, as in Proposition 1.1.1, if we set $U = U_0 \cup U_1 \cup \cdots \cup U_l$, then we have our desired semiabelian scheme $\mathcal{A} \to U$. □

Lemma 1.2.2 (Gabber's lemma). *Let U be a dense Zariski open set of an integral, normal and excellent scheme S and A an abelian scheme over U. Then there is a proper, surjective and generically finite morphism $\pi : S' \to S$ of integral, normal and excellent schemes such that the abelian scheme $A \times_U \pi^{-1}(U)$ over $\pi^{-1}(U)$ extends to a semiabelian scheme over S'*

Proof. In [12, Théorème and Proposition 4.10 in Exposé V], the existence of $\pi : S' \to S$ and the extension of the abelian scheme is proved under the assumption $\pi : S' \to S$ is proper and surjective. Let S'_η be the generic fiber of π. Let z be the closed point of S'_η and Z the closure of z in S'. Moreover, let S_1 be the normalization of Z. Then $\pi_1 : S_1 \to Z \to S$ is our desired morphism. $\square$

1.3. The Hodge sheaf of an abelian variety

Let $G \to S$ be a smooth group scheme over S. Then *the Hodge line bundle* $\lambda_{G/S}$ *of* $G \to S$ is given by

$$\lambda_{G/S} = \det\left(\epsilon^*\left(\Omega_{G/S}\right)\right),$$

where $\epsilon : S \to G$ is the identity of the group scheme $G \to S$.

Let B be a noetherian, normal, integral scheme and K its function field. Let A be an abelian variety over K and let $\mathcal{A} \to U$ be the Néron model over B in codimension one (see Section 1.1). The *Hodge sheaf* $\lambda(A/K; B)$ *of* A *with respect to* B is defined by

$$\lambda(A/K; B) = \iota_*\left(\lambda_{\mathcal{A}/U}\right),$$

where $\iota : U \to B$ be the natural inclusion map. Note that $\lambda(A/K; B)$ is a reflexive sheaf of rank one on B.

From now on, we assume that the characteristic of K is zero. Let $\phi : A \to A'$ be an isogeny of abelian varieties over K. Since there is an injective homomorphism

$$\phi^* : \lambda(A'/K; B) \to \lambda(A/K; B),$$

we can find an effective Weil divisor D_ϕ such that

$$c_1(\lambda(A'/K; B)) + D_\phi = c_1(\lambda(A/K; B)).$$

The ideal sheaf $\mathcal{O}_B(-D_\phi)$ is denoted by $\mathcal{I}_\phi$.

Lemma 1.3.3. *Let* $\phi^\vee : {A'}^\vee \to A^\vee$ *be the dual of* $\phi : A \to A'$*. We assume that* B *is the spectrum of a discrete valuation ring* R *and that* A, A' *have semiabelian reduction over* B*. Then* $\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee} = \deg(\phi)R$.

Proof. Let R' be an extension of R such that R' is a complete discrete valuation ring and the residue field of R' is algebraically closed (cf. [7, Theorem 29.1]). Then, by [12, Exposé VII, Théroèm 2.1.1], $(\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee})R' = \deg(\phi)R'$. Here R' is faithfully flat over R. Thus $\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee} = \deg(\phi)R$. $\square$

1.4. Locally integrable hermitian metric

Let M be a complex manifold and L a line bundle on M. A *singular hermitian metric* $\|\cdot\|$ of L is a C^∞-hermitian metric of $L|_U$, where U is a certain dense Zariski open subset of M. If $\|\cdot\|_0$ is an arbitrary C^∞-hermitian metric of L and $\sigma \neq 0$ is a local section of L around x, the ratio $\mu = \|\sigma\|/\|\sigma\|_0$ of the two norms is independent of σ, and hence μ is a positive C^∞-function defined on U. A *locally integrable hermitian metric* (or L^1_{loc}*-hermitian metric*) is a singular hermitian metric such that the function $\log(\mu)$ on U extends to a locally integrable function on M (of course this definition does not depend on the choice of the C^∞-hermitian metric $\|\cdot\|_0$).

Lemma 1.4.1. *Let M be a complex manifold and $(L, \|\cdot\|)$ a hermitian line bundle on M. Let s be a non-zero meromorphic section of L over M. Then the hermitian metric $\|\cdot\|$ is locally integrable if and only if so is* $\log\|s\|$.

Proof. Let $\|\cdot\|_0$ be a C^∞-hermitian metric of L. Then

$$\log\|s\| = \log(\|\cdot\|/\|\cdot\|_0) + \log\|s\|_0.$$

Note that $\log\|s\|_0$ is locally integrable. Thus $\log\|s\|$ is locally integrable if and only if so is $\log(\|\cdot\|/\|\cdot\|_0)$. □

Lemma 1.4.2. *Let $f : Y \to X$ be a surjective, proper and generically finite morphism of non-singular varieties over $\mathbb{C}$. Let $(L, \|\cdot\|)$ be a singular hermitian line bundle on X. Assume that there are a non-empty Zariski open set U of X and a hermitian line bundle $(L', \|\cdot\|')$ on Y such that $(L', \|\cdot\|')$ is isometric to $f^*(L, \|\cdot\|)$ over $f^{-1}(U)$. If $\|\cdot\|'$ is locally integrable, then so is $\|\cdot\|$.*

Proof. Shrinking U if necessarily, we may assume that f is étale over U. We set $V = f^{-1}(U)$. Let s be a non-zero rational section of L. Note that there is a divisor D on Y such that $L' = f^*(L) \otimes \mathcal{O}_Y(D)$ and $\mathrm{Supp}(D) \subseteq Y \setminus V$. Thus $f^*(s)$ gives rise to a rational section s' of L'. Then $\log\|s'\|'$ is locally integrable by Lemma 1.4.1. Since $f^*(\log\|s\|)|_V = \log\|s'\|'|_V$, we can see that $f^*(\log\|s\|)$ is locally integrable. Let $[f^*(\log\|s\|)]$ be a current associated to the locally integrable function $f^*(\log\|s\|)$. Then, by [5, Proposition 1.2.5], there is a locally integrable function g on X with $f_*[f^*(\log\|s\|)] = [g]$. Since f is étale over U, we can easily see that

$$(f|_V)_*[(f|_V)^*(\log\|s\||_U)] = \deg(f)[\log\|s\||_U].$$

Thus $g = \deg(f) \log \|s\|$ almost everywhere over U. Therefore so is over X because U is a non-empty Zariski open set of X. Hence $\log \|s\|$ is locally integrable on X. $\square$

1.5. Hermitian metric with logarithmic singularities

Let X be a normal variety over $\mathbb{C}$ and Y a proper closed subscheme of X. Let $(L, \|\cdot\|)$ be a hermitian line bundle on X. We say that $(L, \|\cdot\|)$ is a *C^∞-hermitian line bundle with logarithmic singularities along* Y if the following conditions are satisfied:

(1) $\|\cdot\|$ is C^∞ over $X \setminus Y$.

(2) Let $\|\cdot\|_0$ be a C^∞-hermitian metric of L. For each $x \in Y$, let $f_1, \ldots, f_m$ be a system of local equations of Y around x, i.e., Y is given by $\{z \in X \mid f_1(z) = \cdots = f_m(z) = 0\}$ around x. Then there are positive constants C and r such that

$$\max\left\{\frac{\|\cdot\|}{\|\cdot\|_0}, \frac{\|\cdot\|_0}{\|\cdot\|}\right\} \leq C\left(-\sum_{i=1}^{m} \log |f_i|\right)^r$$

around x.

Note that the above definition does not depend on the choice of the system of local equations $f_1, \ldots, f_m$. Moreover it is easy to see that if $(L, \|\cdot\|)$ is a C^∞-hermitian line bundle with logarithmic singularities along Y, then $\|\cdot\|$ is locally integrable.

Lemma 1.5.1. *Let $\pi : X' \to X$ be a proper morphism of normal varieties over $\mathbb{C}$ and Y a proper closed subscheme of X. Let $(L, \|\cdot\|)$ be a hermitian line bundle on X such that $\|\cdot\|$ is C^∞ over $X \setminus Y$. If $\pi(X') \not\subseteq Y$ and $(L, \|\cdot\|)$ has logarithmic singularities along Y, then so does $\pi^*(L, \|\cdot\|)$ along $\pi^{-1}(Y)$. Moreover, if π is surjective and $\pi^*(L, \|\cdot\|)$ has logarithmic singularities along $\pi^{-1}(Y)$, then so does $(L, \|\cdot\|)$ along Y.*

Proof. Let $\{f_1, \ldots, f_m\}$ be a system of local equations of Y. Then $\{\pi^*(f_1), \ldots, \pi^*(f_m)\}$ is a system of local equation of $\pi^{-1}(Y)$. Thus our assertion is obvious. $\square$

1.6. Faltings' metric

Let X be a normal variety over $\mathbb{C}$ and let $f : A \to X$ be a g-dimensional semiabelian scheme over X. We assume that there is a non-empty Zariski open set U of X such that f is an abelian scheme over U. Let $\lambda_{A/X}$ be the Hodge line bundle of $A \to X$, i.e.,

$$\lambda_{A/X} = \det\left(\epsilon^*\left(\Omega_{A/X}\right)\right),$$

where $\epsilon : X \to A$ is the identity of the semiabelian scheme $A \to X$. Via the natural isomorphism $\rho : \lambda_{A_x} \xrightarrow{\sim} f_{x*}(\det(\Omega_{A_x}))$ at each $x \in U$, we define *Faltings' metric* $\|\cdot\|_{\mathrm{Fal}}$ of $\lambda_{A/X}$ by

$$(\|\alpha\|_{\mathrm{Fal},x})^2 = \left(\frac{\sqrt{-1}}{2}\right)^g \int_{A_x} \rho(\alpha) \wedge \overline{\rho(\alpha)}.$$

Faltings' metric is a C^∞-hermitian metric on U and hence it is a singular hermitian metric on X. Furthermore this metric is known to have logarithmic singularities along the boundary $X \setminus U$ (cf. [12, Théorèm 3.2 in Exposé I]) and in particular a locally integrable hermitian metric.

Lemma 1.6.1. *Let X be a smooth variety over $\mathbb{C}$ and X_0 a non-empty Zariski open set of X. Let $A_0 \to X_0$ be an abelian scheme over X_0. Let λ be a line bundle on X such that $\lambda|_{X_0}$ coincides with the Hodge line bundle λ_{A_0/X_0} of $A_0 \to X_0$. Then Faltings' metric $\|\cdot\|_{\mathrm{Fal}}$ of λ_{A_0/X_0} over X_0 extends to a locally integrable metric of λ over X.*

Proof. By virtue of Lemma 1.2.2 (Gabber's lemma), there is a proper, surjective and generically finite morphism $\pi : X' \to X$ of smooth varieties over $\mathbb{C}$ such that the abelian scheme $A_0 \times_{X_0} \pi^{-1}(X_0)$ over $\pi^{-1}(X_0)$ extends to a semiabelian scheme $f' : A' \to X'$. Let $\lambda_{A'/X'}$ be the Hodge line bundle of $A' \to X'$ and $\|\cdot\|'_{\mathrm{Fal}}$ Faltings' metric of $\lambda_{A'/X'}$. Then $(\lambda_{A'/X'}, \|\cdot\|'_{\mathrm{Fal}})\big|_{X'_0}$ is isometric to $\pi_0^*(\lambda_{A_0/X_0}, \|\cdot\|_{\mathrm{Fal}})$, where $X'_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi|_{X'_0}$. Therefore, by Lemma 1.4.2, $\|\cdot\|_{\mathrm{Fal}}$ extends to a locally integrable metric of λ over X. □

1.7. The moduli of abelian varieties

In order to deal with the bounding property of the modular height, we need a reasonable compactification of the moduli space of polarized abelian varieties. For simplicity, an abelian variety with a polarization of degree l^2 is called an *l-polarized abelian variety.*

Theorem 1.7.1. *Let g, l and m be positive integers with $m \geq 3$. Let $\mathbb{A}_{g,l,m,\mathbb{Q}}$ be the moduli space of g-dimensional and l-polarized abelian varieties over $\mathbb{Q}$ with level m structure. Then there exist*

(a) *normal and projective arithmetic varieties $\mathbb{A}^*_{g,l,m}$ and Y^* (i.e., $\mathbb{A}^*_{g,l,m}$ and Y^* are normal and integral schemes flat and projective over $\mathbb{Z}$),*
(b) *a surjective and generically finite morphism $f : Y^* \to \mathbb{A}^*_{g,l,m}$,*
(c) *a positive integer n,*
(d) *a line bundle L on $\mathbb{A}^*_{g,l,m}$, and*
(e) *a semiabelian scheme $G \to Y^*$*

with the following properties:

(1) $\mathbb{A}_{g,l,m,\mathbb{Q}}$ *is a Zariski open set of* $\mathbb{A}^*_{g,l,m,\mathbb{Q}} = \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$ *and* L *is very ample on* $\mathbb{A}^*_{g,l,m}$.

(2) *Let* λ_{G/Y^*} *be the Hodge line bundle of the semiabelian scheme* $G \to Y^*$. *Then* $f^*(L) = \lambda_{G/Y^*}^{\otimes n}$ *on* $Y^*_{\mathbb{Q}} = Y^* \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$.

(3) *Let* $U_{\mathbb{Q}} \to \mathbb{A}_{g,l,m,\mathbb{Q}}$ *be the universal* g*-dimensional and* l*-polarized abelian scheme with level* m *structure. Let* $Y_{\mathbb{Q}}$ *be the pull-back of* $\mathbb{A}_{g,l,m,\mathbb{Q}}$ *by* $f_{\mathbb{Q}} : Y^*_{\mathbb{Q}} \to \mathbb{A}^*_{g,l,m,\mathbb{Q}}$, *i.e.,* $Y_{\mathbb{Q}} = (f_{\mathbb{Q}})^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}})$. *Then* $G_{\mathbb{Q}} \to Y^*_{\mathbb{Q}}$ *is an extension of the abelian scheme* $U_{\mathbb{Q}} \times_{\mathbb{A}_{g,l,m,\mathbb{Q}}} Y_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. *(Note that* $G|_{Y_{\mathbb{Q}}} \to Y_{\mathbb{Q}}$ *is naturally a* g*-dimensional and* l*-polarized abelian scheme with level* m *structure.)*

(4) L *has a metric* $\|\cdot\|$ *over* $\mathbb{A}_{g,l,m,\mathbb{Q}}(\mathbb{C})$ *such that* $f^*((L, \|\cdot\|))$ *is isometric to* $\left(\lambda_{G/Y^*}, \|\cdot\|_{\mathrm{Fal}}\right)^{\otimes n}$ *over* $Y_{\mathbb{Q}}(\mathbb{C})$. *Moreover,* $\|\cdot\|$ *has logarithmic singularities along* $\mathbb{A}^*_{g,l,m,\mathbb{Q}}(\mathbb{C}) \setminus \mathbb{A}_{g,l,m,\mathbb{Q}}(\mathbb{C})$.

Proof. Let $U_{\mathbb{Q}} \to \mathbb{A}_{g,l,m,\mathbb{Q}}$ be the universal l-polarized abelian scheme with level m structure. By [12, Théorème 2.2 in Exposé IV], there are a normal and projective variety $\mathbb{A}^*_{g,l,m,\mathbb{Q}}$, a positive integer n and a very ample line bundle $L_{\mathbb{Q}}$ on $\mathbb{A}^*_{g,l,m,\mathbb{Q}}$ with the following properties:

(i) $\mathbb{A}_{g,l,m,\mathbb{Q}}$ is an Zariski open set of $\mathbb{A}^*_{g,l,m,\mathbb{Q}}$.

(ii) By Gabber's lemma (cf. Lemma 1.2.2), there is a surjective and generically finite morphism $h_{\mathbb{Q}} : S'_{\mathbb{Q}} \to \mathbb{A}^*_{g,l,m,\mathbb{Q}}$ of normal and projective varieties over $\mathbb{Q}$ such that the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g,l,m,\mathbb{Q}}} h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}}) \to h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}})$ extends to a semiabelian scheme $G'_{\mathbb{Q}} \to S'_{\mathbb{Q}}$. Then $h^*_{\mathbb{Q}}(L_{\mathbb{Q}}) = \lambda^{\otimes n}_{G'_{\mathbb{Q}}/S'_{\mathbb{Q}}}$.

Since $L_{\mathbb{Q}}$ is very ample, there is an embedding $\mathbb{A}^*_{g,l,m,\mathbb{Q}} \hookrightarrow \mathbb{P}^N_{\mathbb{Q}}$ in terms of $L_{\mathbb{Q}}$. Let $\mathbb{A}^*_{g,l,m}$ be the closure of the image of

$$\mathbb{A}^*_{g,l,m,\mathbb{Q}} \hookrightarrow \mathbb{P}^N_{\mathbb{Q}} \to \mathbb{P}^N_{\mathbb{Z}}.$$

Let L be the pull-back of $\mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)$ by the embedding $\mathbb{A}^*_{g,l,m} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$. We have obvious isomorphisms $\mathbb{A}^*_{g,l,m,\mathbb{Q}} \simeq \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$ and $L_{\mathbb{Q}} \simeq L|_{\mathbb{A}^*_{g,l,m,\mathbb{Q}}}$. Let S' denote the normalization of $\mathbb{A}^*_{g,l,m}$ in the function field of $S'_{\mathbb{Q}}$. There exists an open subset S'_0 of S' such that G' is an abelian scheme over S'_0 and $G' \times_{S'} S'_0 \to S'_0$ coincides with the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g,l,m,\mathbb{Q}}} h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}}) \to h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}})$ over $\mathbb{Q}$. Thus, using Gabber's lemma again, there are a surjective and generically finite morphism of normal and projective arithmetic varieties $h_2 : Y^* \to S'$ and a semiabelian scheme $G \to Y^*$ such that $G \to Y^*$ is an extension of $G' \times_{S'} h_2^{-1}(S'_0) \to h_2^{-1}(S'_0)$. Thus, over $Y^*_{\mathbb{Q}} = Y^* \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$, the semiabelian variety G is equal to $G'_{\mathbb{Q}} \times_{S'_{\mathbb{Q}}} Y^*_{\mathbb{Q}} \to Y^*_{\mathbb{Q}}$ by the uniqueness of semiabelian extensions. Thus, if we set $f = h \cdot h_1$, then $f^*(L) = \lambda_{G/Y^*}^{\otimes n}$ over $Y^*_{\mathbb{Q}}$.

Finally, since $L_{\mathbb{Q}}|_{\mathbb{A}_{g,l,m,\mathbb{Q}}} = \lambda^{\otimes n}_{U_{\mathbb{Q}}/\mathbb{A}_{g,l,m,\mathbb{Q}}}$, if we give $L_{\mathbb{Q}}$ a metric arising from Faltings' metric of $\lambda_{U_{\mathbb{Q}}/\mathbb{A}_{g,l,m,\mathbb{Q}}}$, then assertion of (4) follows from Lemma 1.5.1 and [12, Théorèm 3.2 in Exposé I]. □

1.8. Arakelov geometry

In this paper, an *arithmetic variety* means an integral scheme flat and quasi-projective over $\mathbb{Z}$. If it is smooth over $\mathbb{Q}$, then it is said to be *generically smooth.*

Let X be a generically smooth arithmetic variety. A pair (Z, g) is called an *arithmetic cycle of codimension* p if Z is a cycle of codimension p and g is a current of type $(p-1, p-1)$ on $X(\mathbb{C})$. We denote by $\widehat{Z}^p(X)$ the set of all arithmetic cycles on X. We set

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\sim,$$

where $\sim$ is the arithmetic linear equivalence.

Let $\overline{L} = (L, \|\cdot\|)$ be a C^∞-hermitian line bundle on X. Then a homomorphism

$$\widehat{c}_1(\overline{L})\cdot : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p+1}(X)$$

is define by

$$\widehat{c}_1(\overline{L}) \cdot (Z, g) = \left(\mathrm{div}(s) \text{ on } Z, [-\log(\|s\|_Z^2)] + c_1(\overline{L}) \wedge g\right),$$

where s is a rational section of $L|_Z$ and $[-\log(\|s\|_Z^2)]$ is a current given by $\phi \mapsto -\int_{Z(\mathbb{C})} \log(\|s\|_Z^2)\phi$.

When X is projective, we can define the canonical arithmetic degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{\dim X}(X) \to \mathbb{R}$$

given by

$$\widehat{\deg}\left(\sum_P n_P P, g\right) = \sum_P n_P \log(\#(\kappa(P))) + \frac{1}{2}\int_{X(\mathbb{C})} g.$$

Thus, if C^∞-hermitian line bundles $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ are given, then we can get the number

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{\dim X})\right),$$

which is called the *arithmetic intersection number of* $\overline{L}_1, \ldots, \overline{L}_{\dim X}$.

Let X be a projective arithmetic variety. Note that X is not necessarily generically smooth. Let $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ be C^∞-hermitian line bundles on X. By [6], we can find a generic resolution of singularities $\mu : Y \to X$, i.e., $\mu : Y \to X$ is a projective and birational morphism such that Y is a generically smooth projective arithmetic variety. Then we can see that the arithmetic intersection number

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{L}_1)) \cdots \widehat{c}_1(\mu^*(\overline{L}_{\dim X}))\right)$$

does not depend on the choice of the generic resolution of singularities $\mu : Y \to X$. Thus we denote this number by

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{\dim X})\right).$$

Let $\overline{L}_1, \ldots, \overline{L}_l$ be C^∞-hermitian line bundles on a projective arithmetic variety X. Let V be an l-dimensional integral closed subscheme on X. Then $\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid V\right)$ is defined by

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1\big|_V) \cdots \widehat{c}_1(\overline{L}_l\big|_V)\right).$$

Moreover, for an l-dimensional cycle $Z = \sum_i n_i V_i$ on X,

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid Z\right)$$

is defined by

$$\sum_i n_i \widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid V_i\right).$$

1.9. Notions concerning the positivity of $\mathbb{Q}$-line bundles on an arithmetic variety

Let X be a projective arithmetic variety and $\overline{L}$ a C^∞-hermitian $\mathbb{Q}$-line bundle on X. Let us introduce several kinds of the positivity of C^∞-hermitian $\mathbb{Q}$-line bundles.

• **ample**: $\overline{L}$ is *ample* if L is ample on X, $c_1(\overline{L})$ is positive form on $X(\mathbb{C})$, and there is a positive number n such that $L^{\otimes n}$ is generated by the set $\{s \in H^0(X, L^{\otimes n}) \mid \|s\|_{\sup} < 1\}$.

• **nef**: $\overline{L}$ is *nef* if $c_1(\overline{L})$ is a semipositive form on $X(\mathbb{C})$ and, for all one-dimensional integral closed subschemes Γ of X, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}) \mid \Gamma\right) \geq 0$.

• **big**: $\overline{L}$ is *big* if $\operatorname{rk}_{\mathbb{Z}} H^0(X, L^{\otimes m}) = O(m^{\dim X_{\mathbb{Q}}})$ and there is a non-zero section s of $H^0(X, L^{\otimes n})$ with $\|s\|_{\sup} < 1$ for some positive integer n.

• **$\mathbb{Q}$-effective**: $\overline{L}$ is *$\mathbb{Q}$-effective* if there is a positive integer n and a non-zero $s \in H^0(X, L^{\otimes n})$ with $\|s\|_{\sup} \leq 1$.

• **pseudo-effective**: $\overline{L}$ is *pseudo-effective* if there are (1) a sequence $\{\overline{L}_n\}_{n=1}^{\infty}$ of $\mathbb{Q}$-effective C^∞-hermitian $\mathbb{Q}$-line bundles, (2) C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{E}_1, \ldots, \overline{E}_r$ and (3) sequences

$$\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{r,n}\}_{n=1}^{\infty}$$

of rational numbers such that

$$\widehat{c}_1(\overline{L}) = \widehat{c}_1(\overline{L}_n) + \sum_{i=1}^{r} a_{i,n}\widehat{c}_1(\overline{E}_i)$$

in $\widehat{\mathrm{CH}}^1(X) \otimes \mathbb{Q}$ and $\lim_{n\to\infty} a_{i,n} = 0$ for all i. If $\overline{L}_1 \otimes \overline{L}_2^{\otimes -1}$ is pseudo-effective for C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \overline{L}_2$ on X, then we denote this by $\overline{L}_1 \succsim \overline{L}_2$.

1.10. Polarization of a finitely generated field over $\mathbb{Q}$

Let K be a field of finite type over the rational number field $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. A *polarization* $\overline{B}$ *of* K is a collection of data $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$, where

(1) B is a normal and projective arithmetic variety whose function field is isomorphic to K;

(2) $\overline{H}_1, \ldots, \overline{H}_d$ are nef C^∞-hermitian line bundles on B.

Here $\deg(\overline{B})$ is given by

$$\int_{B(\mathbb{C})} c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d).$$

Namely,

$$\deg(\overline{B}) = \begin{cases} [K : \mathbb{Q}] & \text{if } d = 0, \\ \deg((H_1)_{\mathbb{Q}} \cdots (H_d)_{\mathbb{Q}}) \text{ on } B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q}) & \text{if } d > 0. \end{cases}$$

If B is generically smooth, then the polarization $\overline{B}$ is said to be *generically smooth.* Moreover, we say the polarization $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ is *fine* (resp. *strictly fine*) if there are (a) a generically finite morphism $\pi : B' \to B$ of normal projective arithmetic varieties, (b) a generically finite morphism $\mu : B' \to (\mathbb{P}^1_{\mathbb{Z}})^d$ and (c) ample C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \ldots, \overline{L}_d$ on $\mathbb{P}^1_{\mathbb{Z}}$ such that $\pi^*(\overline{H}_i) \otimes \mu^*(p_i^*(\overline{L}_i))^{\otimes -1}$ is pseudo-effective (resp. $\mathbb{Q}$-effective) for every i, where $p_i : (\mathbb{P}^1_{\mathbb{Z}})^d \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. Note that if $\overline{H}_1, \ldots, \overline{H}_d$ are big, then the polarization $(B; \overline{H}_1, \ldots, \overline{H}_d)$ is strictly fine. Moreover, if $\overline{B}$ is fine, then $\deg(\overline{B}) > 0$.

Proposition 1.10.1. *Let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a strictly fine polarization of K. Then, for all h, the number of prime divisors Γ on B with*

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid \Gamma) \le h$$

is finite.

Proof. Let us begin with the following lemma.

Lemma 1.10.2. *Let $\pi : X' \to X$ be a generically finite morphism of normal and projective arithmetic varieties. Let $\overline{H}_1, \ldots, \overline{H}_d$ be nef C^∞-hermitian line bundles on X, where $d = \dim X_{\mathbb{Q}}$. Then the following two statements are equivalent:*

(1) *For all h, the number of prime divisors Γ on X with*

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid \Gamma) \le h$$

is finite

(2) *For all h', the number of prime divisors Γ' on X' with*

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi^*(\overline{H}_d)) \mid \Gamma') \le h'$$

is finite.

Proof. Let X_0 be the maximal Zariski open set of X such that X_0 is regular and π is finite over X_0. Then $\operatorname{codim}(X \setminus X_0) \ge 2$. We set $X'_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi|_{X'_0}$. Let $\operatorname{Div}(X)$ and $\operatorname{Div}(X')$ be the groups of Weil divisors on X and X' respectively. Define the homomorphism $\pi^\star : \operatorname{Div}(X) \to \operatorname{Div}(X')$ as the composition of natural homomorphisms:

$$\operatorname{Div}(X) \to \operatorname{Div}(X_0) \xrightarrow{\pi_0^*} \operatorname{Div}(X'_0) \to \operatorname{Div}(X'),$$

where $\operatorname{Div}(X) \to \operatorname{Div}(X_0)$ is the restriction map and $\operatorname{Div}(X'_0) \to \operatorname{Div}(X')$ is defined by taking the Zariski closure of divisors. Note that $\pi_*\pi^\star(D) = \deg(\pi)D$ for all $D \in \operatorname{Div}(X)$.

First we assume (1). Note that the number of prime divisors in $X' \setminus X'_0$ is finite, so that it is sufficient to show that the number of prime divisors Γ' on X' with $\Gamma' \not\subseteq X' \setminus X'_0$ and

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi^*(\overline{H}_d)) \mid \Gamma') \le h'$$

is finite. By the projection formula,

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi^*(\overline{H}_d)) \mid \Gamma') = \widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid \pi_*(\Gamma')).$$

Thus, by (1), the number of $(\pi_*(\Gamma'))_{\mathrm{red}}$ is finite. On the other hand, the number of prime divisors in $\pi^{-1}(\pi_*(\Gamma)_{\mathrm{red}})$ is finite. Hence we get (2).

Next we assume (2). Let Γ be a prime divisor on X with

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\,|\,\Gamma) \leq h.$$

Then

$$\begin{aligned}\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\pi^\star(\Gamma))\\ = \deg(\pi)\widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\,|\,\Gamma) \leq \deg(\pi)h.\end{aligned}$$

Thus, by (2), the number of $\pi^\star(\Gamma)$'s is finite. Therefore we get (1). □

Let us go back to the proof of Proposition 1.10.1. We use the notation in the above definition of strict finiteness. By Lemma 1.10.2, it is sufficient to show that the number of prime divisors Γ' on B' with

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma') \leq h$$

is finite for all h.

There are $\mathbb{Q}$-effective C^∞-hermitian line bundles $\overline{Q}_1,\ldots,\overline{Q}_d$ on B' with

$$\pi^*(\overline{H}_i) = \mu^*(p_i^*(\overline{L}_i))\otimes\overline{Q}_i$$

for all i. Note that

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma')\\ &\quad= \widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_d^*(\overline{L}_d)))\,|\,\Gamma')+\\ &\quad\sum_{i=1}^{d}\widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_{i-1}^*(\overline{L}_{i-1})))\cdot\widehat{c}_1(\overline{Q}_i)\cdot\\ &\qquad\widehat{c}_1(\pi^*(\overline{H}_{i+1}))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma').\end{aligned}$$

Moreover, since $\overline{Q}_i$ is $\mathbb{Q}$-effective, the number of prime divisors Γ' with

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_{i-1}^*(\overline{L}_{i-1})))\cdot\widehat{c}_1(\overline{Q}_i)\cdot\\ &\qquad\widehat{c}_1(\pi^*(\overline{H}_{i+1}))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma') < 0\end{aligned}$$

is finite for every i. Thus we have

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma')\\ &\qquad\geq \widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_d^*(\overline{L}_d)))\,|\,\Gamma')\end{aligned}$$

except finitely many Γ'. On the other hand, by [10, Proposition 5.1.1], the number of prime divisors Γ'' on $(\mathbb{P}^1_{\mathbb{Z}})^d$ with

$$\widehat{\deg}(\widehat{c}_1(p_1^*(\overline{L}_1)) \cdots \widehat{c}_1(p_d^*(\overline{L}_d)) \mid \Gamma'') \leq h$$

is finite. This completes the proof. $\square$

Remark 1.10.3. Let X be a normal and projective arithmetic variety of dimension n. Let $\overline{H}_1, \ldots, \overline{H}_{n-2}$ be nef C^∞-hermitian line bundles on X and $\overline{L}$ a C^∞-hermitian line bundle on X. If $\overline{L}$ is pseudo-effective, then we can expect the number of prime divisors Γ on X with

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_{n-2}) \cdot \widehat{c}_1(\overline{L}) \mid \Gamma) < 0$$

to be finite. If it is true, then Proposition 1.10.1 holds under the weaker assumption that the polarization is fine.

§2. Height functions in terms of hermitian line bundles with logarithmic singularities

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. Let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a fine polarization of K. Let X be a projective variety over K and L an ample line bundle on X. Moreover, let Y be a proper closed subset of X. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a pair of a projective arithmetic variety $\mathcal{X}$ and a hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ with the following properties:

(1) There is a morphism $f : \mathcal{X} \to B$ whose generic fiber is X.
(2) The restriction of $\mathcal{L}$ to the generic fiber of f coincides with L.
(3) $\mathcal{L}$ is ample with respect to the morphism $f : \mathcal{X} \to B$.
(4) Let $\mathcal{Y}$ be a closed set of $\mathcal{X}$ such that $\mathcal{Y}$ gives rise to Y on the generic fiber of $\mathcal{X} \to B$. Then the hermitian metric of $\overline{\mathcal{L}}$ has logarithmic singularities along $\mathcal{Y}(\mathbb{C})$.

For $x \in X(\overline{K}) \setminus Y(\overline{K})$, we denote by Δ_x the Zariski closure of the image of $\operatorname{Spec}(\overline{K}) \to X \to \mathcal{X}$. The height of x with respect to $\overline{B}$ and $\overline{\mathcal{L}}$ is defined by

$$h_{\overline{\mathcal{L}}}^{\overline{B}}(x) = \frac{\widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)|_{\Delta_x}) \cdots \widehat{c}_1(f^*(\overline{H}_d)|_{\Delta_x}) \cdot \widehat{c}_1(\overline{\mathcal{L}}|_{\Delta_x}))}{[K(x) : K]}.$$

Note that since $\overline{\mathcal{L}}|_{\Delta_x}$ has logarithmic singularities along $\mathcal{Y}(\mathbb{C}) \cap \Delta_x(\mathbb{C})$, the number

$$\widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)|_{\Delta_x}) \cdots \widehat{c}_1(f^*(\overline{H}_d)|_{\Delta_x}) \cdot \widehat{c}_1(\overline{\mathcal{L}}|_{\Delta_x}))$$

is well defined. Then we have the following proposition.

Proposition 2.1. (1) *Given a positive integer e, there exists a constant C such that*

$$\#\{x \in X(\overline{K}) \setminus Y(\overline{K}) \mid h^{\overline{B}}_{\overline{\mathcal{L}}}(x) \leq h,\ [K(x):K] \leq e\} \leq C \cdot h^{d+1}$$

for $h \gg 0$.

(2) *There is a constant C' such that $h^{\overline{B}}_{\overline{L}}(x) \geq C'$ for all $x \in X(\overline{K}) \setminus Y(\overline{K})$.*

Proof. We denote by $\|\cdot\|$ the hermitian metric of $\overline{\mathcal{L}}$. Let $\overline{Q}$ be an ample C^∞-hermitian line bundle on B. Then

$$h^{\overline{B}}_{\overline{\mathcal{L}}\otimes f^*(\overline{Q}^{\otimes n})}(x) = h^{\overline{B}}_{\overline{\mathcal{L}}}(x) + n\widehat{\deg}(\widehat{c}_1(\overline{Q}) \cdot \widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d)).$$

and we may assume that $\mathcal{L}$ is ample on $\mathcal{X}$ without loss of generality. Replacing $\overline{\mathcal{L}}$ by a suitable $\overline{\mathcal{L}}^{\otimes n}$, we may furthermore assume that $\mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L}$ is generated by global sections, where $\mathcal{I}_{\mathcal{Y}}$ is the defining ideal sheaf of $\mathcal{Y}$. Let $s_1, \ldots, s_r$ be generators of $H^0(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L})$. We may view $s_1, \ldots, s_r$ as global sections of $H^0(\mathcal{X}, \mathcal{L})$. Then $\mathcal{Y} = \{x \in \mathcal{X} \mid s_1(x) = \cdots = s_r(x) = 0\}$. Here we choose a C^∞-hermitian metric $\|\cdot\|_0$ of $\mathcal{L}$ such that $\|s_i\|_0 < 1$ for all $i = 1, \ldots, r$. We denote $(\mathcal{L}, \|\cdot\|_0)$ by $\overline{\mathcal{L}}^0$.

We claim

$$\begin{aligned}[K(x):K]&h^{\overline{B}}_{\overline{\mathcal{L}}^0}(x) \\ &\geq -\int_{\Delta_x(\mathbb{C})} \log\left(\max_i\{\|s_i\|_0\}\right) c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)).\end{aligned}$$

Indeed we can find s_j with $s_j|_{\Delta_x} \neq 0$, so that

$$\begin{aligned}[K(x):K]h^{\overline{B}}_{\overline{\mathcal{L}}^0}(x) &= \widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \operatorname{div}(s_j|_{\Delta_x})) \\ &\quad - \int_{\Delta_x(\mathbb{C})} \log(\|s_j\|_0)\, c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)).\end{aligned}$$

Then our claim follows from the following two inequalities:

$$\widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \operatorname{div}(s_j|_{\Delta_x})) \geq 0$$

and

$$\|s_j\|_0 \leq \max_i\{\|s_i\|_0\} < 1.$$

Set $g = \|\cdot\|/\|\cdot\|_0$. Since $\|\cdot\|$ has logarithmic singularities, there are positive constants a, b such that

$$|\log(g)| \leq a + b\log\left(-\log(\max_i\{\|s_i\|_0\})\right).$$

Moreover

$$\begin{aligned}&\left|h^{\overline{B}}_{\mathcal{L}}(x) - h^{\overline{B}}_{\mathcal{L}^0}(x)\right| \\ &\qquad \leq \frac{1}{[K(x):K]}\int_{\Delta_x(\mathbb{C})} |\log(g)| c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)).\end{aligned}$$

Note that

$$\int_{\Delta_x(\mathbb{C})} c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)) = [K(x):K]\deg(\overline{B}),$$

where $\deg(\overline{B}) = \displaystyle\int_{B(\mathbb{C})} c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d)$ as in Section 1.10. Thus

$$\begin{aligned}&\frac{\left|h^{\overline{B}}_{\mathcal{L}}(x) - h^{\overline{B}}_{\mathcal{L}^0}(x)\right|}{\deg(\overline{B})} \leq a + \\ &\quad b\int_{\Delta_x(\mathbb{C})} \log\left(-\log(\max_i\{\|s_i\|_0\})\right) \frac{c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d))}{[K(x):K]\deg(\overline{B})}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{\Delta_x(\mathbb{C})} \log\left(-\log(\max_i\{\|s_i\|_0\})\right) \frac{c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d))}{[K(x):K]\deg(\overline{B})} \\ &\leq \log\left(\int_{\Delta_x(\mathbb{C})} -\log(\max_i\{\|s_i\|_0\}) \frac{c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d))}{[K(x):K]\deg(\overline{B})}\right).\end{aligned}$$

Hence we obtain

$$\frac{\left|h^{\overline{B}}_{\mathcal{L}}(x) - h^{\overline{B}}_{\mathcal{L}^0}(x)\right|}{\deg(\overline{B})} \leq a + b\log\left(\frac{h^{\overline{B}}_{\mathcal{L}^0}(x)}{\deg(\overline{B})}\right).$$

Note that there is a real number t_0 such that $a + b\log(t) \leq t/2$ for all $t \geq t_0$. Thus

$$h^{\overline{B}}_{\mathcal{L}^0}(x) \leq \max\left\{\deg(\overline{B})t_0, 2h^{\overline{B}}_{\mathcal{L}}(x)\right\}.$$

Therefore, if $h \geq \deg(\overline{B})t_0/2$, then $h^{\overline{B}}_{\overline{\mathcal{L}}}(x) \leq h$ implies $h^{\overline{B}}_{\overline{\mathcal{L}}^0}(x) \leq 2h$. Hence we get the first assertion by virtue of [10, Theorem 6.4.1].

Next let us check the second assertion. Since

$$
\begin{aligned}
\|s_i\| = g\|s_i\|_0 \leq \exp(a)\|s_i\|_0 &\left(-\log(\max_j\{\|s_j\|_0\})\right)^b \\
&\leq \exp(a)\|s_i\|_0 \left(-\log(\|s_i\|_0)\right)^b
\end{aligned}
$$

and the function $t(-\log(t))^b$ is bounded from above for $0 < t \leq 1$, there is a constant C such that $\|s_i\| \leq C$ for all i. Thus, if we choose s_i with $s_i|_{\Delta_x} \neq 0$, then

$$
\begin{aligned}
[K(x):K]h^{\overline{B}}_{\overline{\mathcal{L}}}(x) &= \widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \operatorname{div}(s_i|_{\Delta_x})) \\
&\quad - \int_{\Delta_x(\mathbb{C})} \log(\|s_j\|)\, c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)) \\
&\geq -\log(C) \int_{\Delta_x(\mathbb{C})} c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)) \\
&= -\log(C) \deg(\overline{B})[K(x):K].
\end{aligned}
$$

Thus we get (2). □

§3. The Faltings modular height

Let K be a field of finite type over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a generically smooth polarization of K. Let A be a g-dimensional abelian variety over K. Let $\lambda(A/K; B)$ be the Hodge sheaf of A with respect to B (cf. Section 1.3). Note that $\lambda(A/K; B)$ is invertible over $B_{\mathbb{Q}}$ because $B_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$. Let $\|\cdot\|_{\mathrm{Fal}}$ be Faltings' metric of $\lambda(A/K; B)$ over $B(\mathbb{C})$. Here we set

$$
\overline{\lambda}^{\mathrm{Fal}}(A/K; B) = (\lambda(A/K; B), \|\cdot\|_{\mathrm{Fal}}),
$$

which is called *the metrized Hodge sheaf of A with respect to B*. In the case where a Néron model $\mathcal{A} \to U$ over B in codimension one is specified, $\overline{\lambda}^{\mathrm{Fal}}(A/K; B)$ is often denoted by $\overline{\lambda}^{\mathrm{Fal}}(\mathcal{A}/U)$. By Lemma 1.6.1, the metric of $\overline{\lambda}^{\mathrm{Fal}}(A/K; B)$ is locally integrable. The *Faltings modular height of A with respect to the polarization $\overline{B}$* is defined by

$$
h^{\overline{B}}_{\mathrm{Fal}}(A) = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \cdot \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; B))\right).
$$

Even if we do not assume that $\overline{B}$ is generically smooth, we can define the Faltings modular height with respect to $\overline{B}$ as follows: Let $\mu : B' \to B$ be a generic resolution of singularities of B. We set $\overline{B}' = (B'; \mu^*(\overline{H}_1), \ldots, \mu^*(\overline{H}_d))$. Then, by (1) of the following Proposition 3.1, $h^{\overline{B}'}_{\mathrm{Fal}}(A)$ does not depend on the choice of the generic resolution $\mu : B' \to B$, so that $h^{\overline{B}}_{\mathrm{Fal}}(A)$ is defined to be $h^{\overline{B}'}_{\mathrm{Fal}}(A)$. In the following, $\overline{B}$ is always assumed to be generically smooth.

Proposition 3.1. *Let $\pi : X' \to X$ be a generically finite morphism of normal and projective generically smooth arithmetic varieties. Let K and K' be the function field of X and X' respectively. Let A be an abelian variety over K. Then there is an effective divisor E on X which has the following two properties:*

(1) $$\pi_* \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_K \operatorname{Spec}(K')/K'; X')) + (E, 0) = \deg(\pi)\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; X)).$$
Further, if π is birational, then $E = 0$.

(2) *For a scheme S, we denote by $S^{(1)}$ the set of points of codimension one in S. Then*
$$\{x \in X^{(1)} \mid A \text{ has semiabelian reduction at } x\} \subseteq (X \setminus \operatorname{Supp}(E))^{(1)}.$$
Moreover, if $A \times_K \operatorname{Spec}(K')$ has semiabelian reduction over X' in codimension one, then
$$\{x \in X^{(1)} \mid A \text{ has semiabelian reduction at } x\} = (X \setminus \operatorname{Supp}(E))^{(1)}.$$

Proof. (1) Let X_0 be the maximal Zariski open set of X such that X_0 is regular and π is finite over X_0. Then $\operatorname{codim}(X \setminus X_0) \geq 2$. We set $X'_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi|_{X'_0}$. Let $\operatorname{Div}(X)$ and $\operatorname{Div}(X')$ be the groups of Weil divisors on X and X' respectively. A homomorphism $\pi^\star : \operatorname{Div}(X) \to \operatorname{Div}(X')$ is defined as the composition of the natural homomorphisms:

$$\operatorname{Div}(X) \to \operatorname{Div}(X_0) \xrightarrow{\pi_0^*} \operatorname{Div}(X'_0) \to \operatorname{Div}(X'),$$

where $\operatorname{Div}(X) \to \operatorname{Div}(X_0)$ is the restriction map and $\operatorname{Div}(X'_0) \to \operatorname{Div}(X')$ is defined by taking the Zariski closure of divisors. Note that $\pi_*\pi^\star(D) = \deg(\pi)D$ for all $D \in \operatorname{Div}(X)$.

Let X_1 (resp. X'_1) be a Zariski open set of X (resp. X') such that $\operatorname{codim}(X \setminus X_1) \geq 2$ (resp. $\operatorname{codim}(X' \setminus X'_1) \geq 2$) and that the Néron model G (resp. G') exists over X_1 (resp. X'_1). Clearly we may assume that $X_1 \subseteq X_0$ and $\pi^{-1}(X_1) \subseteq X'_1$. We set $U' = \pi^{-1}(X_1)$ and $G'_{U'} = G'|_{U'}$. Since $G'_{U'}$ is the Néron model of $A \times_K \operatorname{Spec}(K')$ over

U', there is a homomorphism $G \times_{X_1} U' \to G'_{U'}$ over U'. Thus we get a homomorphism

$$\alpha : \epsilon'^* \left(\bigwedge^g \Omega_{G'_{U'}/U'} \right) \to \pi^* \epsilon^* \left(\bigwedge^g \Omega_{G/X_1} \right)\Bigg|_{U'}, \tag{3.1.1}$$

where ϵ and ϵ' are the zero sections of G and G' respectively.

Let s be a non-zero rational section of $\lambda(A/K; X)$. Then

$$\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; X)) = (\mathrm{div}(s), -\log \|s\|_{\mathrm{Fal}}).$$

Moreover, since $\pi^*(s)$ gives rise to a non-zero rational section of $\lambda(A \times_K \mathrm{Spec}(K')/K'; X')$,

$$\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_K \mathrm{Spec}(K')/K'; X')) = (\mathrm{div}(\pi^*(s)), -\pi^*(\log \|s\|_{\mathrm{Fal}})),$$

where $\pi^*(\log \|s\|_{\mathrm{Fal}})$ is the pull-back of $\log \|s\|_{\mathrm{Fal}}$ by π as a function on a dense open set of $X(\mathbb{C})$. Let $\Gamma_1, \ldots, \Gamma_r$ be all prime divisors in $X' \setminus U'$. Note that $\pi_*(\Gamma_i) = 0$ for all i. Then, since (3.1.1) is injective, there is an effective divisor E' and integers $a_1, \ldots, a_r$ such that

$$\mathrm{div}(\pi^*(s)) + E' = \pi^\star(\mathrm{div}(s)) + \sum_{i=1}^r a_i \Gamma_i.$$

Note that $E' = \sum_{x'} \mathrm{length}_{\mathcal{O}_{X',x'}}(\mathrm{Coker}(\alpha)_{x'})\overline{\{x'\}}$, where x''s run over all points of codimension one in U'. Thus, since

$$\pi_*(\pi^\star(\mathrm{div}(s)), -\pi^*(\log \|s\|_{\mathrm{Fal}})) = \deg(\pi)(\mathrm{div}(s), -\log \|s\|_{\mathrm{Fal}}),$$

we have

$$\begin{aligned} \pi_*\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_K \mathrm{Spec}(K')/K'; X')) + (\pi_*(E'), 0) \\ = \deg(\pi)\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; X)), \end{aligned}$$

yielding the first assertion of (1). If π is birational, then $U' \to X_1$ is an isomorphism, so that $E' = 0$.

(2) Assume that there is an open neighborhood U of x such that $G^{\mathrm{o}}|_U$ is semiabelian. Then $G^{\mathrm{o}}|_U \times_U \pi^{-1}(U)$ is semiabelian so that it is isomorphic to $\left(G'|_{\pi^{-1}(U)}\right)^{\mathrm{o}}$. This shows that $x \notin E_{\mathrm{red}}$. Conversely, if $x \notin E_{\mathrm{red}}$ and $A \times_K \mathrm{Spec}(K')$ has semiabelian reduction in codimension

one, then there exists an open neighborhood $U \subset X_1$ of x such that the homomorphism

$$\alpha : \epsilon'^{*}\left(\bigwedge^{g} \Omega_{G'_{U'}/U'}\right) \to \pi^{*}\epsilon^{*}\left(\bigwedge^{g} \Omega_{G/X_1}\right)\Bigg|_{U'}$$

is an isomorphism over $\pi^{-1}(U)$. Thus the natural homomorphism

$$\epsilon'^{*}\left(\Omega_{G'_{U'}/U'}\right) \to \pi^{*}\epsilon^{*}\left(\Omega_{G/X_1}\right)\big|_{U'}$$

must be an isomorphism over $\pi^{-1}(U)$ and so is the morphism

$$G^{\mathrm{o}} \times_{X_1} U' \to (G'_{U'})^{\mathrm{o}}$$

over $\pi^{-1}(U)$, which means that G^{o} is semiabelian over U. $\square$

Proposition 3.2. *Let $\phi : A \to A'$ be an isogeny of abelian varieties over K. Then*

$$\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A'/K;B))\right) \\
&\qquad - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K;B))\right) \\
&\qquad = \frac{1}{2}\log(\deg(\phi))\deg(\overline{B}) - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\mid D_\phi\right),
\end{aligned}$$

where D_ϕ is an effective divisor given in Section 1.3 *and*

$$\deg(\overline{B}) = \int_{B(\mathbb{C})} c_1(\overline{H}_1)\wedge\cdots\wedge c_1(\overline{H}_d)$$

as in Section 1.10.

Proof. This follows from the fact that

$$\overline{\lambda}^{\mathrm{Fal}}(A'/K;B)\otimes(\mathcal{O}_B(D_\phi),\deg(\phi)|\cdot|_{\mathrm{can}})$$

is isometric to $\overline{\lambda}^{\mathrm{Fal}}(A/K;B)$. $\square$

Proposition 3.3. *If an abelian variety A over K has semiabelian reduction in codimension one over B. Then*

$$\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K;B))\right) \\
&\qquad = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A^\vee/K;B))\right),
\end{aligned}$$

where $A^\vee$ is the dual abelian variety of A.

Proof. Let $\phi : A \to A^\vee$ be an isogeny over K in terms of ample line bundle on A. Let $\phi^\vee : A \to A^\vee$ be the dual of ϕ. Then, by Proposition 3.2,

$$\begin{aligned}
&2\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A^\vee/K;B))\right)\\
&\qquad - 2\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K;B))\right)\\
&\qquad = \log(\deg(\phi))\deg(\overline{B}) - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\mid D_\phi + D_{\phi^\vee}\right).
\end{aligned}$$

On the other hand, by Lemma 1.3.3, $\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee} = \deg(\phi)\mathcal{O}_B$. $(\mathcal{O}_B(D_\phi + D_{\phi^\vee}), |\cdot|_{\mathrm{can}})$ is thus isometric to $(\mathcal{O}_B, \deg(\phi)^{-2}|\cdot|_{\mathrm{can}})$, proving the assertion. □

Let A be an abelian variety over a finite extension field K' of K. Let m be a positive integer such that m has a decomposition $m = m_1 m_2$ with $(m_1, m_2) = 1$ and $m_1, m_2 \geq 3$. Let us consider a natural homomorphism

$$\rho(A, m) : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(A[m](\overline{K})) \simeq \mathrm{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}).$$

Then there is a Galois extension $K(A, m)$ of K' with $\mathrm{Ker}\,\rho(A, m) = \mathrm{Gal}(\overline{K}/K(A, m))$. Note that

$$\mathrm{Gal}(K(A, m)/K') = \mathrm{Gal}(\overline{K}/K)/\mathrm{Ker}\,\rho(A, m) \hookrightarrow \mathrm{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}).$$

Let B'' be a generically smooth, normal and projective arithmetic variety with the following properties:

(i) The function field K'' of B'' is an extension of $K(A, m)$.
(ii) The natural rational map $f : B'' \to B$ induced by $K \hookrightarrow K''$ is actually a morphism.

Then we have the following.

Proposition 3.4. (1) *The number*

$$\frac{\widehat{\deg}\left(\widehat{c}_1(\lambda(A \times_{K'} \mathrm{Spec}(K'')/K''; B'')\cdot\widehat{c}_1(f^*(\overline{H}_1))\cdots\widehat{c}_1(f^*(\overline{H}_1))\right)}{[K'' : K]}$$

does not depend on the choice of m and B'', so that we denote it by $h^{\overline{B}}_{\mathrm{mod}}(A)$.

(2) $h^{\overline{B}}_{\mathrm{mod}}(A) \leq h^{\overline{B}}_{\mathrm{Fal}}(A)$.

Proof. These are consequences of Proposition 1.2.1, Proposition 3.1 and the projection formula. □

Proposition 3.5 (Additivity of heights). *For abelian varieties A, A' over K, we have*

$$h^{\overline{B}}_{\mathrm{Fal}}(A \times_K A') = h^{\overline{B}}_{\mathrm{Fal}}(A) + h^{\overline{B}}_{\mathrm{Fal}}(A'),$$
$$h^{\overline{B}}_{\mathrm{mod}}(A \times_K A') = h^{\overline{B}}_{\mathrm{mod}}(A) + h^{\overline{B}}_{\mathrm{mod}}(A').$$

Proof. Let $\mathcal{A}$ and $\mathcal{A}'$ be the Néron models of A and A' over B_0, where B_0 is a big open set of B. Then $\mathcal{A} \times_{B_0} \mathcal{A}'$ is the Néron model of $A \times_K A'$ over B_0. Thus

$$\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{\mathcal{A} \times_{B_0} \mathcal{A}'/B_0}) = \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{\mathcal{A}/B_0}) + \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{\mathcal{A}'/B_0}).$$

Hence we get our lemma. □

§4. Weak finiteness

Let us fix positive integers g, l and m such that m has a decomposition $m = m_1 m_2$ with $(m_1, m_2) = 1$ and $m_1, m_2 \geq 3$. Let $\mathbb{A}_{g,l,m,\mathbb{Q}}$, $f : Y \to \mathbb{A}^*_{g,l,m}$, $\overline{L}$, n and $G \to Y$ be as in Theorem 1.7.1.

Let K be a field of finite type over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a generically smooth polarization of K.

Let A be a g-dimensional and l-polarized abelian variety over a finite extension K' of K with level m structure. The abelian variety A naturally induces a morphism $x_A : \operatorname{Spec}(K') \to \mathbb{A}^*_{g,l,m}$, which in turn induces $y_A : \operatorname{Spec}(K') \to \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} \operatorname{Spec}(K)$. Let Δ_A be the closure of the image of y_A in $\mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B$. Let $p : \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B \to \mathbb{A}^*_{g,l,m}$ and $q : \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B \to B$ be the projections to the first factor and the second factor respectively. The number

$$h^{\overline{B}}_{\overline{L}}(A) = \frac{\widehat{\deg}\left(\widehat{c}_1(q^*(\overline{H}_1)|_{\Delta_A}) \cdots \widehat{c}_1(q^*(\overline{H}_d)|_{\Delta_A}) \cdot \widehat{c}_1(p^*(\overline{L})|_{\Delta_A})\right)}{\deg(\Delta_A \to B)}$$

is the height of $y_A \in (\mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} \operatorname{Spec}(K))(\overline{K})$ with respect to $\overline{L}$ and $\overline{B}$, of which the behavior is controlled by the following proposition.

Proposition 4.1. *There is a constant $N(g,l,m)$ depending only on g, l, m such that*

$$|h^{\overline{B}}_{\overline{L}}(A) - n h^{\overline{B}}_{\mathrm{mod}}(A)| \leq \log(N(g,l,m)) \deg(\overline{B}).$$

for every g-dimensional and l-polarized abelian variety A over $\overline{K}$ with level m structure, where

$$\deg(\overline{B}) = \int_{B(\mathbb{C})} c_1(\overline{H}_1) \wedge \cdots c_1(\overline{H}_d).$$

Proof. Let A be a g-dimensional and l-polarized abelian variety over $\overline{K}$ with level m structure. Let K' be the minimal finite extension of K such that A, the polarization of A, the level m structure of A are defined over K'. Let $x_A : \mathrm{Spec}(K') \to \mathbb{A}^*_{g,l,m}$ be the morphism induced by A. Moreover let $y_A : \mathrm{Spec}(K') \to \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B$ be the induced morphism by x_A.

Let $\mathrm{Spec}(K_1)$ be a closed point of $Y \times_{\mathbb{A}^*_{g,l,m}} \mathrm{Spec}(K')$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Y & \longleftarrow & \mathrm{Spec}(K_1) \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathbb{A}^*_{g,l,m} & \xleftarrow{x_A} & \mathrm{Spec}(K') \end{array}$$

Here, two l-polarized abelian varieties $A \times_{K'} \mathrm{Spec}(K_1)$ and $G \times_Y \mathrm{Spec}(K_1)$ with level m structures gives rise to the same K_1-valued point of $\mathbb{A}^*_{g,l,m}$. Thus $A \times_{K'} \mathrm{Spec}(K_1)$ is isomorphic to $G \times_Y \mathrm{Spec}(K_1)$ over K_1 as l-polarized abelian varieties with level m structures because $m \geq 3$. The above commutative diagram induces to the commutative diagram:

$$\begin{array}{ccc} Y \times_{\mathbb{Z}} B & \longleftarrow & \mathrm{Spec}(K_1) \\ \downarrow & & \downarrow \\ \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B & \xleftarrow{y_A} & \mathrm{Spec}(K') \end{array}$$

Let B_1 be a generic resolution of singularities of the normalization of B in K_1. Note that a generic resolution of singularities (a resolution of singularities over $\mathbb{Q}$) exists by Hironaka's theorem [6]. Then we have rational maps $B_1 \dashrightarrow Y \times_{\mathbb{Z}} B$ and $B_1 \dashrightarrow \Delta_A$ such that a composition $B_1 \dashrightarrow \Delta_A \to \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B$ of rational maps is equal to $B_1 \dashrightarrow Y \times_{\mathbb{Z}} B \to \mathbb{A}^*_{g,m} \times_{\mathbb{Z}} B$. Thus there are a birational morphism $B_2 \to B_1$ of projective and generically smooth arithmetic varieties, a morphism $B_2 \to \Delta_A$ and a morphism $B_2 \to Y \times_{\mathbb{Z}} B$ with the following commutative diagram:

$$\begin{array}{ccccc} B_1 & \xleftarrow{\gamma} & B_2 & \xrightarrow{\beta} & Y \times_{\mathbb{Z}} B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f\times \mathrm{id}} \\ B & \longleftarrow & \Delta_A & \xrightarrow{\iota} & \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B \end{array}$$

Then

$$h_{\overline{L}}^{\overline{B}}(A) = \frac{\widehat{\deg}\left(\widehat{c}_1(\iota^*(p^*(\overline{L}))) \cdot \widehat{c}_1(\iota^*(q^*(\overline{H}_1))) \cdots \widehat{c}_1(\iota^*(q^*(\overline{H}_1)))\right)}{\deg(\Delta_A \to B)}$$

$$= \frac{\widehat{\deg}\left(\widehat{c}_1(\alpha^*(\iota^*(p^*(\overline{L})))) \cdot \widehat{c}_1(\alpha^*(\iota^*(q^*(\overline{H}_1)))) \cdots \widehat{c}_1(\alpha^*(\iota^*(q^*(\overline{H}_1))))\right)}{\deg(B_2 \to B)}$$

$$= \frac{\widehat{\deg}\left(\widehat{c}_1(\beta^*((f \times \mathrm{id})^*(p^*(\overline{L})))) \cdot \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \cdots \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\right)}{\deg(B_2 \to B)}.$$

On the other hand, since $f^*(L) = \lambda_{G/Y}^{\otimes n}$ over $Y \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Q})$, there is an integer N depending only on g, l and m such that

$$N f^*(L) \subseteq \lambda_{G/Y}^{\otimes n} \subseteq (1/N) f^*(L)$$

on Y. Thus

$$N\beta^*(f \times \mathrm{id})^*(L) \subseteq (\lambda_{G\times_{\mathbb{Z}} B/Y \times_{\mathbb{Z}} B})^{\otimes n} \subseteq (1/N)\beta^*(f \times \mathrm{id})^*(L).$$

Therefore

$$-\frac{\widehat{\deg}(\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \cdots \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \mid (N))}{\deg(B_2 \to B)} + h_{\overline{L}}^{\overline{B}}(A)$$

$$\leq \frac{n\widehat{\deg}\left(\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{G\times_Y B_2/B_2})) \cdot \widehat{c}_1(\gamma^*(\pi_1{}^*(\overline{H}_1))) \cdots \widehat{c}_1(\gamma^*(\pi_1{}^*(\overline{H}_1)))\right)}{\deg(B_2 \to B)}$$

$$\leq \frac{\widehat{\deg}(\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \cdots \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \mid (N))}{\deg(B_2 \to B)} + h_{\overline{L}}^{\overline{B}}(A).$$

Note that

$$\widehat{\deg}(\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \cdots \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \mid (N)) = \log(N) \deg(B_2 \to B) \deg(\overline{B}).$$

By Proposition 1.2.1, we can see that $A \times_{K'} \mathrm{Spec}(K_1)$ has semiabelian reduction in codimension one over B_1. On the other hand, by Proposition 3.1,

$$\gamma_*(\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{G\times_Y B_2/B_2})) = \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_{K'} \mathrm{Spec}(K_1)/K_1; B_1)).$$

Therefore we get

$$|h_{\overline{L}}^{\overline{B}}(A) - n h_{\mathrm{mod}}^{\overline{B}}(A)| \leq \log(N) \deg(\overline{B}).$$

□

Corollary 4.2. *Let l and e be positive integers and let K be a field finitely generated over $\mathbb{Q}$. Put $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and fix a generically smooth and fine polarization $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ of K. Then*

(1) *There exists a constant $C = C(\overline{B}, l, g)$ such that $h^{\overline{B}}_{\mathrm{mod}}(A) \geq C$ for an arbitrary l-polarized abelian variety A of dimension g over $\overline{K}$.*

(2) *There exists a constant $C' = C'(\overline{B}, l, e, g)$ such that the set*

$$\left\{ A \times_{K'} \operatorname{Spec}(\overline{K}) \;\middle|\; \begin{array}{l} A \textit{ is a } g\textit{-dimensional and} \\ l\textit{-polarized abelian variety} \\ \textit{over a finite extension} \\ K' \textit{ of } K \textit{ with } [K' : K] \leq e \\ \textit{and } h^{\overline{B}}_{\mathrm{mod}}(A) \leq h. \end{array} \right\} \Big/ \simeq_{\overline{K}}$$

has cardinality $\leq C' \cdot h^{d+1}$ for $h \gg 0$.

Proof. Let us fix a positive number m such that m has a decomposition $m = m_1 m_2$ with $(m_1, m_2) = 1$ and $m_1, m_2 \geq 3$. Then any l-polarized abelian variety over $\overline{K}$ has a level m structure. Thus (1) is a consequence of Proposition 2.1 and Proposition 4.1.

Let A be an l-polarized abelian variety over a finite extension K' of K. Let K'' be the minimal extension of K' such that $A[m](\overline{K}) \subseteq A(K'')$. Then $[K'' : K'] \leq \#(\operatorname{Aut}(\mathbb{Z}/m\mathbb{Z})^{2g})$. Thus, by using Proposition 2.1 and Proposition 4.1, we get (2). □

§5. Galois descent

Let A be a g-dimensional abelian variety over a field k. Let m be a positive integer prime to the characteristic of k. Note that a level m structure α of A over a finite extension k' of k is an isomorphism $\alpha : (\mathbb{Z}/m\mathbb{Z})^{2g} \to A[m](k')$. If k' is a finite Galois extension over k, then we have a homomorphism

$$\epsilon(k'/k, A, \alpha) : \operatorname{Gal}(k'/k) \to \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g})$$

given by $\epsilon(k'/k, A, \alpha)(\sigma) = \alpha^{-1} \cdot \sigma_A \cdot \alpha$, where

$$\sigma_A : A \times_k \operatorname{Spec}(k') \xrightarrow{\mathrm{id}_A \times (\sigma^{-1})^a} A \times_k \operatorname{Spec}(k')$$

is the natural morphism arising from σ. Note that $(\sigma \cdot \tau)_A = \sigma_A \cdot \tau_A$.

Lemma 5.1. *Let k be a field, k' a finite Galois extension of k and $m \geq 3$ an integer prime to the characteristic of k. Let (A, ξ) and (A', ξ')*

be two polarized abelian varieties over k and let α, α' be level m structures of A, A' defined over k'. If a k'-isomorphism

$$\phi : (A, \xi) \times_k \operatorname{Spec}(k') \to (A', \xi') \times_k \operatorname{Spec}(k')$$

as polarized abelian varieties over k' satisfies

(a) $\phi \cdot \alpha = \alpha'$ *and*
(b) $\epsilon(k'/k, A, \alpha) = \epsilon(k'/k, A', \alpha')$,

then ϕ descends to an isomorphism $(A, \xi) \to (A', \xi')$ over k.

Proof. For $\sigma \in \operatorname{Gal}(k'/k)$, let us consider a morphism

$$\phi_\sigma = \sigma_{A'}^{-1} \cdot \phi \cdot \sigma_A : A \times_k \operatorname{Spec}(k') \to A' \times_k \operatorname{Spec}(k').$$

First of all, ϕ_σ is a morphism over k'. We claim that $\phi_\sigma \cdot \alpha = \alpha'$. Indeed, since $\alpha^{-1} \cdot \sigma_A \alpha = {\alpha'}^{-1} \cdot \sigma_{A'} \cdot \alpha'$, we have

$$\phi_\sigma \cdot \alpha = \sigma_{A'}^{-1} \cdot \phi \cdot \alpha \cdot \alpha^{-1} \cdot \sigma_A \cdot \alpha = \sigma_{A'}^{-1} \cdot \alpha' \cdot {\alpha'}^{-1} \cdot \sigma_{A'} \cdot \alpha' = \alpha'.$$

Thus ϕ_σ preserves the level structures of $A \times_k \operatorname{Spec}(k')$ and $A' \times_k \operatorname{Spec}(k')$. Hence, since $m \geq 3$ and $\phi_\sigma \cdot \phi^{-1}$ preserve the polarization ξ of A over k' (hence $(\phi_\sigma \cdot \phi^{-1})^N = \mathrm{id}$ for $N \gg 1$), by virtue of Serre's theorem, we have $\phi_\sigma = \phi$, that is,

$$\phi \cdot \sigma_A = \sigma_{A'} \cdot \phi$$

for all $\sigma \in \operatorname{Gal}(k'/k)$. Therefore ϕ descends to an isomorphism $(A, \xi) \to (A', \xi')$ over k. $\square$

Proposition 5.2. *Let B be an irreducible normal scheme of finite type over $\mathbb{Z}$ and let K denote its function field. Fix a polarized abelian variety (C, ξ_C) of dimension g defined over $\overline{K}$. Then the set*

$$\mathcal{S} = \left\{ (A, \xi) \;\middle|\; \begin{array}{l} (A, \xi) \textit{ is a polarized abelian variety over } K \textit{ with} \\ (A, \xi) \times_K \operatorname{Spec}(\overline{K}) \simeq (C, \xi_C) \textit{ and } A \textit{ has semiabelian} \\ \textit{reduction over } B \textit{ in codimension one.} \end{array} \right\}$$

modulo K-isomorphisms is finite.

Proof. For $(A, \xi) \in \mathcal{S}$, let B_A be a big open set of B over which we have a semiabelian extension $\mathcal{X}_A \to B_A$ of A. Let $BR(A)$ denote the set of points x of codimension one in B_A such that the fiber of $\mathcal{X}_A$ over x is not an abelian variety.

Claim 5.2.1. *For any $(A, \xi), (A', \xi') \in \mathcal{S}$, $BR(A) = BR(A')$.*

Since $A \times_K \operatorname{Spec}(\overline{K}) \simeq A' \times_K \operatorname{Spec}(\overline{K})$, there is a finite extension K' of K with $A \times_K \operatorname{Spec}(K') \simeq A' \times_K \operatorname{Spec}(K')$. Let $\pi : B' \to B$ be the normalization of B in K'. Then $\mathcal{X}_A \times_{B_A} \pi^{-1}(B_A)$ is isomorphic to $\mathcal{X}_{A'} \times_{B_{A'}} \pi^{-1}(B_{A'})$ over $\pi^{-1}(B_A \cap B_{A'})$, so that $\pi^{-1}(BR(A)) = \pi^{-1}(BR(A'))$, yielding the claim.

Let us fix a positive integer $m \geq 3$ and $(A_0, \xi_0) \in \mathcal{S}$. We set

$$U = B \setminus \left((B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}/m\mathbb{Z})) \cup \operatorname{Sing}(B) \cup \bigcup_{x \in BR(A_0)} \overline{\{x\}} \right).$$

Then U is regular and of finite type over $\mathbb{Z}$. The characteristic of the residue field of any point of U is prime to m. For $(A, \xi) \in \mathcal{S}$, let U_A be the maximal Zariski open set of U over which $\mathcal{X}_A$ is an abelian scheme. By the above claim, $\operatorname{codim}(U \setminus U_A) \geq 2$.

Claim 5.2.2. *There exists a finite Galois extension K' of K such that every m-torsion point of A is defined over K' whenever $(A, \xi) \in \mathcal{S}$.*

For $(A, \xi) \in \mathcal{S}$, let K_A be the finite extension of K obtaining by adding all m-torsion points of A to K. Let V_A be the normalization of U in K_A. It is well-known that V_A is étale over U_A. Moreover, by virtue of the purity of branch loci (cf. SGA 1, Exposé X, Thérème 3.1), V_A is étale over U because $\operatorname{codim}(U \setminus U_A) \geq 2$. Let M be the union of the finite extensions K' of K such that the normalization of U in K' is étale over U. By construction, M is a Galois extension of K. Since $K_A \subseteq M$, we have a continuous homomorphism

$$\rho_A : \operatorname{Gal}(M/K) \to \operatorname{Aut}(A[m](\overline{K})) \simeq \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g})$$

such that $\ker(\rho_A) = \operatorname{Gal}(M/K_A)$. Since $\operatorname{Gal}(M/K) = \pi_1(U)$, by [3, Hermite-Minkowski theorem in Chapter VI], we have only finitely many continuous homomorphisms

$$\rho : \operatorname{Gal}(M/K) \to \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}).$$

Thus there are only finitely many choices of Galois subgroups $\operatorname{Gal}(M/K_A) \subseteq \operatorname{Gal}(M/K)$ and of subfields $K_A \subseteq M$. This shows our claim.

Claim 5.2.3. *For any $(A, \xi), (A', \xi') \in \mathcal{S}$, $(A, \xi) \times_K \operatorname{Spec}(K') \simeq (A', \xi') \times_K \operatorname{Spec}(K')$.*

There is a finite Galois extension K'' of K' such that an isomorphism

$$\phi : (A, \xi) \times_K \operatorname{Spec}(K'') \to (A', \xi') \times_K \operatorname{Spec}(K'')$$

is given over K''. Let α be a level m structure of A over K'' and $\alpha' = \phi\cdot\alpha$. Then $\epsilon(K''/K', A\times_K \operatorname{Spec}(K'), \alpha) = \epsilon(K''/K', A'\times_K \operatorname{Spec}(K'), \alpha') = 1$ because all m-torsion points of A and A' are defined over K'. Thus $A\times_K \operatorname{Spec}(K'') \to A'\times_K \operatorname{Spec}(K'')$ descends to an isomorphism $(A,\xi)\times_K \operatorname{Spec}(K') \to (A',\xi')\times_K \operatorname{Spec}(K')$ by Lemma 5.1.

Finally, let us see the number of isomorphism classes in $\mathcal{S}$ is finite. Fix $(A_0,\xi_0)\in\mathcal{S}$ and a level m structure α_0 of A_0 over K'. Let $\phi_A : (A_0,\xi_0)\times_K \operatorname{Spec}(K') \to (A,\xi)\times_K \operatorname{Spec}(K')$ be an isomorphism over K'. We set $\alpha_A = \phi_A\cdot\alpha_0$ and $\phi^A_{A'} = \phi_{A'}\cdot\phi_A^{-1} : A\times_K \operatorname{Spec}(K') \to A'\times_K \operatorname{Spec}(K')$ for $(A,\xi),(A',\xi')\in\mathcal{S}$. Then $\alpha_{A'} = \phi^A_{A'}\cdot\alpha_A$. Here let us consider a map

$$\gamma : \mathcal{S} \to \operatorname{Hom}(\operatorname{Gal}(K'/K), \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}))$$

given by $\gamma(A) = \epsilon(K'/K, A, \alpha_A)$. By Lemma 5.1, if $\gamma(A) = \gamma(A')$, then $(A,\xi)\simeq(A',\xi')$ over K, while $\operatorname{Hom}(\operatorname{Gal}(K'/K), \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}))$ is a finite group. This completes the proof. $\square$

§6. Strong finiteness

In this section, we give the proof of the main result of this paper.

Theorem 6.1. *Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. Let $\overline{B} = (B;\overline{H}_1,\ldots,\overline{H}_d)$ be a strictly fine polarization of K. Then, for any numbers c, the number of isomorphism classes of abelian varieties defined over K with $h^{\overline{B}}_{\mathrm{Fal}}(A)\le c$ is finite.*

Proof. Considering a generic resolution of singularities $\mu : B'\to B$, we may assume that $\overline{B}$ is generically smooth.

Let us consider the following two sets:

$$\mathcal{S}_0(c) = \left\{ (A,\xi) \;\middle|\; \begin{array}{l} (A,\xi) \text{ is a principally polarized abelian variety} \\ \text{over } K \text{ with } h^{\overline{B}}_{\mathrm{mod}}(A)\le 8c \end{array} \right\}$$

$$\mathcal{S}(c) = \left\{ A \mid A \text{ is an abelian variety over } K \text{ with } h^{\overline{B}}_{\mathrm{Fal}}(A)\le c \right\}$$

By Corollary 4.2, $\left\{(A,\xi)\times\operatorname{Spec}(\overline{K}) \mid (A,\xi)\in\mathcal{S}_0(c)\right\}/\simeq_{\overline{K}}$ is finite. If A is an abelian variety over K, then $(A\times A^{\vee})^4$ is a principally polarized abelian variety over K (Zarhin's trick; see [12, Exposé VIII, Proposition 1]). By Proposition 3.3 and Proposition 3.5,

$$h^{\overline{B}}_{\mathrm{mod}}((A\times A^{\vee})^4) = 8h^{\overline{B}}_{\mathrm{mod}}(A).$$

Thus, if $A \in \mathcal{S}(c)$, then $(A \times A^\vee)^4 \in \mathcal{S}_0(c)$. Here the number of isomorphism classes of direct factors of $(A \times A^\vee)^4 \times_K \operatorname{Spec}(\overline{K})$ is finite (cf. [12, Exposé VIII, Proposition 2]). Thus $\{A \times_K \operatorname{Spec}(\overline{K}) \mid A \in \mathcal{S}(c)\}/\simeq_{\overline{K}}$ is finite. In particular, there is a constant C such that $C \leq h^{\overline{B}}_{\mathrm{mod}}(A)$ for all $A \in \mathcal{S}(c)$.

Let K_A be the minimal finite extension of K such that $A[12](\overline{K}) \subseteq A(K_A)$. Then $[K_A : K] \leq \#\operatorname{Aut}((\mathbb{Z}/12\mathbb{Z})^{2g})$. Let B_A be a generic resolution of singularities of the normalization of B in K_A. By Proposition 1.2.1, $A \times_K \operatorname{Spec}(K_A)$ has semiabelian reduction over B_A in codimension one. Thus, by Proposition 3.1, there is an effective divisor E_A on B with

$$h^{\overline{B}}_{\mathrm{Fal}}(A) - h^{\overline{B}}_{\mathrm{mod}}(A) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid E_A)}{[K_A : K]}.$$

Here $h^{\overline{B}}_{\mathrm{mod}}(A) \geq C$ for all $A \in \mathcal{S}(c)$. Thus we can find a constant C' such that

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid E_A) \leq C'$$

for all $A \in \mathcal{S}(c)$. Therefore, by virtue of Proposition 1.10.1, there is a reduced effective divisor D on B such that, for all $A \in \mathcal{S}(c)$, A has semiabelian reduction over $B \setminus D$ in codimension one. Hence, by Proposition 5.2, we have our assertion. □

Remark 6.2. If the problem in Remark 1.10.3 is true, then Theorem 6.1 holds even if the polarization $\overline{B}$ is fine.

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