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Green functions attached to limit symbols

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To Ryoshi Hotta on his 60th birthday

Abstract.

Hall-Littlewood functions and Green functions associated to complex reflection groups $W = G(r,1,n)$ were constructed in [S1] by means of symbols, which are a generalization of partitions. In this paper, we consider such functions in the case where the symbols are of very special type, called "limit symbols". The situation becomes simple, and is close to the case of symmetric groups when the symbols tends to the "limit". In the case where W is a Weyl group of type B_n, we give a closed formula for Hall-Littlewood functions, and verify some of the conjectures stated in [S1] for the case of Green functions attached to limit symbols.

§0. Introduction

Green functions associated to symmetric groups $\mathfrak{S}_n$ were originally introduced by Green [G] in connection with the representation theory of finite general linear groups $GL_n(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. Later Deligne and Lusztig [DL] constructed Green functions for any finite reductive groups. The algorithm of computing Green functions, in particular in the case of classical groups, shows that Green functions are determined by the information on Weyl groups and some combinatorial data centering u-symbols. Note that u-symbols are combinatorial objects introduced by Lusztig [L] describing unipotent classes in $G(\mathbb{F}_q)$, which is a natural generalization of the notion of partitions in the case of $GL_n(\mathbb{F}_q)$.

In [S1], Green functions associated to complex reflection groups $W = G(e,1,n)$ were introduced, and it was shown that there exists a combinatorial framework for such Green functions based on the theory of symmetric functions as in the case of symmetric groups. In particular, the notion of u-symbols were generalized to a various type of symbols, and Hall-Littlewood functions associated to such symbols were

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constructed. Green functions are essentially given by the matrix $K_\pm(t)$ of Kostka functions, which is defined as the transition matrix between the set of Schur functions and Hall-Littlewood functions (both are associated to W). The set of symbols are divided into similarity classes, and accordingly, $K_\pm(t)$ is regarded as a block matrix. Then $K_\pm(t)$ has the lower triangular shape as a block matrix, with the identity matrix on each diagonal block. These results were generalized in [S2] to the case of complex reflection groups $G(e,p,n)$.

In this paper, we consider the case of limit symbols (see section 1 for the precise definition). The limit symbol is, in some sense, a limit of the symbols discussed in [S1], and Hall-Littlewood functions turn out to be independent of the choice of symbols when it tends to the limit. In this limit, the situation becomes drastically simple, and is close to the case of symmetric groups. For example, each similarity class consists of one element, and so $K_\pm(t)$ is just a lower unitriangular matrix. We further restricts ourselves to the case where $e = 2$ (i.e., W is the Weyl group of type B_n), and give a closed formula for Hall-Littlewood functions, just as in the case of $\mathfrak{S}_n$. This enables us (in the case where $e = 2$) to show that Hall-Littlewood functions and Green functions are polynomials with integral coefficients, which verifies some conjectures in [S1] in this case. Note that even in the case where $e = 2$, Green functions given here (associated to limit symbols) are different from Green functions associated to $Sp_{2n}(\mathbb{F}_q)$ or $SO_{2n+1}(\mathbb{F}_q)$.

It is likely that Green functions associated to limit symbols have rich structures from geometric and combinatorial point of view. For example, one can expect that they are described in terms of Poincaré polynomials of the quotient of the coinvariant algebras of W, just as in the case of symmetric groups (see 3.14 for details). Yamada [Y] has computed such Poincaré polynomials in some small rank cases, which supports our conjecture.

This paper grew up from the discussion with H.-F. Yamada. The author is very grateful to him.

§1. Limit symbols

1.1. We review some notations from [S1]. We denote by $\mathcal{P}_{n,e}$ the set of e-partitions $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ such that $|\boldsymbol{\alpha}| = \sum_{k=0}^{e-1} |\alpha^{(k)}| = n$. Let W be the complex reflection group $G(e,1,n) \simeq \mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$. The set of irreducible characters of W is in bijection with $\mathcal{P}_{n,e}$. We denote by $\chi^{\boldsymbol{\alpha}}$ the irreducible character of W corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. In particular, the unit character corresponds to $(n; -; \dots; -)$ and $\overline{\det}_V$

corresponds to $(-;\dots;-;1^n)$, where $\overline{\det}_V$ denotes the one dimensional representation arising from the complex conjugate of the determinant of the reflection representation V of W.

Let $m_0,\dots,m_{e-1}$ be positive integers such that $m_k \ge n$, and put $\mathbf{m} = (m_0,\dots,m_{e-1})$. We denote by $Z_n^{0,0} = Z_n^{0,0}(\mathbf{m})$ the set of e-partitions $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ such that each $\alpha^{(k)}$ is regarded as an element in $\mathbb{Z}^{m_k}$, written in the form $\alpha^{(k)} : \alpha_1^{(k)} \ge \cdots \ge \alpha_{m_k}^{(k)} \ge 0$. We fix an integer $r > 0$ and an e-tuple of non-negative integers $\mathbf{s} = (s_0,\dots,s_{e-1})$ such that $s_k \le r$. Let us define an e-partition $\boldsymbol{\Lambda}^0 = \boldsymbol{\Lambda}^0(\mathbf{m},\mathbf{s},r) = (\Lambda^{(0)},\dots,\Lambda^{(e-1)})$ as follows.

$$\Lambda^{(k)} : s_k + (m_i - 1)r \ge \cdots \ge s_k + 2r \ge s_k + r \ge s_k \tag{1.1.1}$$

for $k = 0,\dots,e-1$. We denote by $Z_n^{r,\mathbf{s}} = Z_n^{r,\mathbf{s}}(\mathbf{m})$ the set of e-partitions of the form $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, where $\boldsymbol{\alpha} \in Z_n^{0,0}$ and the sum is taken entry-wise. We denote by $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ if $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, and call it the e-symbol of type $(r,\mathbf{s})$ corresponding to $\boldsymbol{\alpha}$. We often denote the symbol $\boldsymbol{\Lambda} = (\Lambda^{(0)},\dots,\Lambda^{(e-1)})$ in the form $\boldsymbol{\Lambda} = (\Lambda_j^{(k)})$ with $\Lambda^{(k)} : \Lambda_1^{(k)} > \cdots > \Lambda_{m_k}^{(k)}$ for $k = 0,\dots,e-1$.

Put $\mathbf{m}' = (m_0+1,\dots,m_{e-1}+1)$, and we define a shift operation $Z_n^{r,\mathbf{s}}(\mathbf{m}) \to Z_n^{r,\mathbf{s}}(\mathbf{m}')$ by associating $\boldsymbol{\Lambda}' = (\Lambda'_0,\dots,\Lambda'_{e-1}) \in Z_n^{r,s}(\mathbf{m})$ to $\boldsymbol{\Lambda} = (\Lambda_0,\dots,\Lambda_{e-1}) \in Z_n^{r,s}(\mathbf{m})$, where $\Lambda'_k = (\Lambda_k + r) \cup \{s_k\}$ for $k = 0,\dots,e-1$. In other words, for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, $\boldsymbol{\Lambda}'$ is obtained as $\boldsymbol{\Lambda}' = \boldsymbol{\alpha}+\boldsymbol{\Lambda}^0(\mathbf{m}',\mathbf{s},r)$, where $\boldsymbol{\alpha}$ is regarded as an element of $Z_n^{0,0}(\mathbf{m}')$ by adding 0 in the entries of $\boldsymbol{\alpha}$. We denote by $\bar{Z}_n^{r,\mathbf{s}}$ the set of classes in $\coprod_{\mathbf{m}'} Z_n^{r,\mathbf{s}}(\mathbf{m}')$ under the equivalence relation generated by shift operations. Note that $\mathcal{P}_{n,e}$ coincides with the set $\bar{Z}_n^{0,0}$. Also note that $\boldsymbol{\Lambda}^0$ is regarded as a symbol in $Z_n^{r,\mathbf{s}}$ with $n = 0$.

Two elements $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ in $\bar{Z}_n^{r,\mathbf{s}}$ ares said to be similar if there exist representatives in $Z_n^{r,\mathbf{s}}(\mathbf{m})$ such that all the entries of them coincide each other with multiplicities. The set of symbols which are similar to a fixed symbol is called a similarity class in $Z_n^{r,\mathbf{s}}$.

We shall define a function $a : \bar{Z}_n^{r,\mathbf{s}} \to \mathbb{Z}_{\ge 0}$. For $\boldsymbol{\Lambda} \in Z_n^{r,\mathbf{s}}$, we put

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda,\lambda' \in \boldsymbol{\Lambda}} \min(\lambda,\lambda') - \sum_{\mu,\mu' \in \boldsymbol{\Lambda}^0} \min(\mu,\mu'). \tag{1.1.2}$$

The function a on $Z_n^{r,\mathbf{s}}$ is invariant under the shift operation, and it induces a function a on $\bar{Z}_n^{r,\mathbf{s}}$. Clearly, the a-function takes a constant value on each similarity class in $Z_n^{r,\mathbf{s}}$.

Remark 1.2. The definition of symbols given here is slightly more general than the one in [S1], where it is assumed that $\mathbf{s}$ is of the form

$(0, s, \dots, s)$. The symbols of this type appear in [Ma] in parameterizing unipotent characters associated to W. However, the arguments in [S1] can be applied without change to the setting as above (except the last paragraph of section 1, see Remark 3.2), and we shall refer to the results in [S1] freely.

1.3. From now on we assume that $\mathbf{m}$ is of the type

$$m_0 = \cdots = m_a = m+1, m_{a+1} = \cdots = m_{e-1} = m$$

for some integers $m \geq n$ and $0 \leq a \leq e-1$. A symbol $\boldsymbol{\Lambda} = (\Lambda_j^{(k)})$ is called special if it satisfies the relation

$$\begin{aligned} \Lambda_j^{(k)} &\geq \Lambda_j^{(k+1)} \qquad \text{for } 1 \leq j \leq m, 0 \leq k \leq e-2, \\ \Lambda_j^{(e-1)} &\geq \Lambda_{j+1}^{(0)} \qquad \text{for } 1 \leq j \leq m. \end{aligned}$$

If $\boldsymbol{\Lambda}^0$ is special, each similarity class in $Z_n^{r,\mathbf{s}}$ contains a unique special symbol, and the set of special symbols is in a bijective correspondence with the set of similarity classes in $Z_n^{r,\mathbf{s}}$.

We now assume that $\boldsymbol{\Lambda}^0 = (\Lambda_j^{(k)})$ itself is special and satisfies the condition that

$$\Lambda_j^{(k)} - \Lambda_j^{(k+1)} \geq n, \qquad \Lambda_j^{(e-1)} - \Lambda_{j+1}^{(0)} \geq n. \tag{1.3.1}$$

For example, we may choose that $r = en, s_0 = 0, s_k = (e-k)n$ for $k = 1, \dots, e-1$. Symbols in $Z_n^{r,\mathbf{s}}$ determined by $\boldsymbol{\Lambda}^0$ satisfying (1.3.1) are called limit symbols. In this case any symbol is special, and so each similarity class consists of one element. The combinatorics concerning Hall-Littlewood functions and Green functions turn out to be drastically simple, and the situation becomes quite similar to the case of symmetric groups, though it is related to W. In the remainder of this section, we shall discuss Hall-Littlewood functions and Green functions associated to limit symbols.

1.4. From now on, we assume that $Z_n^{r,\mathbf{s}}$ is the set of limit symbols. One can identify a symbol $\boldsymbol{\Lambda} \in Z_n^{r,\mathbf{s}}$ (resp. an e-partition $\boldsymbol{\alpha} \in Z_n^{0,0}$) with an element in $\mathbb{Z}_{\geq 0}^M$, where $M = \sum m_i$, by arranging $\boldsymbol{\Lambda} = (\Lambda_j^{(k)})$ as in 1.3,

$$\Lambda_1^{(0)}, \cdots, \Lambda_1^{(e-1)}, \Lambda_2^{(0)}, \cdots, \Lambda_2^{(e-1)}, \Lambda_3^{(0)}, \cdots \tag{1.4.1}$$

and similarly for $\boldsymbol{\alpha} = (\alpha_j^{(k)})$. In particular, symbols give rise to partitions of M by this identification. For $\lambda = (\lambda_i) \in \mathbb{Z}^M$, we define an integer $n(\lambda)$ by

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

If λ is a partition, we have $n(\lambda) = \sum_{i \neq j} \min(\lambda_i, \lambda_j)$. Then it is easy to see, by (1.1.2), that

$$a(\boldsymbol{\Lambda}) = n(\boldsymbol{\Lambda}) - n(\boldsymbol{\Lambda}^0) = n(\boldsymbol{\alpha}), \tag{1.4.2}$$

where $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^0, \boldsymbol{\alpha}$ are regarded as elements in $\mathbb{Z}^M$ under the above identification. Let us introduce a partial order $\boldsymbol{\Lambda} \geq \boldsymbol{\Lambda}'$ on $Z_n^{r,\mathbf{s}}$ by using the dominance order $\geq$ on $\mathbb{Z}^M$, i.e., for $\lambda = (\lambda_i), \mu = (\mu_i) \in \mathbb{Z}^M$, we define $\lambda \geq \mu$ if

$$\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$$

for $k = 1, \dots, M$. By (1.4.2), one can check that the partial order on $Z_n^{r,\mathbf{s}}$ is compatible with the a-function, i.e., we have

$$a(\boldsymbol{\Lambda}) > a(\boldsymbol{\Lambda}') \quad \text{if} \quad \boldsymbol{\Lambda} < \boldsymbol{\Lambda}'. \tag{1.4.3}$$

Under the bijection $Z_n^{0,0} \simeq Z_n^{r,\mathbf{s}}$ by $\boldsymbol{\alpha} \leftrightarrow \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, the partial order on $Z_n^{r,\mathbf{s}}$ and a-function on it are inherited to $Z_n^{0,0} \simeq \mathcal{P}_{n,e}$. This partial order on $Z_n^{0,0}$ is nothing but the order obtained from the dominance order on $\mathbb{Z}^M$ under the embedding $Z_n^{0,0} \subset \mathbb{Z}^M$. Combining this with (1.4.2), we see that

(1.4.4) a-functions and the partial orders on $Z_n^{0,0}$ defined by limit symbols are independent of the choice of $\boldsymbol{\Lambda}^0$ as far as $\boldsymbol{\Lambda}^0$ satisfies (1.3.1).

In the rest of the paper, we express the set of limit symbols $Z_n^{r,\mathbf{s}}$ as Z_n^∞, and always consider the a-functions and partial orders on $Z_n^{0,0}$ inherited from Z_n^∞.

§2. Hall-Littlewood functions attached to limit symbols

2.1 For a given $\mathbf{m} = (m_0, \dots, m_{e-1})$, we introduce a set of indeterminate $x_j^{(k)}$ $(0 \leq k \leq e-1, 1 \leq m_k)$. We denote by x the whole variables $(x_j^{(k)})$, and also denote by $x^{(k)}$ the variables $x_1^{(k)}, \dots, x_{m_k}^{(k)}$. For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$, one can define the Schur function $s_{\boldsymbol{\alpha}}(x)$ and monomial symmetric function $m_{\boldsymbol{\alpha}}(x)$ by

$$s_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} s_{\alpha^{(k)}}(x^{(k)}), \qquad m_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} m_{\alpha^{(k)}}(x^{(k)})$$

where $s_{\alpha^{(k)}}$ (resp. $m_{\alpha^{(k)}}$) denotes the usual Schur function (resp. the monomial symmetric function) associated to the partition $\alpha^{(k)}$ with respect to the variables $x^{(k)}$.

In what follows we regard the variables $x_i^{(k)}$ defined for $k \in \mathbb{Z}/e\mathbb{Z} \simeq \{0, 1, \dots, e-1\}$. We now introduce a new variable t, and define a function $\widetilde{q}^{(k)}_{r,\pm}(x;t)$ associated to $+$ or $-$, for each $0 \le k \le e-1$ and an integer $r \ge 0$, by

$$\widetilde{q}^{(k)}_{r,\pm}(x;t) = \sum_{i \ge 1} (x_i^{(k)})^{r+\delta} \frac{\prod_j x_i^{(k)} - t x_j^{(k\mp 1)}}{\prod_{j \ne i} x_i^{(k)} - x_j^{(k)}} \qquad (r \ge 1), \tag{2.1.1}$$

where $\delta = m_k - 1 - m_{k\pm 1}$, In the product of the denominator, $x_j^{(k)}$ runs over all the variables in $x^{(k)}$ except $x_i^{(k)}$, while in the numerator, $x_j^{(k\pm 1)}$ runs over all the variables in $x^{(k\pm 1)}$. $\widetilde{q}^{(k)}_{r,\pm}$ is a polynomial in $\mathbb{Z}[x;t]$ if $\delta \ge 0$, and lies in $\mathbb{Z}[x, x^{-1};t]$ in general. We define $q^{(k)}_{r,\pm}(x;t)$ as follows. If $\delta \ge 0$ i.e., $m_k \ge m_{k\pm 1} + 1$, put $q^{(k)}_{r,\pm} = \widetilde{q}^{(k)}_{r,\pm}$. If $\delta < 0$, we add $m_{k\pm 1} + 1 - m_k$ variables $\mathbf{x}' = x^{(k)}_{m_k+1}, \dots, x^{(k)}_{m_{k\pm 1}+1}$ to $x^{(k)}$, and consider the polynomial $\widetilde{q}^{(k)}_{r,\pm}$ for such variables with $\delta = 0$, and put $q^{(k)}_{r,\pm} = \widetilde{q}^{(k)}_{r,\pm}|_{\mathbf{x}'=0}$. Hence $q^{(k)}_{r,\pm} \in \mathbb{Z}[x;t]$ in all cases, and we have $q^{(k)}_{0,\pm} = 1$.

For an e-partition $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$, we define a function $q_{\boldsymbol{\alpha},\pm}(x;t)$ by

$$q_{\boldsymbol{\alpha},\pm}(x;t) = \prod_{k=0}^{e-1} \prod_{j=1}^{m_k} q^{(k)}_{\alpha_j^{(k)},\pm}(x;t). \tag{2.1.2}$$

Remark 2.2. In [S1, 2.2], the function $q^{(k)}_{r,\pm}$ was defined by the formula (2.1.1). But since it is not a polynomial, its definition should be modified as above. Then this $q^{(k)}_{r,\pm}$ coincides with the polynomial obtained from the generating function (2.3.1) in [S1], and the properties stated in Lemma 2.3 in [S1] holds for $q^{(k)}_{r,\pm}$. Accordingly, the definition of $R^{\pm}_{\boldsymbol{\alpha},<}(x;t)$, etc. in [S1, 3.2] must be modified appropriately. (However, the notations below have some discrepancies with [S1]. See Reamrk 5.7 in [S2] for details.)

2.3 We denote by $\Xi_{\mathbf{m}} = \bigotimes_{k=0}^{e-1} \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$ the ring of symmetric polynomials (with respect to $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \cdots \times \mathfrak{S}_{m_{e-1}}$) with variables $x = (x_j^{(k)})$. $\Xi_{\mathbf{m}}$ has a structure of a graded ring $\Xi_{\mathbf{m}} = \bigoplus_{i \geq 0} \Xi_{\mathbf{m}}^i$, where $\Xi_{\mathbf{m}}^i$ consists of homogeneous symmetric polynomials of degree i, together with the zero polynomial. As given in [S1, 3.15], one can define the ring of symmetric functions $\Xi = \oplus_{i \geq 0} \Xi^i$ as the direct sum of the inverse limit Ξ^i of $\Xi_{\mathbf{m}}^i$. The Schur function $s_{\boldsymbol{\alpha}}(x)$ with infinitely many variables $x_1^{(k)}, x_2^{(k)}, \dots$ is regarded as an element in Ξ^n with $n = |\boldsymbol{\alpha}|$, and the set $\{s_{\boldsymbol{\alpha}}(x)\}$ with $\boldsymbol{\alpha} \in Z_n^{0,0}$ forms a $\mathbb{Z}$-basis of Ξ^n. It is also shown (see [S1, 3.15]) that $\{q_{\boldsymbol{\alpha},\pm} \mid \boldsymbol{\alpha} \in Z_n^{0,0}\}$ gives rise to a basis of the $\mathbb{Q}(t)$-space $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$ (according to $+$ or $-$, respectively). A similar property holds if one replaces Ξ^n by $\Xi_{\mathbf{m}}^n$.

We now define a scalar product $\langle \, , \, \rangle$ on $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$ by the property that

$$\langle q_{\boldsymbol{\alpha},+}(x;t), m_{\boldsymbol{\beta}}(x) \rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$. Then we have $\langle m_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\beta},+}(x;t) \rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by [S1, (4.7.2)] (But there are some discrepancies with the formulas in [S1] in the discussion below because of some errors in [S1]. For this see Remarks 5.7 in [S2]).

Hall-Littlewood functions $P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ and $Q_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ associated to symbols were constructed in [S1]. $\{P_{\boldsymbol{\Lambda}}^{\pm}\}, \{Q_{\boldsymbol{\Lambda}}^{\pm}\}$ give bases of $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$. In the case of Z_n^{∞}, $\{P_{\boldsymbol{\Lambda}}^{\pm}\}$ are characterized by the following two properties (cf. [S1, Proposition 4.7]).

(2.3.1) For $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z_n^{\infty}$, $P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ can be expressed in terms of Schur functions $s_{\boldsymbol{\beta}}(x)$ as

$$P_{\boldsymbol{\Lambda}}^{\pm}(x;t) = s_{\boldsymbol{\alpha}}(x) + \sum_{\boldsymbol{\beta} < \boldsymbol{\alpha}} u_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}(t) s_{\boldsymbol{\beta}}(x) \qquad (u_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}(t) \in \mathbb{Q}(t)),$$

(2.3.2) $\langle P_{\boldsymbol{\Lambda}}^{+}, P_{\boldsymbol{\Lambda}'}^{-} \rangle = 0$ for $\boldsymbol{\Lambda} \neq \boldsymbol{\Lambda}'$,

Then $Q_{\boldsymbol{\Lambda}}^{\pm}$ are determined as the dual of $P_{\boldsymbol{\Lambda}}^{\pm}$, i.e., we have

(2.3.3) $\langle P_{\boldsymbol{\Lambda}}^{+}, Q_{\boldsymbol{\Lambda}'}^{-} \rangle = \langle Q_{\boldsymbol{\Lambda}}^{+}, P_{\boldsymbol{\Lambda}'}^{-} \rangle = \delta_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}$.

Here the partial order $\boldsymbol{\beta} < \boldsymbol{\alpha}$ in $Z_n^{0,0}$ is the one given in 1.4. We note that $P_{\boldsymbol{\Lambda}}^{\pm}$ coincides with $Q_{\boldsymbol{\Lambda}}^{\pm}$ up to scalar by (2.3.2) and (2.3.3). So one can write it, for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, as

$$Q_{\boldsymbol{\Lambda}}^{\pm}(x;t) = b_{\boldsymbol{\alpha}}^{\pm}(t) P_{\boldsymbol{\Lambda}}^{\pm}(x;t) \tag{2.3.4}$$

for some $b^{\pm}_{\boldsymbol{\alpha}}(t) \in \mathbb{Q}(t)$.

Let $\mathcal{A}$ be the subring of $\mathbb{Q}(t)$ consisting of functions which has no pole at $t = 0$. Then $\mathcal{A}$ is the local ring with the unique maximal ideal $t\mathcal{A}$, and $\mathcal{A}^* = \mathcal{A} - t\mathcal{A}$ is the set of units in $\mathcal{A}$. $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ and $Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are also characterized by the expansions in terms of $s_{\boldsymbol{\beta}}(x)$ and $q_{\boldsymbol{\beta},\pm}(x)$ as follows.

Theorem 2.4 ([S1, Th. 4.4]). (i) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are the unique functions having the following expansions.*

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;t) = \sum_{\boldsymbol{\beta}\geq\boldsymbol{\alpha}} c^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t)$$

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;t) = s_{\boldsymbol{\alpha}}(x) + \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $c^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$ in the first formula, and $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathcal{A}$ in the second formula.

(ii) *$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are the unique functions having the following expansions.*

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t) = q_{\boldsymbol{\alpha},\pm}(x;t) + \sum_{\boldsymbol{\beta}>\boldsymbol{\alpha}} d^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t),$$

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t) = \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} w^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$ in the first formula, and $w^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathcal{A}$ for $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$ and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \in \mathcal{A}^*$ in the second formula.

2.5 We shall give a closed formula for $Q^{\pm}_{\boldsymbol{\Lambda}}$ and $P^{\pm}_{\boldsymbol{\Lambda}}$ in the special case where $e = 2$. So, in what follows we assume that $e = 2$. In this case, $Q^{+}_{\boldsymbol{\Lambda}}$, $P^{+}_{\boldsymbol{\Lambda}}$, $q^{+}_{\boldsymbol{\alpha}}$, etc. coincide with $Q^{-}_{\boldsymbol{\Lambda}}$, $P^{-}_{\boldsymbol{\Lambda}}$, $q^{-}_{\boldsymbol{\alpha}}$, etc., and so we omit the signature $\pm$ and express them simply as $Q_{\boldsymbol{\Lambda}}$, $P_{\boldsymbol{\Lambda}}$, $q_{\boldsymbol{\alpha}}$, etc. In order to obtain the closed forms of $P_{\boldsymbol{\Lambda}}$ and $Q_{\boldsymbol{\Lambda}}$, we recall here another type of symmetric functions $R_{\boldsymbol{\Lambda}} = R^{\pm}_{\boldsymbol{\Lambda}}$ introduced in [S1]. Let $\mathcal{M} = \{(i,k) \mid 0 \leq k \leq 1, 1 \leq i \leq m_k\}$ be the set of pairs (i,k) corresponding to $x^{(k)}_i$. We define a total order on $\mathcal{M}$ compatible with the embedding $Z^{\infty}_n \subset \mathbb{Z}^M$, as in (1.4.1), i.e.,

$$(1,0) < (1,1) < (2,0) < \cdots < (m,1) < (m+1,0) < \cdots .$$

For a fixed $\boldsymbol{\alpha} = (\alpha^{(k)}_j) \in Z^{0,0}_n$, we denote by $\nu_0 = (i_0, k_0)$ the largest element in $\mathcal{M}$ such that $\alpha^{(k_0)}_{i_0} \neq 0$. We assume that $m \geq i_0 + 1$. Put

$$\delta_k = \sharp\{j \mid (i,k) < (j,k+1)\} - \sharp\{j \mid (i,k) < (j,k)\}, \tag{2.5.1}$$

which is independent of the choice of i. We define a function $I_i^{(k)}(x;t)$ attached to $\boldsymbol{\alpha}$ for $0 \le k \le e-1, 1 \le i \le m_k$ by

$$
(2.5.2) \quad I_i^{(k)}(x;t) = \begin{cases} \displaystyle\prod_{\substack{1 \le j \le m_{k+1}-\delta_k \\ (i,k)<(j,k+1)}} (x_i^{(k)} - t x_j^{(k+1)}) & \text{if } (i,k) \le \nu_0, \\ \displaystyle\prod_{\substack{1 \le j \le m_k \\ (i,k)<(j,k)}} (x_i^{(k)} - t x_j^{(k)}) & \text{if } (i,k) > \nu_0. \end{cases}
$$

For $\boldsymbol{\alpha} \in Z_n^{0,0}$, we define a polynomial $v_{\boldsymbol{\alpha}}(t)$ by

$$
v_{\boldsymbol{\alpha}}(t) = \prod_{k=0}^{e-1} v_{\mu_k}(t),
$$

where $\mu_k = \sharp\{j \mid (j,k) > \nu_0\}$ and

$$
v_r(t) = \prod_{i=1}^{r} \frac{1-t^i}{1-t}
$$

for each $r \ge 1$. For a sequence $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{m_k})$, we write as

$$
(x^{(k)})^{\beta} = (x_1^{(k)})^{\beta_1} (x_2^{(k)})^{\beta_2} \cdots (x_{m_k}^{(k)})^{\beta_{m_k}}.
$$

Finally, put $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \mathfrak{S}_{m_1}$ as before. We now define a function $R_{\boldsymbol{\alpha}}(x;t)$ associated to $\boldsymbol{\alpha}$ by

(2.5.3)

$$
R_{\boldsymbol{\alpha}}(x;t) = v_{\boldsymbol{\alpha}}(t)^{-1} \times
$$

$$
\sum_{w \in \mathfrak{S}_{\mathbf{m}}} w \Big\{ \prod_k (x^{(k)})^{\alpha^{(k)}} \prod_{k,i} I_{i,\pm}^{(k)}(x;t) \Big/ \prod_k \prod_{(i,k)<(j,k)} (x_i^{(k)} - x_j^{(k)}) \Big\}.
$$

Since $R_{\boldsymbol{\alpha}}$ can be expressed as

$$
R_{\boldsymbol{\alpha}}(x;t) = v_{\boldsymbol{\alpha}}(t)^{-1} \prod_k \prod_{(i,k)<(j,k)} (x_i^{(k)} - x_j^{(k)})^{-1} \times
$$

$$
\sum_{w \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w) w \Big\{ \prod_k (x^{(k)})^{\alpha^{(k)}} \prod_{k,i} I_{i,\pm}^{(k)}(x;t) \Big\},
$$

it is a polynomial in x.

Remark 2.6. $R_{\boldsymbol{\alpha}}^{\pm}$ was defined in [S1, (3.1.2)] by using a slightly different formula. But the function defined there is a Laurent polynomial in general, and not necessarily a polynomial in x. So, it should be modified to the above form. The results in [S1, 3] remain valid for this $R^{\pm}_{\boldsymbol{\alpha}}$ under an appropriate modification.

2.7. We regard $\mathfrak{S}_{\mu_k}$ as a subgroup of $\mathfrak{S}_{m_k}$ as a permutation group with respect to the letters $\{1 \le j \le m_k \mid (j,k) > \nu_0\}$. In this way, we regard $\mathfrak{S}^0_{\boldsymbol{\alpha}} = \mathfrak{S}_{\mu_0} \times \mathfrak{S}_{\mu_1}$ as a subgroup of $\mathfrak{S}_{\mathbf{m}}$. It is shown in [S1, (3.2.1)] that $R_{\boldsymbol{\alpha}}(x;t)$ can be expressed in the following form also.

(2.7.1)

$$R_{\boldsymbol{\alpha}}(x;t) =$$
$$\sum_{w \in \mathfrak{S}_{\mathbf{m}}/\mathfrak{S}^0_{\boldsymbol{\alpha}}} w\Big\{\prod_k (x^{(k)})^{\alpha^{(k)}} \prod_{\substack{k,i \\ (i,k)\le \nu_0}} I^{(k)}_{i,\pm}(x;t) \Big/ \prod_k \prod_{\substack{(i,k)<(j,k) \\ (i,k)\le\nu_0}} (x^{(k)}_i - x^{(k)}_j)\Big\}.$$

We denote by $\mathfrak{S}_{\boldsymbol{\alpha}}$ the stabilizer of $\boldsymbol{\alpha}$ in $\mathfrak{S}_{\mathbf{m}}$. Let us define a function $n : \mathfrak{S}_{\boldsymbol{\alpha}} \to \mathbb{Z}_{\ge 0}$ by

$$n(w) = \sharp\{(\nu,\nu') \in \mathcal{M}^2 \mid \nu < \nu', w^{-1}(\nu) > w^{-1}(\nu'), b(\nu') \ne b(\nu)\},$$

and define a polynomial $b_{\boldsymbol{\alpha}}(t)$ by

$$b_{\boldsymbol{\alpha}}(t) = \sum_{w \in \mathfrak{S}_{\boldsymbol{\alpha}}} \varepsilon(w)(-t)^{n(w)}.$$

The following result gives an explicit description of Hall-Littlewood functions $Q_{\boldsymbol{\Lambda}}$ and $P_{\boldsymbol{\Lambda}}$.

Theorem 2.8. *Assume that $e = 2$. The for each $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z^{\infty}_n$, we have*

$$Q_{\boldsymbol{\Lambda}}(x;t) = R_{\boldsymbol{\alpha}}(x;t), \qquad P_{\boldsymbol{\Lambda}}(x;t) = b_{\boldsymbol{\alpha}}(t)^{-1} R_{\boldsymbol{\alpha}}(x;t).$$

2.9. The theorem will be proved in 2.12 after some preliminaries. We define an operator R_{ij} on the set $\mathbb{Z}^M$ by $R_{ij}(\lambda) = \lambda'$, where if $\lambda = (\lambda_1, \dots, \lambda_M) \in \mathbb{Z}^M$, then $\lambda' \in \mathbb{Z}^M$ is given by

$$\lambda'_i = \lambda_i + 1, \qquad \lambda'_j = \lambda_j - 1$$

and $\lambda'_l = \lambda_l$ for $l \ne i,j$. A raising operator (resp. a lowering operator) R on $\mathbb{Z}^M$ is defined as a product of various R_{ij} with $i < j$ (resp. $i > j$). In the following, we identify $\mathcal{M}$ with the set $\{1, 2, \dots, M\}$ via the total order on $\mathcal{M}$ and express the operator R_{ij} as $R_{\nu,\nu'}$ for $\nu, \nu' \in \mathcal{M}$. We define the action of raising operators R on $q_{\boldsymbol{\alpha},\pm}$ by $R(q_{\boldsymbol{\alpha},\pm}) = q_{R(\boldsymbol{\alpha}),\pm}$. Note that $q_{\boldsymbol{\beta},\pm}$ makes sense for $\boldsymbol{\beta} \in \mathbb{Z}^M_{\geq 0}$, and we regard $q_{\boldsymbol{\beta},\pm} = 0$ if $\boldsymbol{\beta}$ contains a negative factor. For $\nu = (i,k) \in \mathcal{M}$, we put $b(\nu) = k$. By [S1, Cor. 3.7], $R_{\boldsymbol{\alpha}}$ can be expressed in terms of raising operators as follows.

$$R_{\boldsymbol{\alpha}} = \Big\{\prod_{(*)}(1 - R_{\nu\nu'})/\prod_{(**)}(1 - tR_{\nu\nu'})\Big\}q_{\boldsymbol{\alpha},\pm}, \tag{2.9.1}$$

where the conditions (*) and (**) are given by

(*) $\quad \nu = (i,k), \nu' = (j,k), \nu < \nu'$,

(**) $\quad \nu = (i,k), \nu' = (j,k+1), \nu < \nu', 1 \leq j \leq m_{k+1} - \delta_k$.

Using this, we have

Lemma 2.10. *$R_{\boldsymbol{\alpha}}(x;t)$ can be expressed as*

$$R_{\boldsymbol{\alpha}}(x;t) = q_{\boldsymbol{\alpha}}(x;t) + \sum_{\boldsymbol{\beta} > \boldsymbol{\alpha}} d_{\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta}}(x;t)$$

with $d_{\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$ for $\boldsymbol{\beta} \in Z^{0,0}_n$.

Proof. By (2.9.1), $R_{\boldsymbol{\alpha}}(x;t)$ can be written as a $\mathbb{Z}[t]$-linear combination of $R(q_{\boldsymbol{\alpha}})$ by various raising operators R. It is known (e.g. [M, I]) that $R(\boldsymbol{\alpha}) \geq \boldsymbol{\alpha}$ for a raising operator R and $\boldsymbol{\alpha} \in Z^{0,0}_n \subset \mathbb{Z}^M$. If $R(\boldsymbol{\alpha})$ is not an e-partition, we must replace $R(\boldsymbol{\alpha})$ by $\boldsymbol{\beta} \in Z^{0,0}_n$ by permuting the entries of $R(\boldsymbol{\alpha})$. But then $\boldsymbol{\beta} \geq R(\boldsymbol{\alpha})$, and so we can write $R(q_{\boldsymbol{\alpha}}) = q_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in Z^{0,0}_n$ such that $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$. It is clear that $\boldsymbol{\beta} = \boldsymbol{\alpha}$ if and only if $R = 1$. The lemma follows from this. □

Next we show that

Lemma 2.11. *$R_{\boldsymbol{\alpha}}(x;t)$ can be expressed as*

$$R_{\boldsymbol{\alpha}}(x;t) = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathbb{Z}[t]$ for $\boldsymbol{\alpha} \neq \boldsymbol{\beta} \in Z^{0,0}_n$, and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t) = b_{\boldsymbol{\alpha}}(t)$. Moreover $b_{\boldsymbol{\alpha}}(0) = 1$.

Proof. We shall prove the lemma by using a similar argument as in the case of usual Hall-Littlewood functions ([M, III, 1]). First note that the definition of Schur functions $s_{\boldsymbol{\alpha}}$ given in 1.5 can be extended to the case where $\boldsymbol{\alpha}$ is not necessary an e-partition. If $\alpha^{(k)}_j + (m_k - j)$ are positive and all distinct for $j = 1, \dots, m_k$ (for a fixed k), then $s_{\boldsymbol{\alpha}}$

coincides with the usual Schur function $s_{\boldsymbol\beta}(x)$ up to sign, where $\boldsymbol\beta = (\beta_j^{(k)})$ is obtained by permuting the sequence $\{\alpha_j^{(k)} + (m_k - j)\}$ (for a fixed k) in the decreasing order and by writing it as $\{\beta_j^{(k)} + (m_k - j)\}$. If $\alpha_j^{(k)} + (m_k - j)$ are not all distinct for a fixed k, then $s_{\boldsymbol\alpha} = 0$.

In the description of $R_{\boldsymbol\alpha}$ by (2.7.1), the product $\prod_{i,k} I_i^{(k)}(x;t)$ gives a contribution

$$\prod_{\nu_0 \geq \nu, \nu' > \nu} (x_i^{(k)})^{r_{\nu,\nu'}}(-tx_j^{(k+1)})^{r_{\nu',\nu}}$$

where $\nu = (i,k), \nu' = (j, k+1)$. Here $(r_{\nu,\nu'})$ is an integral matrix indexed by $\mathcal{M}$ consisting of 0 and 1 satisfying the relation

$$r_{\nu,\nu'} + r_{\nu',\nu} = \begin{cases} 1 & \text{if } \nu \leq \nu_0, \nu' \in \mathcal{M}_\nu \\ 0 & \text{otherwise.} \end{cases} \tag{2.11.1}$$

where for each $\nu = (i,k)$, $\mathcal{M}_\nu$ is defined by

$$\mathcal{M}_\nu = \{\nu' = (j, k+1) \mid \nu < \nu', 1 \leq j \leq m_{k+1} - \delta_k\}.$$

Put, for a fixed choice of the matrix $(r_{\nu,\nu'})$ as above,

$$\lambda_i^{(k)} = \alpha_i^{(k)} + \sum_{\nu' \in \mathcal{M}} r_{\nu,\nu'} \tag{2.11.2}$$

for $\nu = (i,k)$. Then the e-composition $\boldsymbol\lambda = (\lambda_i^{(k)})$ yields the "Schur function" $a_{\boldsymbol\lambda}/a_{\boldsymbol\delta}$, where $a_{\boldsymbol\lambda} = \sum_{w \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w) w(x^{\boldsymbol\lambda})$ and $\boldsymbol\delta = (\delta^{(0)}, \delta^{(1)})$ with $\delta^{(k)} = (m_k - 1, \dots, 1, 0)$. $R_{\boldsymbol\alpha}$ can be written as a $\mathbb{Z}$-linear combination of $(-t)^d a_{\boldsymbol\lambda}/a_{\boldsymbol\delta}$ attached to various matrices $(r_{\nu,\nu'})$, where $d = \sum_{\nu<\nu'} r_{\nu',\nu}$.

Now $a_{\boldsymbol\lambda}(x) = 0$ if the composition $\lambda^{(k)}$ is not all distinct for some k. Hence we may assume that all the entries of $\lambda^{(k)}$ are distinct. Then by rearranging its entries in the descending order, we can write as

$$\lambda_{w_k(i)}^{(k)} = \beta_i^{(k)} + (m_k - i) \qquad (1 \leq j \leq m_k)$$

with some $w_k \in \mathfrak{S}_{m_k}$ for $0 \leq k \leq 1$. Then $\boldsymbol\beta = (\beta_i^{(k)}) \in Z_n^{0,0}$ and $a_{\boldsymbol\lambda}/a_{\boldsymbol\delta}$ coincides with $\varepsilon(w)s_{\boldsymbol\beta}$. Thus $R_{\boldsymbol\alpha}$ is written as a sum of $\varepsilon(w)(-t)^d s_{\boldsymbol\beta}$ for such $\boldsymbol\beta$. We shall show that

$$\boldsymbol\beta \leq \boldsymbol\alpha \tag{2.11.3}$$

Let us define a matrix $(s_{\nu,\nu'})$ by $s_{\nu,\nu'} = r_{w(\nu),w(\nu')}$, where $w(\nu) = (w_k(i), k)$ for $\nu = (i,k) \in \mathcal{M}$. Hence $w(\nu) \in \mathcal{M}$, and the matrix $(s_{\nu,\nu'})$ satisfies a similar condition as in (2.11.1). We can write

$$\beta_i^{(k)} + (m_k - i) = \alpha^{(k)}_{w_k(i)} + \sum_{\nu' \in \mathcal{M}} s_{\nu,\nu'} \tag{2.11.4}$$

We want to show that

$$\sum_{k=0}^{1}\sum_{i=1}^{t} \beta_i^{(k)} + \sum_{k=0}^{p} \beta_{t+1}^{(k)} \le \sum_{k=0}^{1}\sum_{i=1}^{t} \alpha^{(k)}_{w_k(i)} + \sum_{k=0}^{p} \alpha^{(k)}_{w_k(t+1)} \tag{2.11.5}$$

for $0 \le p \le 1$ and $1 \le t \le m$. Note that (2.11.5) implies (2.11.3) since $w(\boldsymbol{\alpha}) \le \boldsymbol{\alpha}$ for any $w \in \mathfrak{S}_{\mathbf{m}}$. Now by (2.11.4) we have

$$\begin{aligned}
\sum_{k=0}^{1}\sum_{i=1}^{t} \beta_i^{(k)} &+ \sum_{k=0}^{p} \beta_{t+1}^{(k)} \\
&= \sum_{k=0}^{1}\sum_{i=1}^{t} \alpha^{(k)}_{w_k(i)} + \sum_{k=0}^{p} \alpha^{(k)}_{w_k(t+1)} \\
&\quad - \sum_{k=0}^{1}\sum_{i=1}^{t} (m_k - i) - \sum_{k=0}^{p} (m_k - (t+1)) \\
&\quad + \sum_{\nu \in \mathcal{B}, \nu' \in \mathcal{M}} s_{\nu,\nu'}
\end{aligned}$$

where

$$\mathcal{B} = \{(i,k) \mid 1 \le i \le t, 0 \le k \le 1\} \cup \{(t+1,k) \mid 0 \le k \le p\}.$$

Hence, in order to show (2.11.5), it is enough to see that

$$\begin{aligned}
\sum_{\nu \in \mathcal{B}, \nu' \in \mathcal{M}} s_{\nu,\nu'} &\le \sum_{k=0}^{1}\sum_{i=1}^{t} (m_k - i) + \sum_{k=0}^{p} (m_k - (t+1)) \\
&= tM + \sum_{k=0}^{p} m_k - (t+1)(t+1+p),
\end{aligned} \tag{2.11.6}$$

where $M = m_0 + m_1$ as before. First we note that

$$\delta_k = \begin{cases} m_1 - m_0 + 1 & \text{if } k = 0, \\ m_0 - m_1 & \text{if } k = 1. \end{cases} \tag{2.11.7}$$

One can write

(2.11.8) $$\sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}} s_{\nu,\nu'} = \sum_{\nu,\nu'\in\mathcal{B}} s_{\nu,\nu'} + \sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}-\mathcal{B}} s_{\nu,\nu'}.$$

We shall compute the right hand side of (2.11.8). On the one hand, we have

(2.11.9) $$\begin{aligned}\sum_{\nu,\nu'\in\mathcal{B}} s_{\nu,\nu'} &= \sharp\{(\nu,\nu')\in\mathcal{B}^2 \mid b(\nu)=0, b(\nu')=1\} \\ &= (t+1)(t+p)\end{aligned}$$

by (2.11.1). On the other hand, if $p=0$, we have

(2.11.10)
$$\begin{aligned}\sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}-\mathcal{B}} s_{n,\nu'} &\le (m_1-\delta_0-t)(t+1)+(m_0-\delta_1-(t+1))t \\ &= tM+m_0-(t+1)(2t+1)\end{aligned}$$

by (2.11.7). Then it is easy to see that the sum of (2.11.9) and the right hand side of (2.11.10) coincides with the right hand side of (2.11.6). If $p=1$, we have

(2.11.11) $$\begin{aligned}\sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}-\mathcal{B}} s_{\nu,\nu'} &\le \sum_{k=0}^{1}(m_k-\delta_{k-1}-(t+1))(t+1) \\ &= (t+1)M-(t+1)-2(t+1)^2,\end{aligned}$$

and again the sum of (2.11.9) and the right hand side of (2.11.11) coincides with the right hand side of (2.11.6). Hence (2.11.6) holds and we have proved (2.11.3).

The above computation shows that $\boldsymbol{\beta}=\boldsymbol{\alpha}$ if and only if $w=(w_0,w_1)\in\mathfrak{S}_{\boldsymbol{\alpha}}$ and that $s_{\nu,\nu'}=1$ for all $\nu<\nu'$ such that $b(\nu')\neq b(\nu)$. Then we have $d=n(w)$ since

$$d=\sum_{\nu<\nu'} r_{\nu',\nu}=\sum_{\nu<\nu'} s_{w^{-1}(\nu'),w^{-1}(\nu)},$$

and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t)$ is given as $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t)=\sum_{w\in\mathfrak{S}_{\boldsymbol{\alpha}}}\varepsilon(w)(-t)^d=b_{\boldsymbol{\alpha}}(t)$. Now it is easily checked by the definition that $R_{\boldsymbol{\alpha}}(x;0)=s_{\boldsymbol{\alpha}}(x)$ (see [S1, (3.13.2)]). This implies that $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(0)=b_{\boldsymbol{\alpha}}(0)=1$ and that $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)\in t\mathbb{Z}[t]$ for $\boldsymbol{\beta}\neq\boldsymbol{\alpha}$. The lemma is now proved. □

2.12. We are now ready to prove Theorem 2.8. By Lemma 2.10 and Lemma 2.11, $R_{\boldsymbol{\alpha}}(x;t)$ satisfies the condition in Theorem 2.4 for $Q_{\boldsymbol{\alpha}}(x;t)$. Also $(b_{\boldsymbol{\alpha}})^{-1}R_{\boldsymbol{\alpha}}(x;t)$ satisfies the condition for $P_{\boldsymbol{\alpha}}(x;t)$. Hence Theorem 2.8 holds.

Remarks 2.13. (i) Theorem 2.8 together with Lemma 2.10 implies that $Q_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$. It is shown in the next section that $P_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$ also.

(ii) It is known by [S1, (3.13.1)] that the expansion of $R^{\pm}_{\boldsymbol{\alpha}}$ by Schur functions has an interpretation in terms of lowering operators. Hence in view of Theorem 2.8, we have

$$Q_{\boldsymbol{\Lambda}} = v_{\boldsymbol{\alpha}}(t)^{-1} \prod_{\substack{\nu<\nu', \nu\le\nu_0 \\ b(\nu')\ne b(\nu)}} (1-tR_{\nu'\nu}) \prod_{\substack{\nu<\nu', \nu>\nu_0 \\ b(\nu')=b(\nu)}} (1-tR_{\nu'\nu}) s_{\boldsymbol{\alpha}}$$

for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, where $R(s_{\boldsymbol{\alpha}})$ is defined as $a_{R(\boldsymbol{\alpha}+\boldsymbol{\delta})}/a_{\boldsymbol{\delta}}$ for a lowering operator R.

(iii) Lemma 2.11 (i.e. the property that $\boldsymbol{\beta} \le \boldsymbol{\alpha}$) does not hold in general for $R^{\pm}_{\boldsymbol{\alpha}}$ if $e \ge 3$. For example, assume that $e = 3$, and consdier $W = G(3,1,2)$. Then for $\boldsymbol{\alpha} = (1^2;-;-) \in \mathcal{P}_{2,3}$, we have

$$R^{+}_{\boldsymbol{\alpha}} = s_{(1^2;-;-)} - t^2 s_{(1;-;1)} - t^2 s_{(-;1^2;-)} - t^3 s_{(-;1;1)},$$

and $(1;-;1) > (1^2;-;-) = \boldsymbol{\alpha}$. Hence $R^{+}_{\boldsymbol{\alpha}}$ does note coincide with $Q^{+}_{\boldsymbol{\Lambda}}$ in this case.

§3. Green functions attached to limit symbols

3.1. Although we shall treat the case where $e = 2$ in later discussions, first we review some results from [S1, 5] for general e. Let us define $K_{\pm}(t) = (K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$ as the transition matrix $M(s, P^{\pm})$ between the basis $s = \{s_{\boldsymbol{\alpha}}(x)\}$ of Schur functions and the basis $P^{\pm} = \{P^{\pm}_{\boldsymbol{\Lambda}}(x;t)\}$ of Hall-Littlewood functions in $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$, i.e.,

$$s_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta}} K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}(x;t). \tag{3.1.1}$$

We fix a total order on $Z^{0,0}_n \simeq \mathcal{P}_{n,e}$ which is compatible with the partial order $\boldsymbol{\beta} \le \boldsymbol{\alpha}$ on it. Then $K_{\pm}(t)$ is a lower unitriangular matrix with entries $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$, and $K_{\pm}(0)$ is the identity matrix. We define the matrix $\widetilde{K}_{\pm}(t) = (\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$ by

$$\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = t^{n(\boldsymbol{\beta})} K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t^{-1}),$$

where $n(\boldsymbol{\beta}) = a(\boldsymbol{\Lambda}(\boldsymbol{\beta}))$ is the function given in (1.4.2). $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ (resp. $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$) are called Kostka functions (resp. modified Kostka functions). Green functions are defined as a linear combination of modified Kostka functions. The determination of Green functions is equivalent to that of Kostka functions once we know the character table of W. Here we concentrate ourselves to (modified) Kostka functions rather than Green functions themselves.

Let

$$\Omega(x,y;t) = \prod_{k=0}^{e-1}\prod_{i,j}\frac{1 - tx_i^{(k)}y_j^{(k+1)}}{1 - x_i^{(k)}y_j^{(k)}}. \tag{3.1.2}$$

By Corollary 4.6 in [S1], combined with (2.3.4), $\Omega(x,y;t)$ has the following expansion in terms of Hall-Littlewood functions.

$$\begin{aligned}\Omega(x,y;t) &= \sum_{\boldsymbol{\alpha}} b^-_{\boldsymbol{\alpha}}(t)P^+_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t)P^-_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(y;t) \\ &= \sum_{\boldsymbol{\alpha}} P^+_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t)Q^-_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(y;t),\end{aligned} \tag{3.1.3}$$

where $\boldsymbol{\alpha}$ runs over e-partitions of any size. Let N^* be the number of complex reflections in W. We define a polynomial $\mathbb{G}(t) \in \mathbb{Z}[t]$ by $\mathbb{G}(t) = (t-1)^n t^{N^*}P_W(t)$, where $P_W(t)$ is the Poincaré polynomial associated to W (see. [S1, 1.1]). We denote by $\Lambda'(t)$ the diagonal matrix indexed by $Z_n^{0,0}$, whose $\boldsymbol{\alpha}\boldsymbol{\alpha}$-entry is given by $b^+_{\boldsymbol{\alpha}}(t^{-1})$. We put $\widetilde{\Lambda}(t) = t^{-n}\mathbb{G}(t)\Lambda'(t)$. Let $\Omega' = (\omega'_{{}_B a,\boldsymbol{\beta}})$ be the matrix defined by

$$\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*}R(\chi^{\boldsymbol{\alpha}} \otimes \overline{\chi^{\boldsymbol{\beta}}} \otimes \overline{\det}_V),$$

where $\chi^{\boldsymbol{\alpha}}$ is the irreducible character of W associated to $\boldsymbol{\alpha}$. In general, we denote by $R(f)$, for a class function f of W, the graded multiplicity of f in the coinvariant algebra $R = \oplus R_i$ of W, i.e.,

$$R(f) = \sum_{i\geq 0}\langle f, R_i\rangle_W\, t^i \tag{3.1.4}$$

(see [S1, 1.1]). Then it is known by [S1, Th. 5.4] that $\widetilde{K}^{\pm}(t)$ and $\widetilde{\Lambda}(t)$ are determined as a unique solution for the following matrix equation.

$$\widetilde{K}_-(t)\widetilde{\Lambda}(t)\,{}^t\widetilde{K}_+(t) = \Omega'. \tag{3.1.5}$$

Remark 3.2. Let $\Omega = (\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the matrix defined by

$$\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*}R(\chi^{\boldsymbol{\alpha}} \otimes \chi^{\boldsymbol{\beta}} \otimes \overline{\det}_V).$$

In [S1, 1.4, 1.5] it is shown that the equation $P'\Lambda'\,{}^tP'' = \Omega'$ such as (3.1.5) is equivalent to the equation $P\Lambda\,{}^tP = \Omega$ with $P' = P, P'' = \sigma P \sigma$, where σ is a permutation matrix arising from the complex conjugates of irreducible characters of W. Although it is not written explicitly there, we note that this equivalence works only when $\mathbf{m} = (m+1, m, \dots, m)$ and $s_1 = \cdots = s_{e-1}$ for $\mathbf{s} = (s_0, \dots, s_{e-1})$ in 1.1. So a simple relation between $\widetilde{K}_+(t)$ and $\widetilde{K}_-(t)$ as given in [S1, 1.5] can not be found in our situation.

We now restrict ourselves to the case where $e = 2$, and write $\widetilde{K}_\pm(t)$, etc. as $\widetilde{K}(t)$, etc. as before by omitting the signature $\pm$. The following fact holds.

Proposition 3.3. *Assume that $e = 2$. Then $K_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$, which is a monic of degree $n(\boldsymbol{\beta}) - n(\boldsymbol{\alpha})$, and so $\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$. Moreover, we have $P_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$.*

Proof. We remark that $q = \{q_{\boldsymbol{\alpha},\pm}(x;t)\}$ and $m = \{m_{\boldsymbol{\alpha}}(x)\}$, $Q = \{Q_{\boldsymbol{\Lambda}}(x;t)\}$ and $P = \{P(x;t)\}$ are dual bases of each other with respect to the scalar product $\langle\,,\,\rangle$ on $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$. It follows that

$$M(Q,q) = M(P,m)^* = (K(t)^{-1}K)^* = {}^tK(t)K^*,$$

where $K = M(s,m)$ is the Kostka matrix, and K^* denotes the transposed inverse of K. In view of Lemma 2.10 and Theorem 2.8, $M(Q,q)$ are the matrices with entries in $\mathbb{Z}[t]$. Since K is a matrix with entries in $\mathbb{Z}$, we see that $K(t)$ is a matrix with entries in $\mathbb{Z}[t]$. Since $K(t)$ is unitriangular, $K(t)^{-1}$ is also a matrix with entries in $\mathbb{Z}[t]$. This implies that $P_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$.

It remains to show the formula for $\deg K_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. The following argument is similar to [S1, Cor. 6.8]. By [S1, (6.7.3)], K^* coincides with the matrix of the operator $\prod_{\nu<\nu'} \prod_{b(\nu')=b(\nu)} (1 - R_{\nu,\nu'})$. This fact together with (2.9.1) implies, by a similar argument as in [M, III, (6.3)], that $K_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is the coefficient of $s_{\boldsymbol{\alpha}}$ in

$$\begin{aligned}(3.3.1)\qquad & \prod_{\nu<\nu'} \prod_{b(\nu')\neq b(\nu)} (1 - tR_{\nu,\nu'})^{-1} s_{\boldsymbol{\beta}} \\ &= \prod_{\nu<\nu'} \prod_{b(\nu')\neq b(\nu)} (1 + tR_{\nu,\nu'} + t^2R^2_{\nu,\nu'} + \cdots) s_{\boldsymbol{\beta}}.\end{aligned}$$

Let $\varepsilon_1, \dots, \varepsilon_M$ be the standard basis of $\mathbb{Z}^M$. We denote by R^+ the set of positive roots of type A_{M-1}, i.e., $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le M\}$. For any $\xi = (\xi_1, \dots, \xi_M) \in \mathbb{Z}^M$ such that $\sum \xi_i = 0$, we define a polynomial $P(\xi;t)$ in t by

$$P(\xi;t) = \sum_{(m_\gamma)} t^{\sum m_\gamma},$$

where (m_γ) runs over all the choices such that $\xi = \sum_{\gamma\in R^+} m_\gamma\gamma$ with $m_\gamma \in \mathbb{Z}_{\geq 0}$. We also define $P^*(\xi;t)$ by a similar formula as $P(\xi;t)$, but this time, (m_γ) runs over only the expression such that $\xi = \sum_\gamma m_\gamma\gamma$ and that $\gamma = \varepsilon_i - \varepsilon_j$ corresponds to the raising operator R_{ν_i,ν_j} occurring in the expression in (3.3.1). Then $P(\xi;t)$ is non-zero only when $\xi = \sum \eta_i(\varepsilon_i - \varepsilon_{i+1})$ with $\eta_i \geq 0$, and in that case, $P(\xi;t)$ is a monic of degree $\sum \eta_i = \langle \xi, \delta\rangle$. (See [M, III, 6, Ex.4], here $\delta = (M,\dots,1,0) \in \mathbb{Z}^M$ and $\langle\, ,\rangle$ denotes the standard inner product in $\mathbb{Z}^M$.) Clearly $\deg P^*(\xi;t) \leq \deg P(\xi;t)$ and since the choice $(m_\gamma) = (\eta_i)$ is allowed, we see that $\deg P^*(\xi;t) = \deg P(\xi;t)$.

Hence, by a similar argument as in [loc. cit.], we see that $K_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ coincides with

$$\sum_{w\in\mathfrak{S}_{\mathbf{m}}} \varepsilon(w)P^*(w^{-1}(\boldsymbol{\alpha}+\delta) - (\boldsymbol{\beta}+\delta);t),$$

where $\boldsymbol{\alpha}+\delta, \boldsymbol{\beta}+\delta$ are sums as elements in $\mathbb{Z}^M$. We have

$$\begin{aligned}\langle w^{-1}(\boldsymbol{\alpha}+\delta) - (\boldsymbol{\beta}+\delta), \delta\rangle &= \langle \boldsymbol{\alpha}+\delta, w(\delta)\rangle - \langle \boldsymbol{\beta}+\delta,\delta\rangle \\ &\leq \langle \boldsymbol{\alpha}+\delta,\delta\rangle - \langle\boldsymbol{\beta}+\delta,\delta\rangle \\ &= n(\boldsymbol{\beta}) - n(\boldsymbol{\alpha}).\end{aligned}$$

The equality holds only when $w = 1$. This proves the proposition. □

We shall now compute certain values of $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$. The following fact holds for any $e \geq 1$.

Proposition 3.4. *Let $\boldsymbol{\beta}_0 = (-;\cdots;-;1^n)$ be the smallest element in $Z_n^{0,0}$. (Hence $\chi^{\boldsymbol{\beta}_0}$ coincides with the character $\overline{\det}_V$ of W.) Then we have*

$$\widetilde{K}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = R(\chi^{\boldsymbol{\alpha}}), \qquad \widetilde{K}^{+}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = R(\overline{\chi}^{\boldsymbol{\alpha}} \otimes (\overline{\det}_V)^2).$$

(See (3.1.4) for the definition of $R(\ )$. Note that $R(\chi^{\boldsymbol{\alpha}})$ coincides with the fake degree of $\boldsymbol{\alpha}$).

Proof. Although the argument is similar to, and much simpler than the proof of Lemma 7.2 in [S1], we give it below for the sake of completeness. We consider the equation (3.1.5). Let $\tilde{b}_{\boldsymbol{\beta}_0}$ be the first entry of the diagonal matrix $\widetilde{\Lambda}(t)$. Since $n(\boldsymbol{\beta}_0) = \sum_{i=1}^{n}(ei-1) = N^*$, the equation (3.1.5) implies that $\tilde{b}_{\boldsymbol{\beta}_0} = 1$. Again by (3.1.5), we have

$$\widetilde{K}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t)\tilde{b}_{\boldsymbol{\beta}_0}t^{n(\boldsymbol{\beta}_0)} = \omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}_0} = t^{N^*}R(\chi^{\boldsymbol{\alpha}}),$$

and so $\widetilde{K}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = R(\chi^{\boldsymbol{\alpha}})$. Similarly, we have

$$t^{n(\boldsymbol{\beta}_0)}\tilde{b}_{\boldsymbol{\beta}_0}\widetilde{K}^{+}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = \omega'_{\boldsymbol{\beta}_0,\boldsymbol{\alpha}} = t^{N^*}R(\overline{\chi}^{\boldsymbol{\alpha}} \otimes (\overline{\det}_V)^2).$$

□

Next we shall show that

Proposition 3.5. *Assume that $e = 2$, and let $\boldsymbol{\beta}_1 = (n; -)$ be the largest element in $Z_n^{0,0}$. (Hence $\chi^{\boldsymbol{\beta}_1}$ is the unit character of W). Then for any $\boldsymbol{\alpha} \in Z_n^{0,0}$, we have*

$$K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t) = t^{n(\boldsymbol{\alpha})}.$$

In particular, we have $\widetilde{K}_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t) = 1$.

3.6. The proof of Proposition 3.5 will be done in 3.9. We consider the substitution of $\mathbf{t} = (1, t, t^2, \cdots)$ into the variables $y = \{y_j^{(k)} \mid 1 \le j \le m_k, 0 \le k \le 1\}$ by $y_j^{(k)} = t^{2(j-1)+k}$. Then we have

Lemma 3.7. *$R_{\boldsymbol{\alpha}}(y;t)|_{y=\mathbf{t}}$ is a polynomial in t of the form $t^{n(\boldsymbol{\alpha})}+$ higher degree terms.*

Proof. We consider $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \mathfrak{S}_{m_1}$ as a subgroup of $\mathfrak{S}_M$ along the total order in $\mathcal{M}$ in 2.5. Suppose that $\nu_0 = (i_0, k_0) \in \mathcal{M}$ corresponds to a number b $(1 \le b \le M)$. We define a subset X of $\mathfrak{S}_{\mathbf{m}}$ as follows. If $m_0 = m_1 + 1$, put $X = \mathfrak{S}^0_{\boldsymbol{\alpha}}$ (see 2.7). If $m_0 = m_1$, put

$$X = \left\{ w = \begin{pmatrix} 1 & 2 & \cdots & b & \cdots & M \\ 2a+1 & 2a+2 & \cdots & 2a+b & * & 2a \end{pmatrix} \middle| w \in \mathfrak{S}_{\mathbf{m}}, 1 \le 2a+b \le M \right\} \cup \mathfrak{S}^0_{\boldsymbol{\alpha}}.$$

Then it is easy to check by the definition (2.5.2) that

$$w\left\{\prod_{i,k} I_i^{(k)}(y;t)\right\}\bigg|_{y=\mathbf{t}} = 0$$

unless $w \in X$. If $w \in X$, we see that

$$w\Big\{\prod_k (y^{(k)})^{a^{(k)}}\Big\}\Big|_{y=\mathbf{t}} = t^{n(w(\boldsymbol{\alpha}))}. \tag{3.7.1}$$

Moreover, $w(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ if $w \in \mathfrak{S}^0_{\boldsymbol{\alpha}}$. If $w \notin \mathfrak{S}^0_{\boldsymbol{\alpha}}$, then $w(\boldsymbol{\alpha}) < \boldsymbol{\alpha}$, and so $n(w(\boldsymbol{\alpha})) > n(\boldsymbol{\alpha})$. We also note that

$$w\Big\{\prod_{\substack{k,i\\ (i,k)\le\nu_0}} I^{(k)}_{i,\pm}(x;t)\Big/\prod_k \prod_{\substack{(i,k)<(j,k)\\ (i,k)\le\nu_0}} (x^{(k)}_i - x^{(k)}_j)\Big\}\Big|_{y=\mathbf{t}} = 1$$

for $w \in X$. It follows, by [M, III, 1.4], that

$$\sum_{w\in\mathfrak{S}^0_{\boldsymbol{\alpha}}} w\Big\{\prod_{k,i} I^{(k)}_i(y;t)\Big/\prod_k \prod_{(i,k)<(j,k)} (y^{(k)}_i - y^{(k)}_j)\Big\}\Big|_{y=\mathbf{t}} = v_{\boldsymbol{\alpha}}(t).$$

Then one can write as

$$R_{\boldsymbol{\alpha}}(y;t)\,|_{y=\mathbf{t}} = t^{n(\boldsymbol{\alpha})} + v_{\boldsymbol{\alpha}}(t)^{-1}A(t),$$

where

$$A(t) = \sum_{w\in X\backslash\mathfrak{S}^0_{\boldsymbol{\alpha}}} w\Big\{\prod_k (y^{(k)})^{\alpha^{(k)}} \prod_{k,i} I^{(k)}_i(y;t)\Big/\prod_k \prod_{(i,k)<(j,k)} (y^{(k)}_i - y^{(k)}_j)\Big\}\Big|_{y=\mathbf{t}}.$$

One can check that $A(t)$ has an expansion as a formal power series of t whose initial term is strictly bigger than $t^{n(\boldsymbol{\alpha})}$ by (3.7.1). Since $R_{\boldsymbol{\alpha}}(y;t)|_{y=\mathbf{t}}$ is a polynomial in t, $A(t)$ is a polynomial divisible by $v_{\boldsymbol{\alpha}}(t)$. This implies that $v_{\boldsymbol{\alpha}}(t)^{-1}A(t)$ is a polynomial in t whose lowest degree term is strictly bigger than $t^{n(\boldsymbol{\alpha})}$. The lemma is proved. □

We now consider the substitution of $\mathbf{t} = (1, t, t^2, \dots)$ into the infinitely many variables $y = \{y^{(k)}_j \mid j = 1, 2, \dots\}$. Then we have

Lemma 3.8. $\Omega(x, y; t)\,|_{y=\mathbf{t}} = \prod_j \dfrac{1}{1 - x^{(0)}_j}$.

Proof. We consider the second expression of $\Omega(x, y; t)$ in (3.1.2). For each $i, j \ge 1$, we have

$$\frac{1 - ty^{(0)}_j x^{(1)}_i}{1 - y^{(1)}_j x^{(1)}_i} \cdot \frac{1 - ty^{(1)}_j x^{(0)}_i}{1 - y^{(0)}_j x^{(0)}_i} = \frac{1 - t^{2j}x^{(0)}_i}{1 - t^{2(j-1)}x^{(0)}_i}$$

by substituting $y = \mathbf{t}$. It follows that

$$\Omega(x,y;t)|_{y=\mathbf{t}} = \prod_j \left\{ \frac{1-t^2x_i^{(0)}}{1-x_i^{(0)}} \cdot \frac{1-t^4x_i^{(0)}}{1-t^2x_i^{(0)}} \cdots \right\}$$
$$= \prod_j \frac{1}{1-x_i^{(0)}}.$$

□

3.9. We shall prove Proposition 3.5. By substituting $y = \mathbf{t}$ in the both sides of (3.1.3), and by using Lemma 3.8, we have

$$\prod_j \frac{1}{1-x_j^{(0)}} = \sum_{\boldsymbol{\alpha}} P_{\boldsymbol{\Lambda}}(x;t)Q_{\boldsymbol{\Lambda}}(y;t)\ |_{y=\mathbf{t}},$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$. By taking the degree n parts on both sides (cf. [M, I, (2.5)]),

$$h_n(x^{(0)}) = \sum_{|\boldsymbol{\alpha}|=n} Q_{\boldsymbol{\Lambda}}(y;t)|_{y=\mathbf{t}}P_{\boldsymbol{\Lambda}}(x;t), \tag{3.9.1}$$

where $h_n(x^{(0)})$ is a complete symmetric function of degree n with respect to the variables $x^{(0)}$. Since $h_n(x^{(0)}) = s_{(n)}(x^{(0)})$, we see that $h_n(x^{(0)})$ coincides with $s_{\boldsymbol{\beta}_1}(x)$. Comparing (3.9.1) with (3.1.1), we see that $K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t)$ is obtained as the limit of the polynomials $Q_{\boldsymbol{\Lambda}}(y;t)$ with finitely many variables $y = (y_j^{(k)})$ under the substitution $y = \mathbf{t}$. (Here the limit of $Q_{\boldsymbol{\Lambda}}(y;t)$ is taken in the sense of [S1, 3.15].) On the other hand, $Q_{\boldsymbol{\Lambda}} = R_{\boldsymbol{\alpha}}$ by Theorem 2.8. Hence by Lemma 3.7, we see that $K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t)$ is obtained as the limit of the polynomials of the form $t^{n(\boldsymbol{\alpha})}+$ higher terms. But Proposition 3.3 implies that $\deg K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}} = n(\boldsymbol{\alpha}) - n(\boldsymbol{\beta}_1) = n(\boldsymbol{\alpha})$. This shows that $K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t) = t^{n(\boldsymbol{\alpha})}$, and Proposition 3.5 follows.

As a corollary, we have

Corollary 3.10. *Assume that $e = 2$. Let the $\boldsymbol{\alpha}\boldsymbol{\alpha}$-entry of the diagonal matrix $\widetilde{\Lambda}(t)$ in 3.1 be $\widetilde{b}_{\boldsymbol{\alpha}}(t)$. Then we have*

$$\sum_{\boldsymbol{\alpha}\in\mathcal{P}_{n,r}} \widetilde{b}_{\boldsymbol{\alpha}}(t) = t^{2N^*}.$$

Proof. The equation (3.1.4) together with Proposition 3.5 implies that

$$
\begin{aligned}
\sum_{\boldsymbol{\alpha}} \widetilde{b}_{\boldsymbol{\alpha}}(t) &= \omega_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_1} \\
&= t^{N^*} R(\chi^{\boldsymbol{\beta}_1} \otimes \chi^{\boldsymbol{\beta}_1} \otimes \varepsilon) \\
&= t^{N^*} R(\varepsilon) \\
&= t^{2N^*}.
\end{aligned}
$$

□

Remark 3.11. Let $G(\mathbb{F}_q)$ be a (split) finite reductive group over a finite field of q elements and W its Weyl group. To each irreducible character χ of W, Green function $Q_\chi(q)$ of $G(\mathbb{F}_q)$ is associated by Deligne-Lusztig [DL]. They are determined as a solution of the matrix equation of the form $P\Lambda\,{}^tP = \Omega$. It is known that Λ is a block diagonal matrix, and the sum of the 11-entries of each block is equal to q^{2N^*}, which coincides with the number of unipotent elements in $G(\mathbb{F}_q)$. In the case of $GL_n(\mathbb{F}_q)$, the matrix Λ is a diagonal matrix indexed by partitions of n, and the $\lambda\lambda$-entry of Λ coincides with the number of elements in the unipotent class in $G(\mathbb{F}_q)$ corresponding to λ.

In [GM], Geck and Malle formulated a different matrix equation $P\Lambda\,{}^tP = \Omega$ for each $G(\mathbb{F}_q)$ by making use of parameter set of unipotent characters of $G(\mathbb{F}_q)$ instead of unipotent classes. They conjectured that the sum of 11-entries (which correspond to special characters of W) of each block of Λ is again equal to q^{2N^*}, and verified it in the case of exceptional groups.

In our situation, limit symbols are related neither to unipotent classes nor to unipotent characters. Corollary 3.10 shows that, even so, a similar fact holds in our case.

3.12. Here we give some examples of Green functions associated to limit symbols in the case where $e = 2$. Below is the tables of modified Kostka functions $\widetilde{K}(t) = (\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$. In each of the tables, first column denotes double partitions $\boldsymbol{\beta} \in \mathcal{P}_{n,2}$, under the order compatible with the values of a functions.

Our Green functions associated to limit symbols are different from original Green functions associated to u-symbols, even in the case of Weyl groups of type B_n. We give below the table of modified Kostka functions associated to u-symbols in $W(B_2)$ which is related to the original Green functions of $SO_5(\mathbb{F}_q)$ for the sake of comparison.

Table 1. $\widetilde{K}(t)$ for $W(B_2)$

$(-;1^2)$	t^4				
$(1^2;-)$	t^2	t^2			
$(-;2)$	t^2		t^2		
$(1;1)$	t^3+t	t	t	t	
$(2;-)$	1	1	1	1	1

Table 2. $\widetilde{K}(t)$ for $W(B_3)$

$(-;1^3)$	t^9									
$(1^3;-)$	t^6	t^6								
$(-;21)$	t^7+t^5		t^5							
$(1;1^2)$	$t^8+t^6+t^4$	t^4	t^4	t^4						
$(-;3)$	t^3		t^3		t^3					
$(1^2;1)$	$t^7+t^5+t^3$	t^5+t^3	t^3	t^3		t^3				
$(1;2)$	$t^6+t^4+t^2$	t^2	t^4+t^2	t^2	t^2	t^2	t^2			
$(21;-)$	t^4+t^2	t^4+t^2	t^2	t^2		t^2		t^2		
$(2;1)$	t^5+t^3+t	t^3+t	t^3+t	t^3+t	t	t	t	t	t	
$(3;-)$	1	1	1	1	1	1	1	1	1	1

Table 3. $\widetilde{K}(t)$ for $W(B_2)$, the case of u-symbols

$(-;1^2)$	t^4				
$(1^2;-)$	t^2	t^2			
$(1;1)$	t^3+t	t	t		
$(-;2)$	t^2			t	
$(2;-)$	1	1	1		1

3.13. Let $\mathfrak{gl}_n$ be the Lie algebra of $GL_n(\mathbb{C})$, and $\mathfrak{t}$ the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of diagonal matrices. Let $\mathfrak{o}_\lambda$ be the nilpotent orbit in $\mathfrak{gl}_n$ corresponding to a partition λ of n. We consider the scheme theoretic intersection $\mathfrak{t} \cap \bar{\mathfrak{o}}_\lambda$ of $\mathfrak{t}$ with the closure $\bar{\mathfrak{o}}_\lambda$ of $\mathfrak{o}_\lambda$. Then the coordinate ring $\mathbb{C}[\mathfrak{t} \cap \bar{\mathfrak{o}}_\lambda]$ is a finite dimensional $\mathbb{C}$-algebra, equipped with a structure of graded $\mathfrak{S}_n$-modules. We denote it by $R^\lambda = \bigoplus_i R_i^\lambda$. De Concini and Procesi [DP], and Tanisaki [T] showed that the polynomial

$$R^\beta(\chi^\alpha) = \sum_i \langle \chi^\alpha, R_i^\beta \rangle_{\mathfrak{S}_n} t^i$$

coincides with the modified Kostka polynomial $\widetilde{K}_{\alpha,\beta}(t)$ associated to $\mathfrak{S}_n$. R^λ is also interpreted as the quotient ring of $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ by I_λ, where I_λ is the ideal generated by $p(x) \in \mathbb{C}[x]$ such that $p(\partial)f = 0$ for any $f \in V^\lambda$, (here V^λ is the Specht module of $\mathfrak{S}_n$ realized in $\mathbb{C}[x]$). Note that the map $\mathbb{C}[x] \to R^\lambda$ factors through the surjection $\mathbb{C}[x] \to R$ (R is the coinvariant algebra of $\mathfrak{S}_n$) and we have a surjective algebra homomorphism $R \to R^\lambda$.

This latter construction of R^λ makes sense even in the case of complex reflection groups $W = G(e,1,n)$, and we get the graded W-module $R^{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in \mathcal{P}_{n,e}$. One might expect that $R^{\boldsymbol{\beta}}(\chi^{\boldsymbol{\alpha}})$ coincides with our modified Kostka function $\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ associated to limit symbols. (Note that this does not hold in the case of original Green functions of type B_n since the counter part of the map $R \to R^{\boldsymbol{\alpha}}$ for Green functions is no longer surjective). H.-F. Yamada [Y] has computed some examples of $R^{\boldsymbol{\beta}}(\chi^{\boldsymbol{\alpha}})$ for small rank cases, which supports our conjecture.

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