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Hypersurfaces and uniqueness of holomorphic mappings

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Abstract.

— It is possible to determine meromorphic functions on $\mathbb{C}$ by inverse images of some sets since R. Nevanlinna. However, analogous problems to holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ are complicated. In this paper some results for such problems are given. —

§1. Inroduction

Let $\mathcal{F}$ be a family of nonconstant holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and $S_1, \cdots, S_q$ hypersurfaces of $\mathbb{P}^n(\mathbb{C})$. Then, what S_j have the property that $f^*S_j = g^*S_j$ $(1 \leq j \leq q)$ imply $f = g$ for $f, g \in \mathcal{F}$? Here, we consider S_j as divisors and f^*S_j are pull-backs. Also, we say that a hypersurface S has the uniquness property for $\mathcal{F}$ if $f^*S = g^*S$ implies $f = g$ for $f, g \in \mathcal{F}$.

The origin of this problem is Nevanlinna's unicity theorems:

Theorem N.1 ([N]). *Let a_j $(1 \leq j \leq 5)$ be distinct points in $\overline{\mathbb{C}}$. If nonconstant meromorphic functions f and g satisfy*

$$f^{-1}(a_j) = g^{-1}(a_j) \quad (1 \leq j \leq 5),$$

then $f = g$.

Theorem N.2 ([N]). *Let $a_1, \cdots, a_4$ be distinct points in $\overline{\mathbb{C}}$ such that the nonharmonic ratio is not -1 in each permutaion. If nonconstant meromorphic functions f and g satisfy*

$$f^{-1}(a_j) = g^{-1}(a_j) \quad (\textit{counting multiplicity}) \quad (1 \leq j \leq 4),$$

then $f = g$.

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§2. Uniqueness range sets

A uniqueness range set for entire (meromorphic) functions which has abbreviation URSE(URSM) is a discrete subset $S \subset \overline{\mathbb{C}}$ which has the property that entire (meromorphic) functions f and g such that $f^*S = g^*S$ are identical. For example, the zero set of $e^z + 1$ is not a URSE, but the zero set of $e^z + z$ is a URSE.

Theorem Y.1 ([Y2]). *Let p and d be relatively prime integers such that $d > 2p + 4, p \geq 1$ and a, b nonzero complex constant such that $P(w) := w^d + aw^{d-p} + b = 0$ has no multiple root. Then, the zero set S of $P(w)$ is a URSE.*

The smallest d which satisfies the condition is 7 ($p = 1$). Therefore, there is a URSE with seven elements.

Also, Fujimoto showed a clss of URSM and one of URSE in [F3].

§3. Hypersurfaces with the uniqueness property

Now we consider hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, and $w_0, \cdots, w_n$ represent homogeneous coordinates of the space. Let $v_j = (a_{j0}, \cdots, a_{jn})$ $(0 \leq j \leq n+1)$ be vectors in general position. We consider the hypersurface S defined by

$$\sum_{j=0}^{n+1} \left(\sum_{k=0}^{n} a_{jk} w_k \right)^d = 0.$$

We denote by A_j the $(n+1) \times (n+1)$ matrix which is obtained by omitting the row v_j from $(n+2) \times (n+1)$ matrix $\begin{pmatrix} v_0 \\ \vdots \\ v_{n+1} \end{pmatrix}$, and assume that

$$\left(\frac{\det A_j}{\det A_k} \right)^d \neq \left(\frac{\det A_\mu}{\det A_\nu} \right)^d$$

for $0 \leq j, k, \mu, \nu \leq n+1$ such that $j \neq k, \mu \neq \nu, (j,k) \neq (\mu, \nu)$.

Theorem S.2([S]). *Assume $d \geq (2n+1)^2$. Then the hypersurface S has the uniqueness property for the family of linearly non-degenerate holomorphic mappings.*

Example. Let $v_0 = (1, 0, \cdots, 0), \cdots, v_n = (0, \cdots, 0, 1)$, $v_{n+1} = (a_0, \cdots, a_n)$, where $a_0 \cdots a_n \neq 0$. Then $\det A_j = (-1)^{n-j} a_j$, $\det A_{n+1}$

$= 1$. If we assume that

$$(-1)^{k-j}\frac{a_j}{a_k} \neq (-1)^{\nu-\mu}\frac{a_\mu}{a_\nu} \quad \text{for} \quad j \neq k, \mu \neq \nu, (j,k) \neq (\mu,\nu),$$

then the assumption of the theorem is satisfied, where $a_{n+1} = -1$. Now our hypersurface is defined by

$$w_0{}^d + \cdots + w_n{}^d + (a_0 w_0 + \cdots + a_n w_n)^d = 0.$$

Moreover, if $a_0\eta_0 + \cdots + a_n\eta_n \neq 1$ for any $(d-1)$-st roots η_j of $-a_j$, the hypersurface is non-singular.

§4. Some hypersurfaces case

Now the problem of uniqueness by inverse images of some hypersurfaces are treated.

Let n and m be positive integers and put $w = \exp(2\pi i/n)$, $u = \exp(2\pi i/m)$.

Theorem Y.2 ([Y1]). *Let $S_1 = \{a+b, a+bw, \cdots, a+bw^{n-1}\}$ and $S_2 = \{c\}$ with $n > 4, b \neq 0, c \neq a, (c-a)^{2n} \neq b^{2n}$. If $f^*S_j = g^*S_j$ $(j = 1,2)$ for nonconstant entire functions f and g, then $f = g$.*

Theorem Y.3 ([Y1]). *Let $S_1 = \{a_1 + b_1, a_1 + b_1 w, \cdots, a_1 + b_1 w^{n-1}\}$ and $S_2 = \{a_2 + b_2, a_2 + b_2 u, \cdots, a_2 + b_2 u^{m-1}\}$ with $n > 4, m > 4, b_1 b_2 \neq 0, a_1 \neq a_2$. If $f^*S_j = g^*S_j$ $(j = 1,2)$ for nonconstant entire functions f and g, then $f = g$.*

Let f and g be holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and H_j $(1 \leq j \leq q)$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Assume that

$$(*) \qquad f^{-1}(H_j) = g^{-1}(H_j) \text{ (counting multiplicity) } (1 \leq j \leq q).$$

Theorem F.1 ([F1]). *If f and g are linearly non-degenerate and $q \geq 3n+2$, then $f = g$.*

Theorem F.2 ([F2]). If f and g are algebraically non-degenerate and $q \geq 2n+3$, then $f = g$.

Take $(a_{jk})_{0 \leq j,k \leq n} \in GL(n+1, \mathbb{C})$. Let p_1 and p_2 be positive integers and p the least common multiple of them. Consider hypersurfaces

$$S_1 : w_0{}^{p_1} + \cdots + w_n{}^{p_1} = 0,$$

$$S_2 : \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_{jk} w_k \right)^{p_2} = 0.$$

As an anologue of Theorem Y.3 we have

Theorem SU([SU]). *Assume that* $p_1,\ p_2 \geq (2n+1)^2$ *and that* $(a_{jk})^{2p} \neq (a_{\mu\nu})^{2p}$ *for any* (j,k) *and* (μ,ν) *with* $(j,k) \neq (\mu,\nu)$.*If linearly non-degenerate holomorphic mappings* f *and* g *of* $\mathbb{C}$ *into* $\mathbb{P}^n(\mathbb{C})$ *satisfy* $f^*S_j = g^*S_j$ $(j=1,2)$, *then* $f=g$.

Under the same condition of Theorem F1 and Theorem F2, the following was concluded without the nondegeneracy of f and g but with the addtional conditions $f(\mathbb{C}) \not\subset H_j,\ g(\mathbb{C}) \not\subset H_j$:

Theorem F.4 ([F1]). *If* $q = 3n+1$, *then* $g = Lf$ *by some projective linear transformation* L.

For $n=2$ and any $q \geq 6$, however, Fujimoto gave an example of hyperplanes in general position $H_1, \cdots, H_q$ such that there exist distinct f and g which satisfy $(*)$ and $f(\mathbb{C}) \not\subset H_j,\ g(\mathbb{C}) \not\subset H_j$. Of course, f and g are linearly degenerate, and one is a projective linear transformation of the other.

Problem. Do there exist hypersurfaces $S_1, \cdots, S_q$ such that non-constant holomorphic mapping $f,\ g$ satisfying $f^*S_j = g^*S_j$ $(1 \leq j \leq q)$ are identical?

Next, we consider the case that the family $\mathcal{F}$ is the family of non-constant holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. We consider the case of $n=2$.

Take $v_j = (a_{j0}, a_{j1}, a_{j2}) \in \mathbb{C}^3 (1 \leq j \leq q)$. Assume the following conditions:

(1) $a_{jk} \neq 0 \quad (1 \leq j \leq q,\ 0 \leq k \leq 2)$;

(2) $v_1, \cdots, v_q$ are in general position;

(3) for distinct $1 \leq j_1, j_2, j_3, j_4 \leq q$ and $k = 0,1,2$,

$$\frac{a_{j_1 k}}{a_{j_2 k}} \neq \frac{\det\,({}^t v_{j_1}, {}^t v_{j_3}, {}^t v_{j_4})}{\det\,({}^t v_{j_2}, {}^t v_{j_3}, {}^t v_{j_4})};$$

(4) for distinct $1 \leq j_1, \ldots, j_6 \leq q$ and distinct $1 \leq k_1, \cdots, k_6 \leq q$, and for d-th roots of one $\omega_1, \cdots, \omega_6$, if

$$\det \begin{pmatrix} a_{j_1 0} & a_{j_1 1} & a_{j12} & \omega_1 a_{k_1 0} & \omega_1 a_{k_1 1} & \omega_1 a_{k_1 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j_6 0} & a_{j_6 1} & a_{j62} & \omega_6 a_{k_6 0} & \omega_1 a_{k_1 1} & \omega_1 a_{k_1 2} \end{pmatrix} = 0,$$

then $j_1 = k_1, \cdots, j_6 = k_6, \omega_1 = \cdots = \omega_6$.

Moreover we assume $p \geq 4,\ q \geq 10,\ d \geq (2q-1)^2$ and consider the hypersurface

$$S\ :\ \sum_{j=1}^{q} (a_{j0}w_0{}^p + a_{j1}w_1{}^p + a_{j2}w_2{}^p)^d = 0.$$

Theorem S.3. *Let $f = (f_0 : f_1 : f_2)$ and g be nonconstant holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^2(\mathbb{C})$. If $f^*S = g^*S$, then $g = (f_0 : \omega_1 f_1 : \omega_2 f_2)$, where ω_1, ω_2 are d-th roots of one.*

Corollary S.4. *There exist hypersurfaces S_1 and S_2 with the proprety that nonconstant holomorphic mappings f and g of $\mathbb{C}$ into $\mathbb{P}^2(\mathbb{C})$ satisfying $f^*S_j = g^*S_j$ $(j = 1, 2)$ are identical.*

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