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On the Prime Graph of a Finite Simple Group An Application of the Method of Feit-Thompson-Bender-Glauberman

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Introduction The theorem alluded to in the subtitle is the Odd Order Theorem of Feit-Thompson [FT] which states that all finite groups of odd order are solvable. For the remarkable proof, they invented a revolutionary new method which was influential to the development of finite group theory in the last 30 odd years. Recently, Bender and Glauberman [BG] have published a highly polished proof covering the group theoretical portion of the proof of the Odd Order Theorem.

By design, their proof is by contradiction. From the start they work on the hypothetical minimal simple group of odd order and study its properties. Thus, all the wonderful intermediate results are properties of the hypothetical group, and hence they may be vacuous. One of the goals of this paper is to show that this is not so; their method does give positive results and all the intermediate results are in fact properties of some real groups.

We consider the prime graph $\Gamma(G)$ of a finite group G. This is the graph defined as follows. The set of vertices of $\Gamma(G)$ is the set $\pi(G)$ of the primes dividing the order $|G|$ of G. If $p, q \in \pi(G)$, we join p and q by an edge in $\Gamma(G)$ if and only if $p \neq q$ and G has an element of order pq.

The classification of finite simple groups has several interesting consequences on the prime graph of a finite group. The following is one of them.

Theorem A. *Let Δ be a connected component of the prime graph $\Gamma(G)$ of a finite group G, and let ϖ be the set of primes in Δ. Assume that $\Delta \neq \Gamma(G)$ and $2 \notin \varpi$. Then, Δ is a clique.*

Usually, we identify Δ with ϖ and abuse the terms, saying ϖ is a connected component of the graph $\Gamma(G)$. Theorem A has not been stated

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in the literature in this form. But, the works of Gruenberg and Kegel [GK] and Williams [W] together with properties of Frobenius groups yield Theorem A. The classification of finite simple groups is used in two separate places of its proof. The first is in the proof of the following theorem.

Theorem B. *Theorem A holds for a finite simple group.*

The second use of the classification is to prove the following lemma.

Lemma. *Let G be a finite simple group. Then, $\pi(\operatorname{Out} G)$ is contained in the connected component of the prime graph $\Gamma(G)$ that includes the prime* 2.

This is fairly easy to check because $\operatorname{Out} G$ for a simple group G is not too complicated. The checking of Theorem B is more complex.

The purpose of this work is to show that the method of Feit, Thompson, Bender, and Glauberman can be adapted to give a proof of Theorem B without using the classification of finite simple groups.

Actually, Williams [W] has checked the following result for a finite simple group.

Theorem C. *Let Δ be a connected component of the prime graph $\Gamma(G)$ of a finite simple group G. Let ϖ be the set of primes in Δ. Assume that $\Delta \neq \Gamma(G)$ and $2 \notin \varpi$. Then, G contains a nilpotent Hall ϖ-subgroup H that is isolated in G.*

A subgroup H of any group G is called *isolated* in G if $1 \neq H \neq G$ and for every element $x \in H^\sharp$, we have

$$C_G(x) \subseteq H.$$

Theorem B is weaker than Theorem C which may be considered a local version of the Odd Order Theorem. It would be nice if our method would be able to prove Theorem C.

Originally, Gruenberg and Roggenkamp [GR] are led to study the prime graph, in particular its connectivity, through their work on the decomposition of the augmentation ideal of the integral group ring of a finite group. Specifically they considered the following three conditions on a finite group G.

(1) G has an isolated subgroup.
(2) The augmentation ideal decomposes as a right $\mathbb{Z}G$-module.
(3) The prime graph $\Gamma(G)$ is not connected.

Gruenberg and Roggenkamp [GR] proved that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Using Theorem C, Williams [W] was able to prove that (3) $\Rightarrow$ (1). If ϖ is a connected component of the prime graph $\Gamma(G)$ such that $2 \notin \varpi$ and $\varpi \neq \Gamma(G)$, it is not necessarily true that G has a Hall ϖ-subgroup that is isolated.

1. The Beginning of the Proof

Let G be a finite group and let $\varpi \subseteq \pi(G)$ be the set of primes of a connected component Δ of the prime graph $\Gamma(G)$. Assume that

$$\varpi \neq \pi(G) \quad \text{and} \quad 2 \notin \varpi.$$

These conditions and notation are used throughout this paper. The starting point of the proof is the following proposition.

Proposition 1. *Let P be a nonidentity ϖ-subgroup of G. If $N_G(P)$ is of even order, then G has an abelian Hall ϖ-subgroup that is isolated in G.*

Proof. Since $2 \notin \varpi$, P is of odd order. By assumption, there is an element t of order 2 that normalizes P. Since ϖ is a connected component and $2 \notin \varpi$, the element t acts regularly on P. This yields that

$$x^t = x^{-1} \quad \text{for} \quad x \in P.$$

Thus, P is abelian. If $x \in P^\sharp$, $C_G(x)$ is a ϖ-group and is normalized by t. It follows that $A = C_G(x)$ is abelian and the element t inverts every element of A. If $y \in A^\sharp$, the same argument proves that $C_G(y)$ is abelian. Since $A = C_G(x) \subseteq C_G(y)$, we have $C_G(y) = A$. Therefore, A is an abelian subgroup that is isolated in G. It is known that every isolated subgroup is a Hall subgroup. Q.E.D.

Therefore, to prove Theorem B, we may assume that every ϖ-local subgroup is of odd order. From now on we use the following notation and assumptions in addition to the ones already stated.

Let G be a finite simple group. Let

$$\mathcal{M} = \{M \mid M \text{ is a maximal } \varpi\text{-local subgroup of } G\},$$

define

$$\mathcal{M}(H) = \{M \in \mathcal{M} \mid H \subseteq M\}$$

for any subgroup H of G, and assume that *every subgroup $M \in \mathcal{M}$ is of odd order.*

The set of subgroups $\mathcal{M}$ satisfies properties which are similar to the properties of the set of all maximal subgroups of the hypothetical minimal simple group of odd order studied by [FT] and [BG]. We remark that the situation considered here does occur in real groups. For example, if p is a prime such that $p \equiv 3 \pmod 4$, the alternating group A_p satisfies the condition for $\varpi = \{p\}$.

2. The Local Analysis of $\mathcal{M}$

We can apply the method of Bender and Glauberman to study the subgroups in $\mathcal{M}$. The subgroups in $\mathcal{M}$ are of odd order; hence, they are solvable by the Odd Order Theorem. By definition, $M \in \mathcal{M}$ is a ϖ-local subgroup. It follows that $F(M)$, the Fitting subgroup of M, is a ϖ-subgroup. Let $p \in \pi(G)$ and let $P \in Sy\ell_p(M)$. If P is not cyclic, P contains an elementary abelian p-subgroup A of order p^2. Then, A normalizes $N = O_\varpi(M)$ which is not 1. By a well-known proposition (Proposition 1.16 [BG]),

$$N = \langle C_N(x) \mid x \in A^\sharp \rangle.$$

It follows that $p \in \varpi$. Thus, if M is not a ϖ-group, M has a cyclic Sylow p-subgroup for every $p \in \pi(M) \setminus \varpi$, Thus, $M \in \mathcal{M}$ is almost a ϖ-subgroup. However, I call attention to the following point. For $M \in \mathcal{M}$, the set $\sigma(M)$ of primes is defined in [BG] as

$$\sigma(M) = \{p \in \pi(M) \mid N_G(P) \subseteq M \text{ for some } P \in Sy\ell_p(M)\}$$

(p.70 [BG]). The important set in our case is

$$\sigma_0(M) = \sigma(M) \cap \varpi$$

and the subgroup we should study is

$$M_{\sigma_0} = O_{\sigma_0(M)}.$$

It is proved that M_{σ_0} is a Hall $\sigma_0(M)$-subgroup of M.

Proposition 2. *All the statements of the sections* $7-15$ *of* [*BG*] *hold with proper changes in the hypotheses and conclusions.*

The *types* of subgroups in $\mathcal{M}$ are defined as in pp.128–129 [BG] with the following three changes.

(II*iv*) should read: $V \neq 1$ and if V is a ϖ-group, then

$$N_G(V) \not\subseteq M.$$

(IIv) should read: $N_G(A) \subseteq M$ for every nonidentity subgroup A of M' such that $C_H(A) \neq 1$.

(IIIiii) should read: V is an abelian ϖ-group and $N_G(V) \subseteq M$.

Then, $M \in \mathcal{M}$ is of type I, II, III, IV, or V. We have the following two theorems which are the goal of the local analysis.

Theorem I. *Either every subgroup in $\mathcal{M}$ is of type I or all the following conditions are true.*

(1) *G contains a cyclic subgroup $W = W_1 \times W_2$ with the property that $N_G(W_0) = W$ for every nonempty subset W_0 of $W - \{W_1, W_2\}$. Also, $W_i \neq 1$ for $i = 1, 2$.*
(2) *There are two subgroups S and T in $\mathcal{M}$ such that S and T are of type II, III, IV, or V, $S \cap T = W$, S is not conjugate to T in G, and either S or T (it may be both) is of type II.*
(3) *Every $M \in \mathcal{M}$ is either of type I or conjugate to S or T in G.*

There are other conditions which S and T must satisfy. For each $M \in \mathcal{M}$, two particular subsets $A(M)$ and $A_0(M)$ of M are defined (cf. p.124 and p.131 [BG]). The notation M_F for each $M \in \mathcal{M}$ denotes the normal nilpotent Hall subgroup of maximal order of M.

Theorem II. *For a subgroup $M \in \mathcal{M}$, let $X = A(M)$ or $A_0(M)$, and let*

$$D = \{x \in X^\sharp \mid C_G(x) \nsubseteq M\}.$$

Then, $D \subseteq M_{\sigma_0}$, $|\mathcal{M}(C_G(x))| = 1$ for all $x \in D$, and the following conditions are satisfied.

(Ti) *Whenever two elements of X are conjugate in G, they are conjugate in M.*
(Tii) *If D is not empty, there are ϖ-subgroups $M_1, \ldots, M_n$ in $\mathcal{M}$ of type I or II such that with $H_i = (M_i)_F$,*
   (a) $(|H_i|, |H_j|) = 1$ *for* $i \neq j$,
   (b) $M_i = H_i(M \cap M_i)$ *and* $M \cap H_i = 1$,
   (c) $(|H_i|, |C_M(x)|) = 1$ *for all* $x \in X^\sharp$,
   (d) *$A_0(M_i) - H_i$ is a nonempty TI-set in G with normalizer M_i, and*
   (e) *if $x \in D$, then there is a conjugate y of x in D and an index i such that*

$$C_G(y) = C_{H_i}(y) C_M(y) \subseteq M_i.$$

If $y \in D$ with $C_G(y) \subseteq M_i$, then $y \in A(M_i)$.

(T*iii*) *If some M_i in (Tii) has type II, then M is a ϖ-group and is a Frobenius group with cyclic Frobenius complement, and M_F is not a TI-set in G.*

3. Application of Character Theory

We can study subgroups of $\mathcal{M}$ using character theory as in [FT]. The following are the major steps.

Proposition 3. *There is no subgroup $M \in \mathcal{M}$ of type V.*

Proposition 4. *Every subgroup $M \in \mathcal{M}$ of type I is a Frobenius group.*

This is very powerful. Suppose that $M \in \mathcal{M}$ is not a Frobenius group. Then, any supporting subgroup M_i for M in Theorem II is of type I by (*Tiii*). Then, Proposition 4 yields that M_i is a Frobenius group. However, it is easy to see that $A_0(M_i) = H_i$ for a Frobenius group. This contradicts (*Tii*)(d). Therefore, there is no supporting subgroup. It follows that X is a TI-set in G. This gives a very tight control on the imbedding of any M that is not a Frobenius group. In particular, we can study the subgroups in $\mathcal{M}$ which are of type II, III, or IV. The final result is the following.

Theorem III. *Let G be a finite simple group with disconnected prime graph $\Gamma(G)$. Let ϖ be a connected component such that $2 \notin \varpi$. Then, one of the following two cases occurs.*

(1) *G contains a nilpotent Hall ϖ-subgroup that is isolated in G.*
(2) *We have $\varpi = \{p, q\}$ for some primes p and q, and G has a self-normalizing cyclic subgroup of order pq.*

If the second case occurs, there are many more conditions the primes p and q must satisfy. It may be possible to eliminate the case (2) without referring to the classification of finite simple groups. In any case, Theorem III implies Theorem B.

Theorem IV. *Let G be a finite simple group with disconnected prime graph $\Gamma(G)$. Let Δ be a connected component consisting of odd primes. Then, Δ is a clique.*

4. Lemmas

For the most part, we will follow the notation and terminology in [BG]. Exceptions are noted in the body of the paper. As usual, for a prime $p \in \pi(G)$, we denote by

$$Sy\ell_p(G)$$

the set of all Sylow p-subgroups of G.

Let X be a group and Y a subgroup of X. As in [BG], a *complement* Z of Y in X is defined to be a subgroup Z of X satisfying the two conditions

$$Y \cap Z = 1 \quad \text{and} \quad X = YZ.$$

We have the following well-known lemma.

Lemma. *Let X be a group and Y a subgroup of X. Suppose that Y has a complement Z in X. If U is a subgroup of X such that $Y \subseteq U \subseteq X$, then Y has a complement in U.*

Proof. We will show that $U \cap Z$ is a complement of Y in U. Let $W = U \cap Z$. Then, clearly $Y \cap W = 1$. We have

$$U = X \cap U = YZ \cap U.$$

Since $Y \subseteq U$, the Dedekind law yields that

$$YZ \cap U = Y(Z \cap U) = YW.$$

This proves that Y has a complement W in U. Q.E.D.

Next, we will state five lemmas which are used freely throughout this paper. Their proofs can be found at the end of the introduction.

Lemma A. *If $P \neq 1$ is a ϖ-group, so is $C_G(P)$.*

Lemma B. *Suppose a noncyclic elementary abelian p-group E acts on a subgroup $H \neq 1$.*

(1) *If H is a ϖ-group, then $p \in \varpi$.*
(2) *If $p \in \varpi$ and H is a p'-group, then H is a ϖ-group.*

Lemma C. *Assume $2 \notin \varpi$. If there exists a ϖ-local subgroup of even order, then G contains an isolated abelian Hall ϖ-subgroup.*

Lemma D. *A ϖ-group $\neq 1$ is contained in a ϖ-local subgroup.*

Lemma E.

(1) *If $M \in \mathcal{M}$ and a pi-subgroup $K \neq 1$ is normal in M, then $M = N_G(K)$.*
(2) *If $M \in \mathcal{M}$, then $N_G(M) = M$.*
(3) *If $M \in \mathcal{M}$ normalizes a ϖ-group $H \neq 1$, then $H \subseteq M$ and $M = N_G(H)$.*

Furthermore, we will collect here a few fundamental lemmas which are explicitly stated in the body of the paper. The terminology and notation are given there, as are the proofs.

Lemma F (See §4, page 9). *If $M \in \mathcal{M}$ and $p \in \pi(M) \cap \varpi'$, then M has a cyclic p-Sylow subgroup.*

Lemma G (See §4, page 12). *If $M \in \mathcal{M}$, then M is a ϖ-group except when*

(1) *M is a Frobenius group such that the Frobenius kernel of M is a Hall ϖ-subgroup of M, or*
(2) *M has the following structure: M/M' is a cyclic ϖ-group, $M_\alpha = M_\beta = M_{\sigma_0} \neq 1$ is a nilpotent ϖ-group, and M'/M_β is a non-identity cyclic ϖ'-group that is a Hall subgroup of M. Both M' and M/M_β are Frobenius groups.*

In the case (1), the Frobenius kernel of M is M_{σ_0} and it is either M' or $M_\alpha = M_\beta$.

Lemma H (See §6, page 17). *Let $M \in \mathcal{M}$. If $\tau_2(M) \neq \emptyset$, then M is a ϖ-group. If M is not a ϖ-group, then $r_p(M) \leq 1$ for all $p \notin \sigma_0(M)$.*

We also need some lemmas about the fusion of elements (§11, page 66). Our hypotheses are weaker than those in [BG], and these lemmas guarantee that the same results still hold.

Lemma I. *Let $M \in \mathcal{M}$ and let X be an F-set of M. Every element of $X^\sharp$ is conjugate to an element of D^* in M.*

Lemma J.

(1) *Every element g of M_i is conjugate in M_i to an element of the form $xh = hx$ where $x \in M \cap M_i$ and $h \in H_i$.*
(2) *Suppose that g is an element of M_i with $C_{H_i}(g) \neq 1$. Assume that g is conjugate in M_i to an element of the form hx where $x \in M \cap M_i$ and $h \in C_{H_i}(x)$, and at the same time, g is conjugate to an element of the annex $A(y)$ with $y \in D_j$. Then, $j = i$ and the element x is conjugate to y in M_i. In particular, $x \in D_i$ and $g \in A(M_i)$.*

Finally, we prove Lemmas A through E.

The first three lemmas give a few basic properties of connected components of prime graphs.

Lemma A. *Let G be a group and let ϖ be a connected component (or a union of connected components) of the prime graph $\Gamma(G)$ of G. If $P \neq 1$ is a ϖ-subgroup of G, then $C_G(P)$ is a ϖ-group.*

Proof. Let $p \in \pi(P)$. Then, $p \in \varpi$. Take any $q \in \pi(C_G(P))$. We will show $q \in \varpi$. We may assume $q \neq p$. There are elements x and y such that x is an element of P of order p and y is an element of $C_G(P)$ of order q. Since x and y commute, the product xy has order pq. Thus, (p, q) is an edge of the prime graph $\Gamma(G)$. This proves that q lies in the same connected component as the prime $p \in \varpi$. Hence, $q \in \varpi$. Q.E.D.

Lemma B. *Let G be a group and let ϖ be a connected component* (or a union of connected components) *of the prime graph $\Gamma(G)$ of G. Let p be a prime. Suppose that a noncyclic elementary abelian p-subgroup E normalizes a subgroup H of G.*

(1) *If H is a ϖ-group $\neq 1$, then $p \in \varpi$.*
(2) *If $p \in \varpi$ and H is a p'-group, then H is a ϖ-group.*

Proof. By our assumptions, $K = HE$ is a subgroup and $H \lhd K$.

(1) Suppose that H is a ϖ-group $\neq 1$. If $p \in \pi(H)$, then $p \in \varpi$. Suppose that $p \notin \pi(H)$. Then, H is a p'-group. By Proposition 1.16 [BG],

$$H = \langle C_H(x) \mid x \in E^\sharp \rangle.$$

Since $H \neq 1$, $P = C_H(x) \neq 1$ for some $x \in E^\sharp$. Then, $x \in C_G(P)$ where $P \neq 1$ is a ϖ-group. By Lemma A, $C_G(P)$ is a ϖ-group, so in particular, the order of x is a ϖ-number. This proves $p \in \varpi$.

(2) Let $q \in \pi(H)$ and $Q \in Sy\ell_q(H)$. Then, $q \neq p$ and Q is a Sylow q-subgroup of K. By the Frattini argument, $K = HN_K(Q)$. Therefore, a conjugate of E normalizes Q. We may replace E by a conjugate (in K) and assume that E normalizes Q. Since Q is a p'-group, Proposition 1.16 [BG] yields

$$Q = \langle C_Q(x) \mid x \in E^\sharp \rangle.$$

Since $\langle x \rangle$ is a ϖ-group by assumption, Lemma A implies that $C_Q(x)$ is a ϖ-group. Therefore, $q \in \varpi$. This proves that H is a ϖ-group. Q.E.D.

Lemma C. *Let G be a group and ϖ a connected component of the prime graph $\Gamma(G)$. Suppose that the prime 2 is not contained in ϖ and that there is a ϖ-local subgroup of even order. Then, G contains an abelian Hall ϖ-subgroup A that is isolated. Furthermore, any ϖ-element is conjugate to an element of A and the centralizer of any ϖ-element is abelian.*

Proof. By assumption, there is a pair (H, t) of a ϖ-subgroup H and an element t of order 2 that normalizes H. We have a lemma: For any pair (H, t) consisting of a ϖ-subgroup H and an element t of order 2 that normalizes H, t inverts every element of H and consequently, H is abelian. This follows from the lemma of Burnside ((1.9) [S II] p. 131). Note that since $2 \notin \varpi$, $C_H(t) = 1$ by Lemma A. By a first application of the above lemma, the element t inverts every element x of $H^\sharp$, i.e. $txt^{-1} = x^{-1}$. It follows that t normalizes $A = C_G(x)$. By Lemma A, A is a ϖ-subgroup of G. Take $y \in A^\sharp$. A second application of the lemma proves that $C_G(y)$ is abelian. Since A is abelian, $A \subseteq C_G(y)$. By the definition of A, $A = C_G(x)$ for some $x \in A^\sharp$. Thus, the abelian group $C_G(y)$ must coincide with A, i.e. A satisfies the property that if $y \in A^\sharp$, then $C_G(y) = A$. An easy application of Sylow's Theorem yields that A is a Hall subgroup of G.

If $A \cap uAu^{-1} \neq 1$ for some $u \in G$, then take a nonidentity element y of $A \cap uAu^{-1}$ and consider $C_G(y)$. Then, $A = C_G(y) = uAu^{-1}$. Thus, A is isolated.

We will show that $\varpi = \pi(A)$. Suppose that $\varpi \neq \pi(A)$. Then, there is a pair of primes (p, q) such that $p \in \pi(A)$, $q \in \varpi - \pi(A)$, and (p, q) is an edge of the prime graph $\Gamma(G)$. Therefore, there are elements a, b such that a is of order p, b is of order q, $a \in A^\sharp$, and a commutes with b. It follows that $b \in C_G(a) = A$. This contradicts the choice of q with $q \notin \pi(A)$. We have shown that A is a *varpi*-Hall subgroup of G that is isolated.

If z is any ϖ-element, $\langle z \rangle$ is conjugate to a subgroup of A by a theorem of Wielandt [W 1954]. The last assertion follows. Q.E.D.

We will also need the following properties of $\mathcal{M}$.

Lemma D. *Let G be a group and π a set of primes. Any π-subgroup $H \neq 1$ is contained in a maximal π-local subgroup.*

Proof. By definition, $K = N_G(H)$ is a π-local subgroup of G. Let M be a π-local subgroup of maximal order that contains K. Then, M is a maximal π-local subgroup such that $H \subseteq M$. Q.E.D.

Lemma E.

(1) *If $M \in \mathcal{M}$ and a π-subgroup $K \neq 1$ is normal in M, then $N_G(K) = M$.*
(2) *If $M \in \mathcal{M}$, then $N_G(M) = M$.*
(3) *If $M \in \mathcal{M}$ normalizes a π-subgroup $H \neq 1$ of G, then $H \subseteq M$ and $M = N_G(H)$.*

Proof. (1) By assumption, $N_G(K)$ is a π-local subgroup that contains M. Since $M \in \mathcal{M}$, we get $N_G(K) = M$.

(2) Let $K = O_\pi(M)$. Then, K is a π-subgroup $\neq 1$ of G. Hence, $M = N_G(K)$ by (1). Since $K \operatorname{char} M$, $N_G(M) \subseteq N_G(K)$. Hence, $N_G(M) = M$.

(3) By assumption, $N_G(H)$ is a π-local subgroup that contains M. Since $M \in \mathcal{M}$, we have $N_G(H) = M$. Q.E.D.

Chapter I. Local Analysis

We begin the local analysis of the simple groups G that satisfies the basic assumptions. We need the following notation.

Notation. Let G be a simple group with disconnected prime graph $\Gamma = \Gamma(G)$. Let ϖ be a connected component of Γ that consists of odd primes. We fix the following notation:

$$\begin{aligned} \mathcal{M} &= \text{the set of all maximal } \varpi\text{-local subgroups,} \\ \mathcal{M}(H) &= \text{the set of } M \in \mathcal{M} \text{ such that } H \subseteq M, \\ \mathcal{U} &= \text{the set of all proper subgroups } H \subseteq G \text{ such that } |\mathcal{M}(H)| = 1. \end{aligned}$$

The **basic assumptions** are

$$2 \notin \varpi$$

and

the set $\mathcal{M}$ *consists of subgroups of odd order.*

Thus, if $M \in \mathcal{M}$, then M is a solvable group of odd order.

The above notation and the basic assumptions are in force throughout this paper, not just in Chapter I.

Chapter I contains 10 sections and is organized as follows. Section m of this chapter corresponds to Section $m+6$ of [BG]. Lemma (Theorem, Proposition, or Corollary) $m.k$ of Section m of this paper corresponds to Lemma (Theorem, Proposition, or Corollary) $(m+6).k$ in [BG]. Proof may sometimes be obtained from the proof of the corresponding lemma in [BG] simply by changing the reference to Lemma $n.k$ to Lemma $(n-6).k$ of this paper when $n > 6$. If this is the case, the proof is usually omitted by referring to [BG].

§1. The Transitivity Theorem

Hypothesis 1.1. (1) The group A is a noncyclic subgroup of G with $O_{\varpi}(A) \neq 1$, and $\pi = \pi(A)$.

(2) Whenever X is a ϖ-local subgroup of the group G such that $A \subseteq X$, we have

$$O_{\pi'}(X) = \langle И_X(A;\pi') \rangle.$$

Let $K = O_{\pi'}(C_G(A))$ as in [BG]. Then, *K is the set of all π'-elements in $C_G(A)$.* This is proved as follows. Let $B = O_{\varpi}(A)$. By Hypothesis 1.1 (1), $B \neq 1$. Hence, $C_G(A) \subseteq C_G(B) \subseteq N_G(B)$. This implies that $N_G(B) = X$ is a ϖ-local subgroup that contains A. Let x be a π'-element of $C_G(A)$. Then, $\langle x \rangle$ is a π'-subgroup of X that is A-invariant. By (2), $\langle x \rangle \subseteq O_{\pi'}(X)$. Therefore,

$$\langle x \rangle \subseteq C_G(A) \cap O_{\pi'}(X) \subseteq O_{\pi'}(C_G(A)) = K.$$

Conversely, if $x \in K$, then x is a π'-element of $C_G(A)$. Q.E.D.

Lemma 1.1. *Assume Hypothesis* 1.1. *Suppose, for a prime $q \in \pi' \cap \varpi$, that Q_1, $Q_2 \in И^*_G(A;q)$ and that there exists a ϖ-local subgroup H of G such that*

$$A \subseteq H, H \cap Q_1 \neq 1, \quad \textit{and} \quad H \cap Q_2 \neq 1.$$

Then, $Q_2 = {Q_1}^k$ for some $k \in K$.

Proof. We proceed by induction on $|G|_q/|Q_1 \cap Q_2|$. If this number is 1, then Q_1 and Q_2 are Sylow subgroups of G with $|Q_1 \cap Q_2| = |Q_1| = |Q_2|$. This implies $Q_1 = Q_2 = {Q_1}^k$ with $k = 1 \in K$. Proceed by induction. By the basic assumptions, H is a solvable group. Hence, the A-invariant q-subgroup $H \cap Q_1$ is contained (in $O_{\pi'}(H)$ by Hypothesis 1.1 and so) in an A-invariant Sylow q-subgroup R_1 of $O_{\pi'}(H)$. Similarly, $H \cap Q_2 \subseteq R_2$ where R_2 is an A-invariant Sylow q-subgroup of $O_{\pi'}(H)$. By Proposition 1.5 [BG], ${R_1}^h = R_2$ for some $h \in O_{\pi'}(H) \cap C_G(A)$. Since h is a π'-element of $C_G(A)$, the remark after Hypothesis 1.1 yields $h \in K$.

Take $Q_3 \in И^*_G(A;q)$ such that $R_2 \subseteq Q_3$. Since $h \in K$, $Q_1^h \in И_G(A;q)$. We have $1 \neq (Q_1 \cap H)^h = {Q_1}^h \cap H \subseteq {R_1}^h = R_2 \subseteq Q_3$ and $1 \neq Q_2 \cap H \subseteq R_2 \subseteq Q_3$. Therefore, $1 \neq {Q_1}^h \cap H \subseteq {Q_1}^h \cap Q_3$ and $1 \neq Q_2 \cap H \subseteq Q_2 \cap Q_3$.

If $Q_1 \cap Q_2 = 1$, we are done as in [BG]. Suppose that $Q = Q_1 \cap Q_2 \neq 1$. Since q is assumed to be in ϖ, $N_G(Q)$ is a ϖ-local subgroup that contains A and we may assume $H = N_G(Q)$. The proof of Lemma 7.1 [BG] applies now without change. Q.E.D.

Theorem 1.2. *Assume Hypothesis* 1.1 *and let* $q = \pi' \cap \varpi$. *Suppose* $m(Z(A)) \geq 3$. *Then, K acts transitively on* $\mathcal{U}^*_G(A; q)$.

Proof. By hypothesis, $Z(A)$ contains an elementary abelian p-subgroup B of order p^3 for some prime p. Since B centralizes $O_\varpi(A)$ and $O_\varpi(A) \neq 1$, we have $p \in \varpi \cap \pi$. So, $p \neq q$. The proof of Theorem 7.2 [BG] yields the result if we apply Lemma 1.1 to the ϖ-local subgroup $N_G(\langle z \rangle)$ at the end. Q.E.D.

Theorem 1.3. *Assume Hypothesis* 1.1 *and let* $q \in \pi' \cap \varpi$. *Suppose* $r(Z(A)) \geq 2$ *and* $q \in \pi(C_G(A))$. *Then, K acts transitively on* $\mathcal{U}^*_G(A; q)$.

Proof. The proof of Theorem 7.3 [BG] applies here if we use Lemma 1.1 with the ϖ-local subgroup $N_G(\langle x \rangle)$ for some $x \in B$ with $C_{Q_1}(x) \neq 1$. Q.E.D.

Theorem 1.4. *Assume Hypothesis* 1.1 *and let* $q \in \pi' \cap \varpi$. *Suppose that P is a π-subgroup of G that contains A as a subnormal subgroup and that K acts transitively on* $\mathcal{U}^*_G(A; q)$. *Then,*

(a) $C_K(P) = O_{\pi'}(C_G(P))$,
(b) $O_{\pi'}(C_G(P))$ *acts transitively on* $\mathcal{U}^*_G(P; q)$,
(c) $\mathcal{U}^*_G(P; q) \subseteq \mathcal{U}^*_G(A; q)$, *and*
(d) *for every* $Q \in \mathcal{U}^*_G(P; q)$, *we have* $P \cap N_G(P)' \subseteq N_G(Q)'$ *and* $N_G(P) = O_{\pi'}(C_G(P))(N_G(P) \cap N_G(Q))$.

Proof. Since A is subnormal in P, we have $O_\varpi(A) \subseteq O_\varpi(P)$. Therefore, $O_\varpi(P) \neq 1$ and P is contained in a ϖ-local subgroup. Note that, by the basic assumptions, $|P|$ is odd so P is solvable. The subgroup P satisfies the condition that is obtained from Hypothesis 1.1 replacing A by P.

Since $C_G(P) \subseteq C_G(A)$, $O_{\pi'}(C_G(P))$ is a set of π'-elements of $C_G(A)$. Hence, $O_{\pi'}(C_G(P)) \subseteq K \cap C_G(P) = C_K(P)$. On the other hand, $C_K(P) = O_{\pi'}(C_G(A)) \cap C_G(P) \subseteq O_{\pi'}(C_G(P))$. We have proved (a).

To prove the parts (b) and (c) we use induction on $|P : A|$. Let

$$1 = P_0 \lhd P_1 \lhd \cdots \lhd P_{n-1} \lhd P_n = P$$

be a composition series of P through A. If $A = P_{n-1}$ (or $A = P_n$), the proof of Theorem 7.4 [BG] for the case $k > n-2$ yields (b) and (c). If $A \neq P_{n-1}$, let $B = P_{n-1}$. Note that B satisfies the condition obtained from Hypothesis 1.1 by replacing A by B. The parts (b) and (c) follow by induction as in [BG].

In order to prove (d), take any $Q \in И_G^*(P;q)$ and let $L = N_G(P) \cap N_G(Q)$. If $x \in N_G(P)$, then $Q^x \in И_G^*(P;q)$. By (b), $Q^x = Q^y$ for some $y \in O_{\pi'}(C_G(P))$. Hence, $xy^{-1} \in N_G(Q) \cap N_G(P) = L$. Therefore,

$$N_G(P) = LO_{\pi'}(C_G(P)) = LC_K(P).$$

Since $O_{\pi'}(C_G(P)) = C_K(P) \lhd N_G(P)$, we have $N_G(P) = C_K(P)L$.

Note that $N_G(P)$ is contained in a ϖ-local subgroup, so by the basic assumptions, $N_G(P)$ is solvable of odd order. Lemma 6.5 [BG] with (G,K,U,H) replaced by $(N_G(P), O_{\pi'}(C_G(P)), L, P)$ yields $P \cap N_G(P)' = P \cap L' \subseteq L' \subseteq N_G(Q)'$. Q.E.D.

Proposition 1.5. *Suppose $p \in \varpi$ and A is an abelian p-subgroup of G. Assume that either* (1) $A = \{x \in C_G(A) \mid x^p = 1\}$ *and every ϖ-local subgroup of G has p-length* 1, *or* (2) $A \in SCN_2(P)$ *for some $P \in Syl_p(G)$. Then, A satisfies Hypothesis* 1.1.

Proof. We can use the same method as in the proof of Theorem 7.5 [BG]. The proof in [BG] utilizes the centralizer $C_G(b)$ of an element b of order p. This subgroup need not be ϖ-local. However, it is contained in the ϖ-local subgroup $N_G(\langle b\rangle)$. Since the index $|N_G(\langle b\rangle) : C_G(\langle b\rangle)|$ is prime to p, we may replace $C_G(b)$ by $N_G(\langle b\rangle)$ without affecting the argument. Q.E.D.

Theorem 1.6 (Transitivity Theorem). *Suppose $p \in \varpi$, $A \in SCN_3(p)$, and $q \in p' \cap \varpi$. Then, $O_{p'}(C_G(A))$ acts transitively on $И_G^*(A;q)$ by conjugation.*

§2. The Fitting Subgroup of a Maximal ϖ-Local Subgroup

This section corresponds to Section 8 of [BG]. We begin with the following remark. Let H be a ϖ-local subgroup of G. By the basic assumptions, H is a solvable group of odd order. Let $F = F(H)$ be the Fitting subgroup of H. Since $O_\varpi(H) \neq 1$, we have $O_\varpi(F) \neq 1$. This implies that $\pi(F) \subseteq \varpi$ as F is nilpotent and is the direct product of its Sylow subgroups.

Theorem 2.1. *Suppose $M \in \mathcal{M}$, $p \in \pi(F(M))$, and $A_0 \in \mathcal{E}_p^*(F(M))$. Assume that $m(A_0) \geq 3$. Let $P \in Syl_p(M)$.*

(a) *If $F(M)$ is not a p-group, then $C_{F(M)}(A_0) \in \mathfrak{U}$.*
(b) *If $F(M)$ is a p-group, then $P \in Syl_p(G)$ and every element of $SCN_3(P)$ is contained in $F(M)$ and belongs to $\mathfrak{U}$.*

Proof. (a) Let $F = F(M)$, $\pi = \pi(F)$ and $A = C_F(A_0)$. Then $\pi(A) = \pi$ because $Z(F) \subseteq C_F(A_0) = A \subseteq F$. Note that for every $q \in \pi$,

$$C_G(A) \subseteq C_G(A_q) \subseteq C_G(Z(F)_q) \subseteq N_G(Z(F)_q) = M.$$

The last equality comes from Lemma E (1) since $Z(F)_q$ is a nonidentity normal ϖ-subgroup of M. The notation N_π stands for $O_\pi(N)$ of a nilpotent group N as in [BG].

We will show that $C_G(A)$ is a π-subgroup. Suppose that x is a π'-element of $C_G(A)$. Let $C = C_F(x)$. By the first paragraph, $x \in M$. Since $A \subseteq C$, $C_F(C) \subseteq C_F(A) \subseteq C_F(A_0) = A \subseteq C$. By Proposition 1.10 [BG] , $x \in C_M(F) = C_M(F(M)) \subseteq F$. Since x is a π'-element with $\pi = \pi(F)$, we get $x = 1$. Thus, $C_G(A)$ is a π-subgroup of M.

We prove the following lemma. *Let p be any prime, X a solvable subgroup of G and P a p-subgroup of X. Then,*

$$O_{p'}(N_G(P)) \cap X \subseteq O_{p'}(X).$$

Proof. Let $Y = O_{p'}(N_G(P)) \cap X$. Then,

$$Y = O_{p'}(N_G(P)) \cap N_X(P) \subseteq O_{p'}(N_X(P)).$$

Since $P \subseteq X$, we have $[O_{p'}(N_X(P)), P] \subseteq O_{p'}(N_X(P)) \cap P = 1$. Hence,

$$O_{p'}(N_X(P)) \subseteq O_{p'}(C_X(P)).$$

By Proposition 1.15 [BG], $O_{p'}(C_X(P)) \subseteq O_{p'}(X)$.
This proves $Y \subseteq O_{p'}(X)$. Q.E.D.

With this lemma on hand, we verify Hypothesis 1.1 for A. Take an arbitrary ϖ-local subgroup X that contains A and $Y \in И_X(A; \pi')$. Take any $q \in \pi$. By the first paragraph of the proof, $C_Y(A_q) \subseteq M$. Since Y is an A-invariant π'-subgroup,

$$[C_Y(A_q), A] \subseteq Y \cap [M, A] = Y \cap F = 1.$$

Hence, $C_Y(A_q) \subseteq C_G(A)$. Since $C_G(A)$ is a π-group, we have $C_Y(A_q) = 1$. Thus, by Proposition 1.6 [BG], $Y = C_Y(A_q)[Y, A_q] = [Y, A_q]$. By hypothesis, $|\pi| \geq 2$. Take $r \neq q$ in π. Since $N_G(Z(F)_q) = M$ by Lemma E (1), $A_r \subseteq F_r \subseteq O_{q'}(M)$: and: $A_r \subseteq O_{q'}(N_G(Z(F)_q)) \cap X$.

Since $Z(F) \subseteq A \subseteq X$, Lemma implies

$$A_r \subseteq O_{q'}(X) \quad \text{for any } q \neq r \text{ in } \pi. \tag{1}$$

Since $Y = [Y, A_r]$, we have $Y \subseteq O_{q'}(X)$ for all $q \in \pi$. Hence, $Y \subseteq \bigcap_{q \in \pi} O_{q'}(X) = O_{\pi'}(X)$. This proves Hypothesis 1.1 for A.

We will prove that $\mathcal{U}_G^*(A;q) = \{1\}$ for every $q \in \pi' \cap \varpi$. Take $q \in \pi' \cap \varpi$. Since $m(Z(A)) \geq m(A_0) \geq 3$, Theorem 1.2 implies that $O_{\pi'}(C_G(A))$ acts transitively on $\mathcal{U}_G^*(A;q)$. But, $C_G(A)$ is a π-group, so $O_{\pi'}(C_G(A)) = 1$. Thus, $\mathcal{U}_G^*(A;q) = \{Q\}$ for some q-subgroup Q of G. Since F is nilpotent, $A \lhd\lhd F$. By Theorem 1.4 (c), $\mathcal{U}_G^*(F;q) \subseteq \mathcal{U}_G^*(A;q)$. Therefore, $\mathcal{U}_G^*(F;q) = \{Q\}$ and M normalizes Q. By Lemma E (3), $Q \subseteq M$. Hence, $Q \lhd M$ and $Q \subseteq F(M)$. Since $\pi = \pi(F(M))$ and $q \in \pi'$, we have $Q = 1$. Thus, $\mathcal{U}_G^*(A;q) = \{1\}$ for $q \in \pi' \cap \varpi$.

To prove $A \in \mathfrak{U}$, take $H \in \mathcal{M}(A)$. Let $D = F(H)$ and $\sigma = \pi(D)$. We will prove first $\sigma = \pi$. Since A normalizes D, the last paragraph yields $\sigma \subseteq \pi$. By definition of D, $O_{\sigma'}(H) = 1$. We have

$$O_{\sigma'}(Z(F)) \subseteq O_{\sigma'}(A) = \langle A_r \mid r \in \pi \cap \sigma' \rangle.$$

By (1) for $X = H$, $A_r \subseteq O_{q'}(H)$. Hence

$$\langle A_r \mid r \in \pi \cap \sigma' \rangle \subseteq \bigcap\nolimits_{q \in \sigma} O_{q'}(H) = O_{\sigma'}(H) = 1.$$

It follows that $\pi \cap \sigma'$ is empty, i.e. $\pi \subseteq \sigma$. Thus, $\sigma = \pi$.

For each $q \in \pi$, $O_{q'}(A) = \langle A_r \mid r \neq q \rangle \subseteq O_{q'}(H)$. So,

$$[D_q, O_{q'}(A)] \subseteq [D_q, O_{q'}(H)] = 1. \tag{2}$$

Hence, $D_q \subseteq C_G(O_{q'}(A)) \subseteq N_G(O_{q'}(A)) = M$. The last equality is by Lemma E (1). This proves $D \subseteq M$.

The formula (2) implies that A_p centralizes $O_{p'}(D)$. Since $O_{p'}(D) = F(O_{p'}(H))$, Proposition 1.4 [BG] implies that A_p centralizes $O_{p'}(H)$. Hence, $O_{p'}(H) \subseteq C_G(A_p) \subseteq M$ by the first paragraph of the proof. By Lemma E (1) for H, $O_{p'}(H) \subseteq O_{p'}(N_G(D_p)) \cap M$. Since $D_p \subseteq M$, the lemma applies to get $O_{p'}(H) \subseteq O_{p'}(M)$.

We will prove that $O_{p'}(M) \subseteq O_{p'}(H)$. Since A_0 is a p-subgroup of F, we have $O_{p'}(F) \subseteq C_G(A_0) = A$. Thus, $O_{p'}(F) = O_{p'}(A)$. By (2), D_p centralizes $O_{p'}(A) = O_{p'}(F) = F(O_{p'}(M))$. Proposition 1.4 [BG] shows that D_p centralizes $O_{p'}(M)$, i.e. $O_{p'}(M) \subseteq C_G(D_p) \subseteq N_G(D_p) = H$. The last equality is by Lemma E (1) applied to $H \in \mathcal{M}$. Therefore,

$$O_{p'}(M) \subseteq O_{p'}(N_G(Z(F)_p)) \cap H.$$

We have $Z(F)_p \subseteq O_{q'}(A) \subseteq C_G(D_q) \subseteq H$.

The lemma gives us $O_{p'}(M) \subseteq O_{p'}(H)$. Therefore, $O_{p'}(M) = O_{p'}(H)$ and $M = N_G(O_{p'}(M)) = N_G(O_{p'}(H)) = H$. This proves that $A \in \mathfrak{U}$.

(b) The proof of Part (b) of Theorem 8.1 [BG] is applicable. Note that we must take $q \in p' \cap \varpi$ to apply the Transitivity Theorem 1.6 and

that an A-invariant p'-subgroup is a ϖ-subgroup by Lemma B (2).
Q.E.D.

§3. The Uniqueness Theorem

Theorem 3.1. *Suppose that p is a prime, $M \in \mathcal{M}$, $B \in \mathcal{E}_p(M)$, and B is not cyclic. Assume that* (a) *$C_G(b) \subseteq M$ for all $b \in B^\sharp$ or* (b) *$\langle И_G(B;p')\rangle \subseteq M$. Then, $B \in \mathcal{U}$.*

Proof. Since $O_\varpi(M) \neq 1$ and B normalizes $O_\varpi(M)$, Lemma B (1) implies $p \in \varpi$. If $K \in И_G(B;p')$, Lemma B (2) proves that K is a ϖ-group. In particular, $O_{p'}(M)$ is a ϖ-subgroup. It follows that if $O_{p'}(M) \neq 1$, then $M = N_G(O_{p'}(M))$ by Lemma E (1). With these remarks, the proof of Theorem 9.1 [BG] shows the validity of the conclusion of Theorem 3.1. Q.E.D.

Corollary 3.2. *Suppose that $L \in \mathcal{U}$, K is a subgroup of $C_G(L)$, and $r(K) \geq 2$. Then, $K \in \mathcal{U}$ if one of the following conditions holds:*

(a) *$r_p(K) \geq 2$ for some $p \in \varpi$,*
(b) *$\pi(L) \cap \varpi$ is nonempty, or*
(c) *K is contained in some $M \in \mathcal{M}$.*

Proof. Let $\mathcal{M}(L) = \{H\}$. Take $B \in \mathcal{E}_p^2(K)$ for some prime p. If (a) holds, take $p \in \varpi$. If (b) holds, take $q \in \pi(L) \cap \varpi$ and an element x of L of order q. The element x centralizes B. Since $q \in \varpi$, we have $p \in \varpi$. If (c) holds, B normalizes $O_\varpi(M) \neq 1$. Then, $p \in \varpi$ by Lemma B (1). Thus, we have $p \in \varpi$ in all cases.

For each $b \in B^\sharp$, we have $L \subseteq C_G(b) \subseteq N_G(\langle b\rangle)$. Since $p \in \varpi$, $N_G(\langle b\rangle)$ is a ϖ-local subgroup. Since $\mathcal{M}(L) = \{H\}$, we have

$$C_G(b) \subseteq N_G(\langle b\rangle) \subseteq H$$

for all $b \in B^\sharp$. By Theorem 3.1, $B \in \mathcal{U}$ and $\mathcal{M}(B) = \{H\}$. Since $B \subseteq K$, we have $\mathcal{M}(K) = \{H\}$ and $K \in \mathcal{U}$. Q.E.D.

Corollary 3.3. *Suppose that $p \in \varpi$, A is an abelian p-subgroup of G, and B is a noncyclic p-subgroup of G. Assume that $A \in \mathcal{U}$, $m(A) \geq 3$, and $r_p(C_G(B)) \geq 3$. Then, $B \in \mathcal{U}$.*

Proof. Take $B^* \in \mathcal{E}_p^3(C_G(B))$ and let P be a Sylow p-subgroup of G that contains B^*. Replacing A by a conjugate, if necessary, we can assume that $A \subseteq P$. The proof of Corollary 9.3 [BG] shows $B \in \mathcal{U}$.
Q.E.D.

Lemma 3.4. *Suppose that p is a prime, $M \in \mathcal{M}$, and $r_p(F(M)) \geq 3$. Then, $\mathcal{U}$ contains every abelian p-subgroup of rank at least three.*

Proof. The assumptions imply $p \in \varpi$ by Lemma B (1). Lemma follows from Theorem 2.1 and Corollaries 3.2 and 3.3 as in the proof of Lemma 9.4 [BG] . Q.E.D.

Lemma 3.5. *Suppose $p \in \varpi$ and $A \in SCN_3(p)$. Then, $A \in \mathcal{U}$.*

Proof. Since $p \in \varpi$, $C_G(A)$ is a ϖ-group (Lemma A). By Lemma D, $\mathcal{M}(C_G(A))$ is not empty. Let M be an arbitrary element of $\mathcal{M}(C_G(A))$, and let $F = F(M)$. We assume that $A \notin \mathcal{U}$. By Lemma 3.4, we have $r_p(F) \leq 2$.

Choose a prime q as follows: if $r(F) \leq 2$, let q be the largest primes in $\pi(M)$; if $r(F) \geq 3$, let q be some prime for which $r_q(F) \geq 3$. If $r(F) \leq 2$, Theorem 4.20 (c) [BG] implies $O_q(M) \in Sy\ell_q(M)$. In all cases, $O_q(M) \neq 1$. Then, $q \in \varpi$, for if $q \notin \varpi$, $O_q(M)$ would centralize $O_\varpi(M) \neq 1$ contradicting Lemma A.

Since $q \in \varpi$, we have $M = N_G(O_q(M))$ by Lemma E (1). If $r(F) \leq 2$, then $O_q(M)$ is indeed a Sylow q-subgroup of G. Thus, $r_q(G) \leq 2$. Since $r_p(G) \geq 3$, we have $q \neq p$. If $r(F) \geq 3$, then $r_q(F) \geq 3$ while $r_p(F) \leq 2$. Thus, $q \neq p$ in all cases.

Let P be a Sylow p-subgroup of $N_G(A)$ and let R be a subgroup of $P \cap M$ that contains A. Then R normalizes $O_q(M)$. Take $Q \in И_G^*(R; q)$ such that $O_q(M) \subseteq Q$. We will prove $Q \subseteq N_G(Q) \subseteq M$.

If $r(F) \geq 3$, the definition of the prime q implies $r_q(F(M)) \geq 3$, so Lemma 3.4 applies with q in place of p. Since $O_q(M)$ contains an abelian subgroup of rank at least three, $O_q(M) \in \mathcal{U}$ by Lemma 3.4. Since $O_q(M) \subseteq Q \subseteq N_G(Q)$, we have $N_G(Q) \subseteq M$. On the other hand, if $r(F) \leq 2$, then $Q = O_q(M) \lhd M$. Hence, the claim holds in all cases.

We will prove next $N_G(A) \subseteq M$ and $N_G(P) \subseteq M$.

By definition, R is a p-subgroup so $A \lhd\lhd R$. By Theorem 1.6, $O_{p'}(C_G(A))$ acts transitively on $И_G^*(A; q)$. By Proposition 1.5, A satisfies Hypothesis 1.1. By Theorem 1.4, $O_{p'}(C_G(R))$ acts transitively on $И_G^*(R; q)$. Note that $C_G(R) \subseteq C_G(A) \subseteq M$.

Take $x \in N_G(R)$. Then, $Q^x \in И_G^*(R; q)$. Hence,

$$Q^x = Q^y \quad \text{for some} \quad y \in O_{p'}(C_G(R)) \subseteq M.$$

We have $xy^{-1} \in N_G(Q) \subseteq M$. This implies that $x = (xy^{-1})y \in M$. Thus, $N_G(R) \subseteq M$. By taking $R = A$, we have $P \subseteq N_G(A) \subseteq M$. By taking $R = P$, we have $N_G(P) \subseteq M$.

Let $P_0 = [P, N_G(P)]$ and $D = O_{p'}(F)$. Then, $P_0 \neq 1$ (Theorem 1.18 [BG]). We will prove that P_0 centralizes D. Suppose that P_0 does not centralize D. By Proposition 1.16 [BG],

$$D = \langle C_D(B) \mid B \subseteq \Omega_1(A), \quad \Omega_1(A)/B \text{ cyclic}\rangle.$$

Take $B \subseteq \Omega_1(A)$ such that $\Omega_1(A)/B$ is cyclic and P_0 does not centralize $C_D(B)$. Since $A \in SCN_3(p)$, B is not cyclic. Since $A \notin \mathcal{U}$, we have $B \notin \mathcal{U}$. By Theorem 3.1, there exist $y \in B^\sharp$ and $L \in \mathcal{M}$ such that $C_G(y) \subseteq L$ and $C_G(y) \nsubseteq M$. Since $C_G(A) \subseteq C_G(b) \subseteq L$, we can apply the preceding argument, with L in place of M, to conclude that $N_G(P) \subseteq L$. Hence,

$$N_G(P) \subseteq M \cap L \quad \text{and} \quad P_0 \subseteq (N_G(P))' \subseteq (M \cap L)'.$$

Since $D \cap L \lhd M \cap L$, no subgroup of $D \cap L$ lies in $\mathcal{U}$. As $D = O_{p'}(F(M))$, Lemma 3.4 implies that $r(D \cap L) \leq 2$. Thus, by Corollary 4.19 [BG], P_0 centralizes every chief factor U/V of $L \cap M$ for which $U \subseteq D \cap L$. Since $D \cap L$ is a p'-subgroup, Lemma 1.9 [BG] shows that P_0 centralizes $D \cap L$. However, $D \cap L \supseteq D \cap C_G(y) \supseteq C_D(B)$ and $C_D(B)$ is not centralized by P_0. This contradiction shows that P_0 centralizes D.

We claim that $\{M\} = \mathcal{M}(N_G(P_0))$. Suppose that $r(F) \geq 3$. Since $r_p(F) \leq 2$, we have $r(D) \geq 3$. By Lemma 3.4 applied to a prime q with $r_q(D) \geq 3$, D contains some subgroup in $\mathcal{U}$. Thus, $D \in \mathcal{U}$. Since $M = N_G(D)$, we have $\mathcal{M}(D) = \{M\}$. We have $D \subseteq C_G(P_0) \subseteq N_G(P_0)$ so $\mathcal{M}(N_G(P_0)) = \{M\}$.

Suppose that $r(F) \leq 2$. By Theorem 4.20 [BG], $M' \subseteq F$. We have shown that $P \subseteq N_G(P) \subseteq M$.

Since M/F is abelian, $FP \lhd M$ and $M = O_{p'}(M)N_M(P)$. Since $P_0 = [P, N_G(P)] \lhd N_G(P)$ and $O_{p'}(M)$ centralizes P_0, we have $P_0 \lhd M$. This yields $\{M\} = \mathcal{M}(N_G(P_0))$.

We will complete the proof as in [BG]. Since $A \notin \mathcal{U}$, it follows that $\Omega_1(A) \notin \mathcal{U}$. By Theorem 3.1, there exists $x \in \Omega_1(A)^\sharp$ such that $C_G(x) \nsubseteq M$. Take $H \in \mathcal{M}(C_G(x))$. Then, $C_G(A) \subseteq C_G(x) \subseteq H$. Since M was chosen arbitrary from $\mathcal{M}(C_G(A))$, we can apply the previous argument to H in place of M to conclude

$$\{H\} = \mathcal{M}(N_G(P_0)) = \{M\}$$

that is a contradiction. This completes the proof of Lemma 3.5.

Q.E.D.

Theorem 3.6 (The Uniqueness Theorem). *Suppose that K is a subgroup of G with $r(K) \geq 2$. Assume that $r_p(K) \geq 3$ for some $p \in \varpi$*

or $r_p(C_G(K)) \geq 3$ *for some* $p \in \varpi$. *Then,* $K \in \mathcal{U}$. *In particular, if* $A \in \mathcal{E}_p^2(G) \setminus \mathcal{E}^*(G)$, *for some prime* $p \in \varpi$, *then* $A \in \mathcal{U}$.

Proof. Assume that $r_p(K) \geq 3$ for some $p \in \varpi$. Take $B \in \mathcal{E}_p^3(K)$ of order p^3. Let P be a Sylow p-subgroup of G that contains B. By Lemma 5.1 [BG], there exists $A \in SCN_3(P)$. Since $p \in \varpi$, Lemma 3.5 implies $A \in \mathcal{U}$. Since B is abelian, $B \subseteq C_G(B)$. Corollary 3.3 implies $B \in \mathcal{U}$. Therefore, $K \in \mathcal{U}$.

Assume that $r_p(C_G(K)) \geq 3$ for some $p \in \varpi$. Let $L = C_G(K)$. Then, the first paragraph of the proof shows $L \in \mathcal{U}$. Since $\pi(L) \cap \varpi$ is nonempty, Corollary 3.2 implies $K \in \mathcal{U}$. Q.E.D.

§4. The Subgroups M_α and M_σ

For each $M \in \mathcal{M}$, we define the sets of primes $\alpha(M)$, $\beta(M)$ and $\sigma(M)$, and the subgroups M_α, M_β, and M_σ as in [BG], page 70. In addition, we use the notation

$$\sigma_0(M) = \sigma(M) \cap \varpi \quad \text{and} \quad M_{\sigma_0} = O_{\sigma_0(M)}(M).$$

Lemma F. *Let* $M \in \mathcal{M}$ *and* $p \in \pi(M)$. *If* $p \notin \varpi$, *then* M *has a cyclic Sylow* p*-subgroup.*

Proof. By the basic assumptions, p is odd. If a Sylow p-subgroup S of M is not cyclic, then S contains an elementary abelian p-subgroup E that is not cyclic ([S], II page 59, (4.4)). The group E normalizes $O_\varpi(M)$ which is a nonidentity ϖ-subgroup. By Lemma B (1), we have $p \in \varpi$. Q.E.D.

Theorem 4.1. *Suppose* $M \in \mathcal{M}$, $p \in \sigma(M)$, *and* X *is a nonempty subset of* $G^\sharp$ *such that* $\langle X \rangle$ *is a* p*-subgroup of* G.

(a) *If* $X \subseteq M$, $g \in G$, *and* $X^g \subseteq M$, *then* $g = cm$ *for some* $c \in C_G(X)$ *and* $m \in M$.
(b) *The subgroup* $C_G(X)$ *acts transitively by conjugation on the set* $\{M^g \mid g \in G \text{ and } X \subseteq M^g\}$.
(c) *If* X *is a subgroup of* M, *then* $N_G(X) = N_M(X)C_G(X)$.
(d) *If* X *is a Sylow* p*-subgroup of* M, *then* $X \subseteq M^g$ *implies* $g \in M$ (*so* M *is the only conjugate of* M *that contains* X).
(e) *If* $X \subseteq M$, $C_G(X) \subseteq M$, $g \in G$, *and* $X \subseteq M^g$, *then* $M = M^g$ *and* $g \in M$.

There is only a small difference between our Theorem 4.1 and the corresponding Theorem 10.1 [BG]. To prove (b), we may replace X by

$\langle X \rangle$ and assume that X is a nontrivial p-subgroup of G (as in [BG]). The argument in [BG] proves the result. Note that in the case when $r(P) \geq 3$, we have $p \in \varpi$. This justifies the use of the Uniqueness Theorem on the top of page 72 [BG].

Theorem 4.2. *Let $M \in \mathcal{M}$. Then,*

(a) *M_α is a Hall $\alpha(M)$-subgroup of M and of G,*
(b) *M_σ is a Hall $\sigma(M)$-subgroup of M and of G,*
(c) *$M_\alpha \subseteq M_{\sigma_0} \subseteq M_\sigma \subseteq M'$,*
(d) *$r(M/M_\alpha) \leq 2$ and M'/M_α is nilpotent,*
(e) *$M_{\sigma_0} \neq 1$, and*
(f) *M_{σ_0} is a Hall $\sigma_0(M)$-subgroup of M and of G.*

Proof. The proof is the same as that of Theorem 10.2 [BG]. We will repeat it here because the results are so basic.

The basic assumptions imply that M is a solvable group of odd order. So, M contains a Hall $\alpha(M)$-subgroup $M(\alpha)$. Take $p \in \alpha(M)$ and $P \in Syl_p(M(\alpha))$. By definition of $\alpha(M)$, $r(P) \geq 3$. So, by Lemma F, $p \in \varpi$. The Uniqueness Theorem implies $P \in \mathcal{U}$. In particular, we have $N_G(P) \subseteq M$. Thus, $p \in \sigma(M)$; in fact, $p \in \sigma_0(M)$. Since p is arbitrary in $\alpha(M)$, we have $\alpha(M) \subseteq \sigma_0(M) \subseteq \sigma(M)$. Also, $N_G(P) \subseteq M$ implies that $P \in Syl_p(G)$. Thus, $M(\alpha)$ is a Hall $\alpha(M)$-subgroup of G.

Let $M(\sigma)$ be a Hall $\sigma(M)$-subgroup of M that contains $M(\alpha)$. Take $p \in \sigma(M)$ and $P \in Syl_p(M(\sigma))$. By definition of $\sigma(M)$, we have $N_G(P) \subseteq M$ so $P \in Syl_p(G)$. Hence, $M(\sigma)$ is a Hall $\sigma(M)$-subgroup of G.

By Theorem 1.17 [BG],

$$P \cap G' = \langle x^{-1}y \mid x, y \in P \text{ and } x \text{ is conjugate to } y \text{ in } G \rangle$$
$$P \cap M' = \langle x^{-1}y \mid x, y \in P \text{ and } x \text{ is conjugate to } y \text{ in } M \rangle.$$

Since G is simple, $P \cap G' = P$. If $x, y \in P$ and $y = x^g$ for some $g \in G$, then Theorem 4.1 (a) yields that $g = cm$ where $c \in C_G(x)$ and $m \in M$. This implies $x^g = x^m = y$. It follows that $P = P \cap G' = P \cap M' \subseteq M'$. Since p is arbitrary in $\sigma(M)$, we have $M(\sigma) \subseteq M'$.

Consider the group M/M_α. Since $M_\alpha = O_{\alpha(M)}(M)$, we have

$$M_\alpha \subseteq M(\alpha) \subseteq M(\sigma) \subseteq M'.$$

Consider the normal subgroup F of M such that $M_\alpha \subseteq F$ and $F/M_\alpha = F(M/M_\alpha)$. Then, F/M_α is nilpotent and it is an $\alpha(M)'$-group. The extension of F over M_α splits by the Schur-Zassenhaus Theorem. Hence,

F/M_α is isomorphic to a subgroup of M. Since F/M_α is an $\alpha(M)'$-group, we have $r(F/M_\alpha) \leq 2$. By Theorem 4.20 [BG],

$$M'/M_\alpha = (M/M_\alpha)' \subseteq F(M/M_\alpha) = F/M_\alpha.$$

This implies that M'/M_α is nilpotent. Therefore, any Hall subgroup of M'/M_α is a characteristic subgroup. Since the subgroups $M(\sigma)/M_\alpha$ and $M(\alpha)/M_\alpha$ are normal subgroups of M/M_α, both $M(\sigma)$ and $M(\alpha)$ are normal subgroups of M. It follows that

$$M_\alpha = M(\alpha) \quad \text{and} \quad M_\sigma = M(\sigma).$$

This proves (a) and (b). The last statement (f) is proved in a similar way. We have also (c) and (d). To prove (e), we may assume $M_\alpha = 1$. Then, $r(M) \leq 2$. By Theorem 4.20 [BG], $O_q(M) \in Syl_q(M)$ for the largest prime q of $\pi(M)$. This implies $q \in \sigma(M)$. We need only to note that $q \in \varpi$ as $O_\varpi(M) \neq 1$. Q.E.D.

Lemma 4.3. *Suppose $M \in \mathcal{M}$, X is an $\alpha(M)'$-subgroup of M, and $r(C_{M_\alpha}(X)) \geq 2$. Then, $C_M(X) \in \mathcal{U}$.*

Lemma 4.4. *Suppose $M \in \mathcal{M}$, $p \in \pi(M)$, and $P \in Syl_p(M)$.*

(a) *If p divides $|M/M'|$, then $p \notin \sigma(M)$.*
(b) *Assume $p \notin \sigma(M)$ and $M_\alpha \neq 1$. Then, there exists $x \in \Omega_1(Z(P))^\sharp$ such that $\{M\} \neq \mathcal{M}(C_G(x))$ and $C_{M_\alpha}(x)$ is a Z-group.*
(c) *Assume $p \notin \sigma(M)$ and $r_p(M) = 2$. Then, p is not ideal and $\mathcal{E}_p^2(M) \subseteq \mathcal{E}_p^*(M)$.*

The proof of Lemma 10.4 [BG] applies here. Note that the assumptions of Part (c) imply $p \in \varpi$ by Lemma F. So, the use of the Uniqueness Theorem is justified. On the fourth line of the proof of (b) in [BG], Z stands for $\Omega_1(Z(P))$.

Lemma 4.5. *Suppose that $M \in \mathcal{M}$, $p \in \sigma(M)'$, and X is a non-identity p-subgroup of G with $N_G(X) \subseteq M$. Then, $r_p(M) = 2$, p is not ideal, and if $|X| = p$, there exists $A \in \mathcal{E}_p^2(M)$ that contains X.*

Proof. The assumptions imply $X \subseteq M$. Since $\alpha(M) \subseteq \sigma(M)$, we have $r_p(M) \leq 2$. Let $P \in Syl_p(M)$ that contains X. If $r_p(M) = 1$, P is cyclic. Then, X is a characteristic subgroup of P. So, we have $N_G(P) \subseteq N_G(X) \subseteq M$. This contradicts the assumption that $p \notin \sigma(M)$. Therefore, $r_p(M) = 2$ and p is not ideal by Lemma 4.4. If $X = \Omega_1(Z(P))$, then we have $N_G(P) \subseteq N_G(X) \subseteq M$. So, if $|X| = p$, $X \neq \Omega_1(Z(P))$ and $X\Omega_1(Z(P)) \in \mathcal{E}_p^2(P)$. Q.E.D.

Theorem 4.6. *Let $M \in \mathcal{M}$. Then, M has p-length one for every $p \in \pi(M)$.*

We have followed the usage in [BG] so a group H is said to have p-length one for a given prime p if $H/O_{p',p}(H)$ is a p'-group.

Corollary 4.7. *Suppose that $p \in \pi(G) \cap \varpi$ and $P \in Syl_p(G)$. The following propositions hold.*

(a) *Take V to be any complement of P in $N_G(P)$. Then we have*

$$P = [P, V] \subseteq N_G(P)'.$$

(b) *Suppose $r(P) \leq 2$. Then, either P is abelian or P is the central product of a nonabelian subgroup P_1 of order p^3 and exponent p and a cyclic subgroup P_2 for which $\Omega_1(P_2) = Z(P_1)$.*
(c) *Suppose $Q \subseteq P$, $x \in G$, and $Q^x \subseteq P$. Then, $Q^x = Q^y$ for some element $y \in N_G(P)$.*
(d) *For every subgroup Q of P, the group $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$.*
(e) *Suppose R is a p-subgroup of G and $Q \subseteq P \cap R$ and $Q \lhd N_G(P)$. Then $Q \lhd N_G(R)$.*

Proof. Since $p \in \varpi$, $N_G(P)$ is a ϖ-local subgroup. Take $M \in \mathcal{M}(N_G(P))$. By Theorem 4.6, M has p-length one, so $P \subseteq O_{p',p}(M)$. By the definition of $\sigma(M)$, we have $p \in \sigma(M)$. Theorem 4.2 shows that $P \subseteq M_\sigma \subseteq M'$. The rest of the proof is the same as that of Corollary 10.7 [BG]. Q.E.D.

Lemma 4.8. *Let $M \in \mathcal{M}$. Then the following hold.*

(a) *M_β is a Hall $\beta(M)$-subgroup of M and of G.*
(b) *M' and M_σ have nilpotent Hall $\beta(M)'$-subgroups.*
(c) *For each prime $p \in \pi(M) \setminus \beta(M)$, M' and M_σ have normal p-complements and p is the largest prime divisor of $|M/O_{p'}(M)|$.*

Corollary 4.9. *Let $M \in \mathcal{M}$.*

(a) *Suppose that p and q are distinct primes in $\pi(M) \setminus \beta(M)$ and X is a q-subgroup of M. Assume $X \subseteq M'$ or $p < q$. Then,*
(1) *X centralizes a Sylow p-subgroup of M_σ,*
(2) *if $p \in \alpha(M)$ and $X \neq 1$, then $q \in \varpi$ and $C_M(X) \in \mathcal{U}$, and*
(3) *if $X \in Syl_q(M')$, then $N_M(X)'$ contains a Sylow p-subgroup of M'.*
(b) *If $H \in \mathcal{M} \setminus \{M\}$ and $N_G(S) \subseteq H \cap M$ for some Sylow subgroup S of G, then $M = (H \cap M)M_\beta$ and $\alpha(M) = \beta(M)$.*

The proof of Corollary 10.9 [BG] can be used to prove this corollary. We shall add a few lines to verify the statement (2).

Suppose that $p \in \alpha(M)$ and $X \neq 1$. By (1), X centralizes a Sylow p-subgroup P of M_α. Since $p \in \alpha(M)$, we have $P \neq 1$ and $r(P) \geq 3$. By the Uniqueness Theorem, $P \in \mathfrak{U}$. Note that $p \in \varpi$. Since a nonidentity q-subgroup X centralizes a p-subgroup P, we have $q \in \varpi$. Since $P \subseteq C_M(X)$, $P \in \mathfrak{U}$ implies $C_M(X) \in \mathfrak{U}$.

Lemma G. *If $M \in \mathcal{M}$, then M is a ϖ-group except when*

(1) *M is a Frobenius group such that the Frobenius kernel of M is a Hall ϖ-subgroup of M, or*
(2) *M has the following structure: M/M' is a cyclic ϖ-group, $M_\alpha = M_\beta = M_{\sigma_0}$ is a nilpotent ϖ-group, and M'/M_β is a nonidentity cyclic ϖ'-group.*

In the case (1), the Frobenius kernel is M_{σ_0} and it is either M' or M_β. If it is M_β, then we have $M_\alpha = M_\beta$. In the case (2), both M' and M/M_β are Frobenius groups with Frobenius kernels M_β and M'/M_β, respectively.

Proof. By definition of $\beta(M)$, we have $M_\beta \subseteq M_\alpha \subseteq M'$ and M_β is a ϖ-group. By Lemma 4.8, M'/M_β is nilpotent. Hence, M'/M_β is either a ϖ-group or a ϖ'-group.

Suppose that M'/M_β is a ϖ-group. Then, M' is a ϖ- group. If M/M' is a ϖ-group, so is M. If M/M' is a ϖ'-group, then by Lemma A, $x \in (M')^\sharp$ satisfies $C_G(x) \subseteq M'$. This shows that M is a Frobenius group with Frobenius kernel M'. In this case, M' is nilpotent by a theorem of Thompson. If $p \in \pi(M')$, a Sylow p-subgroup P of M' is a Sylow p-subgroup of M and $P \lhd M$. It follows that $N_G(P) = M$ and $p \in \sigma_0(M)$. This proves that $M' = M_{\sigma_0}$.

Suppose that M'/M_β is a ϖ'-group. If M/M' is a ϖ'-group, so is M/M_β. We see that M is a Frobenius group with Frobenius kernel M_β. In this case, $M_\alpha = M_\beta$ because M_α is a ϖ-group, and $M_\beta = M_{\sigma_0}$ because M_β is nilpotent.

Finally, assume that M/M' is a ϖ-group. Then, M' is a Frobenius group with M_β as Frobenius kernel and M/M_β is a Frobenius group with Frobenius kernel M'/M_β. Thus, M'/M_β is nilpotent (as a Frobenius kernel) and $r(M'/M_\beta) = 1$ (as a Frobenius complement in M'). It follows that M'/M_β is cyclic. The abelian group M/M' satisfies $r(M/M') = 1$ because it is a Frobenius complement in M/M_β. Thus, M/M' is cyclic, too. This proves Lemma G. Q.E.D.

Proposition 4.10. *Suppose that p and q are distinct primes, $A \in$*

$\mathcal{E}_p^2(G) \cap \mathcal{E}_p^*(G)$, *and* $Q \in \mathcal{U}_G^*(A; q)$. *Assume that* $p \in \varpi$ *and* $q \in \pi(C_G(A))$. *Then for some* $P \in Syl_p(G)$ *that contains* A,

(a) $N_G(P) = O_{p'}(C_G(P))(N_G(P) \cap N_G(Q))$,
(b) $P \subseteq N_G(Q)'$, *and*
(c) *if* Q *is cyclic or* $\mathcal{E}^2(Q) \cap \mathcal{E}^*(Q)$ *is not empty, then* P *centralizes* Q.

Proof. Since $p \in \varpi$ and $q \in \pi(C_G(A))$, we have $q \in \varpi$. Since A is a maximal elementary abelian p-subgroup of G, we have $A = \{x \in C_G(A) \mid x^p = 1\}$. Hence, by Proposition 1.5, A satisfies Hypothesis 1.1. Since $m(Z(A)) = 2$, Theorem 1.3 yields that $O_{p'}(C_G(A))$ acts transitively on $\mathcal{U}_G^*(A; q)$. Take $P_1 \in Syl_p(G)$ such that $A \subseteq P_1$. Then, Theorem 1.4 shows

$$\mathcal{U}_G^*(P_1; q) \subseteq \mathcal{U}_G^*(A; q)$$

and for every $Q_1 \in \mathcal{U}_G^*(P_1; q)$, we have $P_1 \cap N_G(P_1)' \subseteq N_G(Q_1)'$ and

$$N_G(P_1) = O_{p'}(C_G(P_1))(N_G(P_1) \cap N_G(Q_1)).$$

Since both Q and Q_1 lie in $\mathcal{U}_G^*(A; q)$, we have $Q_1^x = Q$ for some $x \in O_{p'}(C_G(A))$. Let $P = P_1^x$. Then, P satisfies (a).

Since $p \in \varpi$, Corollary 4.7 shows that $P \subseteq N_G(P)'$. Therefore,

$$P = P \cap N_G(P)' \subseteq N_G(Q)'.$$

This proves (b). To prove (c), note that the hypothesis of (c) implies that Q is narrow. Apply Theorem 5.5 (a) [BG] to the subgroup $N_G(Q)/C_G(Q)$ of $\operatorname{Aut} Q$. It follows that $(N_G(Q)/C_G(Q))'$ is a q-group. Since $P \subseteq N_G(Q)'$, we have $P \subseteq C_G(Q)$. This proves (c). Q.E.D.

Proposition 4.11. *Suppose* $M \in \mathcal{M}$ *and* K *is a* $\sigma_0(M)'$*-subgroup of* M. *Then*

(a) *if* M *is a* ϖ*-group,* $K \notin \mathcal{U}$;
(b) $r(C_K(M_{\sigma_0})) \leq 1$;
(c) $C_K(M_{\sigma_0}) \cap M'$ *is a cyclic normal subgroup of* M; *and*
(d) *if* $p \in \sigma_0(M)'$, $P \in \mathcal{E}_p^1(N_M(K))$, $C_{M_{\sigma_0}}(P) = 1$, *and* K *is an abelian* p'*-group, then* $[K, P]$ *centralizes* M_{σ_0} *and is a cyclic normal subgroup of* M.

Proof. There is a small difference between this and Proposition 10.11 [BG]. If M is a ϖ-group, we have $\sigma_0(M) = \sigma(M)$ and for every subgroup P of M, $N_G(P)$ is a ϖ-local subgroup. The proof of Part (a) in [BG] is valid in our case.

To prove (b), suppose $r_p(C_K(M_{\sigma_0})) \geq 2$ for some prime p. Then,

$$p \in \pi(K) \subseteq \sigma_0(M)'.$$

By Lemma G, M is a ϖ-group. Thus, $\sigma_0(M) = \sigma(M)$ and Part (a) implies $K \notin \mathcal{U}$. The argument in the proof of Part (b) of Proposition 10.11 [BG] gives us $p \in \sigma(M)$. However, $p \in \pi(K) \subseteq \sigma(M)'$. This contradiction proves (b).

For (c) and (d), read σ_0 for σ in the proof of Proposition 10.11 [BG]. The assertions are proved. Q.E.D.

Lemma 4.12. *Suppose M, $H \in \mathcal{M}$ and H is not conjugate to M in G. Then,*

(a) *$M_\alpha \cap H_\sigma = 1$ and $\alpha(M)$ is disjoint from $\sigma(H)$, and*
(b) *if M_σ is nilpotent, then $M_\sigma \cap H_\sigma = 1$ and $\sigma(M)$ is disjoint from $\sigma(H)$.*

Proof. The proof is similar to the one of Lemma 10.12 [BG].

Suppose that $p \in \sigma_0(M) \cap \sigma(H)$. Then some Sylow p-subgroup S of G lies in M and in a conjugate H^g of H. Then, $S \in M \cap H^g$ and $M \neq H^g$ by assumption. Since $p \in \varpi$, the Uniqueness Theorem yields that $r(S) \leq 2$. Thus, $p \notin \alpha(M)$. This proves (a).

Assume that M_σ is nilpotent. Suppose $\sigma(M) \cap \sigma(H)$ is not empty. Take a prime p in $\sigma(M) \cap \sigma(H)$. As before, some Sylow p-subgroup P lies in M and in some conjugate H^x of H. Then, $M \neq H^x$ and $N_G(P) \subseteq M \cap H^x$. In particular, P is not normal in M, so M_σ is not nilpotent. Q.E.D.

Lemma 4.13. *Suppose $p \in \varpi$, $A \in \mathcal{E}_p^2(G) \cap \mathcal{E}_p^*(G)$, and P is a nonabelian p-subgroup of G that contains A. Let $Z_0 = \Omega_1(Z(P))$ and $A_0 \in \mathcal{E}^1(A)$ such that $A_0 \neq Z_0$. Then,*

(a) *$Z_0 \in \mathcal{E}^1(A)$,*
(b) *$C_P(A) = A_0 \times Z$ with Z a cyclic subgroup that contains Z_0, and*
(c) *$N_P(A)$ acts transitively by conjugation on $\mathcal{E}^1(A) \setminus \{Z_0\}$.*

Proof. Let S be a Sylow p-subgroup that contains P. Since $p \in \varpi$, we can apply Lemma 4.7 (b) when $r(S) \leq 2$. The proof of Lemma 10.13 [BG] will prove this lemma. Q.E.D.

Proposition 4.14. *Let $M \in \mathcal{M}$, $p \in \beta(M)$, and $P \in Sy\ell_p(M)$.*

(a) *The sets $\mathcal{E}_p^2(P) \cap \mathcal{E}_p^*(P)$ and $\mathcal{E}_p^2(G) \cap \mathcal{E}_p^*(G)$ are empty.*
(b) *Every p-subgroup R of G such that $r(R) \geq 2$ lies in $\mathcal{U}$.*

(c) *If X is a subgroup of P, then $N_P(X) \in \mathfrak{U}$.*
(d) *For every nonidentity $\beta(M)$-subgroup Y of M, $N_G(Y) \subseteq M$.*

Proof. (a) By the definition of $\beta(M)$, $\mathcal{E}_p^2(G) \cap \mathcal{E}_p^*(P)$ is empty for the Sylow p-subgroup P of M. If $A \in \mathcal{E}_p^2(G) \cap \mathcal{E}_p^*(G)$, take a Sylow p-subgroup Q of G such that $A \subseteq Q$. Then, $Q^g = P$ for some $g \in G$. Thus, $A^g \subseteq Q^g = P$ and $A^g \in \mathcal{E}_p^2(P) \cap \mathcal{E}_p^*(P)$. This is a contradiction.

(b) We can assume $R \subseteq P$ by choosing a conjugate of R. Take $A \in \mathcal{E}_p^2(R)$. By (a), there is $B \in \mathcal{E}_p^*(P)$ such that $A \subseteq B$ and $m(B) \geq 3$. Since $B \subseteq C_G(A)$, we have $r_p(C_G(A)) \geq 3$. Since $p \in \beta(M) \subseteq \varpi$, the Uniqueness Theorem yields $A \in \mathfrak{U}$. Therefore, we have $R \in \mathfrak{U}$.

(c) Let $Q = N_P(X)$. If $r(Q) \geq 2$, then $Q \in \mathfrak{U}$ by (b). Suppose that $r(Q) = 1$. Then, Q is cyclic, X chra Q, and $N_P(Q) \subseteq N_G(X) = Q$. Since P is a p-group, this implies $Q = P$ contrary to the assumption that $p \in \beta(M)$.

(d) Let $q \in \pi(F(Y))$ and $X = O_q(Y)$. We can assume that $q = p$ and $X \subseteq P$. Then, by (c), $N_P(X) \in \mathfrak{U}$. Since $N_G(X)$ is ϖ-local, we have $N_G(X) \subseteq M$ and $N_G(Y) \subseteq N_G(X) \subseteq M$.

§5. Exceptional Subgroups of $\mathcal{M}$

The following conditions and notation are used throughout this section.

Hypothesis 5.1. Suppose $M \in \mathcal{M}$, $p \in \sigma(M)'$, $A_0 \in \mathcal{E}_p^1(M)$, and

$$N_G(A_0) \subseteq M.$$

By Lemma 4.5, $r_p(M) = 2$ and $A_0 \subseteq A$ for some $A \in \mathcal{E}_p^2(M)$. Let P be a Sylow p-subgroup of M that contains A. Since $r_p(M) = 2$ for $p \in \sigma(M)'$, Lemma G implies that M is a ϖ-group. As $p \in \sigma(M)'$, $N_G(P) \not\subseteq M$ and since $C_G(A) \subseteq C_G(A_0) \subseteq N_G(A_0) \subseteq M$, we have $A \in \mathcal{E}_p^*(G)$.

We will fix the subgroups A and P throughout this section.

Lemma 5.1. *Suppose that $g \in G \setminus M$, $A \subseteq M^g$, $q \in \sigma(M)$, and that Q_1 and Q_2 are A-invariant Sylow q-subgroups of M_σ and ${M_\sigma}^g$, respectively. Then,*

(a) *$Q_1 \cap Q_2 = 1$, and*
(b) *if $X \in \mathcal{E}^1(A)$, then $C_{Q_1}(X) = 1$ or $C_{Q_2}(X) = 1$.*

Proof. As remarked at the beginning of this section, Hypothesis 5.1 implies that M is a ϖ-group. Thus, if $Q_1 \cap Q_2 \neq 1$, the subgroup $Q_1 \cap Q_2$ is a ϖ-group $\neq 1$. Also, $C_G(X)$ is a ϖ-group by Lemma A. Thus, if either (a) or (b) is false, there is a ϖ-local subgroup H such that

$$H \cap Q_1 \neq 1 \quad \text{and} \quad H \cap Q_2 \neq 1.$$

(Cf. Lemma D.) By Lemma 1.1, we have $Q_2 = {Q_1}^k$ for some element $k \in C_G(A)$. The rest of the proof is the same as that of Lemma 11.1 [BG]. Q.E.D.

Corollary 5.2. *Suppose $g \in G \setminus M$ and $A \subseteq M^g$. Then,*

(a) $M_\sigma \cap M^g = 1$, *and*
(b) $M_\sigma \cap C_G({A_0}^g) = 1$.

Theorem 5.3. *The group M_σ is nilpotent.*

Corollary 5.4. *Suppose $H \in \mathcal{M}(A)$ and $M_\sigma \cap H_\sigma \neq 1$. Then, $M = H$.*

Theorem 5.5. *The Sylow p-subgroups of M are abelian.*

Corollary 5.6. *We have*

(a) $A = \Omega_1(P)$,
(b) $C_{M_\sigma}(A) = 1$, *and*
(c) *there exist subgroups A_1, $A_2 \in \mathcal{E}_p^1(A)$ such that $A_1 \neq A_2$ and $C_{M_\sigma}(A_1) = C_{M_\sigma}(A_2) = 1$.*

Theorem 5.7. *We have $M_\sigma A \lhd M$.*

§6. The Subgroup E

Let E denote a complement of M_σ in M, which will be fixed for discussion. We use the notation τ_i and E_i as defined in [BG], Section 12.

Lemma 6.1. (a) *E' is nilpotent.*

(b) *$E_3 \subseteq E'$ and $E_3 \lhd E$.*
(c) *If $E_2 = 1$, then $E_1 \neq 1$.*
(d) *E_1 and E_3 are cyclic.*
(e) *$E = E_1E_2E_3$, $E_{12} = E_1E_2$, $E_2E_3 \lhd E$, and $E_2 \lhd E_{12}$.*
(f) $C_{E_3}(E) = 1$.
(g) *If $p \in \tau_2(M)$ and $A \in \mathcal{E}_p^2(M)$, then $A \in \mathcal{E}_p^*(G)$ and $p \notin \beta(G)$.*

Lemma 6.2. *Suppose that $M \in \mathcal{M}$, p is a prime, X is a nonidentity p-subgroup of M, and $M^* \in \mathcal{M}(N_G(X))$. Then,*

(a) $p \in \sigma(M^*) \cup \tau_2(M^*)$, *and*
(b) *if $p \in \sigma(M)$ and $M \neq M^*$, or if $p \in \tau_1(M) \cup \tau_3(M)$, then M^* is not conjugate to M in G.*

Proof. (a) Suppose that $p \notin \sigma(M^*)$. Then, Lemma 4.5 applied to M^* implies that $r_p(M^*) = 2$. This proves $p \in \tau_2(M^*)$.

(b) Suppose that M^* is conjugate to M. Then, $\sigma(M) = \sigma(M^*)$ and $\tau_i(M) = \tau_i(M^*)$ for $i = 1, 2, 3$. Therefore, if $p \in \tau_1(M) \cup \tau_3(M)$, we have a contradiction to (a). Suppose that $p \in \sigma(M)$. Then, by Theorem 4.1(b), M^* and M are conjugate by an element x of $C_G(X)$: $M = (M^*)^x$. Since $C_G(X) \subseteq N_G(X) \subseteq M^*$, we have $M = M^*$. This proves (b). Q.E.D.

Remark. If $p \in \varpi$, a subgroup M^* is available; however, if $p \in \varpi$ is not assumed, Lemma 6.2 holds only when there is a ϖ-local subgroup that contains $N_G(X)$.

Lemma 6.3. *Suppose $M^* \in \mathcal{M} \setminus \{M\}$, p is a prime, $A \in \mathcal{E}_p^2(M \cap M^*)$, and $N_G(A_0) \subseteq M^*$ for some $A_0 \in \mathcal{E}^1(A)$.*

(a) *If $p \notin \sigma(M)$, then A centralizes $M_\sigma \cap M^*$.*
(b) *If $p \in \sigma(M) \setminus \alpha(M)$, then A centralizes $M_\alpha \cap M^*$.*

Proposition 6.4. *Suppose $M \in \mathcal{M}$, p is a prime and $A \in \mathcal{E}_p^2(M)$. Then,*

(a) $C_G(A) \subseteq M$, *and*
(b) *if $\mathcal{M}(N_G(A_0)) \neq \{M\}$ for every $A_0 \in \mathcal{E}^1(A)$, then $p \in \sigma(M)$, $M_\alpha = 1$, and M_σ is nilpotent.*

Proof. By assumption, $r_p(M) \geq 2$ so $p \in \varpi$. Thus, for every $X \in \mathcal{E}^1(A)$, $N_G(X)$ is a ϖ-local subgroup. The proof of Proposition 12.4 [BG] can be adapted to yield the results. However, this is basic so we repeat the argument.

Suppose that $\mathcal{M}(N_G(A_0)) = \{M\}$ for some $A_0 \in \mathcal{E}^1(A)$. Then, $C_G(A) \subseteq C_G(A_0) \subseteq N_G(A_0) \subseteq M$. This proves (a) in this case.

For the remainder of proof, we may assume that $\mathcal{M}(N_G(X)) \neq \{M\}$ for every $X \in \mathcal{E}^1(A)$. For a fixed $X \in \mathcal{E}^1(A)$, choose

$$M^* = M^*(X) \in \mathcal{M}(N_G(X)) \setminus \{M\}.$$

Since $C_M(X) \subseteq M \cap M^*$, the Uniqueness Theorem implies

$$r(C_M(A)) \leq r(C_M(X)) \leq 2.$$

We claim that $p \in \sigma(M)$. Suppose $p \notin \sigma(M)$. Then, Lemma 6.3(a) implies that $C_{M_\sigma}(X) \subseteq M_\sigma \cap M^* \subseteq C_M(A)$. This holds for every $X \in \mathcal{E}^1(A)$. By Proposition 1.16 [BG], $M_\sigma = \langle C_{M_\sigma}(X) \mid X \in \mathcal{E}^1(A)\rangle$. Since $C_{M_\sigma}(X) \subseteq C_M(A)$ for every $X \in \mathcal{E}^1(A)$, we have $M_\sigma \subseteq C_M(A)$ which contradicts Proposition 4.11 (b). Thus, we have $p \in \sigma(M)$.

Let P be a Sylow p-subgroup of M_σ that contains A and let $Z = \Omega_1(Z(P))$. Since $r(C_M(A)) \leq 2$, we have $Z \subseteq A$. Take $X \in \mathcal{E}^1(Z)$. Then, $P \subseteq C_M(X)$ and $r(P) \leq r(C_M(X)) \leq 2$. This proves that $p \in \sigma(M) \setminus \alpha(M)$. We apply the same argument as before to M_α. Again for any $X \in \mathcal{E}^1(A)$, choose $M^* \in \mathcal{M}(N_G(X)) \setminus \{M\}$. Then, Lemma 6.3 (b) implies that $C_{M_\alpha}(X) \subseteq M_\alpha \cap M^* \subseteq C_M(A)$. It follows that $M_\alpha = \langle C_{M_\alpha}(X) \mid X \in \mathcal{E}^1(A)\rangle \subseteq C_M(A)$. This implies $M_\alpha = 1$ because $r(C_M(A)) = 2$. By Theorem 4.2 (d), $M' = M'/M_\alpha$ is nilpotent. Since $M_\sigma \subseteq M'$, M_σ is nilpotent. This proves (b).

Since M_σ is nilpotent, we have $P \lhd M$. Hence,

$$Z = \Omega_1(Z(P)) \lhd M.$$

Since $Z \subseteq A$, we have $C_G(A) \subseteq C_G(Z) \subseteq N_G(Z) = M$. The last equality comes from Lemma E (1). This completes the proof of Proposition 6.4. Q.E.D.

We state a corollary of Lemma G.

Lemma H. *Let $M \in \mathcal{M}$.*

(1) *If $\tau_2(H) \neq \emptyset$, then M is a ϖ-group.*
(2) *If M is not a ϖ-group, then $r_p(M) \leq 1$ for all $p \notin \sigma_0(M)$.*

This follows immediately from the structure of subgroups in $\mathcal{M}$ which are not ϖ-groups given in Lemma G.

Theorem 6.5. *Suppose $M \in \mathcal{M}$ and $\tau_2(M) \neq \emptyset$. Let $p \in \tau_2(M)$ and $A \in \mathcal{E}_p^2(M)$. Then, M is a ϖ-group and the following hold:*

(a) *M_σ is nilpotent,*
(b) *M has abelian Sylow p-subgroups and every Sylow p-subgroup P of M such that $A \subseteq P$ satisfies $\Omega_1(P) = A$ and $N_G(P) \nsubseteq M$,*
(c) *$M_\sigma A \lhd M$,*
(d) *$C_{M_\sigma}(A) = 1$,*
(e) *$M_\sigma \cap M^* = 1$ for every $M^* \in \mathcal{M}(A) \setminus \{M\}$, and*
(f) *there exists $A_1 \in \mathcal{E}^1(A)$ such that $C_{M_\sigma}(A_1) = 1$.*

Proof. Since $\tau_2(M) \neq \emptyset$, M is a ϖ-group by Lemma H. Hence, for any $X \in \mathcal{E}^1(A)$, $N_G(X)$ is a ϖ-local subgroup. Since $p \notin \sigma(M)$, Proposition 6.4 (b) implies that $\mathcal{M}(N_G(A_0)) = \{M\}$ for some $A_0 \in \mathcal{E}^1(A)$. Thus, we have Hypothesis 5.1 for A_0 and M. The results of Section 5 prove Theorem 6.5 except (e).

To prove (e), take $M^* \in \mathcal{M}(A) \setminus \{M\}$. If $N_G(A_0) \subseteq M^*$ for some $A_0 \in \mathcal{E}^1(A)$, Lemma 6.3 (a) shows that A centralizes $M_\sigma \cap M^*$. On the other hand, $C_{M_\sigma}(A) = 1$ by (d). This proves $M_\sigma \cap M^* = 1$. If $N_G(X) \not\subseteq M^*$ for every $X \in \mathcal{E}^1(A)$, the hypothesis of Proposition 6.4 (b) is satisfied for M^*. Hence, we have $p \in \sigma(M^*)$ and $M^*{}_\sigma$ is nilpotent. It follows that $A \subseteq O_p(M^*)$ and $[M_\sigma \cap M^*, A] \subseteq M_\sigma \cap O_p(M^*) = 1$ because $p \notin \sigma(M)$. So, $M_\sigma \cap M^* \subseteq C_{M_\sigma}(A) = 1$. Q.E.D.

Corollary 6.6. *Suppose $M \in \mathcal{M}$ and $\tau_2(M) \neq \emptyset$. Let $p \in \tau_2(M)$ and $A \in \mathcal{E}_p^2(E)$. Then,*

(a) $A \lhd E$ *and* $\mathcal{E}_p^1(E) = \mathcal{E}^1(A)$,
(b) $C_G(A) \subseteq N_M(A) = E$ *and* $N_G(A) \not\subseteq M$,
(c) $\mathcal{M}(C_G(X)) = \{M\}$ *for each* $X \in \mathcal{E}^1(A)$ *such that* $C_{M_\sigma}(X) \neq 1$,
(d) $C_{M_\sigma}(x) = 1$ *for each* $x \in E_3{}^\sharp$,
(e) $C_{M_\sigma}(x) = 1$ *for each* $x \in C_{E_1}(A)^\sharp$, *and*
(f) *if* $M^* \in \mathcal{M}$ *is not conjugate to* M, *then* $M_\sigma \cap M^*{}_\sigma = 1$ *and* $\sigma(M^*)$ *is disjoint from* $\sigma(M)$.

Proof. As before, Lemma H implies that M is a ϖ-group. Since E is a complement of M_σ, Theorem 6.5 (c) implies $A \lhd E$. If $X \in \mathcal{E}_p^1(E)$, then AX is a p-subgroup of E. Let P be a Sylow p-subgroup of E such that $AX \subseteq P$. By Theorem 6.5 (b), we have $\Omega_1(P) = A$. Since $X \subseteq \Omega_1(P)$, $X \subseteq A$. This proves $\mathcal{E}_p^1(E) = \mathcal{E}^1(A)$. This proves (a).

We have $C_G(A) \subseteq M$ by Proposition 6.4 (a). Thus, $C_G(A) \subseteq N_M(A)$. By (a), $E \subseteq N_M(A)$. It follows from the Dedekind law that

$$N_M(A) = N_M(A) \cap M_\sigma E = N_{M_\sigma}(A)E.$$

We have $[N_{M_\sigma}(A), A] \subseteq M_\sigma \cap A = 1$, so $N_{M_\sigma}(A) \subseteq C_{M_\sigma}(A) = 1$ by Theorem 6.5 (d). This proves that $N_M(A) = E$. If P is any Sylow p-subgroup of M that contains A, then $A = \Omega_1(P)$ by Theorem 6.5 (b). Hence, $N_G(P) \subseteq N_G(A)$. Since $N_G(P) \not\subseteq M$ by Theorem 6.5 (b), we have $N_G(A) \not\subseteq M$. This proves (b).

Suppose $C_{M_\sigma}(X) \neq 1$ and $\mathcal{M}(C_G(X)) \neq \{M\}$ for some $X \in \mathcal{E}^1(A)$. Take M^* such that $C_G(X) \subseteq M^* \neq M$. Since $A \subseteq C_G(X)$, Theorem 6.5 (e) implies that $M_\sigma \cap M^* = 1$. Hence,

$$C_{M_\sigma}(X) \subseteq M_\sigma \cap C_G(X) \subseteq M_\sigma \cap M^* = 1.$$

This contradiction proves (c).

For (d) and (e), we may assume that $\langle x\rangle = X$ is a q-group for some prime $q \in \tau_1(M) \cup \tau_3(M)$. As remarked at the beginning of the proof, $q \in \varpi$ so we can take $M^* \in \mathcal{M}(N_G(X))$. By Lemma 6.2, M^* is not conjugate to M. In particular, $M^* \neq M$. Since A and E_3 are normal subgroups of E with $A \cap E_3 = 1$, A centralizes E_3. Thus, $A \subseteq C_G(X) \subseteq M^*$ in all cases. By Theorem 6.5 (e), we have $M_\sigma \cap M^* = 1$. Therefore,

$$C_{M_\sigma}(X) \subseteq M_\sigma \cap C_G(X) \subseteq M_\sigma \cap M^* = 1.$$

This proves (d) and (e).

Since M_σ is nilpotent (cf. Theorem 6.5 (a)), Lemma 4.12 (b) yields (f). Q.E.D.

Theorem 6.7. *Suppose that $M \in \mathcal{M}$, $p \in \tau_2(M)$, $A \in \mathcal{E}_p^2(E)$, and assume that G has nonabelian Sylow p-subgroups. Then,*

(a) $\tau_2(M) = \{p\}$,
(b) $A_0 = C_A(M_\sigma)$ *has order p and satisfies* $F(M) = M_\sigma \times A_0$,
(c) *every* $X \in \mathcal{E}_p^1(E)\backslash\{A_0\}$ *satisfies* $C_{M_\sigma}(X) = 1$ *and* $C_G(X) \not\subseteq M$,
(d) A_0 *has a complement* E_0 *in* E, *and*
(e) $\pi(C_{E_0}(x)) \subseteq \tau_1(M)$ *for every* $x \in M_\sigma{}^\sharp$.

Proof. The assumptions of this theorem imply that M is a ϖ-group (by Lemma H). The argument of the proof of Theorem 12.7 [BG] proves the assertions. We paraphrase a few points in the argument.

The subgroup A_0 was defined as an element of $\mathcal{E}^1(A)$ such that $C_{M_\sigma}(A_0) \neq 1$. It is proved to be the unique element with $C_{M_\sigma}(A_0) \neq 1$. We have $A_0 = C_A(M_\sigma)$. Since $A \lhd E$ by Corollary 6.6 (a), E normalizes A_0. Note that $M_\sigma \lhd M$. Clearly, M_σ normalizes A_0, so $M = M_\sigma E$ normalizes A_0. Thus, $A_0 \lhd M$ and A_0 is a part of the Fitting subgroup $F(M)$. Apply Lemma 6.2 taking each $q \in \pi(F(M))$ and $X = O_q(M)$. Then, $M \in \mathcal{M}(N_G(X))$ and $q \in \sigma(M) \cup \tau_2(M)$. This proves that $\pi(F(M)) = \sigma(M) \cup \{p\}$ as M_σ is nilpotent (Theorem 6.5 (a)) and $\tau_2(M) = \{p\}$ by (a). Q.E.D.

Lemma 6.8. *Suppose that $M \in \mathcal{M}$, $p \in \tau_2(M)$, $A \in \mathcal{E}_p^2(E)$, and S is a Sylow p-subgroup of G that contains A. Assume that S is abelian. Then,*

(a) E_2 *is an abelian normal subgroup of* E,
(b) E_2 *is a Hall* $\tau_2(M)$-*subgroup of* G,
(c) $S \subseteq N_G(S)' \subseteq F(E) \subseteq C_G(S) \subseteq E$ *and* $S = O_p(E)$,
(d) $N_G(A) = N_G(S) = N_G(E_2) = N_G(E_2E_3) = N_G(F(E)) \not\subseteq M$,

(e) *every* $X \in \mathcal{E}^1(E_1)$ *for which* $C_{M_\sigma}(X) = 1$ *lies in* $Z(E)$, *and*
(f) *we have* $C_S(X) \lhd N_G(S)$ *and* $[S, X] \lhd N_G(S)$ *for every subgroup* X *of* $N_G(S)$.

Proof. As before, the assumptions imply that M is a ϖ- group. By Theorem 6.7 (a), each $p \in \tau_2(M)$ satisfies the assumption that G has abelian Sylow p-subgroups. Since $S \subseteq C_G(A) \subseteq E$ by Corollary 6.6 (b), E_2 is a Hall $\tau_2(M)$-subgroup of G. This proves (b).

By Corollary 6.6 (a), we have $E \subseteq N_G(A)$.

Clearly, $A \subseteq O_p(N_G(A)) \subseteq S$. Hence, A is contained in the center of $F(N_G(A))$. Thus,

$$F(N_G(A)) \subseteq C_G(A) \subseteq E \subseteq N_G(A).$$

This proves two properties. One is $F(N_G(A)) \subseteq F(C_G(A)) \subseteq F(E)$, and the other property is $r(F(N_G(A))) \leq r(E) \leq 2$. By Theorem 4.20 [BG], we have $N_G(A)' \subseteq F(N_G(A))$. It follows that $E \lhd N_G(A)$, so $F(E) \subseteq F(N_G(A))$. We have $F(N_G(A)) = F(C_G(A)) = F(E)$. By Theorem 6.5 (b), we have $A = \Omega_1(S)$, so $N_G(S) \subseteq N_G(A)$. Moreover, Corollary 4.7 (a) shows $S \subseteq N_G(S)'$. It follows that

$$S \subseteq N_G(S)' \subseteq N_G(A)' \subseteq F(N_G(A)) = F(E).$$

This implies that $S = O_p(E)$ and $F(E) \subseteq C_G(S) \subseteq C_G(A) \subseteq E$. We have proved (c). As remarked earlier, $S = O_p(E)$ for every $p \in \tau_2(M)$. This implies $E_2 \lhd E$ and (a) holds.

Let $K = E_2E_3$. Then, $E_3 \lhd E$ by Lemma 6.1 (a). Since $E_2 \lhd E$ and $E_2 \cap E_3 = 1$, we have $K = E_2E_3 = E_2 \times E_3$. Since E_3 is cyclic by Lemma 6.1 (d) and E_2 is abelian, K is a Hall subgroup of $F(E)$. Each subgroup in the series

$$A \subseteq S \subseteq E_2 \subseteq E_2E_3 \subseteq F(E)$$

is characteristic in its successor. Since $F(E) = F(N_G(A))$, we have (d).

By (d), $K \lhd N_G(K) = N_G(S)$. Also, $N_G(S)' \subseteq F(E) \subseteq C_G(K)$ by (c). Let $X \in \mathcal{E}^1(E_1)$ be a subgroup such that $C_{M_\sigma}(X) = 1$. Then, $N_G(S)'X \lhd N_G(S)$ and $N_G(S)' \subseteq C_G(K)$. Consider $Y = [K, X]$. Then, it is a subgroup of K and

$$Y = [K, X] = [K, N_G(S)'X] \lhd N_G(S).$$

Thus, $N_G(Y) \supseteq N_G(S)$ so we have $N_G(Y) \not\subseteq M$. On the other hand, Proposition 4.11 (d) applies to X and shows that $Y = [K, X] \lhd M$. If $Y \neq 1$, then Y is a nonidentity normal ϖ-subgroup of M. This would imply $N_G(Y) = M$ by Lemma E (1). However, we have shown that $N_G(Y) \not\subseteq M$. This contradiction proves $[K, X] = 1$. Since $E = E_1K$ and E_1 is cyclic (Lemma 6.1), we have $X \subseteq Z(E)$. This proves (e).

To prove (f), note that for any subgroup X of $N_G(S)$,

$$C_G(S)X \lhd N_G(S)$$

because $N_G(S)' \subseteq C_G(S)$ by (c). Then, $C_S(X) = C_S(C_G(S)X) \lhd N_G(S)$ and $[S, X] = [S, C_G(S)X] \lhd N_G(S)$. Q.E.D.

Corollary 6.9. *Suppose $M \in \mathcal{M}$, $p \in \tau_2(M)$, $A \in \mathcal{E}_p^2(E)$, $q \in \tau_1(M)$, $Q \in \mathcal{E}_q^1(E)$, $C_{M_\sigma}(Q) = 1$, and $[A, Q] \neq 1$. Let $A_0 = [A, Q]$ and $A_1 = C_A(Q)$. Then, G has nonabelian Sylow p-subgroups. We have*

(a) *$A_0 \in \mathcal{E}^1(A)$ and $A_0 = C_A(M_\sigma) \lhd M$,*
(b) *A_0 is not conjugate to A_1 in G, and*
(c) *$A_1 \in \mathcal{E}^1(A)$ and $C_G(A_1) \not\subseteq M$.*

Proof. If G has abelian Sylow p-subgroups, Lemma 6.8 (e) implies either $C_{M_\sigma}(Q) \neq 1$ or $[A, Q] = 1$. Thus, G has nonabelian Sylow p-subgroups.

Since A is abelian, we have $A = A_0 \times A_1$ by Proposition 1.6 [BG]. Proposition 4.11 (d) with (p, P, K) replaced by (q, Q, A) yields that $A_0 = [A, Q] \neq 1$ is a cyclic normal subgroup of M. It follows that $A_0 \subseteq C_A(M_\sigma)$. Theorem 6.7 (b) yields (a).

This implies that $A_1 \in \mathcal{E}^1(A)$. Then, Theorem 6.7 (c) proves (c). Since $r_q(M) = 1$ and Q does not centralize A_0, $C_G(A_0)$ is a q'-group. Therefore, (b) holds. Q.E.D.

Corollary 6.10. *Let $M \in \mathcal{M}$.*

(a) *Every nilpotent $\sigma(M)'$-subgroup of M is abelian.*
(b) *The groups E_2 and E' are abelian.*
(c) *Suppose $p \in \tau_2(M)$ and $A \in \mathcal{E}_p^2(E)$. Then, $E_2E_3 \subseteq C_E(A) \lhd E$ and $\pi(E/C_E(A)) \subseteq \tau_1(M)$.*
(d) *Suppose $p \in \sigma(M)$ and P is a noncyclic p-subgroup of M. Then, $N_G(P) \subseteq M$.*
(e) *Suppose $x \in M^\sharp$, $\pi(\langle x \rangle) \subseteq \tau_2(M)$, and $C_{M_\sigma}(x) \neq 1$. Then, $\mathcal{M}(C_G(x)) = \{M\}$.*

Proof. We will paraphrase the proof of Part (e); the remainder is straightforward (cf. the proof of Corollary 12.10 [BG]).

The group M contains an abelian Hall $\tau_2(M)$-subgroup E_2 (Theorems 6.7 (a) and 6.5 (b), and Lemma 6.8 (a)). This implies that any

$\tau_2(M)$-subgroup of M is conjugate to a subgroup of E_2. Since $\langle x \rangle$ is a $\tau_2(M)$-subgroup of M, $\langle x \rangle$ is conjugate to a subgroup of E_2 in M. Thus, we may assume that $x \in E_2$.

We have $\tau_2(M) \neq \emptyset$. By Lemma H, M is a ϖ-group. So, $C_G(x)$ is contained in a ϖ-local subgroup and contains $A \in \mathcal{E}_p^2(E)$ for some $p \in \tau_2(M)$. If $C_G(x) \subseteq M^* \in \mathcal{M}(C_G(x)) \setminus \{M\}$, Theorem 6.5 (e) yields that $C_{M_\sigma}(x) \subseteq M_\sigma \cap N^* = 1$. This proves (d). Q.E.D.

Lemma 6.11. *Suppose $M \in \mathcal{M}$, $p \in \tau_2(M)$, $A \in \mathcal{E}_p^2(E)$, and $M^* \in \mathcal{M}(N_G(A))$. Then,*

(a) $\tau_2(M) \subseteq \sigma(M^*) \setminus \beta(M^*)$,
(b) $\pi(E/C_E(A)) \subseteq \tau_1(M^*) \cup \tau_2(M^*)$, *and*
(c) *if $q \in \pi(E/C_E(A)) \cap \pi(C_E(A))$, then $q \in \tau_2(M^*)$, some Sylow p-subgroup of G is normal in M^*, and M^* contains an abelian Sylow q-subgroup of G.*

Proof. As before M is a ϖ-group. The proof of (a) and (b) is similar to the corresponding proof of Lemma 12.11 [BG]. We paraphrase the proof of Part (c).

Let $q \in \pi(E/C_E(A)) \cap \pi(C_E(A))$ and $Q \in Syl_q(E)$. Corollary 6.10 (c) yields $q \in \tau_1(M)$. It follows that Q is cyclic. Since $A \lhd E$ by Corollary 6.6 (a), we have $C_E(A) \lhd E$. Hence, $Q \cap C_E(A)$ is a Sylow q-subgroup of $C_E(A)$. Thus, we have $Q_0 = \Omega_1(Q) \subseteq C_E(A)$ and $Q_0 \neq Q$.

By Corollary 6.6 (b), $C_G(A) \subseteq E$ so $C_G(A)$ has a cyclic Sylow q-subgroup. The Frattini argument yields

$$N_G(A) = C_G(A)(N_G(A) \cap N_G(Q_0)).$$

Take $M^{**} \in \mathcal{M}(N_G(Q_0))$. Since $Q_0 \subseteq C_E(A)$, we have $A \subseteq N_G(Q_0)$. Proposition 6.4 applied to M^{**} yields that $C_G(A) \subseteq M^{**}$. The above displayed formula shows $N_G(A) \subseteq M^{**}$. By (b) and Lemma 6.2 (a) both applied to A and M^{**}, the prime q lies in $\sigma(M^{**}) \cup \tau_2(M^{**})$ and in $\tau_1(M^{**}) \cup \tau_2(M^{**})$. Therefore, $q \in \tau_2(M^{**})$. The part (a) applied to M^* and then to M^{**} shows that $p \in \sigma(M^*)$ and $p \in \sigma(M^{**})$. Since $q \in \tau_2(M^{**})$, we can apply Corollary 6.6 for M^{**}. The part (f) implies that M^* is conjugate to M^{**}; otherwise we would have $\sigma(M^*) \cap \sigma(M^{**}) = \emptyset$. Since $A \subseteq M^* \cap M^{**}$, Theorem 4.1 (b) shows that M^{**} is conjugate to M^* by an element of $C_G(A)$. But, $C_G(A) \subseteq M^{**}$ so we have $M^{**} = M^*$. Thus, $q \in \tau_2(M^*)$.

It follows from Theorem 6.5 (a) that $(M^*)_\sigma$ is nilpotent. Since $p \in \sigma(M^*) \setminus \beta(M^*)$ by (a), $O_p(M^*)$ is a Sylow p-subgroup of M^* and of G. This proves the second statement.

Since $q \in \tau_2(M^*)$, Theorem 6.5 (b) applied to M^* yields that M^* has abelian Sylow q-subgroups. Note that $Q_0 \lhd Q$ so $Q \subseteq M^{**} = M^*$. Let E^* be a complement of $(M^*)_\sigma$ in M^* that contains Q. Let S be a Sylow q-subgroup of E^* that contains Q. We will show that $S \in Syl_q(G)$.

Suppose that G has nonabelian Sylow q-subgroups. Theorem 6.7 applied to M^* yields the following. Among the elements of $\mathcal{E}^1(S)$, there is a unique subgroup X_0 such that $C_{M_\sigma^*}(X_0) \neq 1$ (Theorem 6.7 (c)). This subgroup X_0 has a complement E_0 in E^* (Part (d)). We have $A \subseteq M_\sigma^* \cap C_G(Q_0)$. Therefore, we must have $X_0 = Q_0$. Since Q_0 has a complement E_0 in E^*, the Dedekind law shows that $E_0 \cap Q$ must be a complement of Q_0 in Q. Since $Q \neq Q_0$ and Q is cyclic, Q_0 has no complement in Q. This is a contradiction. Thus, G has abelian Sylow q-subgroups. By Lemma 6.8 (b), we have $S \in Syl_q(G)$. This completes the proof. Q.E.D.

Theorem 6.12. *Suppose $M \in \mathcal{M}$ and $C_{M_{\sigma_0}}(e) = 1$ for each $(\tau_1(M) \cup \tau_3(M))$-element $e \in E^\sharp$. Then,*

(a) *E contains an abelian normal subgroup A_0 such that $C_E(x) \subseteq A_0$ for every $x \in (M_{\sigma_0})^\sharp$, and*

(b) *E contains a subgroup E_0 of the same exponent as E such that $E_0M_{\sigma_0}$ is a Frobenius group with Frobenius kernel M_{σ_0}.*

Proof. If $E_2 = 1$, then $E = E_1E_3$ acts regularly on M_{σ_0}. Therefore, with $A_0 = 1$ and $E_0 = E$, (a) and (b) hold.

Assume that $\tau_2(M)$ is not empty. Then, by Lemma H, M is a ϖ-group. Take $p \in \tau_2(M)$. If G has nonabelian Sylow p-subgroups, then Theorem 6.7 provides subgroups A_0 and E_0 as required. Note that (c) implies that $C_{M_\sigma}(x) = 1$ for every p-element x of ${E_0}^\sharp$. Thus, we can assume the hypotheses, notation and conclusions of Lemma 6.8.

By assumptions, $C_E(x)$ is a $\tau_2(M)$-group for every $x \in {M_\sigma}^\sharp$. By Lemma 6.8 (a) and (b), E contains an abelian normal Hall $\tau_2(M)$-subgroup E_2. Hence, we have $C_E(x) \subseteq E_2$ for every $x \in {M_\sigma}^\sharp$. Thus, $A_0 = E_2$ satisfies (a).

For each $p \in \tau_2(M)$, we have a normal abelian subgroup S of rank two such that S is a Sylow p-subgroup of E and of G (Lemma 6.8 (a) and (b)). We will prove that for each $p \in \tau_2(M)$ there is a cyclic normal subgroup $Z = Z_p$ of E having the same exponent as S and satisfying the condition $C_{M_\sigma}(z) = 1$ for every $z \in Z^\sharp$.

We remark that the last centralizer condition is equivalent to

$$C_{M_\sigma}(\Omega_1(Z)) = 1.$$

and this condition is automatically satisfied if Z is a nonidentity cyclic subgroup of S such that $\Omega_1(Z) \lhd N_G(S)$. The first claim is trivial. To prove the second, suppose that $C_{M_\sigma}(\Omega_1(Z)) \neq 1$. Corollary 6.6 (c) for $X = \Omega_1(Z)$ yields $\mathcal{M}(C_G(X)) = \{M\}$. Since $\Omega_1(Z) \lhd N_G(S)$, we have

$$N_G(S) \subseteq N_G(\Omega_1(Z)) \subseteq M.$$

This contradicts Lemma 6.8 (d).

Assume that $C_E(S) = E$. Since S is abelian of rank 2, $S = Y \times Z$ for some cyclic subgroups Y and Z. We choose the notation $|Y| \leq |Z|$. If $|Y| < |Z|$, $\Omega_1(Z)$ is characteristic in S. Then, $\Omega_1(Z) \lhd N_G(S)$ so $Z = Z_p$ satisfies the required property. (Since $C_E(S) = E$, any subgroup of S is normal in E.) If $|Y| = |Z|$, we can take a factor Z in such a way that $\Omega_1(Z)$ is equal to any given $A_1 \in \mathcal{E}^1(S)$ and, by Theorem 6.5 (f), at least one such A_1 satisfies $C_{M_\sigma}(A_1) = 1$. This completes the proof in the case $C_E(S) = E$.

Assume that $C_E(S) \neq E$. Take $q \in \pi(E/C_E(S))$ and let $Q_1 \in Syl_q(E)$ and $Q \in Syl_q(N_G(S))$ such that $Q_1 \subseteq Q$. The definition of q implies $C_S(Q_1) \neq S$. Let $A = \Omega_1(S)$. Then, $A \in \mathcal{E}^2(S)$. By Proposition 1.6 [BG], Q_1 does not centralize A. Therefore, by Corollary 6.10 (c), $q \in \tau_1(M)$ and Q_1 is cyclic. Since $C_S(Q_1) \neq S$ and $C_G \subseteq E$, we have

$$Q_0 = C_Q(S) \subsetneqq Q_1.$$

Suppose that Q/Q_0 acts regularly on S. Then, Proposition 3.9 [BG] shows that Q/Q_0 is cyclic. Hence, $\Omega_1(Q/Q_0) \subseteq Q_1/Q_0$ and $\Omega_1(Q) \subseteq Q_1$. Since Q_1 is cyclic, $\Omega_1(Q) \subseteq Q_1$ implies that Q is cyclic, too. Thus, $r_q(N_G(S)) = 1$. On the other hand, since $q \in \tau_1(M)$, the assumption of this theorem implies that $C_{M_\sigma}(\Omega_1(Q_1)) = 1$. Hence, by Lemma 6.8 (e), $\Omega_1(Q_1)$ lies in $Z(E)$ so $\Omega_1(Q_1)$ centralizes A.

If $M^* \in \mathcal{M}(N_G(A))$, $S \subseteq N_G(A) \subseteq M^*$. So, S is a Sylow p-subgroup of M^*. Now,Lemma 6.11 (c) yields $q \in \tau_2(M^*)$, $S \lhd M^*$, and M^* contains an abelian Sylow q-subgroup of G. This implies that $r_q(N_G(S)) \geq 2$. This contradiction proves that Q/Q_0 does not act regularly on S. Therefore, $1 \neq C_S(X) \neq S$ for some subgroup X of Q. By Proposition 1.6 (d) [BG], we have $S = S_0 \times S_1$ where $S_0 = C_S(X)$ and $S_1 = [S, X]$. Since $r(S) = 2$, both S_0 and S_1 are cyclic. By Lemma 6.8 (f), both S_0 and S_1 are normal in $N_G(S)$. Define $Z = S_0$ if $|S_0| \geq |S_1|$ and $Z = S_1$ if $|S_0| < |S_1|$. Then, Z has the required properties.

Define E_0 to be the product of E_1E_3 and $\prod Z_p$ for all $p \in \tau_2(M)$. Then, E_0 satisfies the requirements of (b). Q.E.D.

Theorem 6.13. *Let $p \in \varpi$. Then, every nonabelian p-subgroup of G lies in $\mathcal{U}$.*

Proof. The proof of Theorem 12.13 [BG] works. We just add some details. Let $p \in \varpi$ and let P be a nonabelian p-subgroup of maximal order that lies in two distinct subgroups M and M^* of $\mathcal{M}$. Then, by Corollary 6.10, $N_G(P) \subseteq M \cap M^*$. It follows that $P \in Syl_p(G)$ and $r(P) = 2$. By Corollary 4.7 (b), P contains a nonabelian subgroup Q of order p^3 and of exponent p and $Z(Q) = \Omega_1(Z(P))$. Let $Z = Z(Q)$ and $K = C_{M_\sigma}(Z)$. It is proved that $K \subseteq M^*$. By Corollary 4.9 (b), $M = (M \cap M^*)M_\alpha$. This implies that $M_\alpha \neq 1$. Similarly, we have $(M^*)_\alpha \neq 1$.

Apply Lemma 6.5 (b) with (K, U, H, G) replaced by $(M_\alpha, M \cap M^*, Z, M)$ to conclude $N_M(Z) = C_{M_\alpha}(Z)(N_M(Z) \cap M^*) \subseteq M^*$. It follows that

$$\mathcal{M}(N_G(Z)) \neq \{M\};$$

otherwise, we would have $N_G(Z) = N_M(Z) \subseteq M \cap M^*$.

Take any $A \in \mathcal{E}_p^2(Q)$ and apply Proposition 6.4 (b) to M, and then M^*. Since $M_\alpha \neq 1$, the hypothesis of Proposition 6.4 (b) does not hold. Thus, there is a subgroup $A_0 \in \mathcal{E}^1(A)$ such that $\mathcal{M}(N_G(A_0)) = \{M\}$. Since Z does not satisfy this condition, we have $A_0 \neq Z$. Similarly, there is a subgroup $A_0^* \in \mathcal{E}^1(A) \setminus \{Z\}$ which satisfies $\mathcal{M}(N_G(A_0^*)) = \{M^*\}$. By the property of the group Q, A_0^* is conjugate to A_0 in Q. This would imply that $\mathcal{M}(N_G(A_0^*))$ would be conjugate to $\mathcal{M}(N_G(A_0))$ by an element of $Q \subseteq M \cap M^*$, so $M^* = M$. This contradiction proves Theorem 6.13. Q.E.D.

Corollary 6.14. *Suppose $M \in \mathcal{M}$, $p \in \sigma(M)$, $X \in \mathcal{E}_p^1(M)$, and $P \in Syl_p(M_\sigma)$. Assume that $p \in \beta(M)$ or $X \subseteq M_\sigma'$. Then, $p \in \varpi$ and $\mathcal{M}(C_G(X)) = \mathcal{M}(P) = \{M\}$.*

Proof. We may assume that X is a subgroup of P. First, we prove a lemma: *under the assumptions of Corollary* 6.14, *if $p \notin \beta(M)$, then we have $X \subseteq P'$.* If $p \notin \beta(M)$, the assumption implies that $X \subseteq M_\sigma'$. The group M_σ/M_β is nilpotent by Lemma 4.8 (b). Since $P \cap M_\beta = 1$, we have $X \subseteq M_\sigma' \cap P = P'$ proving the lemma.

This lemma implies that if $p \notin \beta(M)$, P is nonabelian; in particular, P is not cyclic so $r(P) \geq 2$. Thus, $p \in \varpi$ by Lemma F. If $p \in \beta(M)$, we have $p \in \varpi$. This proves $p \in \varpi$ in all cases.

Suppose that $r(C_P(X)) \geq 3$. By the Uniqueness Theorem, we have $C_P(X) \in \mathcal{U}$. Since $C_P(X) = C_G(X) \cap P$, both $C_G(X)$ and P lie in $\mathcal{U}$. Then, we have

$$\mathcal{M}(C_G(X)) = \mathcal{M}(P) = \{M\}.$$

Suppose that $r(C_P(X)) \leq 2$. If $r(P) \geq 3$, P is narrow by Corollary 5.4 [BG]; so $p \notin \beta(M)$. By the lemma, we have $X \subseteq P'$. On the other hand, if p is narrow and $r(C_P(X)) \leq 2$ for some $X \in \mathcal{E}^1(P)$, Theorem 5.3 (d) shows $X \cap P' = 1$. This contradicts $X \subseteq P'$. Hence, $r(P) \leq 2$ and $p \notin \beta(M)$. The lemma yields that P is nonabelian.

By Corollary 4.7 (b), P is the central product and satisfies $P' \subseteq Z(P)$. We have $P \subseteq C_M(X)$ because $X \subseteq P'$. Since P is nonabelian, $P \in \mathcal{U}$ by Theorem 6.13. This implies that $C_G(X) \in \mathcal{U}$ and completes the proof. Q.E.D.

Proposition 6.15. *Suppose $M \in \mathcal{M}$, $q \in \sigma(M)$, X is a nonidentity q-subgroup of M, and $M^* \in \mathcal{M}(N_G(X)) \setminus \{M\}$. Let S be a Sylow q-subgroup of $M \cap M^*$ that contains X. Then, S, M, and M^* satisfy the following conditions.*

(a) *M^* is not conjugate to M in G.*
(b) *$N_G(S) \subseteq M$.*
(c) *S is a Sylow q-subgroup of M^*.*
(d) *If $q \in \sigma(M^*)$, then (1) $M^* = (M \cap M^*)M^*_\beta$, (2) $\tau_1(M^*) \subseteq \tau_1(M) \cup \alpha(M)$, and (3) $M_\beta = M_\alpha \neq 1$.*
(e) If $q \notin \sigma(M^*)$, then (1) $q \in \tau_2(M^*)$, (2) $\pi(M) \cap \sigma(M^*) \subseteq \beta(M^*)$, and (3) $M \cap M^*$ is a complement to M^*_σ in M^*.

Proof. The assertions follow as in the proof of Proposition 12.15 [BG]; we will paraphrase the proof of (e).

Suppose that $q \notin \sigma(M^*)$. By Lemma 6.2 (a) applied to q, we have $q \in \tau_2(M^*)$. Lemma H shows that M^* is a ϖ-group. Since $S \in Syl_q(M^*)$ by (c), Theorem 6.5 proves $A = \Omega_1(S) \in \mathcal{E}^2(S)$.

Let E^* be a complement of M^*_σ in M^* that contains A. By Theorem 6.5 (e) and Corollary 6.6 (a) with (p, M) replaced by (q, M^*), $M^*_\sigma \cap M = 1$ and $A \lhd E^*$. By Corollary 6.10 (d), we have $N_G(A) \subseteq M$. This implies that $E^* \subseteq N_G(A) \subseteq M$. Thus,

$$M \cap M^* = M \cap M^*_\sigma E^* = (M \cap M^*_\sigma)E^* = E^*.$$

This proves (3).

Suppose that $p \in \pi(M) \cap \sigma(M^*)$ and $p \notin \beta(M^*)$. By Corollary 6.6 (b) applied to M^*, we have $C_G(A) \subseteq E^*$. Since $p \in \sigma(M^*)$, the group $C_G(A)$ is a p'-group.

By (a), M is not conjugate to M^*. Therefore, Corollary 6.6 (f) applied to M^* and q proves that $\sigma(M^*)$ is disjoint from $\sigma(M)$. This implies first $p \neq q$ because $p \in \sigma(M^*)$ and $q \in \sigma(M)$, and secondly $p \notin \beta(M)$ as $\beta(M) \subseteq \sigma(M)$. By (1), $q \in \tau_2(M^*)$ so $q \notin \beta(G)$ by Lemma 6.1 (g) with p replaced by q. We can apply Corollary 4.9 to M^*. If $p < q$, the q-subgroup A of M^* centralizes a Sylow p-subgroup of

M_σ^*. Since $p \in \sigma(M^*)$, $C_G(A)$ has order divisible by p. This contradicts the earlier statement that $C_G(A)$ is a p'-group. Thus, we have $q < p$. We apply Corollary 4.9 to M interchanging p and q. We conclude that a Sylow p-subgroup P of M centralizes a Sylow q-subgroup Q of M_σ. Since $p \in \pi(G)$, we have $P \neq 1$. We may replace Q and P by conjugates and suppose that $A \subseteq Q$. We get a contradiction that $1 \neq P \subseteq C_G(A)$. This completes the proof of (e). Q.E.D.

Corollary 6.16. *Let $M \in \mathcal{M}$ and E a complement of M_σ in M. Suppose that Y is a $\sigma(M)$-subgroup of G such that $O_\varpi(Y) \neq 1$. Then, Y is conjugate to a subgroup of M_σ and for every $p \in \pi(E) \cap \beta(G)'$ and every $H \in \mathcal{M}(Y)$ not conjugate to M in G,*

(a) $r_p(N_H(Y)) \leq 1$, *and*
(b) *if $p \in \tau_1(M)$, then $p \notin \pi(N_H(Y)')$.*

Proof. With the extra condition that $O_\varpi(Y) \neq 1$ Y is contained in a ϖ-local subgroup, so it is solvable. We can take a nonidentity characteristic q-subgroup X of Y for some prime $q \in \sigma_0(M)$. Since M contains a Sylow q-subgroup of G, we may replace Y by some conjugate if necessry, and assume that $X \subseteq M_\sigma$.

First we prove the following lemma as part of the proof of (a). *Let $H \in \mathcal{M}(Y)$. If H is not conjugate to M in G, then for any prime $p \in \pi(E) \cap \beta(G)'$, we have $r_p(H \cap M) \leq 1$.*

Suppose $r_p(H \cap M) \geq 2$ and take $A \in \mathcal{E}_p^2(H \cap M)$. Then $p \in \tau_2(M)$ and by Theorem 6.5 (e) we have $M_\sigma \cap H = 1$ in contradiction to

$$1 \neq X \subseteq M_\sigma \cap H.$$

This proves the lemma.

To prove Corollary 6.16, we assume first that $N_G(X) \subseteq M$. In this case we have $Y \subseteq N_G(X) \subseteq M$. Since M/M_σ is a $\sigma(M)'$-group and Y is a $\sigma(M)$-group, we have $Y \subseteq M_\sigma$. Let $p \in \pi(E) \cap \beta(G)'$ and let $H \in \mathcal{M}(Y)$ such that H is not conjugate to M in G. By the lemma, $r_p(H \cap M) \leq 1$. Since $N_H(Y) \subseteq N_G(Y) \subseteq N_G(X) \subseteq M$, we have $N_H(Y) \subseteq H \cap M$. This implies $r_p(N_H(Y)) \leq r_p(H \cap M) \leq 1$. If $p \in \tau_1(M)$, M' is a p'-group (by definition of $\tau_1(M)$). Then, (b) holds because $N_H(Y)' \subseteq (H \cap M)' \subseteq M'$.

In the remainder of the proof we assume that $N_G(X) \not\subseteq M$. Since X is a ϖ-group, there is $M^* \in \mathcal{M}(N_G(X))$. Since $N_G(X) \not\subseteq M$, we have $M^* \neq M$. By Proposition 6.15, M^* is not conjugate to M in G, $q \in \sigma(M^*) \cup \tau_2(M^*)$, and if $q \in \tau_2(M^*)$, then $\pi(M) \cap \sigma(M^*) \subseteq \beta(M^*)$. Moreover, if K is defined to be M_β^* or M_σ^* according as $q \in \sigma(M^*)$ or

$q \in \tau_2(M^*)$, then $M^* = (M \cap M^*)K$. We claim that K is a $\sigma(M)'$-group. If $q \in \sigma(M^*)$, $K = M^*_\beta \subseteq M^*_\alpha$ and by Lemma 4.12 (a), $\alpha(M^*)$ is disjoint from $\sigma(M)$. If $q \in \tau_2(M^*)$, we have $K = M^*_\sigma$ and $\sigma(M^*) \cap \sigma(M) = \emptyset$ by Corollary 6.6 (f) with M replaced by M^*. Thus, K is a $\sigma(M)'$- group.

Since $Y \subseteq N_G(X)$, the $\sigma(M)$-group Y is contained in $M^* = (M \cap M^*)K$. Since K is a normal $\sigma(M)'$-group, the Schur-Zassenhaus Theorem shows that Y is conjugate to a subgroup of $M \cap M^*$. Since Y is a $\sigma(M)$-group, Y is contained in $M_\sigma \cap M^*$ which is a normal Hall $\sigma(M)$-subgroup of $M \cap M^*$. This proves the first assertion of Corollary 6.16.

Take $p \in \pi(E) \cap \beta(G)'$ and $H \in \mathcal{M}(Y)$ that is not conjugate to M in G. We claim that K is a p'-group. This is clear if $K = (M^*)_\beta$ because $p \notin \beta(G)$. On the other hand, if $K = (M^*)_\sigma$, we have $q \in \tau_2(M^*)$ so p cannot divide $|(M^*)_\sigma|$ because $\pi(M) \cap \sigma(M^*) \subseteq \beta(M^*)$ and $p \notin \beta(M^*)$. Thus, K is a p'-group. Since $N_H(Y) \subseteq N_G(Y) \subseteq M^*$ and M^* is not conjugate to M in G, we may assume $H = M^*$. In this case, we have $H = (H \cap M)K$. Since K is a p'-group, $H \cap M$ contains a Sylow p-subgroup of H. The lemma at the beginning of the proof shows that $r_p(H \cap M) \leq 1$. Thus, we have $r_p(N_H(Y)) \leq r_p(H) \leq 1$. This proves (a).

If $p \in \tau_1(M)$, then $p \notin \pi(M')$. It follows that $(H \cap M)'$ is a p'-group. Clearly, we have $N_H(Y)' \subseteq H' \subseteq (H \cap M)'K$. Therefore, $N_H(Y)'$ is a p'-group. Q.E.D.

Lemma 6.17. *Let $M \in \mathcal{M}$ and E a complement of M_σ in M. Then, we have $C_{M_\sigma}(E) \subseteq (M_\sigma)'$, $[M_\sigma, E] = M_\sigma$, and for every $g \in G \setminus M$, the group $M_\sigma \cap M^g$ is a cyclic $\beta(M)'$-group intersecting $(M_\sigma)'$ trivially.*

Lemma 6.18. *Suppose $M \in \mathcal{M}$, $p \in \tau_1(M)$, $P \in \mathcal{E}_p^1(M)$, $q \in p'$, and Q is a nonidentity P-invariant q-subgroup of M such that $C_Q(P) = 1$ and $\mathcal{M}(N_G(Q)) \neq \{M\}$.*

(a) *If $M_\alpha \neq 1$ and $q \notin \alpha(M)$, then $C_{M_\alpha}(P) \neq 1$ and $C_{M_\alpha}(PQ) = 1$.*
(b) *If $Q \in Syl_q(M)$, then $\alpha(M) = \beta(M)$ and we have the situation of* (a).

Proof. We will rewrite the first paragraph of the proof of Lemma 12.18 [BG]; the remainder of the proof can be adapted directly.

Suppose that $M_\alpha \neq 1$ and $q \notin \alpha(M)$. We will prove that

$$r(C_{M_\alpha}(Q)) \leq 1.$$

Suppose that $r(C_{M_\alpha}(Q)) \geq 2$. Then, $C_{M_\alpha}(Q)$ is a ϖ-group so by Lemma A, Q is a ϖ-group. Lemma 4.3 with X replaced by Q yields that $C_M(Q) \in \mathcal{U}$. Since $q \in \varpi$, $\mathcal{M}(N_G(Q))$ is not empty and, by assumption, contains $H \in \mathcal{M}$ different from M. Thus,

$$C_M(Q) \subseteq N_G(Q) \subseteq H \neq M,$$

so $C_M(Q) \notin \mathcal{U}$. This contradiction proves $r(C_{M_\alpha}(Q)) \leq 1$.

We prove $r(C_{M_\alpha}(P)) \leq 1$. Suppose that $r(C_{M_\alpha}(P)) \geq 2$. The same argument as above yields that $p \in \varpi$ and $C_M(P) \in \mathcal{U}$. Since $p \in \tau_1(M)$, P is contained in a cyclic Sylow p-subgroup S of M. Since $p \notin \sigma(M)$, we have $N_G(S) \nsubseteq M$ Thus, $N_G(S) \subseteq N_G(P) \nsubseteq M$.

We can find $H \in \mathcal{M}(N_G(P))$ because $p \in \varpi$. Then, $H \neq M$ and

$$C_M(P) \subseteq N_G(P) \subseteq H \neq M.$$

Thus, $C_M(P) \notin \mathcal{U}$. This proves $r(C_{M_\alpha}(P)) \leq 1$. Q.E.D.

Lemma 6.19. *Let $M \in \mathcal{M}$ and E a complement of M_σ in M. Then, the group E' centralizes a Hall $\beta(M)'$-subgroup of M_σ.*

§7. Prime Action

This section corresponds Section 13 of [BG]. Troughout this section, s subgroup $M \in \mathcal{M}$ and a complement E of M_σ in M will be fixed.

Lemma 7.1. *Suppose that $M^* \in \mathcal{M}$, $p \in \pi(E) \cap \pi(M^*)$, $p \notin \tau_1(M^*)$, $[M_\sigma \cap M^*, M \cap M^*] \neq 1$, and M^* is not conjugate to M in G. Then,*

(a) *every p-subgroup of $M \cap M^*$ centralizes $M_\sigma \cap M^*$,*
(b) *$p \notin \tau_2(M^*)$, and*
(c) *if $p \in \varpi \cap \tau_1(M)$, then $p \in \beta(G)$.*

Proof. Since $[M_\sigma \cap M^*, M \cap M^*] \subseteq M_\sigma \cap (M^*)'$, there is $q \in \sigma(M) \cap \pi((M^*)')$. Then, $q \neq p$ because $p \in \pi(E)$. Let Y be a Sylow q-subgroup of $(M^*)'$. By Lemma 4.8, $(M^*)'/(M^*)_\beta$ is nilpotent so $(M^*)_\beta Y \lhd M^*$. The Frattini argument yields $M^* = (M^*)_\beta N_{M^*}(Y)$.

In order to prove (b), suppose $p \in \tau_2(M^*)$. Then, $r_p(N_{M^*}(Y)) = 2$ because $N_{M^*}(Y)$ covers $M^*/(M^*)_\beta$. Moreover, M^* is a ϖ-group by Lemma H. It follows that $q \in \varpi$. Lemma 6.1 (g) yields that $p \notin \beta(G)$. Corollary 6.16 (a) can be applied to get $r_p(N_{M^*}(Y)) \leq 1$. This contradiction proves (b).

To prove (c), suppose that $p \in \beta(G)'$. By (b) and the assumptions, $p \in \sigma(M^*) \cup \tau_3(M^*)$. Therefore, $p \in \pi((M^*)')$. We have $(M^*)' =$

$(M^*)_\beta(N_{M^*}(Y))'$. Hence, $N_{M^*}(Y)'$ contains a p-subgroup $P \neq 1$. We will show that this is a contradiction. Let S be a Sylow p-subgroup of $(M^*)'$. Since $(M^*)'/(M^*)_\beta$ is nilpotent, $(M^*)_\beta S \lhd M^*$ and $P \subseteq (M^*)_\beta S$. We claim that $q \in \varpi$. If $q \in \beta(M^*)$, this is trivial. If $q \notin \beta(M^*)$, $(M^*)_\beta S$ is a q'-group. Recall that $p \neq q$. Now, $[Y, P] \subseteq Y \cap (M^*)_\beta S = 1$ because $P \subseteq N_G(Y)$ and $P \subseteq (M^*)_\beta S$. Since $P \neq 1$ is a p-group and $p \in \varpi$, we have $q \in \varpi$. We can apply Corollary 6.16 (b) which yields that if $p \in \tau_1(M)$, then $p \notin \pi(N_{M^*}(Y))$. This contradiction proves (c).

The statement (a) follows as in [BG]. Q.E.D.

Corollary 7.2. *Suppose that $p \in \tau_1(M) \cup \tau_3(M)$, P is a nonidentity p-subgroup of M, and $M^* \in \mathcal{M}(N_G(P))$. Then,*

(a) *every p-subgroup of $M \cap M^*$ centralizes $M_\sigma \cap M^*$,*
(b) *every $\tau_1(M^*)'$-subgroup of $E \cap M^*$ centralizes $M_\sigma \cap M^*$, and*
(c) *if $[M_\sigma \cap M^*, M \cap M^*] \neq 1$, then $p \in \sigma(M^*)$ and in the case $p \in \varpi \cap \tau_1(M)$, we even have $p \in \beta(M^*)$.*

Corollary 7.3. *The following statements hold.*

(a) *Let $P \in Syl_p(E)$ for some $p \in \pi(E) \cap \varpi$. Assume that P is cyclic. Then, P acts in a prime manner on M_σ.*
(b) *If $\varpi \cap \tau_3(M) \neq \emptyset$, E_3 acts in a prime manner on M_σ.*

Proof. Let $P_1 = \Omega_1(P)$. If $x \in P^\sharp$, then $P_1 \subseteq \langle x \rangle \subseteq P$. By assumption, $p \in \tau_1(M) \cup \tau_3(M)$. Therefore, we have $N_G(P) \nsubseteq M$ Since $p \in \varpi$, there exists $M^* \in \mathcal{M}(N_G(P))$. We have $P \subseteq M \cap M^*$. By Corollary 7.2 (a), P centralizes $M_\sigma \cap M^*$. Thus,

$$M_\sigma \cap M^* \subseteq C_{M_\sigma}(P) \subseteq C_{M_\sigma}(x) \subseteq C_{M_\sigma}(P_1).$$

On the other hand, $C_{M_\sigma}(P_1) \subseteq N_G(P_1) \subseteq M^*$, so $C_{M_\sigma}(P_1) \subseteq M_\sigma \cap M^*$. It follows that $C_{M_\sigma}(x) = M_\sigma \cap M^*$ for every $x \in P^\sharp$. This proves (a).

The proof of (b) is similar. Take $X \in \mathcal{E}^1(E_3)$ with $p \in \varpi$ and $M^* \in \mathcal{M}(N_G(X))$. We have $E_3 \subseteq E'$ by Lemma 6.1 (b). Since $E \subseteq N_G(X) \subseteq M^*$, E_3 is a subgroup of $(M^*)'$; in particular, E_3 is a $\tau_1(M^*)'$-subgroup. If $x \in {E_3}^\sharp$ satisfies $X \subseteq \langle x \rangle$, then we have

$$C_{M_\sigma}(E_3) = C_{M_\sigma}(x) = C_{M_\sigma}(X) = M_\sigma \cap M^*.$$

If $p \in \varpi$ for one prime p in $\tau_3(M_3)$, then $\tau_3(M) \subseteq \varpi$. Hence, for any element $x \in {E_3}^\sharp$, we have $C_{M_\sigma}(x) = C_{M_\sigma}(E_3)$. This proves (b).

Q.E.D.

Theorem 7.4. *Suppose that* $p \in \varpi$, $p \in \tau_1(M)$, $P \in \mathcal{E}^1(E)$, $r \in \pi(E)$, *and* $R \in \mathcal{E}_r^1(C_E(P))$. *Then,* $C_{M_\sigma}(P) \subseteq C_{M_\sigma}(R)$.

Proof. By assumption, we have $RP = R \times P$ so $r \in \varpi$. Since $p \in \tau_1(M)$, we have $N_G(P) \not\subseteq M$. We can take $M^* \in \mathcal{M}(N_G(P))$ because $N_G(P)$ is a ϖ-local subgroup. By Lemma 6.2, $p \in \sigma(M^*) \cup \tau_2(M^*)$ (by (a)) and M^* is not conjugate to M in G (by (b)), In particular, $M^* \neq M$.

By Corollary 7.2 (a), P centralizes $M_\sigma \cap M^*$. as in the proof of Corollary 7.3, we have $C_{M_\sigma}(P) = M_\sigma \cap M^*$. This implies that $M_\sigma \cap M^*$ is a ϖ-group by Lemma A. Since $R \subseteq M \cap M^*$, the $\sigma(M)'$-group PR normalizes $M_\sigma \cap M^*$. Therefore, for each $q \in \pi(M_\sigma \cap M^*)$, there is a PR-invariant Sylow q-subgroup S of $M_\sigma \cap M^*$. Then, $S \notin \mathcal{U}$ so S is abelian by Theorem 6.13. Note that $q \in \varpi$.

We have to show that R centralizes S. We will derive a contradiction by assuming that R does not centralize S. Let $Q = [S, R]$ and assume that $Q \neq 1$. Then, $S = Q \times C_S(R)$ and $C_Q(R) = 1$ (because S is abelian). Since $S \subseteq M_\sigma \cap M^*$,

$$Q = [S, R] \subseteq [M_\sigma \cap M^*, M \cap M^*] \neq 1.$$

By Corollary 7.2, we obtain $p \in \beta(M^*)$ from (c) and $r \in \tau_1(M^*)$ from (b).

We check that all the assumptions of Lemma 6.18 (a), except the one about $\mathcal{M}(N_G(Q))$, are satisfied for (M^*, r, R, q, Q) in place of (M, p, P, q, Q). But, since $p \in \beta(M^*)$, one of the conclusions is violated, i.e. $P \subseteq C_{M^*_\alpha}(RQ) \neq 1$. It follows that $\mathcal{M}(N_G(Q)) = \{M^*\}$.

By Lemma 6.2 (a), we have $q \in \sigma(M^*) \cup \tau_2(M^*)$. We can apply Proposition 6.15 for $Q = X$. If $q \in \tau_2(M^*)$, Part (e) applies so $M \cap M^*$ is a complement of $(M^*)_\sigma$ in M^*. However, this is not true because

$$P \subseteq (M^*)_\sigma \cap M = (M^*)_\sigma \cap (M \cap M^*) \neq 1.$$

It follows that $q \in \sigma(M^*)$. Hence, by Proposition 6.15 (d), we have

$$r \in \pi(E) \cap (\tau_1(M) \cup \alpha(M)) = \tau_1(M)$$

and $M_\alpha \neq 1$. Since $q \in \sigma(M^*)$, Lemma 4.12 (a) yields $q \notin \alpha(M)$. Thus, if R does not centralize S, we have $q \notin \alpha(M)$. It follows that $C_{M_\alpha}(P) \subseteq C_{M_\alpha}(R)$ and $r \in \tau_1(M)$. We can interchange p and r to get $C_{M_\alpha}(R) \subseteq C_{M_\alpha}(P)$. Then, $C = C_{M_\alpha}(P) = C_{M_\alpha}(R)$, so

$$C = C_{M_\alpha}(P) = C_{M_\sigma}(P) \cap M_\alpha = M_\alpha \cap M^*$$

because $C_{M_\sigma}(P) = M_\sigma \cap M^*$. The group S normalizes C. Hence,

$$[C, R, S] = [S, C, R] = 1.$$

By the Three Subgroup Theorem we have $[R, S, C] = 1$. Thus, $Q = [R, S]$ centralizes C. It follows that $C = C_{M_\alpha}(R) = C_{M_\alpha}(RQ)$. On the other hand, Lemma 6.18 (a) for M, r, R, q and Q in place of M, p, P, q and Q yields $C_{M_\alpha}(R) \neq C_{M_\alpha}(RQ)$. This contradiction proves Theorem 7.4. Q.E.D.

Theorem 7.5. *Suppose that $\varpi \cap \tau_1(M) \neq \emptyset$. Then, E_1 acts in a prime manner on M_σ.*

Proof. Since E_1 is cyclic, the assumption yields that E_1 is a ϖ-group. For each $p \in \tau_1(M)$, let $P \in \mathcal{E}^1(E_1)$. By Theorem 7.4, the group $C = C_{M_\sigma}(P)$ does not depend on p. If P_1 is any p-subgroup of E_1, we have $C_{M_\sigma}(P_1) = C$ by Corollary 7.3 (a). It follows that $C_{M_\sigma}(X) = C$ for any subgroup X of E_1. Q.E.D.

Lemma 7.6. *Suppose $1 \neq P \subseteq E_1$, $q \in \sigma(M)$, and $X \in \mathcal{E}^1_q(C_{M_\sigma}(P))$. Let $S \in Syl_q(M_\sigma)$. Assume either $P = E_1$ or $\varpi \cap \tau_1(M) \neq \emptyset$. Then, $q \in \varpi$ and $\mathcal{M}(C_G(X)) = \mathcal{M}(S) = \{M\}$.*

Proof. If $q \in \beta(M)$ or $X \subseteq (M_\sigma)'$, Corollary 6.14 yields the conclusion of the lemma. We will derive a contradiction by assuming $q \notin \beta(M)$ and $X \nsubseteq (M_\sigma)'$. If $\varpi \cap \tau_1(M) \neq \emptyset$, E_1 acts in a prime manner on M_σ by Theorem 7.5. Therefore, we may assume that $P = E_1$.

Since $q \notin \beta(M)$, by Lemma 6.19, E' centralizes some Sylow q-subgroup of M_σ. The group E is a $\sigma(M)'$-group that normalizes a $\sigma(M)$-subgroup $C_{M_\sigma}(E')$. Hence, E normalizes some Sylow q-subgroup of $C_{M_\sigma}(E')$. We may replace S by a conjugate without affecting the conclusion. Thus, we may assume that S is normalized by E and centralized by E'.

The group $SE_1 \subseteq SE$ is a Hall $\{q, \tau_1(M)\}$-subgroup of M. Therefore, the subgroup XE_1 of M is conjugate to a subgroup of SE_1. Thus, for some $x \in M$, $(XE_1)^x = X^x E_1^x \subseteq SE_1$ Then, E_1^x and E_1 are Hall subgroups of SE_1, so they are conjugate in SE_1. We may assume that $E_1^x = E_1$. Since $XE_1 = X \times E_1$, we have $X^x \subseteq C_M(E_1^x) = C_M(E_1)$. Also, we have $X^x \subseteq S$ because S is a normal Sylow q-subgroup of SE_1. It follows that $X^x \in \mathcal{E}^1_q(C_{M_\sigma}(P))$ and $X^x \subseteq S$. By replacing X and S by conjugates, we may assume that

$$X \subseteq S \subseteq C_{M_\sigma}(E').$$

By Lemma 6.17, $C_{M_\sigma}(E) \subseteq (M_\sigma)'$. Since $X \nsubseteq (M_\sigma)'$, X does not centralize E, but does centralize E_1 and E'. It follows that $E \neq E_1E'$. Since $E_3 \subseteq E'$ and $E = E_1E_2E_3$ by Lemma 6.1, we have $E_2 \neq 1$. By Lemma H, M is a ϖ-group.

Take $p \in \tau_2(M)$ and $A \in \mathcal{E}_p^2(E)$. We have $A \lhd E$ by Corollary 6.6 (a) and $C_{M_\sigma}(A) = 1$ by Theorem 6.5 (d). Since A is abelian, $A = A_0 \times [A, E_1]$ with $A_0 = C_A(E_1)$. Since $[A, E_1] \subseteq E'$, $[A, E_1]$ centralizes X. Furthermore, Theorem 7.4 shows that A_0 centralizes X. Thus, $X \subseteq C_{M_\sigma}(A)$ and $C_{M_\sigma}(A) \neq 1$. This contradicts Theorem 6.5 (d).

Q.E.D.

Lemma 7.7. *Suppose that $E_1 \neq 1$, $E_3 \neq 1$, and that E_1 does not act regularly on E_3. Then, we have one of the following two cases.*

(1) *We have $\tau_3(M) \cap \varpi = \emptyset$, M is a Frobenius group with Frobenius kernel $M_\alpha = M_\beta = M_{\sigma_0}$, M_α is a ϖ-group, and M/M_α is a ϖ'-group.*

(2) *We have $\tau_3(M) \cap \varpi \neq \emptyset$, M is a ϖ-group and the group E_1E_3 acts in a prime manner on M_σ.*

Proof. By assumption, there exist primes p and r such that $P \in \mathcal{E}^1(E_1)$ centralizes $R \in \mathcal{E}_r^1(E_3)$. These primes p and r lie in the same connected component of the prime graph of G.

Suppose that $\tau_3(M) \cap \varpi = \emptyset$. Then, M is not a ϖ- group. By Lemma G, we have (1).

Suppose that $\tau_3(M) \cap \varpi \neq \emptyset$. Since E_3 is cyclic by Lemma 6.1 (d), we have $\tau_3(M) = \pi(E_3) \subseteq \varpi$. Since M'/M_β is nilpotent by Lemma 4.8 and $\tau_3(M) \subseteq \pi(M'/M_\beta)$by Lemma 6.1 (b), the group M'/M_β is a ϖ-group. The remark at the beginning of the proof shows $\tau_1(M) \cap \varpi \neq \emptyset$. Since $\tau_1(M) \subseteq \pi(M/M')$, M/M' is a ϖ-group. This proves that M is a ϖ-group.

The remainder of the proof is similar to that of Lemma 13.7 [BG]. Since M is a ϖ-group, we can apply Corollary 7.3 and Theorems 7.4 and 7.5. We assume

$$C_{M_\sigma}(P) \neq C_{M_\sigma}(R)$$

and we will obtain a contradiction. We have $1 \neq R \subseteq E_3$ and $C_{M_\sigma}(R) \neq 1$. If $\tau_2(M) \neq \emptyset$, Corollary 6.6 (d) would yield $C_{M_\sigma}(R) = 1$. Therefore, $\tau_2(M) = \emptyset$ and $E = E_1E_3$. Since R char $E_3 \lhd E$ by Lemma 6.1, we have $R \lhd E$. We can take $M^* \in \mathcal{M}(N_G(R))$ since $N_G(R)$ is a ϖ- local subgroup. We have $N_G(R) \nsubseteq M$ so $M^* \neq M$. By our hypothesis,

$$1 \neq [C_{M\sigma}(R), P] \subseteq [M_\sigma \cap M^*, E_1].$$

If $C = C_{E_1}(M_\sigma \cap M^*)$, the above displayed formula yields $C \neq E_1$. On the other hand, C centralizes $M_\sigma \cap M^*$. Since E_1 acts in a prime manner on M_σ by Theorem 7.5, we have $C = 1$, Corollary 7.2 with p and P replaced by r and R yields $\pi(E_1) \subseteq \tau_1(M^*)$ from (b) and $r \in \sigma(M^*)$ from (c). Thus, E_1 is contained in a Hall $\tau_1(M^*)$-subgroup $(E^*)_1$ of M^* and $1 \neq P \subseteq C_{E^*_\sigma}(R)$ where $R \subseteq (M^*)_\sigma$. Since $\tau_1(M^*) \cap \varpi \neq \emptyset$, E^*_1 acts in a prime manner on $(M^*)_\sigma$ by Theorem 7.5. Therefore, E^*_1 centralizes R. Since $E_1 \subseteq E^*_1$, R centralizes E_1. It follows that $R \subseteq C_{E_3}(E)$ because $R \subseteq E_3$ and $E = E_1E_3$. Recall that E_3 is cyclic. However, $C_{E_3}(E) = 1$ by Lemma 6.1 (f). This proves Lemma 7.7. Q.E.D.

Lemma 7.8. *The following configuration is impossible*:

(1) $M, M^* \subseteq \mathcal{M}$ *and* M^* *is not conjugate to* M *in* G,
(2) $p \in \tau_1(M) \cap \tau_1(M^*)$ *and* $P \in \mathcal{E}^1(M \cap M^*)$,
(3) Q *and* Q^* *are* P*-invariant Sylow subgroups (possibly for different primes) of* $M \cap M^*$,
(4) $C_Q(P) = 1$ *and* $C_{Q^*}(P) = 1$, *and*
(5) $N_G(Q) \subseteq M^*$ *and* $N_G(Q^*) \subseteq M$.

Proof. Assume this configuration. It follows from (3) and (5) that Q is a nonidentity Sylow q-subgroup for some prime q different from p and Q^* is a Sylow subgroup of M^*. By Lemma 6.18 (b), we have $\alpha(M) = \beta(M)$, $M_\alpha \neq 1$, and $q \notin \alpha(M)$. Furthermore, by (a) of the same lemma, $C_{M_\alpha}(P) \neq 1$ and $C_{M_\alpha}(PQ) = 1$. Since $C_{M_\alpha}(P) \neq 1$ and $\alpha(M) \subseteq \varpi$, we have $p \in \varpi$ by Lemma A.

Proposition 1.6 [BG] yields that $Q = C_Q(P)[Q, P]$. By (4),

$$Q = [Q, P] \subseteq M' \cap (M^*)'.$$

Theorem 4.2 (d) shows that M'/M_α is nilpotent. It follows that $M_\alpha Q \lhd M$ and the Frattini argument yields $M = M_\alpha N_M(Q)$.

This implies that $N_M(Q)$ contains a Hall $\alpha(M)'$-subgroup K of M. Since $q \notin \alpha(M)$ and $p \in \tau_1(M)$, PQ is an $\alpha(M)'$-subgroup of $N_M(Q)$. We may choose K so that $PQ \subseteq K$. Note that we have

$$M = M_\alpha K, M_\alpha \cap K = 1, \quad \text{and} \quad PQ \subseteq K \subseteq N_M(Q).$$

We claim that $C_M(P) = C_{M_\alpha}(P)C_K(P)$. Take an element of $C_M(P)$ and write it xy with $x \in M_\alpha$ and $y \in K$. This is a unique expression of this sort. For any $z \in P$,

$$xy = z^{-1}(xy)z = (z^{-1}xz)(z^{-1}yz).$$

Since $z^{-1}xz \in M_\alpha \lhd M$ and $z^{-1}yz \in K$, we have $z^{-1}xz = x$ and $z^{-1}yz = y$. This proves $C_M(P) \subseteq C_{M_\alpha}(P)C_K(P)$. The reverse containment is obvious. This proves the claim.

Let H be a Hall $(\beta(M) \cup \beta(M^*))$-subgroup of $C_G(P)$. Recall that $p \in \varpi$, so $C_G(P)$ is contained in a ϖ-local subgroup and it is solvable. Take any $s \in \pi(F(H))$ and $t \in \pi F(C_{M_\beta}(P)))$. By symmetry between M and M^*, we may fix notation and can assume $s \in \beta(M)$. We may choose H so that $C_{M_\beta}(P) \subseteq H$. Let $X = O_s(H)$ and $Y = O_t(C_{M_\beta}(P))$. We will show that $H \subseteq M$.

Since $s \in \beta(M)$, M contains a Sylow s-subgroup of G. Hence, some conjugate M^g with $g \in G$ contains X. By Proposition 4.14 (d), applied to M^g and X, we have $M^g \supseteq N_G(X) \supseteq H \supseteq Y$.

The same argument applied to M and Y yields $M \supseteq N_G(Y) \supseteq C_G(Y)$. Since $Y \subseteq M \cap M^g$, it follows from Theorem 4.1 (b) that $M^g = M^h$ for some element $h \in C_G(Y) \subseteq M$. Thus, $M = M^g \supseteq H$.

Take $r \in \beta(M^*) \cap \pi(H)$. By Lemma 4.12 (a), $r \notin \sigma(M)$. Note that M^* is not conjugate to M by (1). Moreover, since $H \subseteq M$, $r \in \pi(C_M(P))$. Since $C_M(P) = C_{M_\alpha}(P)C_K(P)$, $K \subseteq N_M(Q)$, and $r \notin \alpha(M) \subseteq \sigma(M)$, we have $r \in \pi(C_K(P))$. Therefore, there is a subgroup $R \in \mathcal{E}_r^1(N_M(Q) \cap C_G(P))$. Then, $R \subseteq N_G(Q) \subseteq M^*$ and $r \in \beta(M^*)$. Proposition 4.14 (d) applied to $R \subseteq M^*$ yields $N_G(R) \subseteq M^*$.

The subgroup $PR = P \times R$ is a $\sigma(M)'$-subgroup of M. Hence, PR is conjugate to a subgroup of E in M. Since $p \in \varpi$, we can apply Theorem 7.4 to obtain

$$1 \neq X \subseteq C_{M_\sigma}(P) \subseteq C_{M_\sigma}(R) \subseteq M^*.$$

We claim that $[X, Q] = 1$. We have $X \subseteq M_\alpha \cap M^*$ and $M_\alpha \cap M^*$ is a Q-invariant q'-subgroup because $q \notin \alpha(M)$. Therefore, $[X, Q]$ is a q'-group. We have $Q \subseteq (M^*)'$ and $(M^*)'/(M^*)_\alpha$ is nilpotent by Theorem 4.2 (d). Hence, $(M^*)_\alpha Q \lhd M^*$.

It follows that $[X, Q] \subseteq [X, (M^*)_\alpha Q] \subseteq (M^*)_\alpha Q$. Since $[X, Q]$ is a q'-group, we have $[X, Q] \subseteq (M^*)_\alpha$. On the other hand, $X \subseteq M_\beta$ because $s \in \beta(M)$. Therefore, $[X, Q] \subseteq [M_\beta, Q] \subseteq M_\beta \subseteq M_\alpha$. Lemma 4.12 yields $M_\alpha \cap (M^*)_\alpha = 1$. Thus, we have $[X, Q] = 1$.

Since $X \subseteq H \subseteq C_{M_\alpha}(P)$, we have $1 \neq X \subseteq C_{M_\alpha}(PQ)$. This contradicts the fact that $C_{M_\alpha}(PQ) = 1$. Q.E.D.

Theorem 7.9. *Suppose M, $M^* \in \mathcal{M}$ and M^* is not conjugate to M in G. Then, $\sigma(M)$ is disjoint from $\sigma(M^*)$.*

Theorem 7.10. *Suppose that some $P \in \mathcal{E}^1(E)$ does not centralize E_3. Then, $\tau_1(M) \subseteq \varpi$ and the following hold.*

(a) *E_1 acts regularly on E_3.*

(b) *E_3 acts regularly on M_{σ_0}.*
(c) $C_{M_{\sigma_0}}(P) \neq 1$.

Proof. We remark that the assumption implies $E_3 \neq 1$. Suppose $\tau_1(M) \cap \varpi = \emptyset$. Then, by Lemma G, M is a Frobenius group. The Frobenius kernel is either M' or M_α. In the first case, we have $M' = M_\sigma$ and $E_3 = 1$. On the other hand, if the Frobenius kernel is M_α, the group E is a subgroup of a Frobenius complement. Hence, by the structure of a Frobenius complement, every subgroup of prime order in E is normal in E. In particular, P centralizes E_3. Thus, we have $\tau_1(M) \cap \varpi \neq \emptyset$. In this case, we have $\tau_1(M) \subseteq \varpi$ because E_1 is cyclic.

Suppose that $\tau_1(M) \subseteq \varpi$ but M is not a ϖ-group. Then, Lemma G yields that M/M' is a ϖ-group, M'/M_α is a ϖ'-group and M_α is a ϖ-group. Since $E_3 \subseteq E' \subseteq M'$ by Lemma 6.1 (b), E_3 is a ϖ'-group. Since E_1 is a ϖ-group ($\tau_1(M) \subseteq \varpi$), we have (a).

Lemma G yields $M_\alpha = M_{\sigma_0}$. Hence, M_{σ_0} is a ϖ-group and we have (b). The Frobenius group PE_3 acts on M_α with $C_{M_\alpha}(E_3) = 1$. Theorem 3.10 [BG] yields that $C_{M_\alpha}(P) \neq 1$. This proves (c).

If M is a ϖ-group, the proof of Theorem 13.10 [BG] shows the validity of (a), (b) and (c). Q.E.D.

Corollary 7.11. *Suppose $E_3 \neq 1$ and E_3 does not act regularly on M_{σ_0}. Then, M is a ϖ-group with $\tau_2(M) = \emptyset$. We have* (a) $E_1 \neq 1$, (b) $E = E_1E_3$, (c) E acts in a prime manner on M_σ, and (d) every $X \in \mathcal{E}^1(E)$ is normal in E.

Proof. If $\tau_2(M) \neq \emptyset$, Corollary 6.6 (d) yields that E_3 acts regularly on M_σ. This is false, so we have $\tau_2(M) = \emptyset$. Lemma 6.1 yields (a) and (b). It follows from Theorem 7.10 (b) that every $P \in \mathcal{E}^1(E_1)$ centralizes E_3. This implies (d) because $E = E_1E_3$ and E_1 is cyclic.

By assumption some nonidentity element of E_3 centralizes a ϖ-subgroup. Therefore, $\tau_3(M) \cap \varpi \neq \emptyset$ by Lemma A. By Lemma 7.7 (2), M is a ϖ-group and (c) holds. Q.E.D.

Lemma 7.12. *Suppose $p \in \tau_1(M)$, $P \in \mathcal{E}^1(E)$, $q \in \tau_2(M)$, $A \in \mathcal{E}^2_p(E)$, and $C_A(P) \neq 1$. Then, $C_{M_\sigma}(P) = 1$.*

Proof. Since $\tau_2(M) \neq \emptyset$, M is a ϖ-group by Lemma H. The proof of Lemma 13.12 [BG] may be adapted to this case. Q.E.D.

Lemma 7.13. *Suppose that $p \in \tau_1(M) \cup \tau_3(M)$, $P \in \mathcal{E}^1(E)$, and $C_{M_\sigma}(P) \neq 1$. Then, for every $M^* \in \mathcal{M}(N_G(P))$, we have $p \in \sigma(M^*)$.*

Proof. Once $p \in \tau_2(M^*)$ is assumed, M^* is a ϖ-group by Lemma H. The proof of Lemma 13.13 [BG] works. Q.E.D.

§8. Subgroups of Type $\mathcal{P}$ and Counting Arguments Prime Action

Warning. We will use the notation of [BG] with *one major change.* Let $\kappa(M)$ be the set of primes $p \in \tau_1(M) \cup \tau_3(M)$ such that

$$C_{M_{\sigma_0}}(P) \neq 1 \quad \text{for some} \quad P \in \mathcal{E}_p^1(M).$$

This definition makes $\kappa(M) \subseteq \varpi$. Since we never use the set defined to be $\kappa(M)$ in [BG], we use the same notation for a different meaning. We divide the set $\mathcal{M}$ into three parts $\mathcal{M}_{\mathcal{F}}$, $\mathcal{M}_{\mathcal{P}_1}$, and $\mathcal{M}_{\mathcal{P}_2}$ just as in [BG]. However, the set $\kappa(M)$ is used in the sense defined above.

The notion of σ*-decomposition* and of σ*-length* of an element must also be modified: we replace $\sigma(M)$ used in their definitions in [BG] by $\sigma_0(M)$. For example, we define

$$\mathcal{M}_\sigma(g) = \{M \in \mathcal{M} \mid g \in M_{\sigma_0}\}.$$

However, we use the same notation as that of [BG]. Note that our definition coincides with theirs if g is a ϖ-element. As in [BG], we have $\ell_\sigma(g) = 1$ for a ϖ-element $g \in G$ if and only if $\mathcal{M}_\sigma(g)$ is not empty.

Lemma 8.1. *Suppose that $M \in \mathcal{M} \setminus \mathcal{M}_{\mathcal{P}_1}$. Take any $p \in \pi(M) \setminus \{\sigma(M), \kappa(M)\}$, let $S \in Syl_p(M)$ and let $A = \Omega_1(S)$. Then, $|A| \leq p^2$, $C_{M_{\sigma_0}}(A) = 1$, and M_{σ_0} is nilpotent.*

Proof. We have $\pi(M) \setminus \sigma(M) = \tau_1(M) \cup \tau_2(M) \cup \tau_3(M)$. If $p \in \tau_2(M)$, M is a ϖ-group by Lemma H. Lemma 8.1 follows from (b), (d) and (a) of Theorem 6.5.

If $p \in \tau_1(M) \cup \tau_3(M)$, we have $r_p(M) \leq 1$ so $|A| = p$. Since $p \notin \kappa(M)$, $C_{M_{\sigma_0}}(A) = 1$ and this implies that M_{σ_0} is nilpotent by Thompson's Theorem 3.7 [BG]. Q.E.D.

Proposition 8.2. *Suppose $M \in \mathcal{M}_{\mathcal{P}}$. Let K be a Hall $\kappa(M)$-subgroup of M and define $K^* = C_{M_\sigma}(K)$. Then, $K^* \subseteq M_{\sigma_0}$ and the following hold.*

(a) *The group K acts in a prime manner on M_σ, and acts regularly on some abelian Hall $(\kappa(M) \cup \sigma_0(M))'$-subgroup U of M.*
(b) *For every $X \in \mathcal{E}^1(K)$,*
    (1) *$N_M(X) = N_M(K) = K \times K^*$, and*
    (2) *$X \subseteq (M^*)_\sigma$ for each $M^* \in \mathcal{M}(N_G(X))$. In particular, we have $N_G(X) \not\subseteq M$.*
(c) *$K^* \neq 1$ and every $X \in \mathcal{E}^1(K^*)$ satisfies $\mathcal{M}(C_G(X)) = \{M\}$.*

(d) *Every* $g \in G \setminus M$ *satisfies* $K^* \cap M^g = 1$ *and every* $g \in M \setminus (K \times K^*)$ *satisfies* $K \cap K^g = 1$.

(e) *For every prime* $p \in \pi(K^*)$ *and every* $S \in Syl_p(M_{\sigma_0})$,

$$\mathcal{M}(S) = \{M\} \quad \textit{and} \quad S \nsubseteq K^*.$$

(f) *Every* $\sigma_0(M)$*-subgroup* Y *of* G *satisfying* $Y \cap K^* \neq 1$ *lies in* M_{σ_0}.

(g) *If* $M \in \mathcal{M}_{\mathcal{P}_2}$, *then* $\sigma_0(M) = \beta(M)$, K *has prime order, and* M_{σ_0} *is a nilpotent TI-subgroup of* G.

Proof. Although the proof of Proposition 14.2 [BG] is applicable, we include some details.

We prove (a) and (b1). Take a complement E of M_σ that contains K. Suppose that

$$\kappa(M) \cap \tau_3(M) \neq \emptyset.$$

Then, $E_3 \neq 1$ and E_3 does not act regularly on M_{σ_0}. By Corollary 7.11, M is a ϖ-group, $E_1 \neq 1$, $E = E_1E_3$, E acts in a prime manner on M_σ, and every $X \in \mathcal{E}^1(E)$ is normal in E. Since $E = E_1E_3$ acts in a prime manner on M_σ, we have $\kappa(M) = \pi(E)$. Therefore, $K = E$ and K acts in a prime manner on M_σ. In this case, $\pi(M) = \sigma(M) \cup \kappa(M)$. So, $U = 1$ satisfies (a). If $X \in \mathcal{E}^1(K)$, we have $X \lhd E$. It follows from $M = M_\sigma E$ that

$$N_M(X) = N_{M_\sigma}(X)E = C_{M_\sigma}(X)E.$$

Since K acts in a prime manner on M_σ, we have $C_{M_\sigma}(X) = C_{M_\sigma}(K) = K^*$. Thus, $N_M(X) = K \times K^*$. Therefore, (a) and (b) hold in the case $\kappa(M) \cap \tau_3(M) \neq \emptyset$.

Suppose that $\kappa(M) \cap \tau_3(M) = \emptyset$. Then, $\kappa(M) \subseteq \tau_1(M)$ and $\varpi \cap \tau_1(M) \neq \emptyset$. Theorem 7.5 shows that E_1 acts in a prime manner on M_σ. Thus, $\kappa(M) = \tau_1(M)$ and we may choose $K = E_1$. To prove (a), we need to find U. Suppose that M is not a ϖ-group. Since $\tau_1(M) = \kappa(M) \subseteq \varpi$, M is a group of type (2) in Lemma G. Then, we have

$$\pi(M) \setminus \{\kappa(M), \sigma_0(M)\} = \pi(M) \cap \varpi'.$$

There is an E_1-invariant complement U of M_α in M'. Since $U \cong M'/M_\beta$ is cyclic, U satisfies (a).

Assume that M is a ϖ-group. Then, $\sigma_0(M) = \sigma(M)$ and

$$\pi(M) \setminus \{\kappa(M), \sigma_0(M)\} = \tau_2(M) \cup \tau_3(M).$$

We will show that $U = E_2E_3$ satisfies (a). Since $K = E_1$, U is K-invariant. Assume $E_2 \neq 1$. If E_1 does not act regularly on E_2, some

$P \in \mathcal{E}^1(E_1)$ satisfies $C_A(P) \neq 1$ for some $A \in \mathcal{E}^2(E_2)$. Lemma 7.12 yields $C_{M_\sigma}(P) = 1$ contrary to the fact that $K = E_1$ acts in a prime manner on M_σ. Thus, E_1 acts regularly on E_2. If E_1 does not act regularly on E_3, some $P \in \mathcal{E}^1(E_1)$ centralizes some $R \in \mathcal{E}^1(E_3)$. Since M is assumed to be a ϖ-group, Theorem 7.4 yields that

$$1 \neq C_{M_\sigma}(P) \subseteq C_{M_\sigma}(R).$$

This would imply $\tau_3(M) \cap \kappa(M) \neq \emptyset$ in contradiction to the hypothesis of this case. Thus, $E_1 \neq 1$ acts regularly on E_2E_3. It follows from Theorem 3.7 [BG] that E_2E_3 is nilpotent. By Corollary 6.10 (a), E_2E_3 is abelian. This proves (a).

It follows from the structure of the group M discussed in the proof of (a) that every $X \in \mathcal{E}^1(K)$ is normal in K, M is the semidirect product of M_{σ_0} and UK, and $N_{UK}(X) = K$. We have

$$N_M(X) = N_{M_{\sigma_0}}(X)K = C_{M_{\sigma_0}}(X)K = C_{M_{\sigma_0}}(K)K = K^* \times K.$$

This proves (b1).

Lemma 7.13 yields the first part of (b2). We have $M \notin \mathcal{M}(N_G(X))$, since $X \subseteq E_1$. This proves (b2).

The parts (c), (d), (e) and (f) are proved as in the proof of Proposition 14.2 [BG]. For (f), recall that M_{σ_0} is a normal Hall subgroup of M_σ. Hence, M_{σ_0} contains all $\sigma_0(M)$-subgroup of M_σ.

For the proof of (g), suppose that $U \neq 1$. Then, (a) implies that KU is a Frobenius group with Frobenius kernel U. Suppose that M is not a ϖ-group. Then, by Lemma G, U is a ϖ'-group, so $M_{\sigma_0}U$ is a Frobenius group with Frobenius kernel M_{σ_0}. Hence, $C_{M_\sigma}(U) = 1$ and M_{σ_0} is nilpotent. Thus, the nonidentity Frobenius group KU acts on a nilpotent group M_{σ_0} and K acts in a prime manner on M_{σ_0}. It follows from Theorem 3.10 [BG] that K has prime order. By Lemma G, we have $M_\beta = M_{\sigma_0}$. Lemma 6.17 shows that for every $g \in G \setminus M$, the group $M_{\sigma_0} \cap M^g$ is a $\beta(M)'$-group. Since $M_{\sigma_0} = M_\beta$, $M_{\sigma_0} \cap M^g$ is a $\beta(M)$-group. This proves that $M_{\sigma_0} \cap M^g = 1$ for every $g \in G \setminus M$. Thus, M_{σ_0} is a TI-subgroup of G.

Suppose finally that M is a ϖ-group In this case, we have $U = E_2E_3$. Lemma 8.1 shows that $C_{M_\sigma}(U) = 1$ and M_σ is nilpotent. Since K acts in a prime manner on M_σ by (a), Lemma 3.10 [BG] yields that K has prime order. We have $U = [U, K] \subseteq E'$. By Lemma 6.19, U centralizes a Hall $\beta(M)'$-subgroup of M_σ. Since $C_{M_\sigma}(U) = 1$, a Hall $\beta(M)'$-subgroup equals 1. Therefore, $M_\beta = M_\sigma$ and $\beta(M) = \sigma(M)$. Lemma 6.17 implies that $M_{\sigma_0} \cap M^g = 1$ for every $g \in G \setminus M$. This completes the proof of Proposition 8.2. Q.E.D.

Corollary 8.3. *Suppose $M \in \mathcal{M}$, $x \in M_{\sigma_0}{}^{\sharp}$, and x' is a nonidentity $\sigma(M)'$-element of $C_M(x)$. Then, either*

(1) $\pi(\langle x' \rangle) \subseteq \kappa(M)$ *and* $C_G(x) \subseteq M$, *or*
(2) $\pi(\langle x' \rangle) \subseteq \tau_2(M)$, $\ell_\sigma(x') = 1$, *and* $\mathcal{M}(C_G(x')) = \{M\}$.

Theorem 8.4. *Suppose that x is a ϖ-element of $G^\sharp$ such that $\mathcal{M}_\sigma(x)$ is not empty. Then, $C_G(x)$ has a normal Hall subgroup $R(x)$ that acts sharply transitively on $\mathcal{M}_\sigma(x)$ by conjugation. Furthermore, if $|\mathcal{M}_\sigma(x)| > 1$, then $C_G(x)$ lies in a unique subgroup $N = N(x) \in \mathcal{M}$ and for every $M \in \mathcal{M}_\sigma(x)$,*

(a) $R(x) = C_{N_\sigma}(x) \neq 1$,
(b) $C_G(x) = C_M(x)R(x)$,
(c) $\pi(\langle x \rangle) \subseteq \tau_2(N) \subseteq \sigma_0(M)$,
(d) $\pi(M) \cap \sigma(N) \subseteq \beta(N)$,
(e) $M \cap N$ *is a complement of* N_σ *in* N, *and*
(f) N *is a* ϖ*-group in* $\mathcal{M}_{\mathcal{F}} \cup \mathcal{M}_{\mathcal{P}_2}$.

Proof. The proof of Theorem 14.4 [BG] may be modified with some changes to yield this theorem. We will present the details here. If $|\mathcal{M}_\sigma(x)| = 1$, we can let $R(x) = 1$ and finish the proof. So, we will assume $|\mathcal{M}_\sigma(x)| > 1$ in the remainder of proof.

Since x is a ϖ-element with $\mathcal{M}_\sigma(x) \neq \emptyset$, we can take $M \in \mathcal{M}_\sigma(x)$, $q \in \pi(\langle x \rangle)$, $X \in \mathcal{E}_q^1(\langle x \rangle)$, and $N \in \mathcal{M}(N_G(X))$. Note that $\mathcal{M}_\sigma(x) \subseteq \mathcal{M}_\sigma(X)$ and

$$C_G(x) \subseteq N_G(\langle x \rangle) \subseteq N_G(X) \subseteq N.$$

We will show that $\mathcal{M}_\sigma(X)$ consists of conjugates of M and that $C_G(X)$ acts transitively on $\mathcal{M}_\sigma(X)$ by conjugation. Let $L \in \mathcal{M}_\sigma(X)$. Then, $X \subseteq M_\sigma \cap L_\sigma$. Theorem 7.9 yields that L is conjugate to M. Since $q \in \sigma(M)$ and X is a q-group, Theorem 4.1 (b) yields that $C_G(X)$ acts transitively on $\mathcal{M}_\sigma(X)$. In particular, $C_G(X) \nsubseteq M$ and $N \neq M$.

Since $N \neq M$, Proposition 6.15 (a) applies to N and yields that N is not conjugate to M. Then, by Theorem 7.9, $\sigma(N)$ is disjoint from $\sigma(M)$. It follows that $q \notin \sigma(N)$. Proposition 6.15 (e) now yields that $q \in \tau_2(N)$ and the conditions (d) and (e) of this theorem hold. Since $q \in \tau_2(N)$, $\tau_2(N)$ is not empty. Therefore, $N \notin \mathcal{M}_{\mathcal{P}_1}$ and N is a ϖ-group by Lemma H. This proves (f).

We will prove that $R(x)$ acts sharply transitively on $\mathcal{M}_\sigma(x)$. We have shown that if $L \in \mathcal{M}_\sigma(x)$, then $L = M^u$ with $u \in C_G(X) \subseteq N$. Since $N = (M \cap N)N_\sigma$ by (e), we may choose $u \in N_\sigma$. Then,

$$(x^{-1}ux)^{-1}M(x^{-1}ux) = M^{ux} = L^x = L = M^u.$$

However, if $M^u = M^v$ for $u, v \in N_\sigma$, then $uv^{-1} \in N_G(M) \cap N_\sigma$. Since $N_G(M) = M$ by Lemma E, we have $uv^{-1} \in M \cap N_\sigma = 1$ by (e). We apply this twice. First, the displayed formula yields that if $L = M^u$ with $u \in N_\sigma$, then $u \in R(x)$. Thus, $R(x)$ acts transitively on $\mathcal{M}_\sigma(x)$. Secondly, $M^u = M$ with $u \in R(x)$ implies $u = 1$. Thus, $R(x)$ is sharply transitive. Since $|\mathcal{M}_\sigma(x)| > 1$, we have (a). Since $R(x) \subseteq C_G(x)$ and $R(x)$ is transitive on $\mathcal{M}_\sigma(x)$, we have

$$C_G(x) = (C_G(x) \cap N_G(M))R(x) = C_M(x)R(x).$$

This proves (b).

We prove next $\mathcal{M}(C_G(x)) = \{N\}$. Since $R(x) \neq 1$, there is an element $y \in {N_{\sigma_0}}^\sharp$ such that $y \in C_G(x)$. Apply Corollary 8.3 to (N, y, x) in place of (M, x, x'). Since x is a $\sigma(M)$-element, it is a $\sigma(N)'$-element. Since $q \in \pi(\langle x \rangle) \cap \tau_2(N)$, we have the second case of Corollary 8.3. Thus, $\pi(\langle x \rangle) \subseteq \tau_2(N)$ and $\mathcal{M}(C_G(x)) = \{N\}$.

It remains to prove (c). We have just proved $\pi(\langle x \rangle) \subseteq \tau_2(N)$. Take $p \in \tau_2(N)$. By (e) and Corollary 6.6 (a), there is $A \in \mathcal{E}_p^2(M \cap N)$ such that $A \lhd M \cap N$. Then, $x \in N_{M_\sigma}(A)$. Since $r_p(M) \geq 2$, we have $p \in \sigma(M) \cup \tau_2(M)$. If $p \in \tau_2(M)$, $N_{M_\sigma}(A) = C_{M_\sigma}(A) = 1$ by Corollary 6.5 (d). This contradiction proves $p \in \sigma(M)$. In fact, $p \in \sigma_0(M)$ because N is a ϖ- group by (f). Q.E.D.

We will use the notation $\widetilde{M}$ to mean

$$\{xx' \mid x \in {M_{\sigma_0}}^\sharp \quad \text{and} \quad x' \in R(x)\}.$$

This is slightly different from the usage in [BG].

Lemma 8.5. *The following hold.*

(a) *If x and y are distinct ϖ-elements of $G^\sharp$ of σ-length one, then $xR(x) \cap yR(y) = \emptyset$.*

(b) *If M_1 and M_2 are elements of $\mathcal{M}$ not conjugate in G, then $\widetilde{M}_1 \cap \widetilde{M}_2 = \emptyset$.*

(c) *If $M \in \mathcal{M}$, then $|\mathcal{C}_G(\widetilde{M})| = (|M_{\sigma_0}| - 1)|G : M|$.*

Proof. (a) Suppose that $g = xx'$ with $\ell_\sigma(x) = 1$ and $x' \in R(x)$ lies in $yR(y)$ and $x \neq y$. Write $g = yy'$ with $y' \in R(y)$. Since y is a σ-factor of the element g, we have $y = x'$, so $x' \neq 1$. Therefore, $|\mathcal{M}_\sigma(x)| > 1$. Take $M \in \mathcal{M}(C_G(y))$. Then, $y' = x \in M_\sigma$ and $M \in \mathcal{M}_\sigma(x)$. Take $N \in \mathcal{M}(C_G(x))$. Then, $x' = y \in N_\sigma \cap M$ which is 1 by Theorem 8.4 (e). This contradicts $y \neq 1$.

The parts (b) and (c) follow as in the proof of Lemma 14.4 [BG].
Q.E.D.

Lemma 8.6. *Each nonidentity ϖ-element g satisfies exactly one of the following two conditions*:

(1) $g = xx'$ *with* $\ell_\sigma(x) = 1$ *and* $x' \in R(x)$, *or*
(2) $g = yy'$ *with* $\ell_\sigma(y) = 1$ *and* y' *is a nonidentity* $\kappa(M)$*-element of* $C_M(y)$ *for some* $M \in \mathcal{M}_\sigma(y)$.

Proof. Suppose that both (1) and (2) hold for some ϖ-element $g \neq 1$. We will derive a contradiction. Take $N \in \mathcal{M}(C_G(x))$ and $L \in \mathcal{M}_\sigma(x)$. Since y is a σ-factor of g, we have $y = x$ or $y = x'$. Suppose $y = x$. We may choose $L = M$. Since $y = x$, we have $x' = y' \neq 1$. By Theorem 8.4, $|\mathcal{M}_\sigma(x)| = |R(x)| > 1$ and. by Part (e),

$$x' = y' \in N_\sigma \cap M = 1.$$

This contradicts $y' \neq 1$. Suppose next $y = x'$. Then, we have $y' = x$ and it is a $\kappa(M)$-element and at the same time a $\tau_2(N)$-element by Theorem 8.4 (c). Since $1 \neq y = x' \in M_\sigma \cap N_\sigma$, N is conjugate to M by Theorem 7.9. Therefore, we have $\tau_2(N) = \tau_2(M)$ Since $\kappa(M) \cap \tau_2(M) = \emptyset$, we have a contradiction $y' = 1$.

We will prove that either (1) or (2) holds for every g. Suppose that no decomposition of type (1) or (2) is possible and we will derive a contradiction. We have $\ell_\sigma(g) > 1$ since the choice of $x = g$ and $x' = 1$ provides (1) if $\ell_\sigma(g) = 1$. Let x be a σ-factor of g with $\ell_\sigma(x) = 1$, and write $g = xx'$. We prove a lemma: *under the hypothesis of this paragraph, no subgroup* $M \in \mathcal{M}_\sigma(x)$ *contains* g.

Suppose $g \in M$. Then, $x' \in M$ and $x' \neq 1$ because $\ell_\sigma(g) > 1$. Since x is a σ-factor of the element g, x' is a $\sigma(M)'$-element but not a $\kappa(M)$-element because $g = xx'$ does not satisfy (2). Therefore, we must have the case (2) of Corollary 8.3. Thus, we have $\ell_\sigma(x') = 1$ and $\mathcal{M}(C_G(x')) = \{M\}$. It follows that

$$x \in M_\sigma \cap C_G(x') = R(x').$$

This implies that $g = xx'$ is a decomposition of type (1) with (x', x) in place of (x, x'). This is a contradiction and proves the lemma.

Let x be a σ-factor of the element g with $\ell_\sigma(x) = 1$ and write $g = xx'$. Then, x is a power of g. Take $M \in \mathcal{M}_\sigma(x)$ and $N \in \mathcal{M}(C_G(x))$. Then, $g \in C_G(x) \subseteq N$. By the lemma, none of the σ-factor of g of σ-length one lies in N_σ. It follows that g is a $\sigma(N)'$-element of N. We have $x = x^g \in M \cap M^g$ and $g \notin M$ (by the lemma). Thus, $|\mathcal{M}_\sigma(x)| > 1$ and, by Theorem 8.4 (e), $M \cap N$ is a complement of N_σ in N. Since g is a $\sigma(N)'$-element, $g \in (M \cap N)^u$ for some element $u \in N$. Thus, $g \in M^u$. Since x is a power of g, we have $x \in M^u$. However, x is a $\sigma(M)$-element.

Since $\sigma(M) = \sigma(M^u)$, we have $x \in M_\sigma{}^u$. This contradicts the Lemma. Q.E.D.

Theorem 8.7. *Suppose $M \in \mathcal{M}_\mathcal{P}$ and K is a Hall $\kappa(M)$-subgroup of M. Let $K^* = C_{M_\sigma}(K)$, $k = |K|$, $k^* = |K^*|$, $Z = K \times K^*$, and $\widehat{Z} = Z \setminus (K \cup K^*)$. Then, for some other $M^* \in \mathcal{M}_\mathcal{P}$ that is not conjugate to M, we have*

(a) *$\mathcal{M}(C_G(X)) = \{M^*\}$ for every $X \in \mathcal{E}^1(K)$,*
(b) *K^* is a Hall $\kappa(M^*)$-subgroup of M^* and a Hall $\sigma_0(M)$-subgroup of M^*,*
(c) *$K = C_{M^*_\sigma}(K^*)$ and $\kappa(M) = \tau_1(M)$,*
(d) *Z is cyclic and for every $x \in K^\sharp$ and $y \in (K^*)^\sharp$,*

$$M \cap M^* = Z = C_M(x) = C_{M^*}(y) = C_G(xy),$$

(e) *$\widehat{Z}$ is a TI-subset of G with $N_G(\widehat{Z}) = Z$, $\widehat{Z} \cap M^g$ empty for all $g \in G \setminus M$, and*

$$|\mathcal{C}_G(\widehat{Z})| = (1 - \frac{1}{k} - \frac{1}{k^*} + \frac{1}{kk^*})|G| > \frac{1}{2}|G|,$$

(f) *M or M^* lies in $\mathcal{M}_{\mathcal{P}_2}$ and, accordingly, K or K^* has prime order,*
(g) *every $H \in \mathcal{M}_\mathcal{P}$ is conjugate to M or M^* in G, and*
(h) *M' is a complement of K in M and $M' = M_{\sigma_0}U$ where U is the subgroup defined in Proposition 8.2 (a).*

Proof. Although the proof of Theorem 14.7 [BG] is adequate to cover this theorem, we will paraphrase their proof of this miraculous theorem. By the hypothesis, $M \in \mathcal{M}_\mathcal{P}$. Thus, $\kappa(M)$ is not empty and $K \neq 1$.

We begin the proof with the following lemma which is not really necessary. *If $X \in \mathcal{E}^1(K)$, then $N_G(X) \in \mathcal{U}$.* Suppose $H, L \in \mathcal{M}(N_G(X))$. By Proposition 8.2 (b2), $X \subseteq H_\sigma \cap L_\sigma$. It follows from Theorem 7.9 that $L = H^g$ for some $g \in G$. Since $C_G(X) \subseteq N_G(X) \subseteq H$, Theorem 4.1 (e) with H in place of M yields $L = H^g = H$. This completes the proof of the lemma.

Let $M_1, M_2, \ldots, M_n$ be the distinct subgroups in $\mathcal{M}$ that contain $N_G(X)$ for some $X \in \mathcal{E}^1(K)$. For each i, take $X_i \in \mathcal{E}^1(K)$ such that $M_i \in \mathcal{M}(N_G(X_i))$. By Proposition 8.2 (b), we have $\pi(X_i) \subseteq \sigma_0(M_i)$ and

$$Z = K \times K^* \subseteq N_G(X_i) \subseteq M_i.$$

Since $\pi(X_i) \subseteq \pi(K) = \kappa(M) \subseteq \sigma(M)'$, none of M_i is conjugate to M in G. Therefore, by Theorem 7.9, $\sigma(M)$ is disjoint from $\sigma(M_i)$. Thus, K^* is a $\sigma(M_i)'$-subgroup of M_i.

Take $X^* \in \mathcal{E}^1(K^*)$. By Proposition 8.2 (c), $\mathcal{M}(C_G(X^*)) = \{M\}$. Apply Corollary 8.3 to $M_i \in \mathcal{M}$, $x \in X_i{}^\sharp$, and $x' \in X^{*\sharp}$. All the assumptions of Corollary 8.3 are satisfied. Since $\mathcal{M}(C_G(X^*)) \neq \{M_i\}$, we have the first case: $\pi(X^*) \subseteq \kappa(M_i)$. We can take X^* arbitrary in $\mathcal{E}^1(K^*)$, so $\pi(K^*) \subseteq \kappa(M_i)$.

Let K_i be a Hall $\kappa(M_i)$-subgroup of M_i that contains X^*, and define $K_i^* = C_{M_{i\sigma}}(K_i)$. Recall that K_i is a Z-group and that, by Proposition 8.2 (b1) for M_i, every subgroup in $\mathcal{E}^1(K_i)$ is normal in K_i. We claim that $K^* \subseteq K_i$. This is proved as follows. Since K^* is a $\kappa(M_i)$-subgroup of a solvable group M_i, $K^* \subseteq (K_i)^g$ for some $g \in M_i$. Then, X^* and X^{*g} are normal subgroups of the same prime order in the Z-group $(K_i)^g$. Hence, we have $X^* = (X^*)^g$. Thus, $g \in N_{M_i}(X^*) = N_{M_i}(K_i)$ by Proposition 8.2 (b1) for M_i. This implies $K^* \subseteq (K_i)^g = K_i$.

Since $X^* \subseteq K^*$ and $K \subseteq M_i$, we have

$$K \subseteq C_{M_i}(X^*) \subseteq N_{M_i}(X^*) = K_i \times K_i^*.$$

Therefore, $K \times K^* \subseteq K_i \times K_i^*$. Similarly, with M_i, K_i, X^*; M, K, and X_i in place of M, K, X_i; M_i, K_i, and X^*, we have $K_i \times K_i^* \subseteq K \times K^*$. We need to check a few relations: $X^* \subseteq K_i$, $M \in \mathcal{M}(N_G(X^*))$ and

$$X_i \subseteq K \quad \text{where} \quad X_i \in \mathcal{E}^1(K_i^*).$$

We check the last one. We have $X_i \subseteq K \subseteq K_i \times K_i^*$, $\pi(X_i) \subseteq \sigma_0(M_i)$, and K_i^* is a Hall $\sigma_0(M_i)$-subgroup of $K_i \times K_i^*$. Therefore, $X_i \subseteq \mathcal{E}^1(K_i^*)$. It follows that $K \times K^* = K_i \times K_i^*$ for each i. Let $M_0 = M$, $K_0 = K$, and $K_0^* = K^*$. Take $X_0^* \in \mathcal{E}^1(K^*)$. Then, by Proposition 8.2 (c), $\mathcal{M}(C_G(X_0^*)) = \{M_0\}$. For each $X_i^* \in \mathcal{E}^1(K_i^*)$ we have $\mathcal{M}(C_G(X_i^*)) = \{M_i\}$. It follows that $K_i^* \cap K_j^* = 1$ if $i \neq j$. Otherwise, we would have $\{M_i\} = \mathcal{M}(C_G(X)) = \{M_j\}$ for $X \in \mathcal{E}^1(K_i^* \cap K_j^*)$.

We claim that $Z = K_0^* \times K_1^* \times \cdots \times K_n^*$. Let Z_0 be the subgroup of Z generated by the subgroups K_i^*. For each i, we have $Z = K_i \times K_i^*$ where K_i^* is a $\sigma(M_i)$-group and K_i is a $\sigma(M_i)'$-group. Therefore, K_i^* is a normal Hall subgroup of Z. If $i \neq j$, we have shown $K_i^* \cap K_j^* = 1 (i \neq j)$. Then, K_i^* and K_j^* with $i \neq j$ have relatively prime orders and centralize each other. It follows that

$$Z_0 = K_0^* \times K_1^* \times \cdots \times K_n^*.$$

We will show that $Z = Z_0$. First, we prove that every $X \in \mathcal{E}^1(Z)$ is contained in some K_i^*. Since $Z = K_0 \times K_0^*$ and $(|K_0|, |K_0^*|) = 1$, either

$X \subseteq K_0^*$ or $X \subseteq K_0$. Suppose $X \subseteq K_0 = K$. It follows from the definition of the subgroups M_i that

$$\mathcal{M}(N_G(X)) = \{M_i\}$$

for some i (cf. the lemma at the beginning of the proof). By Proposition 8.2 (b1) for M_i, we have $X \subseteq (M_i)_\sigma$. Since K_i^* is the normal Hall $\sigma(M_i)$-subgroup of Z, we have $X \subseteq K_i^*$. Take any element $x \in Z$ of order p^e where p is a prime. Let $X \in \mathcal{E}^1(\langle x \rangle)$. Then, $X \subseteq K_i^*$ for some i. Hence, $p \in \pi(K_i^*)$ and $\langle x \rangle$ is a $\sigma(M_i)$-subgroup. This implies $x \in K_i^*$ as before.

Finally, let y be an arbitrary element of Z of order n. If $n = \prod {p_i}^{e_i}$ is the canonical decomposition of the integer n into the product of powers of distinct primes $p_1, \ldots, p_m$, we have $y = x_1 \ldots x_m$ where x_i is a power of the element y and the order of x_i is ${p_i}^{e_i}$. Then, $x_i \in Z$ so each x_i lies in Z_0 and we have $Z = Z_0$.

The subgroups K_i^* are distinct and each is a normal Hall subgroup of Z. It follows that the groups M_i are pairwise not conjugate in G. By Theorem 7.9, $\sigma(M_i)$ is disjoint from $\sigma(M_j)$ if $j \neq i$. Since K_j^* is a $\sigma(M_j)$-group, K_j^* is a $\sigma(M_i)'$-group for $j \neq i$. Therefore, we have $K_j^* \subseteq K_i$ for $j \neq i$, so if we let $W = \prod_{j \neq i} K_j^*$, $W \subseteq K_i$. The groups K_i and W are complements of K_i^* in Z. This implies $K_i = W$.

For every element $z \in Z$, the factorization $z = \prod z_i$ with $z_i \in K_i^*$ is the σ-decomposition of z.

Define $T = Z \setminus \{K_0^*, K_1^*, \ldots, K_n^*\}$. Note that $z \in Z$ is in T if and only if $z = yy'$ with $y \in K_i^{*\sharp}$ and $y' \in {K_i}^\sharp$ for some index i. In this case, y' is a nonidentity $\kappa(M_i)$-element of $C_{M_i}(y)$ with $\ell_\sigma(y) = 1$. Thus, we have the case (2) of Lemma 8.6. It follows that $T \cap \widetilde{H} = \emptyset$ for any $H \in \mathcal{M}$. Thus, $\mathcal{C}_G(T) \cap \mathcal{C}_G(\widetilde{M_i}) = \emptyset$ for each i. Since M_i are not conjugate to each other, Lemma 8.5 yields

$$\mathcal{C}_G(\widetilde{M_i}) \cap \mathcal{C}_G(\widetilde{M_j}) = \emptyset \quad \text{if} \quad i \neq j.$$

We will prove that T is a TI-subset of G with $N_G(T) = Z$. Suppose that $t \in T$, $g \in G$, and $t^g \in Z$. Write $t = yy' = y'y$ with $y \in K_i^{*\sharp}$ and $y' \in {K_i}^\sharp$ for some i. Then, y^g and $(y')^g$ are powers of t^g. Hence, $y^g \in K_i^* \cap (M_i)^g$. By Proposition 8.2 (d) for M_i, we have $g \in M_i$. Then, $y'^g \in K_i \cap (K_i)^g$. The same proposition yields that $g \in Z$. This proves that T is a TI-subset of G with $N_G(T) = Z$.

We count the number of elements in $\mathcal{C}_G(T)$. With $z = |Z|$, $k_i = |K_i|$, and $k_i^* = |K_i^*|$, we have

$$\begin{aligned}|\mathcal{C}_G(T)| &= |T||G : N_G(T)| \\ &= (z - 1 - \sum_{i=0}^{n}(k_i^* - 1))|G : Z| \\ &= (1 + \frac{n}{z} - \sum_{i=0}^{n}\frac{1}{k_i})|G|.\end{aligned}$$

Suppose that all the subgroups M_i lie in $\mathcal{P}_1$. Then $M_i = M_{i\sigma_0}K_i$ so $|M_i| = |M_{i\sigma_0}||K_i|$. By Lemma 8.5,

$$\begin{aligned}|\mathcal{C}_G(\widetilde{M_i})| &= (|M_{i\sigma_0}| - 1)|G : M_i| \\ &= (\frac{1}{k_i} - \frac{1}{|M_i|})|G| \geq (\frac{1}{k_i} - \frac{1}{2z})|G|.\end{aligned}$$

The last inequality comes from $Z \subsetneq M_i$. Since the sets $\mathcal{C}_G(T)$, $\mathcal{C}_G(\widetilde{M_0})$, $\ldots$, $\mathcal{C}_G(\widetilde{M_n})$ are pairwise disjoint,

$$\begin{aligned}|G^\sharp| &\geq |\mathcal{C}_G(T)| + \sum_{i=0}^{n}|\mathcal{C}_G(\widetilde{M_i})| \\ &\geq ((1 + \frac{n}{z} - \sum_{i=0}^{n}\frac{1}{k_i}) + \sum_{i=0}^{n}(\frac{1}{k_i} - \frac{1}{2z}))|G| \\ &> |G|\end{aligned}$$

and this contradiction proves that some M_i is of type $\mathcal{P}_2$.

If M_i is of type $\mathcal{P}_2$, Proposition 8.2 (g) yields that K_i is of prime order and $M_{i\sigma_0}$ is nilpotent. Therefore, $K_i = K_j^*$ for $j \neq i$ and we have $n = 1$.

Since $K_i^* \subseteq M_{i\sigma_0}$, K_i^* is nilpotent. Furthermore,

$$Z = K_i \times K_i^* = K_j \times K_j^*,$$

$K_j = K_i^*$, $K_i = K_j^*$ and $r(K_i^*) = 1$. It follows that the nilpotent group K_i is cyclic. Since K_i is of prime order, $Z = K_i \times K_i^*$ is cyclic. This proves the first statement of (d).

Since $n = 1$, we have $T = \widehat{Z}$. Suppose $g \in G \setminus M$ and $T \cap M^g \neq \emptyset$. Take $uv \in T \cap M^g$ with $u \in K^\sharp$ and $v \in K^{*\sharp}$. Then, any power of uv lies in M^g so in particular, $v \in K^{*\sharp} \cap M^g$. This contradicts Proposition

8.2 (d). We have

$$\begin{aligned}|\mathcal{C}_G(T)| &= (1 - \frac{1}{k} - \frac{1}{k^*} + \frac{1}{kk^*})|G| \\ &= (1 - \frac{1}{k})(1 - \frac{1}{k^*})|G| \geq \frac{8}{15}|G| > \frac{1}{2}|G|\end{aligned}$$

because k and k^* are odd integers ≥ 3 and $k \neq k^*$. This proves (e).

With $M^* = M_1$, we have proved (f). We will prove (g). Suppose that $H \in \mathcal{M}_{\mathcal{P}}$. Let L be a Hall $\kappa(H)$-subgroup of H, $L^* = C_{H_\sigma}(L)$, and $S = L \times L^* \setminus \{L, L^*\}$. We have $|\mathcal{C}_G(T)| > |G|/2$ and $|\mathcal{C}_G(S)| > |G|/2$. It follows that $\mathcal{C}_G(T) \cap \mathcal{C}_G(S) \neq \emptyset$.

Replacing M and H by conjugates, we may assume that $T \cap S$ is not empty. Then, $L^* \cap K_i^* \neq 1$ for some i. If $Y \in \mathcal{E}^1(L^* \cap K_i^*)$ then Proposition 8.2 (c) yields $\{H\} = \mathcal{M}(C_G(Y)) = \{M_i\}$. This proves (g).

We will prove (a)¿ If $X \in \mathcal{E}^1(K)$, then $X \in \mathcal{E}^1(K_1^*)$ because $K_1^* = K$. Proposition 8.2 (c) yields $\mathcal{M}(C_G(X)) = \{M_1\} = \{M^*\}$.

Since $K^* = K_1$, K^* is a Hall $\kappa(M^*)$-subgroup of M^*. This is the first statement of (b). Clearly, K^* is a $\sigma_0(M)$-subgroup of M^*. Let H be a Hall $\sigma_0(M)$-subgroup of M^* that contains K^*. The subgroup H is a $\sigma_0(M)$-subgroup such that $H \cap K^* \neq 1$. By Proposition 8.2 (f), we have $H \subseteq M_{\sigma_0}$. Hence, $[H, K] \subseteq [M_{\sigma_0}, K] \subseteq M_{\sigma_0}$. On the other hand, $H \subseteq M^*$ and $K = K_1^* \subseteq (M^*)_\sigma$. It follows that

$$[H, K] \subseteq [M^*, (M^*)_\sigma] \subseteq (M^*)_\sigma,$$

and $[H, K] \subseteq M_{\sigma_0} \cap (M^*)_\sigma$. But, M is not conjugate to M^*, so by Theorem 7.9, $[H, K] = 1$. Therefore, $H \subseteq C_{M_\sigma}(K) = K^*$. This proves $H = K^*$. Thus, (b) holds.

To prove (c) and (h), let U be the subgroup defined in Proposition 8.2 (a). Since K acts regularly on U, we have $U = [U, K] \subseteq M'$. Since $M_{\sigma_0} \subseteq M'$, $M_{\sigma_0}U \subseteq M'$. On the other hand, $M_{\sigma_0}U$ is a normal subgroup of M with $M/M_{\sigma_0} \cong K$. Since K is cyclic by the first part of (d) which we have proved, $M_{\sigma_0}U$ contains M'. Therefore, we have $M_{\sigma_0}U = M'$ and K is a complement of M' in M. This proves (h).

Moreover, K is a cyclic Hall subgroup of M such that $\pi(K) \cap \pi(M') = \emptyset$. By defintion, we have

$$\kappa(K) = \pi(K) = \tau_1(M).$$

Since $K = K_1^*$ and $K^* = K_1$, we have $K = C_{M-\sigma^*}(K^*)$. This proves (c).

It remains to prove the second part of (d). By (b), K^* is a Hall $\sigma_0(M)$-subgroup of M^*. Therefore, $K^* = M_{\sigma_0} \cap M^*$. It follows that

$K^* = M_{\sigma_0} \cap M^* \lhd M \cap M^*$ and, by Proposition 8.2 (b1),

$$M \cap M^* \subseteq N_{M^*}(K^*) = K \times K^*.$$

Since $K \times K^* \subseteq M \cap M^*$, we have $M \cap M^* = K \times K^* = Z$.

If $x \in K^\sharp$ and $y \in K^{*\sharp}$, (a) yields $C_G(x) \subseteq M^*$ so $C_M(x) \subseteq M \cap M^* = Z$. Since Z is cyclic by the first part of (d), we have

$$C_M(x) = Z.$$

Similarly, $C_{M^*}(y) = Z$. Moreover, $C_G(xy) = C_G(x) \cap C_G(y) \subseteq M \cap M^*$. This implies $C_G(xy) = Z$ and completes the proof of (d). Q.E.D.

Remark. From now on, we reserve the notation M^* or K^* to denote the subgroups given in Theorem 8.7 for the subgroup M in $\mathcal{M}_\mathcal{P}$.

Corollary 8.8. *The subgroups in $\mathcal{M}_{\mathcal{P}_1}$, if any, are all conjugate in G and, if $\mathcal{M}_\mathcal{P}$ is not empty, then $\mathcal{M}_\mathcal{P}$ contains exactly two conjugacy classes of subgroups.*

Corollary 8.9. *Choose a system of representatives $M_1, M_2, \ldots, M_n \in \mathcal{M}$ from each conjugacy class of subgroups of $\mathcal{M}$.*

(a) *If $\mathcal{M}_\mathcal{P}$ is empty, then the set of ϖ-elements of $G^\sharp$ is the disjoint union of the sets $\mathcal{C}_G(\widetilde{M_i})$ for $i = 1, 2, \ldots, n$.*

(b) *If $\mathcal{M}_\mathcal{P}$ is not empty, the set of ϖ-elements of $G^\sharp$ is the disjoint union of $\mathcal{C}_G(\widehat{Z})$ and the sets $\mathcal{C}_G(\widetilde{M_i})$ for $i = 1, 2, \ldots, n$ with $\widehat{Z}$ as in Theorem 8.7.*

Corollary 8.10. *For every ϖ-element $g \in G$ we have $\ell_\sigma(g) \leq 2$.*

Lemma 8.11. *Suppose that $M \in \mathcal{M}_\mathcal{F}$, E is a complement of M_σ in M, $q \in \pi(E)$, and $Q \in \mathcal{E}_q^1(E)$. Assume that $Q \not\subseteq F(E)$. Then, M is a ϖ-group with $\tau_2(M) \neq \emptyset$. Take $p \in \tau_2(M)$, $A \in \mathcal{E}_p^2(E)$ and $H \in \mathcal{M}(N_G(A))$. Then, $A_0 = [E, Q] = C_A(M_\sigma) \in \mathcal{E}_p^1(A)$ and $E = A_0 C_E(Q)$. Moreover, either*

(1) *$q \in \tau_2(H)$ and $\mathcal{M}(C_G(Q)) = \{H\}$, or*
(2) *$q \in \kappa(H)$ and $H \in \mathcal{M}_{\mathcal{P}_1}$.*

Proof. Suppose $\tau_2(M) = \emptyset$. Then, M is a Frobenius group and E is a Frobenius complement. From the structure of Frobenius complement, we get $Q \subseteq F(E)$. This contradicts the assumption $Q \not\subseteq F(E)$. Therefore, $\tau_2(M) \neq \emptyset$ and, by Lemma H, M is a ϖ-group.

Let $p \in \tau_2(M)$ and take $A \in \mathcal{E}_p^2(E)$. By Corollary 6.6 (a), we have $A \lhd E$ and $\mathcal{E}_p^1(E) = \mathcal{E}_p^1(A)$. Since $Q \nsubseteq F(E)$, we have $q \notin \tau_2(M)$. By Lemma 6.1, E_3 is a cyclic normal Hall $\tau_3(M)$-subgroup of E. Thus, $q \notin \tau_3(M)$. Therefore, we have $q \in \tau_1(M)$.

Let S be a Sylow p-subgroup of G that contains A. Suppose that S is abelian. Since $q \in \tau_1(M)$ and M is a Frobenius group, we can apply Lemma 6.8 (e) to conclude that Q lies in $Z(E) \subseteq F(E)$. This contradiction proves that S is nonabalian.

By Theorem 6.7 (b), $A_0 = C_A(M_\sigma)$ has order p and satisfies $F(M) = M_\sigma \times A_0$. Let $K = [E, Q]$. Then, $K \subseteq E'$ and E' is abelian by Corollary 6.10 (b). It follows that K is an abelian q'-group. Therefore, because $KQ \lhd E$, the Frattini argument yields $E = KN_E(Q)$. Since $[Q, N_E(Q)]$ is a q'-subgroup of Q, we have $N_E(Q) = C_E(Q)$ and $E = KC_E(Q)$. This implies $K = [E, Q] = [K, Q]$. Now, Proposition 4.11 (d) with q and Q in place of p and P yields that $[K, Q] = K$ is a cyclic normal subgroup of M that centralizes M_σ¿ It follows that $K \subseteq F(M) \cap E = A_0$. We have $K \neq 1$ because $Q \nsubseteq Z(E)$. Therefore, we have $K = A_0$.

Take $H \in \mathcal{M}(N_G(A))$. Since $A = [A, Q] \times C_A(Q)$ and $[A, Q] = A_0$, we have $C_A(Q) \in \mathcal{E}^1(A)$. Lemma 6.11 yields that $p \in \sigma_0(H) \setminus \beta(H)$ and $q \in \tau_1(H) \cup \tau_2(H)$. Recall that $p \in \tau_2(M)$ and $p \in \varpi$.

Suppose $q \in \tau_2(H)$. Since $C_A(Q) \neq 1$ and $A \subseteq H_{\sigma_0}$, Corollary 6.10 (e) for H and Q in place of M and $\langle x \rangle$ yields $\mathcal{M}(C_G(Q)) = \{H\}$. This is the case (1).

If $q \in \tau_1(H)$, $C_A(Q) \neq 1$ and $A \subseteq H_{\sigma_0}$ imply $q \in \kappa(H)$. Since $q \in \sigma_0(H)$, we have $\sigma_0(H) \neq \beta(H)$. Proposition 8.2 (g) for H yields that $H \in \mathcal{M}_{\mathcal{P}_1}$. Thus, we have the case (2). Q.E.D.

Corollary 8.12. *Suppose $M \in \mathcal{M}_{\mathcal{P}_2}$. Let K, M^*, and K^* be as in Theorem 8.7 and U as in Proposition 8.2 (a). Suppose $r \in \pi(U)$ and $R \in Syl_r(U)$.*

(a) *If M is not a ϖ-group, there is no $H \in \mathcal{M}(N_G(R))$ and $H \neq M$.*
(b) *If M is a ϖ-group, $\mathcal{M}(N_G(R))$ is not empty. For any $H \in \mathcal{M}(N_G(R))$, H is a ϖ-group in $\mathcal{M}_{\mathcal{F}}$ such that $U \subseteq H_\sigma$, $M \cap H = UK$, $N_H(U) \nsubseteq M$, $K \subseteq F(H \cap M^*)$, and $H \cap M^*$ is a complement of H_σ in H.*

Proof. Suppose that $N \in \mathcal{M}_{\mathcal{P}_2}$. Then, $U \neq 1$ and there exists $r \in \pi(U)$. Let $H \in \mathcal{M}(N_G(R))$ and $H \neq M$. We will prove that H is not conjugate to M or M^*.

Suppose that $H = M^g$ for some $g \in G$. Then, $R \in Syl_r(H)$. Since $N_G(R) \subseteq H$, we have $r \in \sigma(H)$. Since $C_G(R) \subseteq N_G(R) \subseteq H$, Theorem

4.1 (e) with R and H in place of X and M yields $H = M^g = M$. Thus, H is not conjugate to M in G.

Suppose $H = (M^*)^g$ for some $g \in G$. Then, $K \subseteq (M^*)^g = H$. Proposition 8.2 (d) with M and K^* replaced by M^* and K (cf. Theorem 8.7 (b) and (c)) yields $g \in M^*$. Thus, $H = (M^*)^g = M^*$. The nonabelian group KU is contained in $M \cap H$. However, $H = M^*$ so $M \cap H = M \cap M^*$ that is cyclic by Theorem 8.7 (d). Hence, H is not conjugate to M^* either.

To prove (a), suppose that there is a subgroup $H \neq M$ such that

$$H \in \mathcal{M}(N_G(R)).$$

Then, H is not conjugate to M or M^*. By Theorem 8.7 (g), we have $H \in \mathcal{M}_{\mathcal{F}}$. By assumption, M is not a ϖ-group. Hence, by Lemma G, U is a ϖ'-group. It follows that H is not a ϖ-group. Lemma G yields that H is a Frobenius group with Frobenius kernel that is a ϖ-group. In particular, H is ϖ-closed. However, $UK \subseteq H$ and the subgroup UK is not ϖ-closed. This contradiction proves (a).

To prove (b), take $H \in \mathcal{M}(N_G(R))$. Since M is a ϖ-group, we have $r \notin \sigma(M)$ so $N_G(R) \nsubseteq M$. It follows that $H \neq M$. The first part of the proof shows that H is not conjugate to M or M^*. By Theorem 8.7 (g), we have $H \in \mathcal{M}_{\mathcal{F}}$.

Since $M \in \mathcal{M}_{\mathcal{P}_2}$, we have $U \neq 1$. Proposition 8.2 (g) implies that K has prime order, say q. Note that $q \in \varpi$ because M is a ϖ-group. We will prove that H is a ϖ-group. If H is not a ϖ-group, Lemma G implies that H is a Frobenius group and the Frobenius kernel of H is a Hall ϖ-subgroup. Since UK is a ϖ-subgroup of H, UK is contained in the Frobenius kernel of H. Since UK is not nilpotent, we have a contradiction. Thus, H is a ϖ-group.

Since H is not conjugate to M^*, Theorem 7.9 implies that $\sigma(M^*)$ is disjoint from $\sigma(H)$. By Theorem 8.7 (c), we have $q = |K| \in \sigma(M^*)$. Hence, $q \notin \sigma(H)$. It follows that K lies in some complement D of H_σ in H. We will prove that $K \subseteq F(D)$.

Suppose $K \nsubseteq F(D)$. By Lemma 8.11 with (H, D, K) in place of (M, E, Q), there is a subgroup $L \in \mathcal{M}$ such that either $q \in \tau_2(L)$ and $\mathcal{M}(C_G(K)) = \{L\}$, or $q \in \kappa(L)$ and $L \in \mathcal{M}_{\mathcal{P}_1}$. If $L \in \mathcal{M}_{\mathcal{P}_1}$, then L must be conjugate to M or M^* by Theorem 8.7 (g). Note that L is not conjugate to M^* because $q \in \sigma(M^*)$. Hence, L is conjugate to M. This contradicts the assumption that $M \in \mathcal{M}_{\mathcal{P}_2}$. Therefore, we have $\mathcal{M}(C_G(K)) = \{L\}$. However, Theorem 8.7 (a) yields $C_G(K) \subseteq M^*$. This is a contradiction as $M^* \neq L$. Thus, we have $K \subseteq F(D)$.

It follows that K is subnormal in D. We claim that this implies $D \subseteq M^*$. We will prove that if $K \subseteq L \subseteq M^*$, then $N_G(L) \subseteq M^*$. If

$g \in N_G(L)$, then $K \subseteq L = L^g \subseteq (M^*)^g$. Then, Proposition 8.2 (d) with (M, K^*) replaced by (M^*, K) yields $g \in M^*$. By obvious induction, if K is subnormal in D, then $D \subseteq M^*$.

The subgroup U is a q'-group satisfying $U = [U, K]$. Since

$$K \subseteq O_q(D)H_\sigma \lhd H,$$

we get

$$U = [U, K] \subseteq U \cap O_q(D)H_\sigma \subseteq H_\sigma.$$

We will prove next $M \cap H = UK$. Clearly, $UK \subseteq M \cap H$. Since UK is a complement of M_σ in M, we have $M \cap H = XUK$ where $X = M_\sigma \cap H \lhd M \cap H$. Then, since $U \subseteq H_\sigma$,

$$[X, U] \subseteq M_\sigma \cap H_\sigma = 1$$

because M is not conjugate to H (by Theorem 7.9). On the other hand, Lemma 8.1 with M and $p \in \pi(U)$ yields that $C_{M_\sigma}(U) = 1$. Thus, $X = 1$ and we have $M \cap H = UK$.

By Lemma 8.1 with H and q in place of M and p yields that H_σ is nilpotent. Since $M \cap H = UK$ and $U \subseteq H_\sigma$, we have $M \cap H_\sigma = U$. Thus, if U is a proper subgroup of H_σ, then $N_H(U) \nsubseteq M$ by a fundamental property of nilpotent groups. On the other hand, if $U = H_\sigma$, $N_H(U) = H$ and certainly $N_H(U) \nsubseteq M$.

It remains to show that $D = H \cap M^*$. We have seen that $D \subseteq H \cap M^*$. Suppose that $H \cap M^* \neq D$. Then, $H_\sigma \cap M^* \neq 1$. Since $K \subseteq (M^*)_\sigma$,

$$[H_\sigma \cap M^*, K] \subseteq H_\sigma \cap (M^*)_\sigma = 1.$$

It follows from the definition of the set $\mathcal{M}_\mathcal{F}$ that $H \in \mathcal{M}_\mathcal{F}$ implies $q \in \tau_2(H)$. Then, Theorem 6.5 (e) for H yields $H_\sigma \cap M^* = 1$ contradicting the earlier inequality. This proves $D = H \cap M^*$. Q.E.D.

Lemma 8.13. *Assume that x is a ϖ-element such that $|\mathcal{M}_\sigma(x)| > 1$. Let $N = N(x)$ be as in Theorem 8.4 and $M \in \mathcal{M}_\sigma(x)$.*

- (a) *If $\sigma(N) \cap \pi(M) \neq \emptyset$, then $M \in \mathcal{M}_\mathcal{F}$ and $\tau_2(M) = \emptyset$. In this case, M is a Frobenius group with Frobenius kernel M_{σ_0}.*
- (b) *If $y \in (M_{\sigma_0})^\sharp$ and $C_G(y) \nsubseteq M$, then $|\mathcal{M}_\sigma(y)| > 1$ and $N(y)$ is defined. If $N(y)^g = N$ for some $g \in G$, then $N(y)^m = N$ for some $m \in M$.*

Proof. (a) By Theorem 8.4 (f), N is a ϖ-group in $\mathcal{M}_\mathcal{F} \cup \mathcal{M}_{\mathcal{P}_2}$. Take $q \in \sigma(N) \cap \pi(M)$, $Q \in \mathcal{E}_q^1(M)$, and $H \in \mathcal{M}(N_G(Q))$. Since $\sigma(M)$

is disjoint from $\sigma(N)$, $q \notin \sigma(M)$ so Q lies in some complement E of M_σ in M. By Theorem 8.4 (d),

$$q \in \sigma(N) \cap \pi(M) \subseteq \beta(N) \subseteq \beta(G).$$

Therefore, N contains a Sylow q-subgroup of G. By Sylow's Theorem, $Q \subseteq N^g$ for some $g \in G$. Corollary 6.14 with N^g, q, and Q in place of M, p, and X yields $\mathcal{M}(C_G(Q)) = \{N^g\}$. Note that $q \in \beta(N^g)$. It follows that $H = N^g$. By Lemma 6.1 (g) and Theorem 8.4 (c),

$$\pi(\langle x \rangle) \subseteq \tau_2(N) \subseteq \sigma_0(M) \setminus \beta(M).$$

In particular, $\sigma_0(M) \neq \beta(M)$. Hence, Proposition 8.2 (g) yields $M \notin \mathcal{M}_{\mathcal{P}_2}$. Suppose that $M \in \mathcal{M}_{\mathcal{P}_1}$. Then, $\pi(M) = \sigma(M) \cup \kappa(M)$. Since $q \notin \sigma(M)$, we have $q \in \kappa(M)$. There is a subgroup $M^* \in \mathcal{M}$ with properties stated in Theorem 8.7. We may take a Hall $\kappa(M)$-subgroup of M that contains Q. Define $Q^* = C_{M_\sigma}(Q)$. Then, $Q^* \subseteq M_{\sigma_0}$ and by Proposition 8.2 (b1) and Theorem 8.7 (b), Q^* is a Hall $\sigma_0(M)$-subgroup and a Hall $\kappa(M^*)$-subgroup of M^*. It follows that $M_{\sigma_0} \cap M^* = Q^*$and $\sigma_0(M) \cap \pi(M^*) = \kappa(M^*)$. On the other hand, Theorem 8.7 (a) yields

$$\mathcal{M}(C_G(Q)) = \{M^*\}.$$

Therefore, $M^* = N^g$ and $\pi(M^*) = \pi(N)$. Since $x \in M_{\sigma_0} \cap N$,

$$\pi(\langle x \rangle) \subseteq \sigma_0(M) \cap \pi(N).$$

By Theorem 8.4 (c), $\pi(\langle x \rangle) \subseteq \tau_2(N) \nsubseteq \kappa(N)$. Since $M^* = N^g$, we have $\kappa(N) = \kappa(M^*)$ and $\sigma_0(M) \cap \pi(N) \nsubseteq \kappa(M^*)$. This contradiction proves that $M \notin \mathcal{M}_{\mathcal{P}}$. Thus, $M \in \mathcal{M}_{\mathcal{F}}$.

Suppose that $\tau_2(M)$ is not empty. Take any $p \in \tau_2(M)$. By Lemma 6.1 (g), $p \notin \beta(G)$. Theorem 8.4 yields

$$\pi(M) \cap \sigma(N) \subseteq \beta(N) \quad \text{and} \quad \tau_2(N) \subseteq \sigma_0(M).$$

Therefore, $p \notin \sigma(N) \cup \tau_2(N)$. It follows that $r_p(N) \leq 1$. The rest of proof is as in [BG]. Q.E.D.

§9. The Subgroup M_F

Let $M \in \mathcal{M}$. We will choose a Hall $\kappa(M)$-subgroup K and a complement U of KM_{σ_0} in M that is K-invariant. If $M \in \mathcal{M}_{\mathcal{P}}$, the subgroup U is defined in Proposition 8.2 (a). If $M \in \mathcal{M}_{\mathcal{F}}$, $k = 1$ and U can be any complement of M_{σ_0} in M. We will choose one and fix it throughout the discussion. In addition, M_F denotes the largest normal nilpotent Hall subgroup of M. The notation is fixed in the rest of this paper. The subgroup UK is a complement of M_{σ_0} in M.

Lemma 9.1. *The following conditions hold.*

(a) $UM_{\sigma_0} \lhd M = KUM_{\sigma_0}$, *$K$ is cyclic,* $M_{\sigma_0} \subseteq M'$, *and* M'/M_{σ_0} *is abelian.*

(b) *If* $K \neq 1$, *then* $M' = UM_{\sigma_0}$ *and* U *is abelian.*

(c) *If* X *is a nonidentity subgroup of* U *such that* $C_{M_{\sigma_0}}(X) \neq 1$, *then*

$$\mathcal{M}(C_G(X)) = \{M\}$$

and X *is a cyclic* $\tau_2(M)$*-subgroup.*

(d) *The group* $\langle C_U(x) \mid x \in (M_{\sigma_0})^\sharp \rangle$ *is abelian.*

(e) *If* $U \neq 1$, *then* U *contains a subgroup* U_0 *of the same exponent as* U *such that* $U_0 M_{\sigma_0}$ *is a Frobenius group with Frobenius kernel* M_{σ_0}.

Proof. Since U is K-invariant, $UM_{\sigma_0} \lhd M$. If $K \neq 1$, Theorem 8.7 (d) implies that K is cyclic. By Theorem 4.2 (c),

$$M_\alpha \subseteq M_{\sigma_0} \subseteq M_\sigma \subseteq M'.$$

Part (d) of the same theorem implies that M'/M_{σ_0} is nilpotent. By the definition of the sets $\tau_i(M)$, Theorem 6.5 (b), and Lemma F, the group M'/M_{σ_0} has abelian Sylow subgroups. Therefore, M'/M_{σ_0} is abelian. This proves (a).

If $K \neq 1$, we have $U = [U, K] \subseteq M'$. Then, $M' = UM_{\sigma_0}$ and $U \cong M'/M_{\sigma_0}$. Hence, U is abelian and we have (b).

To prove (c), take nonidentity elements x' and x such that

$$x' \in X \text{ and } x \in C_{M_{\sigma_0}}(X)^\sharp.$$

Since $x' \in U^\sharp$, $\pi(\langle x' \rangle) \not\subseteq \kappa(M)$. By Corollary 8.3, we have $\pi(\langle x' \rangle) \subseteq \tau_2(M)$ and $\mathcal{M}(C_G(x')) = \{M\}$. It follows that X is an abelian $\tau_2(M)$-subgroup. If $r_p(X) > 1$ for some prime p, take $A \in \mathcal{E}_p^2(X)$. Theorem 6.5 (d) yields $C_{M_\sigma}(A) = 1$. Then,

$$C_{M_{\sigma_0}}(X) \subseteq C_{M_\sigma}(A) = 1$$

contrary to the hypothesis. Therefore, $r_p(X) \leq 1$ for all primes and X is cyclic. Taking the element x' to be a generator of X, we have $\mathcal{M}(C_G(X)) = \{M\}$.

If $K \neq 1$, U is abelian by (b). In this case, (d) is trivial. Suppose that $K = 1$. In this case, U is a complement of M_{σ_0}. Let $V = U \cap M_\sigma$. Then, V is a complement of M_{σ_0} in M_σ, and V is a Hall $\sigma(M)$-subgroup of U such that $V \lhd U$. There is a complement E of V in U. It follows that E is a complement of M_σ in M.

If $\sigma_0(M) = \sigma(M)$, we have $U = E$ and Theorem 6.12 yields (d) and (e). If $\sigma_0(M) \neq \sigma(M)$, V is a nontrivial ϖ'-group that acts regularly on M_{σ_0}. By Lemma H, $\tau_2(M) = \emptyset$. Since $K = 1$, the group E acts regularly on M_{σ_0}. It follows that $U = EV$ acts regularly on M_{σ_0}. Thus, $U = U_0$ satisfies (e), while the subgroup defined in (d) is 1.

It remains to prove (e) in the case $K \neq 1$. If M is not a ϖ-group, U is a ϖ'-group by Lemma G. It follows that $U_0 = U$ satisfies (e). Suppose that M is a ϖ-group. Then, $M_{\sigma_0} = M_\sigma$ and $\kappa(M) = \tau_1(M)$ by Theorem 8.7 (c). We may assume that $U = E_2E_3$.

Since $\kappa(M) = \tau_1(M)$, E_3 acts regularly on M_σ. The group E_2 is an abelian group of rank 2. We use the same argument as that of the proof of Theorem 6.12. Take $p \in \tau_2(M)$ and $S \in Syl_p(E_2)$. If G has nonabelian Sylow p-subgroups, then Theorem 6.7 provides a subgroup S_0 of the same exponent as S that acts regularly on M_σ. Furthermore, we have $E_2 = S$ (by Theorem 6.7 (a)). So, $U_0 = S_0E_3$ satisfies the condition (e). We can assume that S is a Sylow p-subgroup of G for every $p \in \tau_2(M)$. We write $S = Y \times Z$ for some cyclic subgroups with $|Y| \leq |Z|$. If $|Y| < |Z|$, then $C_{M_\sigma}(\Omega_1(Z)) = 1$ (cf. the proof of Theorem 6.12). If $|Y| = |Z|$, we can choose Z to satisfy the same condition. Then, the product U_0 of all those cyclic factors and E_3 satisfies the condition (e). Q.E.D.

Theorem 9.2. *For every $M \in \mathcal{M}$, we have*

$$1 \neq M_F \subseteq M_{\sigma_0} \subseteq M_\sigma \subseteq M'.$$

Suppose $M_F \neq M_{\sigma_0}$, and let $p = |K|$, $K^* = C_{M_\sigma}(K)$, and $q = |K^*|$. Then,

(a) *$M \in \mathcal{M}_{\mathcal{P}_1}$ and $M_\sigma = M'$,*
(b) *p and q are primes and $q \in \pi(M_F) \cap \beta(M)$,*
(c) *M has a normal Sylow q-subgroup Q, so $K^* \subseteq Q$,*
(d) *a complement D of Q in M' is nilpotent,*
(e) *$Q_0 = C_Q(D) \lhd M$,*
(f) *$\overline{Q} = Q/Q_0$ is a minimal normal subgroup of M/Q_0 and is elementary abelian of order q^p, and*
(g) *$M'' = (M_\sigma)' \subseteq F(M) = QC_M(Q) = C_M(\overline{Q}) = C_{M_\sigma}(\overline{K^*}) \subseteq M_\sigma$.*

Proof. This theorem has assumptions slightly different from those of Theorem 15.2 [BG]. However, the proof is almost identical. Since M

is a ϖ-local subgroup, $O_{\varpi}(M) \neq 1$. Therefore, the Fitting subgroup $F(M)$ of M is a ϖ-group. It follows that $M_F \subseteq M_{\sigma_0}$. Since $M_{\sigma_0} \neq 1$ by Theorem 4.2 (e), we have $M_F \neq 1$ if $M_F = M_{\sigma_0}$. Therefore, we may assume $M_F \neq M_{\sigma_0}$. Lemma 8.1 yields that $M \in \mathcal{M}_{\mathcal{P}_1}$, i.e. $K \neq 1$ and $M = KM_\sigma$. Then, $M/M_\sigma \cong K$. Since K is cyclic by Theorem 8.7 (d), we have $M' \subseteq M_\sigma$. Therefore, $M_\sigma = M'$ and (a) is proved.

We can continue the proof along the line adapted from the proof of Theorem 15.2 [BG]. Q.E.D.

Corollary 9.3. *Suppose H is a Hall subgroup of M_σ such that $\pi(H) \cap \varpi \neq \emptyset$. Then,*

(a) *$C_M(H) = C_{M_{\sigma_0}}(H)X$ with X a cyclic $\tau_2(M)$-subgroup, and*
(b) *if H is a ϖ-group, any two elements of H conjugate in G are already conjugate in $N_M(H)$.*

Proof. Since H contains a nontdentity ϖ-subgroup, $C_M(H)$ is a ϖ-group by Lemma A. If $x \neq 1$ is a $\kappa(M)$-element, $C_{M_\sigma}(x)$ is conjugate to K^* and does not contain any Hall subgroup of M_σ by Proposition 8.2 (d). It follows that $C_M(H) = C_{M_{\sigma_0}}(H)X$ where X is a $(\sigma_0(M) \cup \kappa(M))'$-subgroup . By Lemma 9.1, X is conjugate to a subgroup of U and, since $C_{M_{\sigma_0}}(X) \neq 1$, X is a cyclic $\tau_2(M)$-subgroup.

Suppose that $x, y \in H$, $g \in G$, and $x = y^g$. Then, $x \in M \cap M^g$ and $M = M^{gc}$ for some element $c \in C_G(x)$ by Theorem 8.4. Then $m = gc \in M$ by Lemma E (2) and $x = y^m$. This proves (b) in the case $H \lhd M$.

Suppose the H is not normal in M. Then $M_F \neq M_{\sigma_0}$ and we can use Theorem 9.2 as in the proof of Corollary 15.3 [BG] to finish the proof. Q.E.D.

Corollary 9.4. *Suppose that H is a nonidentity nilpotent Hall subgroup of G. If H is a ϖ-group, then there is a subgroup $M \in \mathcal{M}$ such that $H \subseteq M_{\sigma_0}$.*

Proof. Let S be a nonidentity Sylow subgroup of H and let $M \in \mathcal{M}(N_G(S))$. Then, we have $S \subseteq M_{\sigma_0}$. By Corollary 9.3 (a), $C_M(S) = C_{M_{\sigma_0}}X$ where X is a cyclic $\tau_2(M)$-subgroup of M. If $p \in \tau_2(M)$, Sylow p-subgroups of M are not cyclic. Hence, X contains no Sylow subgroup of G. A nilpotent Hall subgroup H is written $H = S \times L$ where L is a product of Sylow subgroups of G. Since $H \subseteq N_G(S) \subseteq M$, we have $L \subseteq C_M(S)$. It follows that $L \subseteq C_{M_{\sigma_0}}(S)$ because X is a $\sigma_0(M)'$-subgroup that contains no Sylow subgroup of G. This proves that $H = S \times L \subseteq M_{\sigma_0}$. Q.E.D.

Corollary 9.5. *Let $H = M_F$ and $Y = O_{\sigma_0(M)'}(F(M))$. Then,*

(a) *Y is a cyclic $\tau_2(M)$-subgroup of $F(M)$,*
(b) *$M'' \subseteq F(M) = C_M(H)H = F(M_{\sigma_0}) \times Y = F(M_\sigma) \times Y$,*
(c) *$H \subseteq M'$ and M'/H is nilpotent, and*
(d) *if $K \neq 1$, then $F(M) \subseteq M'$.*

Proof. (a) We have $H \subseteq F(M) \subseteq HC_M(H)$. By Corollary 9.3 (a), a Hall $\sigma_0(M)'$-subgroup of $C_M(H)$ is a cyclic $\tau_2(M)$-subgroup. This implies (a).

(b) Clearly, we have $F(M) = F(M_{\sigma_0}) \times Y = F(M_\sigma) \times Y$. Suppose $H = M_{\sigma_0}$. Then, $M'' \subseteq M_{\sigma_0}$ by Lemma 9.1 (a). Thus,

$$M'' \subseteq F(M) = H \times Y \subseteq HC_M(H) = M_{\sigma_0} \times X \lhd M$$

where X is a cyclic $\tau_2(M)$-group by Corollary 9.3 (a). Since $H = M_{\sigma_0}$, $M_{\sigma_0} \times X$ is a nilpotent normal subgroup of M. Hence, $M_{\sigma_0} \times X \subseteq F(M)$. Thus, $HC_M(H) = F(M)$ in this case. Suppose $H \neq M_{\sigma_0}$. By Theorem 9.2, M has a normal Sylow q-subgroup Q such that $Q \subseteq H$ and

$$M'' \subseteq F(M) = F(M_{\sigma_0}) \times Y \subseteq HC_M(H) \subseteq HC_M(Q) \subseteq QC_M(Q) = F(M).$$

Theorem 9.2 (g) yields the first containment and the last equality.

(c) If $H = M_{\sigma_0}$, Theorem 4.2 (c) and (d) yield the conclusions. If $H \neq M_{\sigma_0}$, Theorem 9.2 yields that $M' = M_\sigma$ contains H and M'/H is nilpotent (Part (d)).

(d) If $K \neq 1$, M' is a complement of K in M by Theorem 8.7 (h). Thus, M/M' is a $\kappa(M)$-group. By (c), we have $H \subseteq M_\sigma \subseteq M'$. By Corollary 9.3, $C_M(H) \subseteq M_\sigma X$ where X is a $\tau_2(M)$-group. It follows that $F(M) = HC_M(H) \subseteq M'$. Q.E.D.

Corollary 9.6. *Suppose $M \in \mathcal{M}_{\mathcal{P}}$. Then, $K^* = C_{M_\sigma}(K)$ is a nonidentity cyclic subgroup of M_F and M''. Furthermore, M_F is not cyclic.*

Proof. By definition, K is a ϖ-group. Therefore, $K^* \subseteq M_{\sigma_0}$. If $M_F = M_{\sigma_0}$, certainly $K^* \subseteq M_F$. If $M_F \neq M_{\sigma_0}$, Theorem 9.2 yields that $K^* \subseteq Q$ for some $Q \subseteq Syl_q(M)$. Since $Q \lhd M$, we have $Q \subseteq M_F$. Thus, $K^* \subseteq M_F$ in all cases.

Since $M \in \mathcal{M}_{\mathcal{P}}$, we have $K \neq 1$. Theorem 8.7 (h) yields that M' is a complement of K. Thus, M' is a normal Hall $\kappa(M)'$-subgroup of M. By Lemma 6.3 [BG], $K^* \subseteq C_{M'}(K) \subseteq M''$.

By Proposition 8.2 (c) and Theorem 8.7 (d), $K^* \neq 1$ and K^* is cyclic. Finally, we will prove that M_F is not cyclic. If M_F is cyclic, Theorem 9.2 yields that $M_{\sigma_0} = M_F$ because Q in Theorem 9.2 is non-cyclic (by (f)). Then, $F(M) = M_F \times Y$ is cyclic by Corollary 9.5 (a) and (b). This implies $M'' = 1$ which contradicts $K^* \subseteq M''$. Hence, M_F is not cyclic. Q.E.D.

Theorem 9.7. *Suppose that $F(M)$ is not a TI-subset of G. Let $H = M_F$ and define*

$$X = F(M) \cap F(M)^g \neq 1 \quad \textit{for some} \quad g \in G \setminus M.$$

Let E, E_1, E_2 and E_3 be as in Section 6. *Then,*

(a) $M \in \mathcal{M}_{\mathcal{F}} \cup \mathcal{M}_{\mathcal{P}_1}$ *and* $H = M_{\sigma_0}$;
(b) $X \subseteq H$, *X is cyclic, and H is a $\beta(M)'$-group*;
(c) $M' \subseteq F(M) = M_{\sigma_0} times Y$ *where Y is as in Corollary* 9.5;
(d) $E_3 = 1$, $E_2 \lhd E = E_1E_2$, *and E_1 is cyclic*; *and*
(e) *one of the following conditions holds*:
  (1) $M \in \mathcal{M}_{\mathcal{F}}$ *and H is abelian of rank* 2,
  (2) $|X| = p$ *is a prime in* $\sigma_0(M) \setminus \beta(M)$, $O_p(H)$ *is not abelian,* $O_{p'}(H)$ *is cyclic, and the exponent of* M/H *divides* $q - 1$ *for every* $q \in \pi(H)$, *or*
  (3) $|X| = p$ *is a prime in* $\sigma_0(M) \setminus \beta(M)$, $O_{p'}(H)$ *is cyclic,* $O_p(H)$ *has order* p^3 *and exponent* p *and is not abelian,* $M \in \mathcal{M}_{\mathcal{P}_1}$, *and* $|M/H|$ *divides* $p + 1$.

Proof. We remark first that $F(M)$ is a ϖ-group so any prime in $\pi(X)$ lies in ϖ. Take $p \in \pi(X)$ and $X_1 \in \mathcal{E}_p^1(X)$. We will show that $O_p(M)$ is not cyclic. If $O_p(M)$ were cyclic, X_1 would be the unique subgroup of order p in $F(M)$ as well as in $F(M)^g$. This would imply $M = N_G(X_1) = M^g$ so $g \in M$. Thus, $O_p(M)$ is not cyclic. Corollary 9.5 (a) yields that $p \in \sigma_0(M)$. Since p is arbitrary in $\pi(X)$, we have $\pi(X) \subseteq \sigma_0(M)$. Hence, $X \subseteq M_{\sigma_0} \cap M^g$. By Lemma 6.17, X is a cyclic $\beta(M)'$-subgroup. In particular, $\sigma_0(M) \neq \beta(M)$ and by Proposition 8.2 (g), we have $M \notin \mathcal{M}_{\mathcal{P}_2}$. Thus, the first part of (a) is proved.

Since $X_1 \subseteq M \cap M^g$, Theorem 4.1 yields $C_G(X_1) \not\subseteq M$. This implies that $C_H(X_1) \notin \mathcal{U}$ where $H = M_F$. Since $\langle H, X_1\rangle \subseteq F(M)$, $O_{p'}(H)$ centralizes X_1. Theorem 6.13 and the Uniqueness Theorem yield that every Sylow subgroup of $O_{p'}(H)$, and hence $O_{p'}(H)$ itself, is abelian of rank ≤ 2. Let $P = O_p(H)$. Then, $X_1 \subseteq P$ and $C_P(X_1)$ is abelian of rank ≤ 2. Therefore, H is a $\beta(M)'$-group. If $H \neq M_{\sigma_0}$, Theorem 9.2 (b) and (c) yield that a Sylow q-subgroup Q is normal in M and $q \in \pi(M_F) \cap \beta(M)$. This contradiction proves that $H = M_{\sigma_0}$. Thus, (a) holds.

If M is a ϖ-group, certainly $M_\sigma = M_{\sigma_0}$. If M is not a ϖ-group, M is not a group of type (2) in Lemma G because a group of type (2) satisfies $\sigma_0(M) = \beta(M)$. Similarly, if M is a group of type (1) in Lemma G, we have $M' = M_\sigma = M_F$. Thus, we have $M_\sigma = M_{\sigma_0}$ even if M is not s ϖ-group. Since $M_{\sigma_0} = H$, M_σ is a nilpotent $\beta(M)'$-subgroup of M. By Lemma 6.19, the group E' centralizes M_σ. Since E' is nilpotent by Lemma 6.1 (a), we have

$$M' = M_\sigma E' = M_\sigma \times E' \subseteq F(M) = M_{\sigma_0} \times Y$$

where Y is a cyclic $\tau_2(M)$-subgroup (by Corollary 9.5). Hence, E' is a $\tau_2(M)$-group. Since $E_3 \subseteq E'$, we have $E_3 = 1$. This proves (c) and (d).

The last part (e) can be proved by adapting the proof of the corresponding part of Theorem 15.7 [BG]. Q.E.D.

Theorem 9.8. *Suppose that we have the situation of Corollary* 8.12 *and assume that M is a ϖ-group. Thus, $M \in \mathcal{M}_{\mathcal{P}_2}$, K, M^*, and K^* are as in Theorem* 8.7 *and U is as in Proposition* 8.2 *(a). Suppose that $R \in Syl_r(U)$ for some $r \in \pi(U)$ and $H \in \mathcal{M}(N_G(R))$. Furthermore, suppose that $\tau_2(H)$ is not empty. Then, for $|K| = q$, q is the unique prime in $\tau_2(M)$ and $\tau_2(M)$ is empty.*

Proof. By Corollary 8.12, H is a ϖ-group such that

$$U \subseteq H_\sigma, M \cap H = UK, K \subseteq F(H \cap M^*),$$

and $H \cap M^*$ is a complement of H_σ in H. By Theorem 8.2 (g), $q = |K|$ is a prime. Let $D = H \cap M^*$. Then, D is a complement of H_σ in H by Corollary 8.12 (b).

By assumption, $\tau_2(H)$ is not empty so we can choose $A \in \mathcal{E}^2(D)$. Corollary 6.6 (a) yields $A \subseteq F(D)$. Since $K \subseteq F(D)$, $[A, K] = 1$ if A is not a q-group. If A is a q-group, Theorem 6.5 (b) implies that $K \subseteq A$ so $[A, K] = 1$ trivially. If $A \subseteq M^*_\sigma$, then $\pi(A) \subseteq \tau_2(M^*)$. Theorem 6.5 (d) for M^* yields $C_{M^*_\sigma}(A) = 1$. This contradicts $[A, K] = 1$ because $K \subseteq M^*_\sigma$. Hence, we have $A \subseteq M^*_\sigma$.

We claim that $F(M^*)$ contains A as well as a Sylow q-subgroup Q of M^*. If $(M^*)_F = (M^*)_{\sigma_0}$, this is certainly true because $(M^*)_{\sigma_0} \subseteq F(M^*)$. On the other hand, if $(M^*)_F \neq (M^*)_{\sigma_0}$, $F(M^*)$ contains a Sylow q-subgroup Q of M^* by Theorem 9.2 (c). Also, the part (g) of the same theorem quoted above yields that $F(M^*) = C_{M^*_\sigma}(\overline{K})$ which contains A. This proves the claim.

We prove next that $q \notin \beta(G)$. If A is a q-group, Lemma 6.1 (g) implies $q \notin \beta(G)$. If A is not a q-group, then we have $[Q, A] = 1$ because both Q and A are subgroups of a nilpotent group $F(M^*)$. Since $A \notin \mathcal{U}$, we have $Q \notin \mathcal{U}$ by Corollary 3.2 (a). This proves $q \notin \beta(G)$.

Theorem 9.2 (b) yields $(M^*)_F = (M^*)_{\sigma_0}$. Since $\sigma_0(M^*) \neq \beta(M^*)$, $M^* \in \mathcal{M}_{\mathcal{P}_1}$ by Proposition 8.2 (g). Therefore, we have $M^* = (M^*)_{\sigma_0}K^*$. By Lemma 6.17,

$$K = C_{M^*_{\sigma_0}}(K^*) \subseteq (M^*)_{\sigma_0}{}'.$$

Since $(M^*)\sigma_0 = (M^*)_F$ is nilpotent, $K \subseteq Q'$. Hence, Q is nonabelian and, by Theorem 6.13, $Q \in \mathcal{U}$. Since $A \notin \mathcal{U}$, Lemma 3.2 yields that $[Q, A] \neq 1$. Therefore, A is a q-group. Since Q is nonabelian, we have $\tau_2(H) = \{q\}$ by Theorem 6.7 (a).

The remaining statements are proved as in [BG]. Q.E.D.

Corollary 9.9. *Let $x \in M^\sharp_{\sigma_0}$ and $N \in \mathcal{M}(C_G(x))$. Assume that $C_G(x) \not\subseteq M$ and $N \notin \mathcal{M}_{\mathcal{F}}$. Take $r \in \pi(\langle x \rangle)$ and $X \in \mathcal{E}^1_r(\langle x \rangle)$. Then, both M and N are ϖ-groups. Furthermore, for a suitable choice of a complement E of M_σ in M,*

(a) *$M \in \mathcal{M}_{\mathcal{F}}$ and $N \in \mathcal{M}_{\mathcal{P}_2}$,*
(b) *E is cyclic and M is a Frobenius group with Frobenius kernel M_σ, and*
(c) *$r \in \tau_2(N)$, $N_E(X) \subseteq E \cap N$ and $|E \cap N| = |N/N'|$.*

Proof. Take $y \in C_G(x) \setminus M$. Then, $M, M^y \in \mathcal{M}_\sigma(x)$ and $M \neq M^y$. Hence, we are in the situation of Theorem 8.4 with $|\mathcal{M}_\sigma(x)| > 1$. Therefore,

$$C_{N_\sigma}(x) \neq 1, \mathcal{M}(C_G(x)) = \{N\}, r \in \tau_2(N) \cap \sigma_0(M),$$

N is a ϖ-group in $\mathcal{M}_{\mathcal{F}} \cup \mathcal{M}_{\mathcal{P}_2}$, and $M \cap N$ is a complement of N_σ in N. By assumption, we have $N \in \mathcal{M}_{\mathcal{P}_2}$.

Let K_1 be a Hall $\kappa(N)$-subgroup of N. Since $M \cap N$ is a complement of N_σ in N, we can take $K_1 \subseteq M \cap N$. By Proposition 8.2 (g) and (a), $|K_1|$ is a prime and there is an abelian complement U_1 of K_1 in $M \cap N$ for which $C_{U_1}(K_1) = 1$ and $U_1 \lhd M \cap N$.

Let $R \in Syl_r(M \cap N)$. Then, $R \subseteq U_1$ and $R \in Syl_r(N)$. Since

$$r \in \tau_2(N) \subseteq \sigma_0(M),$$

R is not cyclic and, by Corollary 6.10 (d), $N_G(R) \subseteq M$. Corollary 8.12 (b) with N, K_1, U_1, and M in place of M, K, U, and H yields that M is a ϖ-group in $\mathcal{M}_{\mathcal{F}}$ with $M \cap N = U_1K_1$. This proves (a).

By Lemma 8.1, M_{σ_0} is nilpotent. Since $R \subseteq U_1 \subseteq M$ and $r \in \sigma(M)$, we have $R \subseteq M_\sigma$. The group RK_1 is not nilpotent. Therefore, $K_1 \not\subseteq M_\sigma$.

Since $|K_1|$ is a prime, we have $K_1 \cap M_\sigma = 1$. We choose E to satisfy $K_1 \subseteq E$. Theorem 9.8 with N and M in place of M and H implies that if $\tau_2(M)$ is not empty, then $\tau_2(N)$ is empty. However, $\tau_2(N)$ is not empty, so $\tau_2(M)$ must be empty, i.e. $E_2 = 1$ for M. The element x is contained in M_σ and $C_G(x) \nsubseteq M$. Since M_σ is nilpotent, $F(M)$ is not a TI-subset of G. By Theorem 9.7 (d), we have $E_3 = 1$. It follows that $E = E_1$ and it is cyclic. Since $M \in \mathcal{M}_{\mathcal{F}}$, $\kappa(M)$ is empty. This implies that E acts regularly on $M_\sigma = M_{\sigma_0}$. Thus, M is a Frobenius group with Frobenius kernel M_σ.

We have shown that $r \in \tau_2(N)$. Since $C_G(x) \subseteq N_G(X)$, we have $N_G(X) \subseteq N$. Therefore, $N_E(X) \subseteq E \cap N$. The choice of E implies

$$K_1 \subseteq E \cap N \subseteq M \cap N = U_1 K_1.$$

Since $C_{U_1}(K_1) = 1$, we have $K_1 = C_{E \cap N}(K_1)$. This implies $K_1 = E \cap N$ because E is cyclic. It follows from Theorem 8.7 (h) that $|E \cap N| = |K_1| = |N/N'|$. Q.E.D.

§10. The Main Results

Theorem A. *Let $M \in \mathcal{M}$. Then, the following conditions are satisfied by M.*

(1) *M has a unique normal Hall $\sigma_0(M)$-subgroup M_{σ_0} which is also a Hall $\sigma_0(M)$-subgroup of G.*
(2) *M has a cyclic Hall $\kappa(M)$-subgroup K.*
(3) *KM_{σ_0} has a K-invariant complement U in M, i.e.*

$$UM_{\sigma_0} \lhd M = KUM_{\sigma_0} \quad \textit{and} \quad U \lhd UK.$$

(4) *$C_U(k) = 1$ for every $k \in K^\sharp$.*
(5) *$K^* = C_{M_{\sigma_0}}(K) \neq 1$ and if $K \neq 1$, then $C_M(k) = K \times K^*$ for every $k \in K^\sharp$.*
(6) *$1 \neq M_F \subseteq M_{\sigma_0} \subseteq M' \subset M$ and M'/M_F is nilpotent.*
(7) *$M'' \subseteq F(M) = C_M(M_F)M_F$ and if $K \neq 1$, then $F(M) \subseteq M'$.*
(8) *If $M_F \neq M_{\sigma_0}$, then $U = 1$, $F(M)$ is a TI-subgroup in G, and K has prime order.*

Proof. The group M_{σ_0} is defined as $O_{\sigma_0(M)}(M)$. Hence, $M_{\sigma_0} \lhd M$. Theorem 4.2 (f) yields (1); a normal Hall subgroup is unique. If $\kappa(M)$ is empty, $K = 1$ and the conditions (2), (3), (4) and (5) are trivially satisfied. If $\kappa(M)$ is not empty, $M \in \mathcal{M}_{\mathcal{P}}$. Then, Proposition 8.2 (a) implies the conditions (3) and (4), and Proposition 8.2 (c) yields $K^* \neq 1$;

while Theorem 8.7 (d) yields (2) and the second part of the condition (5).

Theorem 9.2 proves the first part of (6). If $M_F = M_{\sigma_0}$, Theorem 4.2 (c) and (d) imply the second part of (6). If $M_F \neq M_{\sigma_0}$, Theorem 9.2 (d) yields the result. The condition (7) has been proved in Corollary 9.5 (b) and (d). Theorem 9.2 (a) and (b) yield the first and third conditions of (8), respectively, while Theorem 9.7 (a) implies the second condition. Q.E.D.

To state further results, we need the following notation:

$$\widehat{M_\sigma} = \{a \in M \mid C_{M_{\sigma_0}}(a) \neq 1\}.$$

Note that this definition is slightly different from that in [BG]. We also define $A(M)$ and $A_0(M)$ as in [BG]; however, we use the set $\widehat{M_\sigma}$ defined above in the definition of the sets $A(M)$ and $A_0(M)$. Thus, the sets $A(M)$ and $A_0(M)$ are different from the sets in [BG] even though they are denoted by the same notation. In particular, $\widehat{M_\sigma}$ consists of ϖ-elements.

If $M \in \mathcal{M}$ is not a ϖ-group, we can determine these sets from Lemma G. The result is contained in the following table:

Type	K	U	M_{σ_0}	$\widehat{M_\sigma}$	$A(M)$	$A_0(M)$
(1)	1	a Z-group	M_{σ_0}	M_{σ_0}	M_{σ_0}	M_{σ_0}
(2)	$\neq 1$	*cyclic*	M_{σ_0}	$M_{\sigma_0} \cup \mathcal{C}_M(Z)$	M_{σ_0}	$M_{\sigma_0} \cup \mathcal{C}_M(\widehat{Z})$

Theorem B. *Let $M \in \mathcal{M}$. The following conditions are satisfied by M.*

(1) *Every Sylow subgroup of U is abelian of rank at most* 2.
(2) *$\langle U \cap \widehat{M_\sigma} \rangle$ is abelian.*
(3) *U has a subgroup U_0 that has the same exponent as U and satisfies $U_0 \cap \widehat{M_\sigma} = 1$.*
(4) *$\mathcal{M}(C_G(X)) = \{M\}$ for every nonidentity subgroup X of U such that $C_{M_{\sigma_0}}(X) \neq 1$.*
(5) *The set $A(M) \setminus M_{\sigma_0}$ is either empty or a TI-subset of G with normalizer M.*

Proof. It follows from the definition of the subgroup U (at the beginning of Section 9) that $\pi(U) = \pi(M) \setminus \{\kappa(M), \sigma_0(M)\}$. Take $p \in \pi(U)$ and $S \in Syl_p(U)$. If $p \in \tau_1(M) \cup \tau_3(M)$, then S is cyclic by the definition of the sets $\tau_i(M)$. If $p \in \tau_2(M)$, S is abelian of rank 2 by Theorem 6.5 (b). Finally, if $p \in \sigma(M) \setminus \sigma_0(M)$, S is cyclic by Lemma F. This proves (1).

Lemma 9.1 (d), (e) and (c) imply the conditions (2), (3) and (4), respectively. Suppose that the set $B = A(M) \setminus M_{\sigma_0}$ is not empty. The table before Theorem B shows that M is a ϖ-group. Since U is a Hall subgroup of $M_{\sigma_0}U$, every element g of $M_{\sigma_0}U$ can be written uniquely as a product of a $\sigma_0(M)$-element x and a $\pi(U)$-element v such that $g = xv = vx$. We say that v is the $\pi(U)$-component of the element g. It is a power of g, and v is conjugate to an element of U in M by the Schur-Zassenhaus Theorem.

Suppose that $g \in B$. Then, $g \notin M_{\sigma_0}$ so the $\pi(U)$-component v is not the identity. Also, $g \in B$ implies $C_{M_{\sigma_0}}(g) \neq 1$. It follows that $C_{M\sigma_0}(v) \neq 1$. If v is conjugate to an element u of U in M, we have $v = u^y$ for some $y \in M$ and $u \neq 1$. Then, Theorem B (4) yields that $\mathcal{M}(C_G(u)) = \{M\}$ because $u \neq 1$ and $C_{M_{\sigma_0}}(u) \neq 1$. Since $v = u^y$ with $y \in M$, we have $\mathcal{M}(C_G(v)) = \{M\}$. Thus, $g \in B$ implies $\mathcal{M}(C_G(v)) = \{M\}$ for the $\pi(U)$-component of g. Therefore, if $g \in B \cap B^h$ for some $h \in G$, then

$$\{M\} = \mathcal{M}(C_G(v)) = \{M^h\}.$$

This implies $M = M^h$ and $h \in M$. This proves (5). Q.E.D.

Theorem C. *Let $M \in \mathcal{M}_{\mathcal{P}}$ so $K \neq 1$. The following conditions hold.*

(1) *U is abelian. If M is a ϖ-group, $N_G(U) \not\subseteq M$. If $\sigma(M) \neq \sigma_0(M)$, then $N_G(U) \subseteq M$.*
(2) *K^* is cyclic, $1 \neq K^* \subset M_F$, but M_F is not cyclic.*
(3) *$M' = UM_{\sigma_0}$ and $K^* \subseteq M''$.*
(4) *There exists a unique subgroup $M^* \in \mathcal{M}_{\mathcal{P}}$ such that $K = C_{M^*_\sigma}(K^*)$ and K^* is a Hall $\kappa(M^*)$-subgroup of M^*.*
(5) *$\mathcal{M}(C_G(X)) = \{M\}$ and $\mathcal{M}(C_G(Y)) = \{M^*\}$ for all subgroups $X \subseteq K^*$ and $Y \subseteq K$ of prime order.*
(6) *$M \cap M^* = Z = K \times K^*$ and Z is cyclic.*
(7) *M or M^* is of type $\mathcal{P}_2$ and every subgroup $H \in \mathcal{M}_{\mathcal{P}}$ is conjugate to M or M^* in G.*
(8) *$\widehat{Z}$ is a TI-subset of G with $N_G(\widehat{Z}) = Z$.*
(9) *$\mathcal{C}_M(\widehat{Z})$ is equal to $A_0(M) \setminus A(M)$ and is a TI-subset of G with normalizer M.*
(10) *If $U \neq 1$, then K has prime order and $F(M)$ is a TI-subset of G that contains M_{σ_0}.*
(11) *If $U = 1$, then K^* has prime order.*

Proof. Proposition 8.2 (a) shows that U is abelian. The second statement of (1) follows from Corollary 8.12 (b), while the last one is obvious from the definitions.

Proposition 8.2 (c) implies the second condition of (2) and the first one in (5). Corollary 9.6 proves the remaining conditions of (2) and the last condition of (3). The most of the other conditions (3)—(9) follow from Theorem 8.7. Thus, the first condition of (3) follows from the part (h), (4) from the parts (b) and (c), the uniqueness of M^* and the second part of (5) from (a), the condition (6) from (d), the condition (7) from (f) and (g), the conditions (8) and the second part of (9) from (e) and the first part of (9) follows from the definitions.

Consider the conditions (10). The first one follows from Proposition 8.2 (g). Since $U \neq 1$, we have $M \in \mathcal{M}_{\mathcal{P}_2}$. Theorem 9.7 (a) now yields that $F(M)$ is a TI-subset of G. Then, $M_{\sigma_0} = M_F$ so $M_{\sigma_0} \subseteq F(M)$ by Theorem 9.2 (a).

The assumption $U = 1$ of (11) implies that $M \in \mathcal{M}_{\mathcal{P}_1}$. By Theorem 8.7 (f), $M^* \in \mathcal{M}_{\mathcal{P}_2}$ and K^* has prime order. Q.E.D.

Theorem D. *Let $M \in \mathcal{M}$. The following conditions are satisfied by M.*

(1) *Whenever two elements of M_{σ_0} are conjugate in G, they are conjugate in M.*

(2) *For every $g \in G \setminus M$, the group $M_\sigma \cap M^g = M_\sigma \cap (M_\sigma)^g$ is cyclic.*

(3) *For every $x \in (M_{\sigma_0})^\sharp$, $C_M(x)$ is a Hall subgroup of $C_G(x)$ and has a normal complement $R(x)$ in $C_G(x)$ that acts sharply transitively by conjugation on the set $\{M^g \mid g \in G, x \in M^g\}$.*

(4) *If $x \in (M_{\sigma_0})^\sharp$ and $C_G(x) \not\subseteq M$, then $\mathcal{M}(C_G(x)) = \{N\}$ for some ϖ-group $N = N(x) \in \mathcal{M}$ such that $R(x) = C_{N_\sigma}(x)$, $N_{\sigma_0} = N_F$, $x \in A(N) \setminus N_{\sigma_0}$, $N \in \mathcal{M}_{\mathcal{F}} \cup \mathcal{M}_{\mathcal{P}_2}$, and $M \cap N$ is a complement of N_σ in N. If $N \in \mathcal{M}_{\mathcal{P}_2}$, then M is a ϖ-group in $\mathcal{M}_{\mathcal{F}}$ that is a Frobenius group with cyclic Frobenius complement and Frobenius kernel $M_\sigma = M_F$. Furthermore, M_F is not a TI-subset in G.*

Proof. Corollary 9.3 (b) with H replaced by M_{σ_0} yields the condition (1), while Lemma 6.17 implies (2).

The assumptions of (3) and (4) imply that $\mathcal{M}_\sigma(x)$ is not empty. Therefore, Theorem 8.4 yields (3) and the most parts of (4). In particular, $N \notin \mathcal{M}_{\mathcal{P}_1}$. Then, Theorem 9.2 (a) applied to N proves that $N_{\sigma_0} = N_F$. We have $\pi(\langle x \rangle) \subseteq \tau_2(N)$. Since $N \notin \mathcal{M}_{\mathcal{P}_1}$, either $N = N_{\sigma_0}U$ or $N_{\sigma_0}U$ is a normal complement of K where U and K are defined as a Hall $(\sigma_0(N), \kappa(N))'$-subgroup and a Hall $\kappa(N)$-subgroup of N, respectively. Recall that if $K \neq 1$, K is a Hall $\tau_1(N)$-subgroup of N by Theorem 8.7 (c). Thus, $\pi(\langle x \rangle) \subseteq \tau_2(N)$ implies that $x \in N_{\sigma_0}U$. This proves that $x \in A(N) \setminus N_{\sigma_0}$.

If $N \in \mathcal{M}_{\mathcal{P}_2}$, Corollary 9.9 yields that $M \in \mathcal{M}_{\mathcal{F}}$ and M is a Frobenius group with Frobenius kernel M_{σ_0}. We have $M_{\sigma_0} = M_F$ by Theorem 9.2 (a). Since $C_G(x) \nsubseteq M$ and $x \in M_{\sigma_0} = M_F$. M_F is not a TI-subset of G. Q.E.D.

Theorem E. *For each $x \in (M_{\sigma_0})^\sharp$, let $R(x)$ be as in Theorem D. Define*

$$\widetilde{M} = \{xR(x) \mid x \in (M_{\sigma_0})^\sharp\}.$$

Then,

(1) $|\mathcal{C}_G(\widetilde{M})| = (|M_{\sigma_0}| - 1)|G : M|$.

Let $M_1, \dots, M_n$ be a set of subgroups in $\mathcal{M}$ such that every subgroup of $\mathcal{M}$ is conjugate in G to exactly one of the M_i. Then,

(2) *ϖ is the disjoint union of the sets $\sigma_0(M_i)$.*

(3) *Let $\widetilde{G}$ be the union of the sets $\mathcal{C}_G(\widetilde{M_i})$. Then, $\widetilde{G}$ is the disjoint union of the sets $\mathcal{C}_G(\widetilde{M_i})$.*

If $\mathcal{M}_{\mathcal{P}}$ is empty, $\widetilde{G}$ is the set of the nonidentity ϖ-elements of G. If $\mathcal{M}_{\mathcal{P}}$ is not empty and $M \in \mathcal{M}_{\mathcal{P}}$, then the set of nonidentity ϖ-elements of G is the disjoint union of $\widetilde{G}$ and $\mathcal{C}_G(\widehat{Z})$ where $\widetilde{Z}$ is as defined in Theorem 8.7.

Proof. If $p \in \varpi$, take $P \in Syl_p(G)$ and $M \in \mathcal{M}(N_G(P))$. We have $p \in \sigma_0(M)$. If $H \in \mathcal{M}$ is not conjugate to M, $\sigma(H)$ is disjoint from $\sigma(M)$ by Theorem 7.9. Thus, ϖ is the disjoint union of the sets $\sigma_0(M_i)$.

The remaining assertions of this theorem follow from Lemma 8.5 and Corollary 8.9. Q.E.D.

We define the type of a subgroup as in [BG] pp.128–129 with the following three changes.

We change (IIiv), (IIv) and (IIIiii) to read

(IIiv) $V \neq 1$ and, if V is a ϖ-group, $N_G(V) \nsubseteq M$.

(IIv) $N_G(A) \subseteq M$ for every nonidentity subgroup A of M' such that $C_H(A) \neq 1$.

(IIIiii) V is an abelian ϖ-group and $N_G(V) \subseteq M$.

Proposition 10.1. *Let M be an element of $\mathcal{M}$.*

(a) *$M \in \mathcal{M}_{\mathcal{F}}$ if and only if M is of type I.*

(b) $M \in \mathcal{M}_{\mathcal{P}_2}$ *if and only if* M *is of type II.*
(c) $M \in \mathcal{M}_{\mathcal{P}_1}$ *and* $M_F \neq M_{\sigma_0}$ *if and only if* M *is of type III or IV.*
(d) $M \in \mathcal{M}_{\mathcal{P}_1}$ *and* $M_F = M_{\sigma_0}$ *if and only if* M *is of type V.*
(e) $M' = M_{\sigma_0}U$ *if and only if* M *is not of type I.*
(f) $M_F = M_{\sigma_0}$ *if and only if* M *is of type I, II, or V.*

Proof. (a) Suppose that $M \in \mathcal{M}_{\mathcal{F}}$. This means that $K = 1$ and $U \neq 1$ in the notation of this section. As in [BG], let H denote M_F. Since $U \neq 1$, Theorem A (8) yields that $M_{\sigma_0} = M_F = H$. Thus, U is a complement of H in M. We have $H \neq M$ because $U \neq 1$. By Theorem A (6), $H \neq 1$ so we have the condition (Ii). The conditions (Iii), (Iiii) and (Iiv) are Theorem B (2),(3) and (1), respectively. We need to prove (Iv). Suppose that H is not a TI-subset of G. Then, $F(M)$ is not a TI-subset of G. Since $M \in \mathcal{M}_{\mathcal{F}}$, the case (3) of Theorem 9.7 (e) does not occur. Suppose that neither (a) nor (b) hold in (Iv). Then, we have the case (2) of Theorem 9.7 (e). Then, for every $q \in \pi(H)$, either $q \in \sigma_0(M) \setminus \beta(M)$, or M has a cyclic Sylow q-subgroup. Thus, $q \in \pi^*$. Furthermore, the exponent of M/H divides $q-1$. Since $O_{p'}(H)$ is cyclic for one prime $p \in \pi(H)$, M satisfies the condition (Iv). Therefore, every subgroup in $\mathcal{M}_{\mathcal{F}}$ is of type I.

Conversely, suppose that M is of type I. Suppose that $\kappa(M) \neq \emptyset$. Let K be a Hall $\kappa(M)$-subgroup of M and $K^* = C_{M_\sigma}(K)$. Then, by Theorem C (2), $K^* = C_H(K) \neq 1$. We will prove that $C_H(K) = 1$ contrary to the above inequality.

Since $K \cap H \subseteq K \cap M_{\sigma_0} = 1$, there is a complement E of H in M that contains K. Since K is a cyclic Hall subgroup of M by Theorem A (2), (Iiii) implies that K acts regularly on H by conjugation. Thus, $C_H(K) = 1$. This contradiction proves that every subgroup of type I lies in $\mathcal{M}_{\mathcal{F}}$.

(b) Suppose that $M \in \mathcal{M}_{\mathcal{P}}$, i.e. $K \neq 1$. By Theorem C (3), $M' = UM_{\sigma_0}$, so M' is a normal complement of K. Hence, M' is a Hall $\kappa(M)'$-subgroup of M. It contains H because $H \subseteq M_{\sigma_0} \subseteq M'$ by Theorem A (6). Thus, M satisfies (T1).

Define $W_1 = K$, $W_2 = K^*$, and let V be a K-invariant complement of H in M'. If $M_{\sigma_0} = H$, then choose $V = U$. By Theorem A (6), V ($\cong M'/H$) is nilpotent. This proves the condition (T2) for M. The group H is not cyclic by Theorem C (2). The remaining parts of (T3) follow from Theorem A (7). Since $K^* \subseteq M_{\sigma_0} \subseteq M'$, we have

$$KK^* \cap M' = (K \cap M')K^* = K^*.$$

This, together with Theorem A (5), implies (T4); while Theorem C (8) yields (T5).

Suppose that A_0 and A_1 are subgroups of prime order in V such that

$$(A_0)^g = A_1,\ g \in G \setminus M,\ C_H(A_0) \neq 1, \text{ and } \quad C_H(A_1) \neq 1.$$

If $H = M_{\sigma_0}$, then $V = U$. Hence, by Theorem B (4), we have

$$\{M\} = \mathcal{M}(C_G(A_1)) = \mathcal{M}(C_G(A_0))^g = \{M^g\}.$$

This would imply $g \in M$ by Lemma E. Therefore, we have $H \neq M_{\sigma_0}$. Theorem A (8) yields $U = 1$ and

$$A_i \subseteq V \subseteq M' = UM_{\sigma_0} = M_{\sigma_0}.$$

By Theorem D (1), A_0 and A_1 are conjugate in M. This proves that M satisfies (T6).

Assume that $M \in \mathcal{M}_{\mathcal{P}_2}$. Then, $K \neq 1$ and $U \neq 1$. Theorem A (8) yields $M_{\sigma_0} = H$ so $V = U$. We will check the conditions in (T7) for $M \in \mathcal{M}_{\mathcal{P}_2}$. By Theorem C (10), $W_1 = K$ has prime order and $F(M)$ is a TI-subset in G. Since $F(M) = C_M(H)H$ by Theorem A (7), we have (T7)(*ii*). Theorem C (1), together with Theorem B (1), yields that U is abelian of rank ≤ 2. This is (II*iii*).

Since $M \in \mathcal{M}_{\mathcal{P}_2}$, we have $V = U \neq 1$. Suppose that U is a ϖ-group. Then, M is a ϖ-group and Theorem C (1) yields $N_G(U) \not\subseteq M$. This proves (II*iv*).

To prove (II*v*), let A be a nonidentity subgroup of M' such that $C_H(A) \neq 1$. Since $M' = HU$, we have $A = XY$ where $X = A \cap H$ is a normal Hall subgroup of A and Y is a complement of X in A. Then, $N_G(A) \subseteq N_G(X)$. If $X \neq 1$, we have $N_G(X) \subseteq M$ because $F(M)$ is a TI-subset of G. In this case, $N_G(A) \subseteq M$. Suppose $X = 1$. Then $A = Y$ is a $\sigma_0(M)'$-group and it is conjugate to a subgroup of U in M. We may assume, by replacing A by a conjugate in M if necessary, that $A \subseteq U$. Since $C_H(A) \neq 1$, Theorem B (4) yields that $\mathcal{M}(C_G(A)) = \{M\}$. Therefore, $N_G(A) \subseteq M$. This proves (II*v*). Thus, a group in $\mathcal{M}_{\mathcal{P}_2}$ is of type II.

Assume that $M \in \mathcal{M}_{\mathcal{P}_1}$, i.e. $K \neq 1$ but $U = 1$. In this case, we have $V \subseteq M' = M_{\sigma_0}$. Therefore, V is a ϖ-group. Suppose that $V \neq 1$. Recall that V is defined as a complement of H in M'. Thus, in this case, we have $H \neq M_{\sigma_0}$. Hence, by Theorem A (8), conditions (*i*) and (*ii*) of (T7) hold. Since V is a Hall subgroup, V contains a Sylow p-subgroup P of G. Since $P \subseteq V \subseteq M_{\sigma_0}$, we have $p \in \sigma_0(M)$ and $N_G(P) \subseteq M$. By (T2), V is nilpotent. Hence,

$$N_G(V) \subseteq N_G(P) \subseteq M.$$

Thus, $M \in \mathcal{M}_{\mathcal{P}_1}$ with $V \neq 1$ is of type III or IV according as V is abelian or not.

Finally, suppose that $M \in \mathcal{M}_{\mathcal{P}_1}$ and $V = 1$. In this case, we have $H = M_{\sigma_0} = M'$. Suppose that H is not a TI-subset of G. Then, $F(M)$ is not a TI-subset of G. Theorem 9.7 (e) implies that M satisfies one of the three conditions. Since $M \in \mathcal{M}_{\mathcal{P}_1}$, the first condition does not hold. Hence, M is of type V.

Suppose that M is a group of type II, III, IV, or V. Then, M' is a Hall subgroup of M with a cyclic complement W_1 by (T1) and (T2). The group W_1 is a cyclic Hall subgroup of M such that $C_H(W_1) = W_2 \neq 1$. This implies $\pi(W_1) \subseteq \kappa(M)$. Thus, $\kappa(M) \neq \emptyset$ and $M \in \mathcal{M}_{\mathcal{P}}$. The group M has a series of characteristic subgroups $H \subseteq M_{\sigma_0} \subseteq M'$. The type of M is determined by the properties of this series. The type is V if and only if $H = M'$. The type is IV if and only if the group M'/H is nonabelian.

For the remaining types, M'/H is abelian. The type is III if and only if M'/H is an abelian ϖ-group and $N_G(V) \subseteq M$. Thus, the type of a group in $\mathcal{M}_{\mathcal{P}}$ is uniquely defined. Therefore, the statements (b), (c), and (d) hold. The other parts of Proposition 10.1 are proved as in [BG]. Q.E.D.

Theorem I. *Let H be a nilpotent Hall subgroup of G. Suppose that H is a ϖ-group. Then, two elements of H are conjugate in G if and only if they are conjugate in $N_G(H)$.*

Either every subgroup in $\mathcal{M}$ is of type I or all of the following conditions are true.

(a) *G contains a cyclic subgroup $W = W_1 \times W_2$ with the property that $N_G(W_0) = W$ for every nonempty subset W_0 of $W \setminus \{W_1, W_2\}$. Also, $W_i \neq 1$ for $i = 1, 2$.*

(b) *There are two subgroups S and T in $\mathcal{M}$ not of type I such that*

$$S = W_1 S', \ T = W_2 T', \ S' \cap W_1 = T' \cap W_2 = 1 \text{ and } S \cap T = W.$$

(c) *$M \in \mathcal{M}$ is either of type I or conjugate to S or T.*

(d) *S or T is of type II.*

(e) *Both S and T are of type II, III, IV, or V.*

(f) *The group S is not conjugate to T in G.*

Proof. Let H be a nilpotent Hall subgroup of G, and assume that H is a ϖ-group. In order to prove the first statement, we may assume $H \neq 1$. By Corollary 9.4, there is a subgroup $M \in \mathcal{M}$ such that $H \subseteq M_{\sigma_0}$. We will show that $N_G(H) \subseteq M$. Take a prime $p \in \pi(H)$ and $P \in Syl_p(H)$. Since H is a Hall subgroup of G, we have $P \in Syl_p(G)$.

It follows from the definition of $\sigma_0(M)$ that $N_G(P) \subseteq M$. Since H is nilpotent, $N_G(H) \subseteq N_G(P)$ so $N_G(H) \subseteq M$ as claimed.

Corollary 9.3 (b) implies that any two elements of H which are conjugate in G are already conjugate in $N_M(H)$. Since $N_G(H) \subseteq M$, we have $N_M(H) = N_G(H)$ and the first statement is proved.

Suppose that there is a subgroup $M \in \mathcal{M}$ not of type I. Then, $M \in \mathcal{M}_{\mathcal{P}}$ by Proposition 10.1 (a). Let $M = S$, $M^* = T$, $K = W_1$, and $K^* = W_2$. The group M satisfies the conditions (T1)—(T6) by Proposition 10.1 (a). These conditions imply that W_1 and W_2 are nonidentity cyclic subgroups of relatively prime orders. Condition (T4) yields

$$W = W_1 W_2 = W_1 \times W_2.$$

Hence, W is cyclic. Condition (T5) yields the first condition (a) of Theorem I. By Theorems C (4), C (6), and C (7), together with (T1) and (T2), S and T satisfy the conditions (b), (c) and (d). The last two conditions follow from Proposition 10.1 and Theorem 8.7. Q.E.D.

We state here the definition of the sets $A(M)$ and $A_0(M)$ for each $M \in \mathcal{M}$. Let $H = M_F$.

If M is of type I, then

$$A(M) = A_0(M) = \bigcup_{x \in H^\sharp} C_M(x).$$

If M is of type II,

$$A(M) = \bigcup_{x \in H^\sharp} C_{M'}(x);$$

while if M is of type III, IV, or V,

$$A(M) = M'.$$

If M is not of type I, then

$$A_0(M) = A(M) \cup \mathcal{C}_M(\widehat{W}).$$

Theorem II. *For a subgroup $M \in \mathcal{M}$, let $X = A(M)$ or $X = A_0(M)$, and let*

$$D = \{x \in X^\sharp \mid C_G(x) \nsubseteq M\}.$$

Then, $D \subseteq M_{\sigma_0}$, $|\mathcal{M}(C_G(x))| = 1$ for all $x \in D$, and the following conditions are satisfied.

(Fi) *Whenever elements of X are conjugate in G, they are conjugate in M.*

(Fii) *If D is not empty, then there are ϖ-subgroups $M_1, \ldots, M_n$ of $\mathcal{M}$ of type I or II such that with $H_i = (M_i)_F$,*

(a) $(|H_i|, |H_j|) = 1$ *for* $i \neq j$,
(b) $M_i = H_i(M \cap M_i)$ *and* $M \cap H_i = 1$,
(c) $(|H_i|, |C_M(x)|) = 1$ *for all* $x \in X^\sharp$,
(d) $A_0(M_i) \setminus H_i$ *is a nonempty TI-subset of G with normalizer M_i, and*
(e) *if $x \in D$, then there is a conjugate y of x in D and an index i such that $C_G(y) = C_{H_i}(y)C_M(y) \subseteq M_i$. If $y \in D$ with $C_G(y) \subseteq M_i$, then $y \in A(M_i)$.*

(Fiii) *If some M_i in* (Fii) *has type II, then M is a ϖ-group and is a Frobenius group with cyclic Frobenius complement, and M_F is not a TI-subset in G.*

Proof. For any $M \in \mathcal{M}$, $A_0(M)$ is a disjoint union of the sets

$$M_{\sigma_0}, A(M) \setminus M_{\sigma_0}, \text{ and } A_0(M) \setminus A(M).$$

The order of an element of M_{σ_0} involves only primes in $\sigma_0(M)$, the order of an element of $A(M) \setminus M_{\sigma_0}$ involves no prime of $\kappa(M)$ and some prime in $\pi(U)$ which is disjoint from $\sigma_0(M)$, and the order of an element of $A_0(M) \setminus A(M)$ involves a prime of $\kappa(M)$. Thus, an element of any of these sets is not conjugate to an element of one of the other two sets. By Theorems B (5) and C (9), the latter two sets are TI-subsets of G with normalizer M if not empty. Therefore, we have $D \subseteq M_{\sigma_0}$. Thus, if $x \in D$, then x is a ϖ-element with $M \in \mathcal{M}_\sigma(x)$. In fact, we have $|\mathcal{M}_\sigma(x)| > 1$ because $C_G(x) \not\subseteq M$. Theorem 8.4 yields $|\mathcal{M}(C_G(x))| = 1$.

It follows from the definition of the set X that $X \setminus M_{\sigma_0}$ is either empty or a TI-subset of G with normalizer M as remarked earlier. Therefore, Theorem D(1) implies (Fi).

Assume that D is not empty. For each $x \in D$, let $N(x)$ be the element of $\mathcal{M}(C_G(x))$. By Theorem D (4), $N(x)$ is a ϖ-group of type I or II. Let $\mathcal{A}$ be the collection of all such subgroups $N(x)$ and let $\{M_1, \ldots, M_n\}$ be a subset of $\mathcal{A}$ such that each $N \in \mathcal{A}$ is conjugate in G to exactly one M_i. The last condition (Fiii) follows from Theorem D (4).

We will prove (Fii). Take some M_i. Theorem D (4) yields that $(M_i)_{\sigma_0} = H_i$ and $M \cap M_i$ is a complement of H_i in M_i. This proves (b). By Theorem E (2), the sets $\sigma(M_i)$ are pairwise disjoint which implies (a).

By Theorem D (4), $x \in A(M_i)\backslash H_i$. Thus, $A(M_i)\backslash H_i$ is a nonempty TI-subset of G with normalizer M_i (by Theorem B (5)). If $A_0(M_i) \neq A(M)$, then $A_0(M_i) \setminus A(M)$ is also a TI-subset (by Theorem C (9)) and does not fuse to $A(M)$. This proves that $A_0(M_i) \setminus H_i$ is a nonempty TI-subset of G with normalizer M_i.

To prove (e), let $x \in D$. Then, the subgroup $N(x)$ is conjugate to M_i for some i, so $N(x)^g = M_i$ for some $g \in G$. By Lemma 8.13 (b), we may take $g \in M$. Then, $y = x^g \in D$ and $N(x) = M_i$. By Theorem 8.4 (b), we have

$$C_G(y) = C_{H_i}(y)C_M(y) \subseteq M_i.$$

If M_i is of type I, certainly $y \in A(M_i)$. Suppose that M_i is of type II. Theorem 8.4 (c) yields $\pi(\langle y\rangle) \subseteq \tau_2(M_i)$. Since M_i is of type II, Theorem 8.7 (h) and (c) yield $y \in (M_i)'$. Hence, $y \in A(M_i)$ in this case, too.

It remains to prove (c). Suppose $x \in X^\sharp$ and $(|H_i|, |C_M(x)|) \neq 1$ for some i. Then, $\pi(M) \cap \sigma_0(M_i)$ is not empty. By Lemma 8.13 (a), M is a Frobenius group with Frobenius kernel M_{σ_0}. Hence,

$$A_0(M) = M_{\sigma_0} = X \quad \text{and} \quad C_M(x) \subseteq M_{\sigma_0}.$$

It follows that $\sigma_0(M) \cap \sigma_0(M_i)$ is not empty. By Theorem E (2), M is conjugate to M_i in G. However, this is a contradiction because $\tau_2(M) = \emptyset$ by Lemma 8.13 (a), while $\tau_2(M_i) \neq \emptyset$ by Theorem 8.4 (c). Q.E.D.

Chapter II. Application of Character Theory

We continue to study the structure and embedding of the subgroups in $\mathcal{M}$ and use the notation introduced in Chapter I. We will follow most of the terms and notation of [BG] and [FT]; however, I follow the practice of denoting elements of groups by the lower case letters and subsets by the capitals. For a group H, let

$$\mathrm{Irr}(H)$$

denote the set of all irreducible characters of the group H over the field $\mathbb{C}$ of complex numbers. If X is a subset of H,

$$I(X)$$

denotes the set of virtual characters which vanish outside X. The subset of $I(X)$ consisting of those virtual characters which take zero at the identity will be important and denoted by

$$I_0(X).$$

Sometimes, the set of complex valued class functions which vanish outside X will be considered; it is denoted by $C(X)$. The subset of those class functions taking value zero at the identity is denoted $C_0(X)$.

When H is a subgroup of a group G, the *induction* φ^G and the *restriction* θ_H of class functions are defined as usual. If the groups involved are clear from the context, the notation φ^* for the induction may be used.

Our starting point is Theorems I and II of Section 10, Chapter I. Theorem I asserts that every subgroup $M \in \mathcal{M}$ is of type I, II, III, IV, or V. The definition of groups of each type is stated in [BG] but we have made three changes in Section 10, Chapter I; we will be using the modified definition in Chapter II. Theorem II concerns the embedding of the subgroups of $\mathcal{M}$. We say that a subset X of M is an *F-set* (or satisfies Feit-Thompson-Sibley-Bender-Glauberman conditions) if M and X satisfy the conditions (Fi), (Fii) and (Fiii) of Theorem II. Theorem II simply says that both $X = A(M)$ and $X = A_0(M)$ are F-sets of M. In [BG], it is suggested to call $A(M)$ and $A_0(M)$ tamely imbedded subsets. I choose a different term because there is another tamely imbedded subset in [FT] and I have added two conditions to (Fii).

The set of subgroups $\{H_1, H_2, \ldots, H_n\}$ in (Fii) is called the set of *supporting subgroups* of the F-set X. Sometimes we abuse the term and may call subgroups $\{M_1, \ldots, M_n\}$ are also supporting subgroups.

If X is an F-set of M, we will use the following notation throughout Chapter II. Let D be the set defined by

$$D = \{x \in X^\sharp \mid C_G(x) \not\subseteq M\}.$$

If D is empty, X is a TI-subset of M. If X is either $A(M)$ or $A_0(M)$, and if D is not empty, then Theorem II yields that $D \subseteq M_{\sigma_0}$. Therefore, the set D does not depend on whether $X = A(M)$ or $X = A_0(M)$. The following notation is used.

$$D_0 = \{x \in X \mid C_G(x) \subseteq M\}$$

and for $i > 0$,

$$D_i = \{x \in D \mid C_G(x) \subseteq M_i\}$$

where M_i is one of the supporting subgroups of the set X. We have abused the notation already. It is convenient to define

$$H_0 = \{1\}, M_0 = M, \quad \text{and} \quad D^* = \bigcup_{i=0}^{n} D_i.$$

As in [FT], we define for $i \geq 0$ and $x \in D_i$

$$A_x = A(x) = \{hx \mid hx = xh, h \in H_i\}.$$

Note that each $A(x)$ consists of nonidentity elements. We call a subset of the form $A(x)$ for some $x \in D^*$ an *annex*. For an F-set X of M, we call the set of elements of G which are conjugate to an element of some annex $A(x)$ for $x \in D^*$ the *territory* of X. Sometimes, we abuse the term and call it the territory of M. A class function θ on G is called *well-behaved* if θ takes a constant value on each annex. The well behaved class functions will play an important role in the following discussion.

§11. Preparation from Character Theory

First we paraphrase the proof of Lemma 4.5 [FT] because it is basic to our work. Afterwards, we define the basic character correspondence τ and prove its properties. This part corresponds to Section 9 of [FT].

For convenience, we state Lemma 4.5 [FT]:

Lemma. *Let H be a normal subgroup of the group X and let θ be an irreducible character of H. Suppose X contains a normal subgroup X_0 such that the inertia group $I(\theta) \subseteq X_0$ and such that X_0/H is abelian. Then θ^* is a sum of irreducible characters of X which have the same degree and occur with the same multiplicity in θ^*. This common degree is a multiple of $|X : I(\theta)|$. If furthermore H is a Hall subgroup of X_0, then θ^* is a sum of $|I(\theta) : H|$ distinct irreducible characters of degree $|X : I(\theta)|\theta(1)$.*

We need a lemma.

Lemma. *Let M be a group, $H \lhd M$, $\theta \in \mathrm{Irr}(H)$, and let $I = I(\theta)$ be the inertia group of θ in M. If $\theta^I = \sum a_i\lambda_i$ where a_i are positive integers and λ_i are distinct irreducible characters of the group I, then λ_i^M are distinct irreducible characters of M and*

$$\theta^M = \sum a_i\lambda_i^M.$$

By the reciprocity theorem, $(\lambda_i)_H$ contains the character θ with exact multiplicity a_i. Since $H \lhd I$, $(\lambda_i)_H$ is a sum of the conjugates of θ. It follows from the definition of the inertia group that θ is the only conjugate of θ in I. Thus, we have

$$(\lambda_i)_H = a_i\theta;$$

in particular, $\lambda_i(1) = a_i\theta(1)$.

Let ξ be an irreducible component of λ_i^M. Then, ξ_I involves λ_i. Hence, ξ_H contains $a_i\theta$. Since $H \lhd M$, ξ_H contains all the $|M : I|$ conjugates of θ with the same multiplicity. It follows that

$$\xi(1) \geq a_i|M : I|\theta(1) = |M : I|\lambda_i(1) = \lambda_i^M(1).$$

Since ξ is an irreducible component of λ_i^M, we have $\lambda_i^M = \xi$, i.e. λ_i^M is irreducible. The preceding proof yields that $(\lambda_i^M)_H$ involves θ exactly a_i times. This implies that the character $(\lambda_i^M)_I$ does not involve λ_j for any $j \neq i$. Thus, $\lambda_i^M \neq \lambda_j^M$ for $i \neq j$. This proves the lemma. Q.E.D.

Hypotheses of Lemma 4.5 [FT] are $H \lhd X$, $\theta \in \mathrm{Irr}(H)$, $I = I(\theta)$, the inertia group of θ in X, and I/H is abelian. By the preceding lemma, we need only to prove the assertion for I.

Let λ be an irreducible component of θ^I and let $\{\mu_1, \ldots, \mu_m\}$ be the set of all irreducible characters of I/H. Since I/H is abelian, μ_i are linear and $\{\mu_1, \ldots, \mu_m\}$ is a multiplicative group of order $m = |I/H|$. Suppose that we take notation $\lambda\mu_i = \lambda$ if and only if $1 \leq i \leq n$. For every j, $(\lambda\mu_j)\mu_i = (\lambda\mu_j)$ for $i = 1, 2, \ldots, n$.

We have $(\theta^I)_H = m\theta$ so $\lambda_H = a\theta$ for some positive integer a. Then,

$$\lambda(1_H)^I = (\lambda_H)^I = a\lambda^I.$$

Since $(1_H)^I = \sum \mu_j$, the irreducible components of θ^I are characters of the form $\lambda\mu_j$. This proves that all the irreducible components of θ^I are of the same degree. Also, the equality

$$\lambda(\sum \mu_j) = a\theta^I$$

yields that θ^I contains each irreducible component $\lambda\mu_j$ with the same multiplicity, say b. This proves the first assertion of Lemma 4.5 [FT]. We remark that $n = ab$.

The second part of the lemma asserts that if in addition H is a Hall subgroup of the inertia group I, then θ^X is a sum of exactly $|I : H|$ distinct irreducible characters of degree $|X : I|\theta(1)$. By the lemma, it suffices to prove the case $X = I$.

We can take an abelian complement A of H in I because H is a Hall subgroup of I and I/H is abelian. We will show by induction that if $H \subseteq K \subseteq I$, θ^K is a sum of exactly $|K : H|$ distinct irreducible characters of degree $\theta(1)$. The first part of Lemma yields that

$$\theta^K = b(\lambda_1 + \cdots + \lambda_s)$$

where $\lambda_1, \ldots, \lambda_s$ are distinct irreducible characters of degree $a\theta(1)$.

Suppose that $|K : H| = p$ is a prime. By definition of the induced character, we have

$$(\theta^K)_H = p\theta.$$

Then, the orthogonality relations yield

$$b^2 s = (\theta^K, \theta^K) = (\theta, (\theta^K)_H) = p(\theta, \theta) = p.$$

Since p is a prime, we have $b = 1$ and $s = p$. Thus, $n = n_K = 1$.

Suppose that $H \subseteq K \subseteq L \subseteq I$ and $|L : K| = q$ is a prime. Suppose that $n_K = 1$. Take an irreducible component λ of θ^K and let $\{\mu_1, \ldots, \mu_s\}$ be the set of irreducible characters of the abelian group K/H. Since $n_K = 1$, the characters $\lambda\mu_i$ are distinct. Therefore,

$$\theta^K = \sum \lambda\mu_i.$$

We claim that $L \subseteq I(\lambda)$. If $x \in L$, $(\theta^K)^x = (\theta^x)^K = \theta^K$ because $x \in L \subseteq I$. Thus, λ^x is an irreducible component of θ^K, i.e. $\lambda^x = \lambda\mu_i$ for some i. We need to show that μ_i is the principal character of K/H. We may assume that $x \in K \cap A$. If μ_i is nonprincipal, there is an element $y \in K \cap A$ such that $\mu_i(y) \neq 1$ and the order of y is a power of some prime r. Lemma 4.2 [FT] implies

$$\lambda(y) \equiv \lambda(1) \pmod{\mathfrak{r}}$$

where $\mathfrak{r}$ is a prime ideal dividing r in the ring of integers of a number field. Since $n_K = 1$, we have $\lambda(1) = \theta(1)$. Also, $\lambda(1)$ divides the order $|H|$. The group H is a Hall subgroup of I so r does not divide $\lambda(1)$. It follows from the above congruence that $\lambda(y) \neq 0$. Since A is abelian and $x, y \in A$, we have

$$\lambda(y) = \lambda^x(y) = \lambda(y)\mu_i(y).$$

Therefore, $\mu_i(y) = 1$ because $\lambda(y) \neq 0$. This contradiction proves that $\lambda^x = \lambda$ and $L \subseteq I(\lambda)$.

By the result proved earlier, the induced character λ^L is a sum of $|L : K|$ distinct irreducible characters. This holds for any irreducible component of θ^K. Thus, θ^L is a sum of exactly $|L : H|$ distinct irreducible characters of degree $\theta(1)$. In particular, $n_L = 1$. This completes the proof of Lemma 4.5 [FT]. Q.E.D.

We need some lemmas about the fusion of elements.

Lemma I. *Let $M \in \mathcal{M}$ and let X be an F-set of M. Every element of $X^\sharp$ is conjugate to an element of D^* in M.*

Proof. Let $x \in X^\sharp$. If $x \in D_0$, the assertion is trivial. If $x \notin D_0$, we have $x \in D$. By (Fii, e), there is a conjugate y of x such that $y \in D_i$. Since x and y are two elements of X which are conjugate, (Fi) yields that they are conjugate in M. Q.E.D.

Lemma J. (a) *Every element g of M_i is conjugate in M_i to an element of the form $xh = hx$ where $x \in M \cap M_i$ and $h \in H_i$.*

(b) *Suppose that g is an element of M_i with $C_{H_i}(g) \neq 1$. Assume that g is conjugate in M_i to an element of the form hx where $x \in M \cap M_i$ and $h \in C_{H_i}(x)$, and at the same time g is conjugate to an element of the annex $A(y)$ with $y \in D_j$. Then, $j = i$ and the element x is conjugate to y in M_i. In particular, $x \in D_i$ and $g \in A(M_i)$.*

Proof. (a) If $i = 0$, $M_i = M$ and (a) holds trivially. Assume $i > 0$. The subgroup H_i is a normal Hall subgroup of M_i with complement $M \cap M_i$ by (Fii, b). Let $g = uv = vu$ be the decomposition of the element g into the product of a $\pi(H_i)$-element u and a $\pi(H_i)'$-element v. Since H_i is nilpotent, we can apply the Schur-Zassenhaus Theorem to the subgroup $\langle H_i, g\rangle$. Then, $\langle v\rangle$ is conjugate in M_i to a subgroup of $M \cap M_i$. It follows that g is conjugate in M_i to an element of the form hx where $x \in M \cap M_i$ and $h \in C_{H_i}(x)$.

(b) Suppose that $C_{H_i}(g) \neq 1$ and that g is conjugate to an element ky of $A(y)$ with $y \in D_j$ and $k \in C_{H_j}(y)$. The first assumption implies that $i > 0$. We will prove that $j > 0$. If $j = 0$, we have $C_G(y) \subseteq M$. It follows that

$$C_G(ky) \subseteq C_G(y) \subseteq M.$$

Since $C_{H_i}(g) \neq 1$ for some $i > 0$, $(|C_G(g)|, |H_i|) \neq 1$. We have $|C_G(g)| = |C_G(ky)|$ because g is conjugate to ky. Therefore,

$$(|C_G(y)|, |H_i|) \neq 1.$$

This contradicts (Fii,c) as $|C_G(y)| = |C_M(y)|$. Hence, we have $j > 0$.

It follows that $C_G(y) \subseteq M_j$ and $C_G(y) = C_{H_j}(y)C_M(y)$. Suppose that $j \neq i$. Then, by (Fii) (a) and (c), $|C_G(y)|$ is prime to $|H_i|$. This is a contradiction because

$$|C_G(g)| = |C_G(ky)| \quad \text{and} \quad C_G(ky) \subseteq C_G(y).$$

Therefore, we have $j = i$. Since $y \in X^\sharp$, the order of y is prime to $|H_i|$ by (Fii,c). Hence, y is the $\pi(H_i)'$-part of the element ky. Similarly, the element x is the $\pi(H_i)'$-part of hx. Since hx is conjugate to ky, the element x is conjugate to y in G. By (Fii,e), the element y is in $A(M_i)$. Since hx is conjugate to g in M_i and $C_{H_i}(g) \neq 1$, we have $C_{H_i}(x) \neq 1$.

If M_i is of type I, then $x \in A(M_i)$. If M_i is of type II, $(M_i)'$ is a Hall subgroup of M_i by (T1), and $y \in A(M_i) \subseteq (M_i)'$. Since x is conjugate to y in G, we have

$$|\langle x\rangle| = |\langle y\rangle|.$$

Hence, $x \in (M_i)'$ and $x \in A(M_i)$. Clearly, $y \neq 1$ so $y \notin H_i$. Since H_i is a Hall subgroup of M_i, we have $x \notin H_i$. Thus, x and y are elements of $A_0(M_i) \setminus H_i$ that is a TI-subset in G with normalizer M_i. Since x is conjugate to y, they are conjugate in M_i. It follows that

$$C_G(x) \subseteq M_i \quad \text{and} \quad x \in D_i.$$

By (Fii,e), $hx \in A(M_i)$. Since g is conjugate to hx in M_i, we have $g \in A(M_i)$. Q.E.D.

We will define the fundamental mapping τ.

Definition K. Let $M \in \mathcal{M}$ and let X be an F-set of M. For

$$\alpha \in I_0(X) \quad \text{and} \quad 1 \leq i \leq n,$$

define

$$\alpha_i = \alpha_{M \cap M_i}.$$

Let α_{i1} be the virtual character of M_i/H_i that is the lift of α_i and let α_{i2} be the virtual character of M_i induced by α_i. We define

$$\alpha^\tau = \alpha^G + \sum\nolimits_{i=1}^{n} (\alpha_{i1} - \alpha_{i2})^G.$$

Thus, α^τ is a virtual character of the group G that vanishes at the identity.

Lemma L. (a) *If $g \in G$ is not conjugate to any element of $X^\sharp$ in G, then $\alpha^G(g) = 0$. If $g \in X^\sharp$, then*

$$\alpha^G(g) = |C_G(g) : C_M(g)|\alpha(g).$$

(b) *Let i be one of the integers between* 1 *and n. If $g \in G$ is not conjugate to any element x of M_i with $C_{H_i}(x) \neq 1$, then $(\alpha_{i1} - \alpha_{i2})^G(g) = 0$.*

Proof. (a) The first statement is obvious from the definition of induced characters. Suppose $a \in X^\sharp$. By definition,

$$\alpha^G(g) = \sum \alpha_0(x_i^{-1} g x_i)$$

where α_0 is the function that agrees with α on M but vanishes outside M, and the sum is over a system $\{x_i\}$ of the representatives of the cosets of M. We need to count the number of x_i such that $x_i^{-1}gx_i \in X^\sharp$. If $x^{-1}gx \in X$, (Fi) yields

$$x^{-1}gx = m^{-1}gm$$

for some $m \in M$. Hence, $xm^{-1} = c \in C_G(g)$. We choose c as a representative of the coset xM. Then, $\alpha_0(x_i^{-1}gx_i) = \alpha(g)$. This yields the result.

(b) If $x \in M_i$ is not conjugate to any element of $M \cap M_i$, then it follows from the definition of the induced character that $\alpha_{i2}(x) = 0$. We may take the set H_i as a set of representatives from the cosets of $M \cap M_i$. If $x \in M \cap M_i$ and $h \in H_i$,

$$h^{-1}xh \in M \cap M_i$$

implies $[x, h] = x^{-1}h^{-1}xh \in (M \cap M_i) \cap H_i = 1$. Hence, if $x \in M \cap M_i$, then

$$\alpha_{i2}(x) = |C_{H_i}(x)|\alpha_i(x).$$

Thus, $(\alpha_{i1} - \alpha_{i2})(x) = 0$ if $x \in M_i$ satisfies $C_{H_i}(x) = 1$. This, together with Lemma J(a), proves (b). Q.E.D.

The following lemmas correspond to the lemmas in Section 9 of [FT].

Lemma 11.1. *Let $M \in \mathcal{M}$ and X an F-set of M. For $\alpha \in I_0(X)$, let α^τ be defined as in Definition K. Then, $\alpha^\tau(g) = 0$ if g is not conjugate to an element of $A(x)$ for any $x \in D^*$. If $g \in A(x)$ for $x \in D^*$, then*

$$\alpha^\tau(g) = \alpha(x).$$

In the other words, if $\alpha \in I_0(X)$, the support of the function α^τ is contained in the territory of X, and the function α^τ is well-behaved.

Proof. Suppose that $\alpha^\tau(g) \neq 0$. Then, clearly, g must be conjugate to some element of $M, M_1, \ldots,$ or M_n. In order to have $\alpha^G(g) \neq 0$ or $\alpha_{i2}^G(g) \neq 0$, the element g must be conjugate to an element of $X^\sharp$. By Lemma J, if $g \in M_i$ for $i > 0$, g is conjugate to an element of the form hx such that $x \in M \cap M_i$ and $h \in C_{H_i}(x)$. In order to have $\alpha_{i1}^G(g) \neq 0$, g must be conjugate to an element ky such that $y \in X^\sharp$ and k is a $\pi(H_i)$-element commuting with y.

Every element of $X^\sharp$ is conjugate to an element of D^* by Lemma I. It follows that if g is not conjugate to an element of $A(x)$ for any $x \in D^*$, $\alpha^\tau(g) = 0$. This proves the first part.

Suppose that the element $g \in G$ is not conjugate to an element x of M_i with $C_{H_i}(x) \neq 1$ for any $i > 0$. Then, by Lemma L(b), $\sum(\alpha_{i1} - \alpha_{i2})^G(g) = 0$. Hence, we have

$$\alpha^\tau(g) = \alpha^G(g).$$

If g is not conjugate to any element of $X^\sharp$ then $\alpha^G(g) = 0$. Suppose that g is conjugate to an element $x \in X^\sharp$. If x is conjugate to an element y of D_i for some $i > 0$, then $C_G(y) \subseteq M_i$ and $C_{H_i}(y) \neq 1$. This contradicts the hypothesis. By Lemma I, g is conjugate to an element u of D_0. Then,

$$\alpha^\tau(g) = \alpha^G(u) = \alpha(u)$$

by Lemma L (a).

Suppose that $g \in M_i$ and $C_{H_i}(g) \neq 1$. By Lemma J (a), we may assume $g = hx$ with $x \in M \cap M_i$ and $h \in C_{H_i}(x)$. We may also assume that $\alpha^\tau(g) \neq 0$. By the first paragraph of the proof, g is conjugate to an element of $A(y)$ for some $y \in D_j$ $(j \geq 0)$. By Lemma J (b), $j = i$ and $x \in D_i$. Since x is a power of g,

$$C_G(g) \subseteq C_G(x) \subseteq M_i.$$

The conditions (Fii)(e), (a) and (c) yield that for $j \neq i$,

$$(|C_G(x)|, |H_j|) = 1.$$

Since $C_G(g) \subseteq C_G(x)$, g is not conjugate to any element u of M_j with $C_{H_j}(u) \neq 1$ and $j \neq i$. It follows from Lemma L that

$$\alpha^\tau(g) = \alpha^G(g) + (\alpha_{i1} - \alpha_{i2})^G(g).$$

Suppose that $h \neq 1$ in $g = hx$. Then, $\pi(\langle g \rangle) \cap \pi(H_i) \neq \emptyset$. Hence, by (Fii, c), g is not conjugate to any element of $X^\sharp$. Lemma L yields $\alpha^G(g) = 0$. Also, no conjugate of g lies in $M \cap M_i$ because $M \cap M_i$ is a $\pi(H_i)'$-subgroup. Thus, $\alpha_{i2}^G(g) = 0$. If $g_1 = v^{-1}gv$, $v \in G$, and $g_1 \in M_i$, then Lemma J (b) yields $g_1 \in A(M_i)$. Therefore, by (Fii,d), we have $v \in M_i$. This proves

$$\alpha^\tau(g) = \alpha_{i1}^G(g) = \alpha_{i1}(x).$$

Suppose that $h = 1$ in $g = hx$. All conjugates of g are contained in $A(M_i) \setminus H_i$. By (Fii, d),

$$(\alpha_{i1} - \alpha_{i2})^G(g) = (\alpha_{i1} - \alpha_{i2})(x).$$

Since $x \in M \cap M_i$, we have $\alpha_{i1}(x) = \alpha(x)$ and

$$\alpha_{i2}(x) = |C_{H_i}(x)|\alpha(x).$$

By (Fii, e), we have $|C_{H_i}(x)| = |C_G(x) : C_M(x)|$. Therefore, $\alpha^\tau(g) = \alpha(x)$ by Lemma L. Q.E.D.

Lemma 11.2. *Let $M \in \mathcal{M}$ and X an F-set of M. For $\alpha \in I_0(X)$, let α^τ be defined as in Definition K. Then, for $x \in A(M_i)$,*

$$\alpha^\tau(x) = \alpha_{i1}(x).$$

Furthermore, $(\alpha^\tau)_{M_i}$ is a linear combination of characters of M_i/H_i. If M_i is of type II, elements of $(M \cap M_i) \setminus (M_i)'$ are not contained in X and for $y \in M_i \setminus (M_i)'$

$$\alpha^\tau(y) = 0 = \alpha_{i1}(y).$$

Proof. By Lemma J, an element x of M_i is conjugate in M_i to an element of the form hu with $u \in M \cap M_i$ and $h \in C_{H_i}(u)$. Suppose $x \in A(M_i)$. Then $C_{H_i}(x) \neq 1$. Suppose that x is conjugate to an element of $A(y)$ for some $y \in D_j$ $(j \geq 0)$. By Lemma J (b), $u \in D_i$. Hence, by Lemma 11.1,

$$\alpha^\tau(x) = \alpha(u) = \alpha_{i1}(hu) = \alpha_{i1}(x)$$

because x is conjugate to hu in M_i. On the other hand, if x is not conjugate to any element of $A(y)$ for $y \in D^*$, then $u \notin X^\sharp$ and $\alpha^\tau(x) = 0$ by Lemma 11.1. Thus,

$$\alpha^\tau(x) = 0 = \alpha_{i1}(hu) = \alpha_{i1}(x)$$

because $\alpha_{i1}(hu) = \alpha_i(u) = \alpha(u) = 0$.

Suppose that M_i is of type II and $\alpha^\tau(y) \neq 0$ for some $y \in M_i \setminus (M_i)'$. Since M_i is of type II, $\kappa(M_i) = \{q\}$ and $q \in \pi(\langle y \rangle)$ for some prime q and $C_{H_i}(y) \neq 1$. The element y is conjugate to an element of the form hu with $u \in M \cap M_i$ and $h \in C_{H_i}(u)$. By Lemma 11.1, the assumptions of Lemma J (b) are satisfied. Thus, we have $u \in D_i$ and $y \in A(M_i)$ by Lemma J. This is a contradiction because $A(M_i) \subseteq (M_i)'$ in the group M_i of type II. Therefore, $\alpha^\tau(y) = 0$ for $y \in M_i \setminus (M_i)'$.

Since M_i is of type II, $M \cap M_i$ is a Frobenius group. Thus, if y is an element of $M \cap M_i$ outside $(M_i)'$, the order of y is q and $(|C_G(y)|, |H_i|) \neq 1$. Suppose that $y \in X^\sharp$. Then, y must be conjugate to an element $z \in D_i$. It follows that $z \in A(M_i)$ by (Fii, e). Since M_i is of type II, $A(M_i) \subseteq$

$(M_i)'$ and $(M_i)'$ is a Hall subgroup of M_i. This is a contradiction because both z and y have the same order. This proves

$$\alpha_{i1}(y) = \alpha(y) = 0.$$

It remains to prove that $(\alpha^\tau)_{M_i}$ is a linear combination of characters of M_i/H_i. Let θ be any irreducible character of M_i which does not have H_i in its kernel. By Lemma 4.3 [FT], θ vanishes on those elements x of M_i such that $C_{H_i}(x) = 1$. Compute $((\alpha^\tau)_{M_i}, \theta)$.

Suppose that M_i is of type I. Then, θ vanishes off $A(M_i)$ and $(\alpha^\tau)_{M_i}$ agrees with α_{i1} on $A(M_i)$. Hence,

$$((\alpha^\tau)_{M_i}, \theta) = (\alpha_{i1}, \theta) = 0.$$

This proves the assertion. If M_i is of type II, then both $(\alpha^\tau)_{M_i}$ and α_{i1} vanish outside $(M_i)'$. On $(M_i)'$, θ vanishes off $A(M_i)$ and $(\alpha^\tau)_{M_i} = \alpha_{i1}$ on $A(M_i)$. Therefore, we have

$$((\alpha^\tau)_{M_i}, \theta) = (\alpha_{i1}, \theta) = 0.$$

Lemma 11.3. *Let $M \in \mathcal{M}$ and X an F-set of M. For $\alpha \in I_0(X)$, let α^τ be defined as in Definition K. Then,*

$$(\alpha^\tau, 1_G)_G = (\alpha, 1_M)_M$$

where 1_G and 1_M are the principal characters of G and M, respectively.

Lemma 11.4. *Let $M \in \mathcal{M}$ and X an F-set of M. Let Θ be a virtual character of G that is well-behaved. If $\alpha, \beta \in I_0(X)$, then*

$$(\alpha^\tau, \Theta)_G = (\alpha, \Theta_M)_M, \quad (\alpha^\tau, \beta^\tau)_G = (\alpha, \beta)_M.$$

Lemma 11.5. *Let $M \in \mathcal{M}$ and X an F-set of M. Let Θ be a class function of G that is well-behaved. Let G_0 be the territory of the set X. Then, we have*

$$\frac{1}{|G|}\sum\nolimits_{x \in G_0} \Theta(x) = \frac{1}{|M|}\sum\nolimits_{x \in X^\sharp} \Theta(x).$$

The proof of each of the above three lemmas is similar to the corresponding proof of Lemmas 9.3, 9.4, and 9.5 in [FT]. We mention here that the assumption of Θ being well-behaved is essential in the proof.

§12. Coherent Set of Characters

Let $M \in \mathcal{M}$ and let X be an F-set of M. In the preceding section, we defined a mapping from $I_0(X)$ to the set of virtual characters of the group G. We will denote this mapping $\tau = \tau_M$ in the remainder of this paper. Its main property is stated in Lemma 11.4: *the mapping τ is an isometry on $I_0(X)$*. It is useful to extend the domain of τ so as to include some characters. For this purpose, the concept of coherent subsets has emerged; its definition is in Section 10 of [FT]. For the purpose of reference we state the definition.

If $\mathcal{S}$ is a set of virtual characters, we denote by $I_0(\mathcal{S})$ the set of linear combinations of elements of $\mathcal{S}$ with integer coefficients which take the value zero at the identity.

Definition. A set $\mathcal{S}$ of virtual characters of M is said to be coherent if and only if

(1) $I_0(\mathcal{S}) \neq \{0\}$ and $I_0(\mathcal{S}) \subseteq I_0(X)$, and
(2) It is possible to extend τ form $I_0(\mathcal{S})$ to a linear isometry mapping $\mathcal{S}$ into the set of virtual characters of G.

When $\mathcal{S}$ is a coherent set, an extension of τ to $I(\mathcal{S})$ will be denoted by the same letter τ. The following lemma which corresponds to Lemma 10.4 of [FT] illustrates the usefulness of the concept of coherency and suggests a tight connection between λ and λ^τ when λ^τ is defined.

Lemma 12.1. *Let $M \in \mathcal{M}$ and let X be an F-set of M. Let a be the least common multiple of all the orders of elements in X. Suppose that $\mathcal{S}$ is a coherent set of virtual characters of M such that $\mathcal{S}$ contains at least two irreducible characters. If λ is an irreducible character in $\mathcal{S}$, then the values assumed by λ^τ are contained in the field $\mathbb{Q}_a$ of the primitive ath roots of unity.*

Proof. Let $n = |G|$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_a)$. By assumption, $\mathcal{S}$ contains another irreducible character μ. Then,

$$\mu(1)\lambda - \lambda(1)\mu \in I_0(\mathcal{S})$$

and the values assumed by $(\mu(1)\lambda - \lambda(1)\mu)^\tau$ lie in $\mathbb{Q}_a$ by Lemma 11.1. Therefore,

$$\sigma(\mu(1)\lambda^\tau - \lambda(1)\mu^\tau) = \mu(1)\lambda^\tau - \lambda(1)\mu^\tau.$$

Since $\mathcal{S}$ is coherent, λ^τ and μ^τ are either irreducible characters or the negatives of irreducible characters of G. The same statement holds for $\sigma(\lambda^\tau)$ and $\sigma(\mu^\tau)$. It follows that $\sigma(\lambda^\tau) = \lambda^\tau$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_a)$. Thus, the values assumed by λ^τ lie in $\mathbb{Q}_a$. Q.E.D.

It follows easily from the definition that a subset $\mathcal{T}$ of a coherent set $\mathcal{S}$ is coherent provided $I_0(\mathcal{T}) \neq 0$. It is more difficult to decide whether or not the union of two or more coherent sets is coherent. One of the useful necessary conditions is Theorem 10.1 [FT]. We will state the theorem for the purpose of reference and refer the proof as well as the definition of a subcoherent set to the original paper [FT]. The following set of conditions and definitions is used.

Hypothesis 12.2. (i) Let $M \in \mathcal{M}$ and let X be an F-set of M.

(ii) For $1 \leq i \leq k$, $\mathcal{S}_i = \{\lambda_{is} \mid 1 \leq s \leq n_i\} \subseteq I(X)$.

(iii) $\mathcal{S} = \bigcup \mathcal{S}_i$ consists of pairwise orthogonal characters.

(iv) For any i $(1 \leq i \leq k)$, $\mathcal{S}_i$ is coherent with isomrtry τ_i, $\mathcal{S}_i$ is partitioned into sets $\mathcal{S}_{ij}$ such that each $\mathcal{S}_{ij}$ either consists of irreducible characters of the same degree and $|\mathcal{S}_{ij}| \geq 2$ or $(\mathcal{S}_{ij}, \tau_{ij})$ is subcoherent in $\mathcal{S}$ where τ_{ij} is the restriction of τ_i on $\mathcal{S}_{ij}$.

(v) For $1 \leq i \leq k$, $1 \leq s \leq n_i$, there exist integers ℓ_{is} such that

$$1 = \ell_{11} \leq \ell_{21} \leq \cdots \leq \ell_{k1},$$
$$\lambda_{is}(1) = \ell_{is}\lambda_{11}(1), \quad \text{and} \quad \ell_{i1} \mid \ell_{is}.$$

(vi) λ_{11} is an irreducible character of M.

(vii) For any integer m with $1 < m \leq k$,

$$\sum_{i=1}^{m-1} \sum_{s=1}^{n_i} \frac{\ell_{is}^2}{\|\lambda_{is}\|^2} > 2\ell_{m1}.$$

Theorem 12.3. *Suppose that Hypothesis* 12.2 *is satisfied. Then,* $\mathcal{S}$ *is coherent.*

The isometry on $\mathcal{S}$ is an extension of τ_i and is essentially unique (cf. Theorem 10.1 [FT]). The most important condition is the inequality (vii); we refer it as "the inequality"of Hypothesis 12.2.

For applications in this paper it is convenient to have a specialized set of conditions adapted to our case. To state the results we need further definitions.

Let $\mathcal{S}$ be a set of pairwise orthogonal characters. Define an equivalence relation on $\mathcal{S}$ by the condition that two characters in $\mathcal{S}$ are equivalent if and only if they have the same degree and the same weight. For any normal subgroup A, let $\mathcal{S}(A)$ be the subset of $\mathcal{S}$ consisting of those characters which are equivalent to some character in $\mathcal{S}$ that has A in its kernel.

Consider the following set of conditions.

Hypothesis 12.4. (i) Let $M \in \mathcal{M}$ and let $X = A(M)$.

(ii) Let H be a nilpotent normal subgroup of M such that

$$M_F \subseteq H \subseteq X.$$

Define $K = M$ if M is of type I; otherwise, let $K = M'$.

(iii) $\mathcal{S}$ is a set of characters of M which are induced by nonprincipal irreducible characters of K, each of which vanishes outside X. Assume that $I_0(\mathcal{S}) \neq 0$ and $\mathcal{S}$ consists of pairwise orthogonal characters.

(iv) There exista an integer d such that $d|M : K|$ divides $\lambda(1)$ for every $\lambda \in \mathcal{S}$. Furthermore, $\mathcal{S}$ contains an irreducible character of degree $d|M : K|$.

(v) Define an equivalence relation as before. Then, eqch equivalence class of $\mathcal{S}$ is either subcoherent in $\mathcal{S}$, or consists of irreducible characters and contains at least two characters.

Theorem 12.5. *Suppose that Hypothesis* 12.4 *is satisfied. Let* H_1 *be a normal subgroup of* M *such that* $H_1 \subseteq H$ *and*

$$|H : H_1| > 4d^2|M : K|^2 + 1.$$

If $\mathcal{S}(H_1)$ *is coherent and contains an irreducible character of degree* $d|M : K|$, *then* $\mathcal{S}$ *is coherent.*

This is Theorem 11.1 [FT] of page 817 which is proved under more complex conditions. Actually, we need to consider the case when the group M/H is a Frobenius group with Frobenius kernel K/H and $\mathcal{S}$ is the set of all irreducible characters of M/H that do not contain K/H in their kernel. In this case we will state the following result.

Lemma 12.6. *Let* M *be of type III or IV and let* $\mathcal{S}_0$ *be the set of all irreducible characters of* M/H *that do not contain* K/H *in their kernel. Then,* $\mathcal{S}_0$ *is coherent except possibly if* K/H *is a nonabelian* p*-group for some prime* p *and*

$$|(K/H) : (K/H)'| \leq 4|M : K|^2 + 1.$$

In this case, we have $(K/H)' = \Phi(K/H)$.

This is Lemma 11.2 [FT].

Lemma 12.7. *Let* $M \in \mathcal{M}$, $H \lhd M$, $H_1 \subseteq H$, $e = |M : H|$, *and* $h = |H : H_1|$. *Let* $\mathcal{S}$ *be the set of characters of* M *which are induced by nonprincipal irreducible characters of* H. *Suppose that* H *is an* F*-set of*

M and $\mathcal{S}$ is coherent. Assume further that $H_1 \lhd M$, H/H_1 is abelian, M/H_1 is a Frobenius group with Frobenius kernel H/H_1, and

$$|H : H_1| > (|M : H| + 1)|M : H| + 1.$$

Let $\zeta = (1_H)^M$ and let λ be an irreducible character of M/H_1 with degree e. Then, $\{\mathcal{S}, \zeta\}$ is coherent if we define

$$\zeta^\tau = (\zeta - \lambda)^\tau + \lambda^\tau.$$

Proof. Since M/H_1 is a Frobenius group with Frobenius kernel H/H_1, there are irreducible characters of degree $|M : H| = e$. In fact, there are $n = (h-1)/e$ such characters. Let $\lambda_1 = \lambda, \lambda_2, \dots, \lambda_n$ be those characters. Then, $\lambda_i - \lambda_j \in I_0(H)$. Thus, $\{\lambda_i^\tau\}$ are defined; they are virtual characters of G with weight one and satisfy

$$(\lambda_i - \lambda_j)^\tau = \lambda_i^\tau - \lambda_j^\tau.$$

Since $\alpha = \zeta - \lambda \in I_0(H)$, Lemma 11.4 yields $\|\alpha^\tau\|^2 = e + 1$,

$$(\alpha^\tau, (\lambda_i - \lambda_j)^\tau) = 0$$

and $(\alpha^\tau, (\lambda - \lambda_i)^\tau) = -1$ if $2 \le i, j \le n$. Write

$$\alpha^\tau = \Delta - \lambda^\tau.$$

Then, if $a_i = (\Delta, \lambda_i^\tau)$, then $a_1 = a_i$ for all i and

$$\Delta = 1_G + \sum a_i \lambda_i^\tau + \Delta_1$$

where $(\Delta_1, 1_G) = (\Delta_1, \lambda_i^\tau) = 0$. It follows that

$$1 + (a_1 - 1)^2 + (n-1)a_1^2 + \|\Delta_1\|^2 = \|\alpha\|^2 = e + 1.$$

If $a_1 \ne 0$, then we have $n - 1 \le e$. This contradicts the assumption. Thus, Δ does not involve any λ_i^τ. Hence, we have

$$\|\Delta\|^2 = e = \|\zeta\|^2.$$

Let σ be any character of $\mathcal{S}$. We want to prove $(\Delta, \sigma^\tau) = 0$. By definition, $\sigma = \mu^M$ for some nonprincipal irreducible character μ of H. Since $H \lhd M$, we have $(\zeta, \sigma) = 0$. Suppose that $(\Delta, \sigma^\tau) \ne 0$. Then, $\sigma \ne \lambda_i$. Choose $\lambda_2 \ne \lambda$ and consider

$$\beta = \mu(1)\lambda_2 - \sigma.$$

Then, $\beta \in I_0(H)$ and Lemma 11.4 yields $(\alpha^\tau, \beta^\tau) = (\alpha, \beta) = 0$ because $\lambda_2 \neq \lambda$ and $(\zeta, \sigma) = 0$. Since $\mathcal{S}$ is coherent,

$$\beta^\tau = \mu(1)\lambda_2^\tau - \sigma^\tau$$

and $(\lambda_1^\tau, \sigma^\tau) = 0$. Then,

$$0 = (\alpha^\tau, \beta^\tau) = (\Delta - \lambda^\tau, \mu(1)\lambda_2^\tau - \sigma^\tau) = -(\Delta, \sigma^\tau) \neq 0.$$

This contradiction proves $(\Delta, \sigma^\tau) = 0$ for every $\sigma \in \mathcal{S}$. Since $\|\Delta\|^2 - \|\zeta\|^2$, the set $\{\mathcal{S}, \zeta\}$ is coherent if we define $\zeta^\tau = \Delta$. Q.E.D.

§13. The Self Normalizing Cyclic Subgroup

Suppose that there is a subgroup in $\mathcal{M}$ that is not of type I. Then, by Theorem I, there is a cyclic subgroup $W = W_1 \times W_2$ such that $W_i \neq 1$ for $i = 1, 2$ and $N_G(W_0) = W$ for any nonempty subset W_0 of $\widehat{W} = W \setminus \{W_1, W_2\}$. Consequences of the existence of such a subgroup are very important. They are discussed in Section 13 of [FT]. We will briefly review them and introduce the notation.

Let ω_{10} and ω_{01} be faithful irreducible characters of W/W_2 and W/W_1, respectively. Define

$$\omega_{ij} = \omega_{10}^i \omega_{01}^j$$

for $0 \leq i < w_1 = |W_1|$ and $0 \leq j < w_2 = |W_2|$. Thus, ω_{00} is the principal character of W. The following lemma is the key to applications and serves as introduction of the family of virtual characters $\{\eta_{ij}\}$ of G.

Lemma 13.1. *The set $\widehat{W}$ is a TI-subset with normalizer W in G (in fact, in any subgroup that contains W). There exists an orthonormal set $\{\eta_{ij}\}$ of virtual characters of G such that for $0 \leq i < w_1$ and $0 \leq j < w_2$, the value assumed by η_{ij}, η_{i0}, η_{0j} lie in $\mathbb{Q}_w$, $\mathbb{Q}_{w_1}$, $\mathbb{Q}_{w_2}$, respectively. We have $\eta_{00} = 1_G$, $\eta_{ij}(x) = \omega_{ij}(x)$ for $x \in \widehat{W}$, and*

$$(\omega_{00} - \omega_{i0} - \omega_{0j} + \omega_{ij})^G = 1_G - \eta_{i0} - \eta_{0j} + \eta_{ij}$$

for $1 \leq i < w_1$ and $1 \leq j < w_2$. In particular, the right side of the above equality is a virtual character that vanishes outside $\mathcal{C}_G(\widehat{W})$. Furthermore, every irreducible character of G distinct from $\{\pm\eta_{ij}\}$ vanishes on $\widehat{W}$.

The proof is in Lemma 13.1 [FT]. The set $\{\eta_{ij}\}$ is orthonormal. Therefore, either η_{ij} or $-\eta_{ij}$ is an irreducible character of G and they are distinct.

Lemma 13.2. *Suppose that a virtual character $\alpha = \sum a_{ij}\omega_{ij}$ of W vanishes on $\widehat{W}$. Then, for all s and t, we have*

$$a_{00} - a_{s0} - a_{0t} + a_{st} = 0.$$

If in addition $\alpha = \beta_1 + \beta_2$ with $\|\beta_1\|^2 = \|\beta_2\|^2 = 2$, then $\alpha = 0$.

Proof. The first part is proved as in the proof of Lemma 13.2 [FT]. The second half follows by case-by-case analysis. Q.E.D.

Theorem I yields that the subgroup W is contained in two subgroups S and T of $\mathcal{M}$ such that neither S nor T is of type I,

$$S \cap T = W, \quad S'W_1 = S, \quad T'W_2 = T,$$

and $S' \cap W_1 = T' \cap W_2 = 1$. We can apply Lemma 13.1 to S and T. Thus, each subgroup has a family of orthonormal virtual characters corresponding to the family $\{\omega_{ij}\}$. The following lemma serves to define the notation.

Lemma 13.3. *Let $M = S$ and let $H = M_F$. Suppose that M is not of type I. Then, $W_2 \subseteq H \subseteq M'$ and $W \setminus W_2$ is a TI-subset of M. There is a complement V of H in M' that is normalized by W_1. The group VW_1 is a Frobenius group with Frobenius kernel V. The group V is nilpotent; if M is of type II, V is abelian.*

Proof. All the conditions follow from the conditions (T1)–(T7) in the definition of groups not of type I in [BG], page 128. Thus, (T1) yields $H \subseteq M'$, while (T4) yields $W_2 \subseteq H$ and $C_{M'}(x) = W_2$ for all $x \in W_1{}^\sharp$. It follows that $C_M(x) = W$ if $x \in W \setminus W_2$. Therefore, $W \setminus W_2$ is a TI-subset of M with normalizer W. The remaining conditions also follow from (T1)–(T7). Q.E.D.

Lemma 13.4. *Let $M \in \mathcal{M}$ be not of type I. Use the notation in Lemma* 13.3*. Then, M has a family of irreducible characters μ_{ij} $(0 \le i < w_1, 0 \le j < w_2)$ such that for some $\varepsilon_j = \pm 1$*

$$\mu_{ij}(x) = \varepsilon_j \omega_{ij}(x)$$

for all $x \in \widehat{W}$. The family of virtual characters $\{\varepsilon_j \mu_{ij}\}$ is the one corresponding to $\{\omega_{ij}\}$ in Lemma 13.1. For each k, $(\mu_{ik})_{M'} = (\mu_{jk})_{M'}$ and μ_k defined by $\mu_k = (\mu_{ik})_{M'}$ is an irreducible character of M'. Define $\xi_k = \sum_i \mu_{ik}$. Then

$$\xi_k = (\mu_k)^M = \sum\nolimits_i \mu_{ik}.$$

Proof. By Lemma 13.1, there is a family of irreducible characters $\{\mu_{ij}\}$ such that $\pm\mu_{ij}(x) = \omega_{ij}(x)$ for all $x \in \widehat{W}$. Set $q = |W_1|$. Then, M has exactly q linear characters because $M/M' \cong W_1$. Let ζ be the linear character such that $\zeta_W = \omega_{10}$. Then, $\{\zeta^i\}$ $(0 \leq i < w_1)$ is the set of linear characters of M and $(\zeta^i)_W = \omega_{i0}$. Let $\varepsilon_j = \pm 1$ so that

$$\mu_{0j}(x) = \varepsilon_j \omega_{0j}(x)$$

for $x \in \widehat{W}$. Since ζ^i is a linear character, $\zeta^i \mu_{0j}$ is an irreducible character of M. Consider the restriction of $\zeta^i \mu_{0j}$ on $\widehat{W}$. We have for $x \in \widehat{W}$

$$\zeta^i \mu_{0j}(x) = \zeta^i(x)\mu_{0j}(x) = \varepsilon_j \omega_{i0}(x)\omega_{0j}(x) = \varepsilon_j \omega_{ij}(x).$$

Since the characters ω_{ij} are distinct on $\widehat{W}$, Lemma 13.1 yields

$$\zeta^i \mu_{0j} = \mu_{ij}.$$

Thus, $\mu_{ij}(x) = \varepsilon_j \omega_{ij}(x)$ for $x \in \widehat{W}$. This proves that $\{\varepsilon_j \mu_{ij}\}$ is the family corresponding to $\{\omega_{ij}\}$ in M. Clearly, $\mu_k = (\mu_{ik})_{M'}$ is independent of i. By the tensor product formula, we have

$$\xi_k = (\mu_k)^M = \mu_{0k} \otimes (1_{M'})^M = \sum\nolimits_i \mu_{ik}.$$

Since $(\xi_k)_M = q\mu_k$, the orthogonality relations yield

$$q = (\xi_k, \xi_k) = (\mu_k^M, \xi_k) = (\mu_k, q\mu_k)_M = q\|\mu_k\|^2.$$

Therefore, μ_k is an irreducible character of M'. Q.E.D.

The set $W \setminus W_2$ is a TI-subset of M by Lemma 13.3. For each k $(0 \leq k < w_2)$, the set $\{\omega_{ik} \mid 0 \leq i < w_1\}$ is coherent and the characters $\{\omega_{ik}^\tau\}$ are $\{\varepsilon_k \mu_{ik}\}$ (cf. Lemma 13.3 of [FT]).

Lemma 13.5. *Let $M \in \mathcal{M}$ be not of type I and use the notation in Lemma 13.3. Then, an irreducible character of M' induces either an irreducible character of M or one of the characters ξ_j $(0 \leq j < w_2)$.*

The proof of Lemma 13.7 of [FT] gives the result.

Lemma 13.6. *Let M and $\{\mu_{ij} \mid 0 \leq i < w_1, 0 \leq j < w_2\}$ be as in Lemma 13.4. Suppose that for some i, j, k with $0 \leq i < w_1$, $1 \leq j, k < w_2$, we have $\mu_{ij}(1) = \mu_{ik}(1)$. Then, $\mu_{ij} - \mu_{ik} \in I_0(A_0(M))$ and*

$$(\mu_{ij} - \mu_{ik})^\tau = \pm(\eta_{ij} - \eta_{ik})$$

where η_{ij}, η_{ik} are virtual characters of G defined in Lemma 13.1.

Proof. The factor group M/H is isomorphic to the group VW_1 which is a Frobenius groupwith Frobenius kernel$V$. By 3.16 of [FT], every nonprincipal irreducible character of M'/H induces an irreducible character of M. Therefore, Lemma 13.5 yields that μ_{ij} with positive j does not contain H in its kernel. By Lemma 4.3 [FT], these μ_{ij} vanish on $M' \setminus A(M)$. Thus, if $j, k > 0$ and $\mu_{ij}(1) = \mu_{ik}(1)$, then for $X = A_0(M)$,

$$\mu_{ij} - \mu_{ik} \in I_0(X).$$

Since τ is an isometry on $I_0(X)$, $(\mu_{ij} - \mu_{ik})^\tau$ is the difference of two irreducible characters.

We have $\widehat{W} \subseteq X$. If $x \in \widehat{W}$, then $C_G(x) = W \subseteq M$. Thus, x is not conjugate to any element in $A(y)$ for $y \in D_t$ with $t > 0$. Hence, Lemma 11.1 yields that

$$(\mu_{ij} - \mu_{ik})^\tau(x) = (\mu_{ij} - \mu_{ik})(x)$$

for $x \in \widehat{W}$. It follows that $(\mu_{ij} - \mu_{ik})^\tau$ is the difference of two characters of the form $\pm\eta_{st}$. Since $\eta_{ij}(x) = \omega_{ij}(x)$ for $x \in \widehat{W}$, Lemma 13.2 yields that $(\mu_{ij} - \mu_{ik})^\tau = \pm(\eta_{ij} - \eta_{ik})$. Q.E.D.

Lemma 13.7. *Let M, $\{\mu_{ij}\}$, andξ_k be as in Lemma* 13.4. *Choose k with $1 \leq k < w_2$. Let $\mathcal{S}_1 = \{\xi_j \mid 1 \leq j < w_2, \xi_j(1) = \xi_k(1)\}$. Then, $\mathcal{S}_1$ is coherent and*

$$\xi_j^\tau = \varepsilon \sum\nolimits_i \eta_{ij}$$

for some $\varepsilon = \pm 1$. Furthermore, if $\mathcal{S}$ is the set of characters of M which are induced by the nonprincipal irreducible characters of M' that vanish outside $A(M)$, then $(\mathcal{S}_1, \tau)$ is subcoherent in $\mathcal{S}$.

The proofs of Lemmas 13.9 and 13.10 in [FT] can be adapted to a proof of the above lemma by changing the references suitably (and correcting a misprint).

Lemma 13.8. *Let $M \in \mathcal{M}$ be of type II or III, $H = M_F$, and $q = |W_1|$. For positive integers r and s with $r > 1$, let $A(r,s)$ be the set of nonprincipal irreducible characters α of H such that $|I(\alpha) : H| = qr$ and $\alpha(1) = s$. Let $B(r,s)$ be the set of characters of M induced from the irreducible components of $\alpha^{M'}$ with $\alpha \in A(r,s)$. Then, $B(r,s)$ consists of characters of the same degree and $B(r,s)$ is coherent.*

Proof. Since M is of type II or III, the factor group M'/H is abelian and H is a Hall subgroup of M. We can apply Lemma 4.5 of [FT]. If

$\alpha \in A(r,s)$, then the inertia index of α in M' is r. So, by Lemma 4.5 of [FT], $\alpha^{M'}$ is a sum of exactly r distinct irreducible characters $\theta_1, \dots, \theta_r$ of M' of degree $|M' : H|s/r$.

Lemma 13.5 yields that elements of $B(r,s)$ are irreducible characters or one of the characters ξ_j. Let $\mathcal{S}_1$ be the set of irreducible characters in $B(r,s)$ and let $\mathcal{S}_2$ be the set of ξ_j which are in $B(r,s)$. The characters of $B(r,s)$ have the same degree. By Lemma 4.3 [FT], they vanish outside $A(M)$. If $\beta \in \mathcal{S}_i$ for $i = 1, 2$, then the complex conjugate $\overline{\beta} \in \mathcal{S}_i$. Thus, $I_0(\mathcal{S}_i) \neq \emptyset$. It follows that each subset $\mathcal{S}_i$ is coherent. We want to prove that the union $\mathcal{S}_1 \cup \mathcal{S}_2$ is coherent. Since $\mathcal{S}_1$ consists of irreducible characters of the same degree and $\mathcal{S}_2$ is subcoherent (by Lemma 13.7), Theorem 12.3 yields that $\mathcal{S}_1 \cup \mathcal{S}_2$ is coherent provided the inequality of Hypothesis 12.2 is satisfied. The condition becomes $|\mathcal{S}_1| > 2$ in this case. We need to examine the set of irreducible components $\{\theta_i\}$ of $\alpha^{M'}$. By assumption $|I(\alpha) : H| = qr$ with $r > 1$. We may assume that $Q = W_1 \subseteq I(\alpha)$ because Q is a Hall subgroup of M (by (T1)). Then, $I(\alpha) \cap VW_1 = RQ$ where $R = V \cap I(\alpha)$ is contained in the inertia group I of α in M' and $|R| = r$.

From the proof of Lemma 4.5 [FT] at the beginning of Section 11 of this paper, we have

$$\alpha^I = \gamma_1 + \gamma_2 + \cdots + \gamma_r$$

where γ_i are irreducible characters of I. Since α is Q-invariant, Q permutes these characters $\{\gamma_i\}$. Since $r \equiv 1 \pmod q$, one of them, say γ_1, is Q-invariant. We can write $\gamma_i = \gamma_1\mu_i$ where $\mu_1, \dots, \mu_r$ are the set of linear characters of I/H and μ_1 is the principal character. Since $I/H \cong R$, Q acts on the set of nonprincipal characters $\{\mu_2, \dots, \mu_r\}$ without fixed points. Thus, if the notation is such that $\gamma_i^{M'} = \theta_i$, then θ_1 is Q-invariant and all the other θ_i for $i > 1$ induce irreducible characters of $B(r,s)$. Therefore, each $\alpha \in A(r,s)$ contributes one character of $\mathcal{S}_2$ and $(r-1)/q$ characters of $\mathcal{S}_1$. Since $r > 1$ is odd, we have $(r-1)/q \geq 2$. If $\alpha \in A(r,s)$, then $\overline{\alpha} \in A(r,s)$ and $\overline{\alpha}$ is not conjugate to α in M. It follows that $|\mathcal{S}_1| \geq 4$. This proves that $B(r,s)$ is coherent. Q.E.D.

§14. Further Properties of Coherent Sets

In this section, we use the following notation. Let $M \in \mathcal{M}$ and let X be an F-set of M. Let H be one of the supporting subgroups for the set X with $N = N_G(H) \in \mathcal{M}$. Thus, H is N_F, i.e. the largest normal nilpotent Hall subgroup of N. Define N_0 as follows. If N is of type I, let $N_0 = N$, while if N is not of type I, then $N_0 = N'$. It follows from the definition that $A(N) \subseteq N_0$.

The following two lemmas correspond to Lemmas 10.2 and 10.3 of [FT].

Lemma 14.1. *Let M, X, H, N and N_0 be as above. For each nonprincipal irreducible character α of H, let $S(\alpha)$ be the set of irreducible characters of N_0 which are involved in α^{N_0}, and let $T(\alpha)$ be the set of the virtual characters of G of the form $(\theta_1 - \theta_2)^G$ with θ_1, $\theta_2 \in S(\alpha)$. If Θ is a virtual character of G which is orthogonal to the elements of $T(\alpha)$ for all $\alpha \neq 1_H$, then Θ is constant on the cosets of H which lie in $N_0 \setminus H$.*

Proof. The subgroup H is a Hall subgroup of N by definition of groups of type I or II. If N is of type I, then N satisfies the assumptions of Lemma 4.5 [FT]. If N is of type II, then $N_0 = N'$ and N_0/H is abelian by (IIiii). Lemma 4.5 [FT] is applicable to N_0. In all cases, α^{N_0} is a sum of irreducible characters of the same degree with multiplicity one.

Fix a nonprincipal irreducible character α of H. If θ_1, $\theta_2 \in A(\alpha)$.

$$(\Theta_{N_0}, \theta_1 - \theta_2) = (\Theta, (\theta_1 - \theta_2)^G) = 0.$$

Thus, Θ_{N_0} contains each $\theta \in S(\alpha)$ with the same multiplicity. Since the sum of all $\theta \in S(\alpha)$ is α^{N_0}, we have

$$\Theta_{N_0} = \Theta_1 + \beta^{N_0}$$

where Θ_1 is a virtual character of the group N_0/H and β is a virtual character of H. Since β^{N_0} vanishes outside H, Θ_{N_0} is constant on the cosets of H lying in $N_0 \setminus H$. Q.E.D.

Lemma 14.2. *Suppose that M, X, H and N_0 are as in Lemma 14.1. Let $\mathcal{S}$ be a coherent subset of $I(X)$ that contains at least two irreducible characters. For any $\lambda \in \mathcal{S}$, λ^τ is constant on the cosets of H that lie in $N_0 \setminus H$.*

Proof. Take any nonprincipal irreducible character α of H and let $S(\alpha)$ be the set of irreducible characters of N_0 defined in Lemma 14.1. We will show that for θ_1, $\theta_2 \in S(\alpha)$

$$((\lambda^\tau)_{N_0}, \theta_1 - \theta_2) = 0.$$

Assume that this does not hold. Let λ_1,$\lambda_2 \in \mathcal{S}$ be distinct irreducible characters. Then

$$\beta = \lambda_1(1)\lambda - \lambda(1)\lambda_1 \in I_0(X).$$

By Lemma 11.2, $(\beta^\tau)_N$ is a linear combination of characters of N/H. Hence,

$$((\beta^\tau)_{N_0}, \theta_1 - \theta_2) = 0.$$

It follows that

$$((\lambda_1^\tau)_{N_0}, \theta_1 - \theta_2) \neq 0.$$

Similarly, we have $((\lambda_2^\tau)_{N_0}, \theta_1 - \theta_2) \neq 0$.

Suppose that N is of type I so $N_0 = N$. By Lemma 4.3 of [FT], $\theta_i \in S(\alpha)$ vanishes outside $A(N)$. Since θ_1 and θ_2 are equal on H, $\theta_1 - \theta_2$ vanishes outside $A(N) \setminus H$. Since N is of type I, $A(N) = A_0(N)$ and (Fii,d) yields that $A_0(N) \setminus H$ is a TI-subset of G with normalizer N. Hence, $(\theta_1 - \theta_2)^G = \Theta_1 - \Theta_2$ where Θ_i are irreducible characters of G. We have

$$(\lambda_i^\tau, (\theta_1 - \theta_2)^G) = ((\lambda_i^\tau)_N, \theta_1 - \theta_2) \neq 0.$$

It follows that the irreducible character $\pm\lambda_i^\tau$ is either Θ_1 or Θ_2. We may assume that $\lambda_i^\tau = \varepsilon\Theta_i$ for $\varepsilon = 1$ or -1. It is crucial that

$$\lambda_1(1)\lambda_2^\tau - \lambda_2(1)\lambda_1^\tau$$

vanishes at the identity so both λ_1^τ and λ_2^τ are irreducible characters or both of them are not. Then, $\lambda_1(1) = \lambda_2(1)$ and

$$\lambda_1 - \lambda_2 \in I_0(X).$$

By Lemma 11.2, $((\lambda_1 - \lambda_2)^\tau)_N$ is orthogonal to $\theta_1 - \theta_2$. Thus,

$$\begin{aligned} 0 &= ((\lambda_1^\tau - \lambda_2^\tau)_N, \theta_1 - \theta_2) \\ &= (\lambda_1^\tau - \lambda_2^\tau, (\theta_1 - \theta_2)^G) \\ &= (\varepsilon(\Theta_1 - \Theta_2), (\Theta_1 - \Theta_2)) = 2\varepsilon. \end{aligned}$$

This contradiction yields that λ^τ is orthogonal to every element in $T(\alpha)$ for any $\alpha \neq 1_H$. By Lemma 14.1, λ^τ is constant on a coset of H that lies in $N \setminus H$.

Assume that N is a group of type II. Suppose that

$$((\lambda^\tau)_{N_0}, \theta_1 - \theta_2) \neq 0$$

for some θ_1, $\theta_2 \in S(\alpha)$. Then, θ_1^N and θ_2^N are distinct characters of N. If they are irreducible, then the previous argument can be applied here. In this case, θ_1^N and θ_2^N vanish outside $A_0(N) \setminus H$, and $A_0(N) \setminus H$ is a TI-subset in G by (Fii,d). Hence, $(\theta_1 - \theta_2)^G$ is a difference of two irreducible characters. We obtain a contradiction as before.

Suppose that θ_1^N is not irreducible. In this case, we are in the situation of Lemma 13.8. The set $A(N)$ is an F-set of N. Let $\sigma = \tau_N$ be the isometry associated with N. Then, by Lemma 13.8, the set $S(\alpha) \cup S(\overline{\alpha})$ is coherent with respect to the isometry σ and for some j. We have

$$(\theta_1^N)^\sigma = \varepsilon \sum_i \eta_{ij}$$

where $\{\eta_{ij}\}$ is the family of characters defined in Lemma 13.1. Thus,

$$(\theta_1^N)^\sigma - (\theta_2^N)^\sigma = \varepsilon \sum_i \eta_{ij} - \Theta_2$$

where $\{\Theta_i\}$ is the family of virtual characters $\{\theta_i^N \mid i > 1\}$.

Since N is of type II, Theorem II (F*iii*) yields that M is a Frobenius group and $X = M_F$. Let $q = |M : M'|$. By Lemma 11.2, elements of N of order q are not contained in X. Since $X = M_F$ is a Hall subgroup of M, no element of X has order q. Lemma 12.1 yields that λ_1^τ as well as λ_2^τ is q-rational.

The virtual character $\theta_1 - \theta_2$ vanishes outside $A(N) \setminus H$. If $g \in A(N) \setminus H$ is conjugate to an element of the form hx where $x \in N \cap M$ and $h \in C_{H_i}(x)$, then Lemma J (b) yields that $x \in D_i$. By Theorem II, we have $N = \mathcal{M}(C_G(x))$. Since g is conjugate in N to an element having this property, $C_G(g) \subseteq N$. In other words, no suppoting subgroup contributes any to $(\theta_1^N - \theta_2^N)^\sigma(g)$. It follows that

$$(\theta_1 - \theta_2)^G = (\theta_1^N - \theta_2^N)^\sigma = \varepsilon \sum_i \eta_{ij} - \Theta_2$$

for some j. We have shown that λ_i^τ is not equal to $\pm\Theta_k$. Since both λ_1^τ and λ_2^τ are not orthogonal to $(\theta_1 - \theta_2)^G$, both λ_1^τ and θ_2^τ are one of $\pm\eta_{ij}$. However, at most one of the characters $\pm\eta_{ij}$ is q-rational for a given j. This contradiction proves that λ^τ is constant on the cosets of H that lie in $N_0 \setminus H$. Q.E.D.

Lemma 14.3. *Let $M \in \mathcal{M}$, X an F-set of M, and let $\mathcal{S}$ be a coherent subset of $I(X)$. If $\mathcal{S}$ contains at least two irreducible characters, every $\lambda \in \mathcal{S}$ satisfies the property that λ^τ is constant on the set of the form $A(x)$ for every $x \in D^*$.*

Proof. The sets $A(x)$ and D^* are defined at the beginning of Chapter II. If $x \in D_0$, then $C_G(x) \subseteq M$. In this case, $A(x) = \{x\}$ and the assertion is trivial. Suppose that $x \in D_i$ for some $i > 0$. Then, $C_G(x) \subseteq M_i$ for some supporting subgroup M_i. By (F*ii*, e), we have

$x \in A(M_i)$. It follows from the definition of the subgroup $(M_i)_0$ at the beginning of this section that $A(M_i) \subseteq (M_i)_0$. Since $A(x)$ is contained in a coset of H_i in $(M_i)_0 \setminus H_i$, Lemma 14.2 yields that λ^τ is constant on the set $A(x)$. Q.E.D.

For any virtual character λ, the set of irreducible characters ρ such that $(\lambda, \rho) \neq 0$ is called the *support* of λ.

The following lemma corresponds Lemma 10.5 of [FT].

Lemma 14.4. *Let X be an F-set of $M \in \mathcal{M}$. Let $\mathcal{S}$ be a coherent set consisting of characters of $I(X)$ with disjoint supports and let Θ be a virtual character of M that is well behaved. Suppose that there is a virtual character θ of M such that for every $\alpha \in I_0(\mathcal{S})$,*

$$(\alpha^\tau, \Theta) = (\alpha, \theta).$$

Then, there is a pair (r, β) of a rational number r and a virtual character β of M such that β is orthogonal to every element of $\mathcal{S}$ and

$$\Theta(g) = \theta(g) + r\beta(g) \quad \textit{for} \quad g \in X^\sharp.$$

Suppose that Θ_1 is a well behaved virtual character of G that is orthogonal to every element of $\mathcal{S}^\tau$, then there is a pair (r_1, β_1) of a rational number r_1 and a virtual character β_1 of M such that β_1 is orthogonal to every element of $\mathcal{S}$ and $\Theta_1(g) = r_1\beta_1(g)$ for $g \in X^\sharp$.

Suppose that $\mathcal{S}$ contains at least two irreducible characters of M. Then, for any $\lambda \in \mathcal{S}$, there is a pair (s, γ) of a rational number s and a virtual character γ of M, depending on λ, such that γ is orthogonal to every element of $\mathcal{S}$ and

$$\lambda^\tau(g) = \lambda(g) + s\gamma(g) \quad \textit{for} \quad g \in X^\sharp.$$

Proof. Since $\mathcal{S}$ is coherent, $I_0(\mathcal{S}) \neq 0$. Therefore, $\mathcal{S}$ contains at least two characters. Let λ, $\mu \in \mathcal{S}$. Then, $\alpha = \lambda(1)\mu - \mu(1)\lambda$ is an element of $I_0(X)$. Since Θ is well behaved, Lemma 11.4 yields

$$(\alpha^\tau, \Theta) = (\alpha, \Theta_M).$$

By assumption, there exists a virtual character θ such that $(\alpha, \Theta_M - \theta) = 0$. For each $\sigma \in \mathcal{S}$, let $\theta(\sigma)$ be the portion of $\Theta_M - \theta$ on the support of σ. Thus,

$$\Theta_M - \theta = \sum_{\sigma \in \mathcal{S}} \theta(\sigma) + \Delta_1$$

where Δ_1 is a linear combination of irreducible characters not involved in any $\sigma \in \mathcal{S}$. Since

$$0 = (\alpha, \Theta_M - \theta) = (\lambda(1)\mu - \mu(1)\lambda, \Theta_M - \theta),$$

the orthogonality relations yield

$$\mu(1)(\lambda, \theta(\lambda)) = \lambda(1)(\mu, \theta(\mu)).$$

Thus, for a rational number s,

$$(\lambda, \theta(\lambda)) = s\lambda(1) \text{ for every } \lambda \in \mathcal{S}.$$

Let $\rho(\sigma)$ be the portion of the regular representation ρ on the support of $\sigma \in \mathcal{S}$. If $\sigma = \sum_i a_i\xi_i$ is the decomposition of σ into the sum of irreducible characters ξ_i, then $\rho(\sigma) = \sum_i \xi_i(1)\xi_i$. Hence,

$$(\sigma, \rho(\sigma)) = \sum_i a_i\xi_i(1) = \sigma(1).$$

Let $\rho = \sum_\sigma \rho(\sigma) + \Delta_2$. Then, Δ_2 is a linear combination of irreducible characters not involved in any $\sigma \in \mathcal{S}$. Set $s = m/n$ with integers m, n and define

$$r = 1/n \quad \text{and} \quad \beta = \sum_{\sigma \in \mathcal{S}} (n\theta(\sigma) - m\rho(\sigma)) - m\Delta_2 + n\Delta_1.$$

Then, for $x \in X^\sharp$,

$$r\beta(x) = (\sum \theta(\sigma) + \Delta_1)(x) - s\rho(x) = \Theta_M(x) - \theta(x).$$

We compute (σ, β). Since the supports of elements of $\mathcal{S}$ are disjoint, we have

$$(\sigma, \beta) = n(\sigma, \theta(\sigma)) - m(\sigma, \rho(\sigma)) = 0.$$

This proves the first part.

For the second part, $\theta_1 = 0$ satisfies the assumption of the first part for Θ_1 since $(\alpha^\tau, \Theta) = 0$ for every $\alpha \in I_0(X)$. For the third part, Lemma 11.4 yields with $\Theta = \lambda^\tau$,

$$(\alpha^\tau, \Theta) = (\alpha^\tau, \lambda^\tau) = (\alpha, \lambda)$$

for all $\alpha \in I_0(X)$. The first part applies. Q.E.D.

§15. Characters of Subgroups of Type I

Let $M \in \mathcal{M}$ be a subgroup of type I. Let $H = M_F$, $X = A(M)$, and let $\mathcal{S}$ be the set of irreducible characters of M that do not have H in their kernel. Let E be a complement of H in M. By (I*iii*), E contains a subgroup E_0 of the same exponent as E such that HE_0 is a Frobenius group with Frobenius kernel H. With the notation introduced here we prove the following lemma.

Lemma 15.1. *Let $M \in \mathcal{M}$ be a subgroupof type I. Then the set $\mathcal{S}$ defined above satisfies Hypothesis* 12.4 *with $H = M_F$, $K = M$ and $d = |E_0|$.*

Proof. If $\lambda \in \mathcal{S}$, Lemma 4.3 of [FT] yields that λ vanishes outside X. If $\lambda \in \mathcal{S}$, then the complex conjugate character $\overline{\lambda}$ is different from λ and $\overline{\lambda} \in \mathcal{S}$. Thus, $I_0(\mathcal{S}) \neq 0$ and $\mathcal{S}$ satisfies (*iii*) of Hypothesis 12.4. Since $\mathcal{S}$ consists of irreducible characters, $\mathcal{S}$ satisfies (*v*).

The definition of groups of type I implies that elements of $A(M) \cap E$ are τ_2-elements in the notation of §6. Therefore, $E = AB$ with $(|A|, |B|) = 1$ such that A is abelian and B is a Z-group (cf. Hypothesis 28.1 [FT]). In fact, $E = E_1E_2E_3$, E_2 is abelian, and E_1 and E_3 are cyclic groups of relatively prime order by Lemmas 6.1 and 6.8, and Theorem 6.7. We may take $A = E_2$ and $B = E_1E_3$. Since E_0 has the same exponent as E, the order of B divides $|E_0|$. Conjugacy of Hall subgroups in a solvable group yields that we may assume $B \subseteq E_0$. Furthermore, we may assume that A contains a Hall $\pi(A)$-subgroup of E_0.

Since HE_0 is a Frobenius group with Frobenius kernel H, no element of E_0 stabilizes any nonprincipal irreducible character of H. Thus, for any $\lambda \in \mathcal{S}$, the number of conjugate characters $|M : I(\lambda)|$ is divisible by $|E_0|$. By (I*ii*), the normal closure of $I(\lambda)/H$ is abelian. Therfore, Lemma 4.5 [FT] yields that λ^M is a sum of irreducible characters of degree $|M : I(\lambda)|\lambda(1)$. Thus, $d = |E_0|$ divides $\lambda(1)$ for every $\lambda \in \mathcal{S}$.

It remains to prove that $\mathcal{S}$ contains an irreducible character of degree exactly $d = |E_0|$. This is proved as in Lemma 28.1 of [FT]. Let $E = AB$ as above. Since H is nilpotent, $H/\Phi(H)$ is elementary abelian. Let L be a maximal A-invariant subgroup such that $\Phi \subseteq L \subset H$, and let $A_1 = C_A(H/L)$. Then, A acts on H/L irreducibly and A/A_1 is cyclic. Since E_0 has the same exponent as E and $E_0 \cap A_1 = 1$, $|A/A_1|$ is equal to the exponent of A. This implies that $E_0A_1 = E$. Let λ be a nonprincipal linear character of H/L. Then, $HA_1 = I(\lambda)$. Therefore, Lemma 4.5 [FT] yields that λ^M is a sum of irreducible characters of degree exactly equal to $|E_0| = d$. Q.E.D.

Remark. At the final stage of proof, Lemma 4.5 [FT] yields that λ^M is a sum of exactly $|A_1|$ irreducible characters. We have

$$|A_1| = |E : E_0|.$$

Thus, M has at least $2|E : E_0|$ irreducible characters of degree d.

Theorem 15.2. *Let $M \in \mathcal{M}$ be a group of type I and let $\mathcal{S}$ be as in Lemma* 15.1. *If $H/\Phi(H)$ is not a chief factor of M, then $\mathcal{S}$ is coherent. The assumption on H is satisfied if $Z(E)$ contains an element x such that $C_H(x) \not\subseteq H'$ and $C_H(x) \neq H$.*

Proof. Let $d = |E_0|$. Since E_0 acts regularly on H, each chief factor of M in H has order at least $2d + 1$ It follows that $|H : H'| > 4d^2 + 1$. By Theorem 12.5, $\mathcal{S}$ is coherent if $\mathcal{S}(H')$ is coherent.

Let $\mathcal{S}_1, \dots, \mathcal{S}_k$ be the equivalence classes of characters in $\mathcal{S}(H')$. Each $\mathcal{S}_i$ is a set of irreducible characters of the same degree. If $\lambda \in \mathcal{S}_i$, then $\overline{\lambda} \in \mathcal{S}_i$. Thus, $|\mathcal{S}_i| \geq 2$. Hence, by Lemma 10.1 [FT], each $\mathcal{S}_i$ is coherent. Let $n_i = |\mathcal{S}_i|$ and let $d\ell_i$ be the common degree of the characters of $\mathcal{S}_i$. We may choose the notation so that

$$\ell_1 = 1 < \ell_2 < \cdots < \ell_k.$$

Lemma 4.5 [FT] yields that for any nonprincipal linear character α, the irreducible component of α^M has degree $|M : I(\alpha)|$. Since $d = |E_0|$, we get $\ell_i \leq |E : E_0|$.

If $\mathcal{S}(H')$ is not coherent, Theorem 12.3 yields that the inequality of Hypothesis 12.2 is violated, i.e. we have

$$\sum_{i=1}^{m-1} n_i \ell_i^2 \leq 2\ell_m \leq 2|E : E_0|$$

for some m.

We define $\overline{H} = H/\Phi(H)$ and use the *bar* convention. By assumption, we have a normal subgroup H_1 of H such that $\overline{H_1}$ is a nontrivial proper E-invariant subgroup. Since $(|E|, |H|) = 1$, there is a complementof $\overline{H_1}$ in $\overline{H}$. Thus, there is an E- invariant subgroup H_2 such that $H_1 H_2 = H$ and $H_1 \cap H_2 = \Phi(H)$. Then, H_2 is a normal subgroup of M. We have remarked that there are at least $2|E : E_0|$ irreducible characters of degree d having H_1 in their kernel and at least $2|E : E_0|$ irreducible ones of degree d having H_2 in their kernel. It follows that $4|E : E_0| \leq n_1$. This contradicts the earlier inequality. Therefore, $\mathcal{S}$ is coherent.

Suppose that $Z(E)$ contains an element x such that $C_H(x) \not\subseteq H'$ and $C_H(x) \neq H$. Let $\overline{C} = C_{\overline{H}}(x)$. Then, $\overline{H} = \overline{C} \times [\overline{H}, x]$. If $\overline{C} = \overline{H}$, then $C_H(x)\Phi(H) = H$. This implies $C_H(x) = H$. Therefore, $\overline{C} \neq \overline{H}$. If $\overline{C} = 1$, then $C_H(x)(x) \subseteq \Phi(H)$. This is impossible as $C_H(x)H'$ corresponds to a direct factor of H/H'. Since $x \in Z(E)$, $\overline{C}$ is M-invariant. Hence, $\overline{H} = H/\Phi(H)$ is not a chief factor of M and the first part of Theorem 15.2 yields that $\mathcal{S}$ is coherent. Q.E.D.

§16. Characters of Subgroups of Type III and IV

The following notation is used as in §29 [FT]. Let $S = S'Q^*$ be a subgroup of type II, III, or IV where $q = |Q^*|$ is a prime and Q^* corresponds to the subgroup W_1 in the definition of the groups of type II, III, or IV. Let $H = S_F$ and let V be a Q^*-invariant complement of H in S. We have a subgroup T not of type I paired with S in Theorem I.

Let $\pi(H) = \{p_1, \ldots, p_t\}$ and for $1 \leq i \leq t$, let $P_i \in Syl_{p_i}(H)$, $C_i = C_V(P_i)$, and $C = \bigcap_{i=1}^t C_i$. Let $|H| = h$, $|V| = v$, $|Q^*| = q$, $|C_i| = c_i$ $(1 \leq i \leq t)$, and $|C| = c$.

Let $\mathcal{S}_0$ be the set of characters of S which are induced by non-principal irreducible characters of S'/H and $\mathcal{S}$ the set of characters of S induced by irreducible characters of S' that do not have H in their kernel.

Theorem 16.1. (a) *If S is of type III, then $\mathcal{S}_0 \cup \mathcal{S}$ is coherent except possibly if H is abelian with $|H| = p^q$ for some prime p, VQ^* acts irreducibly on H, and $C = 1$.* (b) *If S is of type IV, then $\mathcal{S}_0 \cup \mathcal{S}$ is coherent except possibly if H is abelian with $|H| = p^q$ for some prime p, VQ^* acts irreducibly on H, $C = V'$, and $\mathcal{S}_0$ is not coherent.*

This is Theorem 29.1 [FT]. We paraphrase a part of their proof.

Throughout this section we assume that S is of type III or IV. By Theorem I, T is of type II. Therefore W_2 is of prime order. Let $p = |W_2|$ and write $p = p_1$, $P = P_1$, and $P^* = W_2$. Since S is of type III or IV, we have $S'' \subseteq F(S) = HC_S(H) = HC \subseteq S'$ by (T3).

We will prove Theorem 16.1 in 6 steps.

Lemma 16.2. *Hypothesis 12.4 is satisfied for S, $F(S)$, and $\mathcal{S}_0 \cup \mathcal{S}$ in place of M, H, and $\mathcal{S}$, respectively, with $d = 1$.*

Proof. By the definition of groups of type III or IV, $H \subseteq F(S)$, $F(S) \neq S'$, and S/H is a Frobenius group with Frobenius kernel S'/H. Thus, $\mathcal{S}_0$ contains an irreducible character of degree $q = |S:S'|$. Every character of $\mathcal{S}_0 \cup \mathcal{S}$ is induced by an irreducible character of S'. So, the degree is a multiple of q. Thus, (iv) of Hypothesis 12.4 is satisfied.

By Lemmas 13.5 and 13.7, each equivalence class of $\mathcal{S}_0 \cup \mathcal{S}$ is either subcoherent or consists of irreducible characters. An equivalence class contains λ as well as $\overline{\lambda}$. Thus, the condition (v) is satisfied. Q.E.D.

Lemma 16.3. *Let $F = F(S)$ and let $\mathcal{S}(F')$ be the subset of $\mathcal{S}_0 \cup \mathcal{S}$ consisting of those characters which are equivalent to some character in $\mathcal{S}_0 \cup \mathcal{S}$ that has F' in its kernel. If $\mathcal{S}(F')$ is coherent, then $\mathcal{S}_0 \cup \mathcal{S}$ is coherent.*

Proof. Since $W_2 \subseteq P$, V does not centralize P. Then, the Frobenius group VQ^* acts nontrivially on $P/\Phi(P)$. This implies $|P\colon \Phi(P)| \geq p^q$. Thus,

$$|F\colon F'| \geq |P\colon \Phi(P)| \geq p^q > 4q^2 + 1$$

by (5.9) of [FT]. Theorem 12.5 yields Lemma 16.3. Q.E.D.

Lemma 16.4. *If $\mathcal{S}(F')$ is not coherent, then $S'' = F$.*

Proof. By Corollary 9.6, W_2 is a subgroup of S''. It follows that S/S'' is a Frobenius group with Frobenius kernel S'/S''. The proof of Lemma 29.3 [FT] proves Lemma 16.4. Q.E.D.

Lemma 16.5. *If $\mathcal{S}(F')$ is not coherent, then $H = P$, $P' = \Phi(P)$, $|P : P'| = p^q$, $P^* \cap \Phi(P) = 1$, and $C = V'$. Furthermore, VQ^* acts irreducibly on P.*

Proof. The proof is the same as that of Lemma 29.4 [FT]. Since $|P : \Phi(P)| = p^q$ and V does not act trivially, VQ^* acts irreducibly on $P/\Phi(P)$. Q.E.D.

Lemma 16.6. *If $\mathcal{S}(F')$ is not coherent, then P is an elementary abelian p-group of order p^q.*

Proof. See the proof of Lemma 29.5 [FT]. I will paraphrase the part of the proof concerning the linear characters s_i of V modulo p.

For $u, v \in V$, we have $s_i(uv) \equiv s_i(u)s_i(v) \pmod p$. Thus, s_i are indeed linear characters modulo p. None of these characters is trivial because $C_{P/P'}(V) = 1$. If we take the notation that a generator w of Q^* shifts the one-dimensional V-modules downwards $i \to i-1$, then

$$s_{i+1}(v) \equiv s_i(w^{-1}vw) \quad \text{for all} \quad v \in V.$$

If $s_i s_j = 1$ for some $i < j = i + k$, then for $x = w^k$, $s_j(v) = s_i(x^{-1}vx)$ so $1 = s_i(v)s_j(v) = s_i(vx^{-1}vx)$ for all v. We claim that the mapping defined by $\theta : v \to vx^{-1}vx$ induces an injection on the group V/V'. Suppose that $\theta(v) \equiv \theta(u) \pmod{V'}$. Then,

$$x^{-1}vu^{-1}x \equiv v^{-1}u \equiv uv^{-1}[v^{-1}, u] \equiv (vu^{-1})^{-1}$$

modulo V'. Since the group VQ^*/V' has odd order, this happens only when $vu^{-1} \equiv 1 \pmod{V'}$. Thus, θ induces an injection on a finite set. Therefore, θ induces a surjective map. Then, for every $v \in V$, there are elements $u \in V$ and $z \in V'$ such that $v = \theta(u)z$. Since $s_i(z) = 1$, we have $s_i(v) = s_i(\theta(u)) = 1$. This contradicts the statement that s_i is not trivial. Thus, $s_i s_j \neq 1$ for any i, j with $1 \leq i, j \leq q$.

The remaining proof is given in [FT]. Q.E.D.

Lemma 16.7. *If $\mathcal{S}(F')$ is not coherent and $C \neq 1$, then $\mathcal{S}_0$ is not coherent.*

Proof. This is a paraphrase of the proof of Lemma 29.6 [FT]. Assume that $\mathcal{S}_0$ is coherent. Note that S/H is a Frobenius group with Frobenius kernel S'/H that is a nonabelian group of order v. Thus, $\mathcal{S}_0$ is the set of all irreducible characters of S/H that do not have S' in their kernel.

Let $\mathcal{S}_1 = \mathcal{S}_0$ and let $\mathcal{S}_2, \ldots, \mathcal{S}_k$ be the equivalence classes of $\mathcal{S}(F') - \mathcal{S}_0$ such that every character of $\mathcal{S}_m$ has degree $l_m q$ for $m \geq 2$ and $l_2 \leq l_3 \leq \cdots \leq l_k$. By assumption, $\mathcal{S}(F')$ is not coherent. We check the validity of Hypothesis 12.2. If $\lambda \in \mathcal{S}_i$, then $\overline{\lambda} \in \mathcal{S}_i$. Now, Lemmas 13.5 and 13.7 yield the condition (iv). Since $\mathcal{S}_0$ contains an irreducible character of degree d, all conditions of Hypothesis 12.2 except the inequality are satisfied. Since $\mathcal{S}(F')$ is not coherent, the inequality must be violated. If $\lambda \in \mathcal{S}_m$ for $m \geq 2$, then λ is equivalent to a constituent μ of a character induced by a linear character of F and $\lambda(1) = \mu(1)$. Since $V' = C$ by Lemma 6.5 and $F = HC$, the degree $l_m q$ of μ satisfies $l_m \leq v/c$.

Consider the contribution to the left side of the inequality from $\mathcal{S}_0$. A character λ_{1s} of $\mathcal{S}_0$ is irreducible of degree $l_{1s}q$. Since S/H is a Frobenius groupwith Frobenius kernelS'/H, $\mathcal{S}_0$ is the set of irreducible characters of S/H that do not have S' in their kernel. There are exactly q other characters of degree 1. Thus,

$$\sum_s (l_{1s}q)^2 + q = qv, \quad \text{or} \quad \sum_s \frac{l_{1s}^2}{\|\lambda_{1s}\|^2} = \frac{v-1}{q}.$$

Thus, we obtain $v - 1 \leq 2l_m q \leq 2qu/c$. Since $1 < c < u$ and $c \equiv 1 \pmod{2q}$, we get a contradiction. Q.E.D.

Suppose that S is of type III or IV. If $\mathcal{S}_0 \cup \mathcal{S}$ is not coherent, $\mathcal{S}(F')$ is not coherent by Lemma 16.3. Then, by Lemmas 16.5 and 16.6, $H = P$ is elementary abelian of order p^q and VQ^* acts irreducibly on P. If S is of type III, then V is abelian so $C = V' = 1$ by Lemma 16.5. If S is of type IV, then $C = V'$ and $\mathcal{S}_0$ is not coherent by Lemma 16.7. This proves Theorem 16.1. Q.E.D.

§17. Characters of Subgroups of Type II, III and IV

We will use the notation introduced at the beginning of §16. In addition, we denote $a = \exp V/V'$.

In the first part of this section, we assume that

(1) S is a subgroup of type II, III, or IV,
(2) $\mathcal{S}$ is not coherent if S is of type III or IV, and
(3) V/V' has exponent a.

In this section, we denote by $\mathcal{S}(A)$ the set of characters in $\mathcal{S}$ which have A in their kernel. This usage is different from the one used in §16. We follow the argument of §30 [FT].

Lemma 17.1. *The degree of every character in $\mathcal{S}$ is divisible by aq.*

Proof. Every character in $\mathcal{S}$ is a constituent of a character of S induced by a nonprincipal character θ of H. Let $V_1 = V \cap I(\theta)$ and let $b = |V : V_1|$. If S is of type II or III, V is abelian by (II*iii*) or (III*iii*). By Lemma 4.5 [FT], it suffices to show that a divides b.

The group V_1 centralizes a section of H. Then, $V_1 \subseteq A(S)$ as shown in the proof of Lemma 30.1 [FT]. Consider V^b and suppose that $V^b \neq 1$. Then, $V^b \subseteq V_1 \subseteq A(S)$. If S is of type II, the modified (II*v*) yields $N_G(V^b) \subseteq S$. Since V^b char V, we get $N_G(V) \subseteq N_G(V^b) \subseteq S$ in contradiction to (II*iv*). If S is of type III, $V^b \neq 1$ is a normal subgroup of the Frobenius group VQ^*. Since $\mathcal{S}$ is not coherent, Theorem 16.1 yields that VQ^* acts irreducibly on H. Then, the abelian group $V^b \neq 1$ acts semisimply and one component is trivial as $V^b \subseteq V_1$. It follows that V^b acts trivially on H. Therefore, $V^b \subseteq C$, contradicting Theorem 16.1.

Suppose that S is of type IV. Then, Theorem 16.1 yields that $\mathcal{S}_0$ is not coherent. By Lemma 12.6, $V(\cong S'/H)$ is a nonabelian r-group for some prime r and $V' = \Phi(V)$. In this case V/V' is an elementary abelian r-group, so $a = r$. Since VQ^* acts irreducibly on H, we have $C_H(V) = 1$. It follows that V does not stabilize any nonprincipal character of H. Hence, the degree of a character in $\mathcal{S}$ is divisible by r. This proves Lemma 17.1. Q.E.D.

Lemma 17.2. *For $1 \le i \le t$, $|P_i : \Phi(P_i)| = p_i{}^q$, VQ^* acts on $P_i/\Phi(P_i)$ irreducibly, and V/C_i has exponent a.*

The proof of Lemma 30.2 [FT] applies.

Lemma 17.3. *For $1 \le i \le t$, either $a \mid (p_i - 1)$ or $a \mid (p_i{}^q - 1)$ and $(a, p_i - 1) = 1$. In the second case, V/C_i is a cyclic group of order a and acts irreducibly on $P_i/\Phi(P_i)$.*

Cf. Lemma 30.3 [FT].

We prove two (known) properties of finite abelian p-groups for some prime p.

Lemma O. (1) *Let A be a finite abelian p-group such that $\Phi(A)$ is a maximal characteristic subgroup of A. Then, A is a direct product of cyclic groups of the same order.*

(2) *Suppose that an abelian group U acts on a finite abelian p-group A. Assume that the exponent of U divides $p - 1$. Then, A is a direct product of U-invariant cyclic subgroups.*

Lemma 17.4. *Suppose $(a, p_i - 1) = 1$ for some i, $1 \le i \le t$. Let $H_1 = P_i' \Pi_{j \ne i} P_j$. Then, $|H : H_1| = |P_i : P_i'| = p^{qm_i}$ for some integer m_i. Furthermore, $\mathcal{S}(H_1)$ contains at least*

$$\frac{1}{q}\{\frac{(p_i^{qm_i} - 1)c_i}{a} - (p_i^{m_i} - 1)\}$$

irreducible characters of degree aq and at least $p_i^{m_i} - 1$ characters of weight q and degree aq.

Proof. I will paraphrase the proof of Lemma 30.4 [FT]. Lemma 17.3 yields that V/C_i is cyclic. Suppose that S is of type IV. Then, by Theorem 16.1, $\mathcal{S}_0$ is not coherent. We showed in the proof of Lemma 17.1 that $V' = \Phi(V) = C = C_i$. Then, $V/\Phi(V)$ is cyclic. Hence, V is cyclic. This is a contradiction because V is nonabelian for type IV. Therefore, S is of type II or III, and V is abelian. Lemma 17.3 yields that V acts irreducibly on $P_i/\Phi(P_i)$. It follows from Lemma M that $H/H_1 (\cong P_i/P_i')$ is a direct product of q cyclic groups of order $p_i^{m_i}$. On each chief factor in H/H_1, Q^* centralizes a subgroup of order p_i. Since $C_H(Q^*)$ is cyclic by Theorem C (2), we have $|C_{H/H'}(Q^*)| = p_i^{m_i}$.

The group HC_i/H_1 is the direct product of H/H_1 and H_1C_i/H_1. Both factors are abelian. Since V/C_i acts regularly on H/H_1, every linear character α of HC_i/H_1 that does not have H/H_1 in its kernel has exactly $a = |V : C_i|$ conjugates. Hence, α induces an irreducible character of degree a. There are at least $(p_i^{m_i q} - 1)c_i/a$ distinct irreducible characters of degree a. Among them, precisely $p_i^{m_i} - 1$ are Q^*-invariant. The assertions of Lemma 17.4 follow from Lemma 13.5. Q.E.D.

Lemma 17.5. *Suppose that $a \mid (p_i - 1)$ for some i with $1 \le i \le t$. Let H_1 be as in Lemma 17.4. Then, $|H : H_1| = |P_i : P_i'| = p_i^{m_i q}$ for some integer m_i and $\mathcal{S}(H_1)$ contains at least*

$$\frac{(p_i^{m_i} - 1)}{a} \frac{v}{av'}$$

irreducible characters of degree aq where $|V'| = v'$.

Proof. The Frattini factor group of H/H_1 is isomorphic to $P_i/\Phi(P_i)$. By Lemma 17.2, VQ^* acts irreducibly on $P_i/\Phi(P_i)$. Since $a \mid (p_i - 1)$, H/H_1 is a direct product of V-invariant cyclic groups of the same order. There are V-invariant subgroups K_1 and K_2 such that H/K_2 is a cyclic group of order p^{m_i}, $K_1K_2 = H$, and $K_1 \cap K_2 = H_1$. Let $V_1 = C_V(H/K_2)$. Then, V/V_1 is a subgroup of $\mathrm{Aut}(H/K_2)$. Hence, V/V_1 is cyclic, so $|V : V_1| \le a$.

Consider the factor group $L = HV_1/K_2V'$. Since $V_1 \supseteq V'$ and V_1 centralizes H/K_2, L is abelian. Let $\mathcal{L}$ be the set of linear characters of L which do not contain H in their kernel. If $\lambda \in \mathcal{L}$, λ induces an irreducible character θ of degree $|V : V_1|$ in $\mathcal{S}$. By Lemma 17.1, $|V : V_1| \ge a$. Hence, $|V : V_1| = a$. Suppose that $\theta^S = \lambda^S$ is not irreducible or for $\lambda, \mu \in \mathcal{L}$, they induce the same irreducible character of $\mathcal{S}$. In the first case, λ is Q^*-invariant, so $H \cap \ker \lambda$ is Q^*-invariant. In the second case, λ and μ are Q^*-conjugate so $H \cap \ker \lambda$ and $H \cap \ker \mu$ are Q^*-conjugate. However, $H \cap \ker \lambda$ and $H \cap \ker \mu$ have the same index and both contain K_2. Since H/K_2 is a cyclic group of order $p_i^{m_i}$, there is a unique subgroup of each index. It follows that $H \cap \ker \lambda$ is Q^*- invariant. Since every subgroup of H that contains K_2 is V-invariant, H contains a subgroup of index p_i that is VQ^*-invariant. This contradicts Lemma 17.2.

We have $|\mathcal{L}| = l = (p^{m_i} - 1)v/av'$ and the characters of $\mathcal{L}$ produce exactly l/a distinct irreducible characters of degree aq in $\mathcal{S}(H_1)$.

Q.E.D.

Lemma 17.6. *If $\mathcal{S}(H')$ contains no irreducible character of degree aq, then $t = 1$, $P_1' = \Phi(P_1)$, $a = u = ({p_1}^q - 1)/(p_1 - 1)$, and $c = c_1 = 1$. Furthermore, $\mathcal{S}(H')$ is coherent and S is not of type IV.*

Proof. See the proof of Lemma 30.6 [FT]. We note that in the following equation $a = ({p_i}^q - 1)/(p_i - 1)$, p_i is determined by a. This remark yields $t = 1$. Since $c = 1$, $V' = C = 1$, and S is not of type IV. Since $\mathcal{S}(H')$ contains all the characters of degree aq and weight q,

$\mathcal{S}(H')$ consists of characters ξ_j, $0 \le j < p$. By Lemma 13.7, $\mathcal{S}(H')$ is coherent. Q.E.D.

In the remainder of this section we assume that

(2)′ $\mathcal{S}$ is not coherent

in place of the condition (2). Note that $S'' \subseteq F(S) = HC \subseteq S'$ by Theorem A (7). Define $F = F(S)$.

Lemma 17.7. *If $\mathcal{S}(H')$ is not coherent, then $H = P_1$, $C_1 = 1$, $a = (p-1)/2$, $p = p_1$, $v \neq a$, and $\Phi(P_1) = P_1'$. The degree of every character in $\mathcal{S}(H')$ is either aq or vq/c, and $\mathcal{S}(H')$ contains exactly $2v/a$ irreducible characters of degree aq. Furthermore, S is not of type IV.*

Proof. The proof of Lemma 30.7 [FT] shows that if $\mathcal{S}(F')$ is not coherent, the degree of any character in $\mathcal{S}(F')$ is either aq or uq where $u = v/c$, and the other conditions in Lemma 17.7 are satisfied. If S is of type II or III, then V is abelian. Hence, $F' = H'$ because $F = H \times C$ with C abelian. Thus, the result is proved if S is of type II or III.

It remains to show that if $\mathcal{S}(H')$ is not coherent, then S is not of type IV. Suppose that S is of type IV. Since $\mathcal{S}(H')$ is not coherent, $\mathcal{S}_0 \cup \mathcal{S}$ is not coherent. Theorem 16.1 (b) yields that $H = P$ is elementary abelian, VQ^* acts irreducibly on P, and $\mathcal{S}_0$ is not coherent. Since $S/H \cong VQ^*$ is a Frobenius group, Lemma 12.6 implies that V is an r-group for some prime r and $V' = \Phi(V)$. It follows that V/V' is an elementary abelian group of order r^n. Thus, the exponent of V/V' is r; we have $r = a$. We claim that $n \le 2$. Suppose that $n > 2$. Since $H = P$ is elementary abelian, so is F/C. Let θ_1 and θ_2 be linear characters of F/C with exactly a conjugates in S', so each induces an irreducible character of degree a. Suppose that $\theta_1\theta_2$ is not the principal character. Then, $I(\theta_1\theta_2) \supseteq I(\theta_1) \cap I(\theta_2)$. Since both $I(\theta_1)$ and $I(\theta_2)$ have index r, the index of $I(\theta_1\theta_2)$ in S' is at most r^2. Since the index of the inertia group of a nonprincipal character is either r or r^n, $|S' : I(\theta_1\theta_2)| = r$. Thus, the set of linear characters with at most r conjugates forms a VQ^*-invariant subgroup of the character group of F/C. Since VQ^* acts irreducibly on F/C, every nonprincipal character of F/C has exactly r conjugates. It follows from the permutation lemma that the number of orbits on the character group by the action of V is the same as the number of orbits on $P^\sharp$. Since each orbit has at least r elements, every element of $P^\sharp$ has exactly r conjugates. Take an element $x \neq 1$ in P^*. Then, $C_V(x) = X$ is a maximal subgroup of V. Since x is Q^*-invariant, so is X. Hence, $C_P(X)$ is VQ^*-invariant. Since VQ^* acts on P irreducibly, we have $C_P(X) = P$. This contradicts Theorem 16.1

because $C = C_V(P) = V'$. Therefore, we have $n \leq 2$. If $n = 1$, $V/\Phi(V)$ is cyclic. This implies that V is cyclic. This contradicts the definition of a group of type IV (IV*iii*). If $n = 2$, Lemma 11.3 [FT] yields that $\mathcal{S}_0$ is coherent. This final contradiction shows that S is not of type IV if $\mathcal{S}(H')$ is not coherent. Q.E.D.

Lemma 17.8. *The family $\mathcal{S}(H')$ is coherent.*

Proof. Suppose that $\mathcal{S}(H')$ is not coherent. Lemma 17.7 yields that $H = P_1$, $C_1 = 1$, $a = (p-1)/2$, $p = p_1$, $v \neq a$, $P_1' = \Phi(P_1)$, and S is of type II or III. The last condition implies that the subgroup V is abelian. Let $\mathcal{S}_1$ be the set of irreducible characters in $\mathcal{S}(H')$ of degree aq. By Lemma 17.7, the degree of every character in $\mathcal{S}(H')$ is either aq or vq, and $|\mathcal{S}_1| = 2v/a$. We will prove some properties of characters of $P = P_1$ having exactly a conjugates. Note that there is such a character because $\mathcal{S}_1 \neq \emptyset$.

We prove a lemma. *Let θ be a nonprincipal character of P/P' with exactly a conjugates. Then, $V_1 = V \cap I(\theta)$ contains no Q^*-invariant subgroup different from* 1.

Proof. Suppose $1 \neq U \subseteq V_1$ and U is Q^*-invariant. Then, V_1 centralizes$P/\ker\theta$. Since V is a p'-group, $C_{P/P'}(V) \neq 1$. Hence, $C_{P/P'}(U) \neq 1$, and it is a direct factor of P/P' because U is a p'-group. Since U is Q^*-invariant and V is abelian, $C_{P/P'}(U)$ is VQ^*-invariant. By Lemma 17.2, VQ^* acts irreducibly on $P/\Phi(P)$. It follows that $C_{P/P'}(U) = P/P'$, and hence $U \subseteq C_V(P) = C_1 = 1$. This contradiction proves the lemma. Q.E.D.

We claim that there is a pair of characters θ_1 and θ_2 of P/P' having exactly a conjugates such that $\theta_1\theta_2$ has v conjugates. Suppose that this does not hold. Then, the characters having at most a conjugates form a subgroup of the character group of P/P' that is VQ^*-invariant. Then, Lemma 17.2 yields that every character of P/P' has at most a conjugates. This gives a contradiction as follows. There is a Q^*-invariant nonprincipal character θ of P/P'. Then, $I(\theta) \cap V$ is Q^*-invariant, contradicting the lemma.

Choose a pair of characters θ_1 and θ_2 each having exactly a conjugates such that $\theta_1\theta_2$ has v conjugates. Then, $|S' \colon I(\theta_i)| = a$ for $i = 1, 2$, and $I(\theta_1) \cap I(\theta_2) = P$. Thus, v divides a^2; in particular, $v \leq a^2$. We will prove that $r(V) \leq 2$. Take an arbitrary prime $r \in \pi(V)$. We will show that V has a Sylow r-subgroup generated by at most two elements. Let x be an element of order a in V, and let w be a generator of Q^*. If $\langle x\rangle \cap \langle x\rangle^w$ is an r'-group, $\langle x, x^w\rangle$ contains a Sylow r-subgroup of V

that is generated by two elements. Suppose that $\langle x\rangle \cap \langle x\rangle^w$ contains a subgroup R of order r. Then, R is the unique subgroup of order r in $\langle x\rangle$ as well as in $\langle x\rangle^w$. Thus, R is Q^*-invariant. By the lemma, $I(\theta_i)$ does not contain R. Let $V_i = I(\theta_i) \cap V$. Then, $V_1 \cap V_2 = 1$. Since $V_i \cap R = 1$, V/V_i has a cyclic Sylow r-subgroup. The Second Isomorphism Theorem yields that V_1 also has a cyclic Sylow r-subgroup. Thus, a Sylow r-subgroup of V is generated by at most two elements. Since V is abelian, we have $r(V) \leq 2$. If $r(V) = 1$, V would be cyclic. Then, $a = v$, contrary to $a \neq v$. Thus, $r(V) = 2$.

We prove that if θ has exactly a conjugates, then $V_1 = I(\theta) \cap V$ is cyclic and $V_1 \cap {V_1}^w = 1$ for any $w \in Q^*$. If V_1 is not cyclic, V_1 contains an elementary abelian group E of order r^2 for some $r \in \pi(V)$. Since $r(V) = 2$, E is a characteristic subgroup of V. Thus, E is Q^*-invariant, contradicting the lemma. Therefore, V_1 is cyclic. If $V_1 \cap {V_1}^w \neq 1$ for some $w \neq 1$ in Q^*, then take a subgroup R of prime order in $V_1 \cap {V_1}^w$. Since V_1 is cyclic, R is the unique subgroup of its order. The same holds for ${V_1}^w$. Then, R is a Q^*-invariant subgroup of V_1. The lemma yields that this is not possible. Thus, $V_1 \cap {V_1}^w = 1$.

The proof of Lemma 30.8 [FT] can be carried over. The Q^*-invariant nonprincipal characters of P have exactly v conjugates as seen from the third paragraph of the present proof. Thus, $\mathcal{S}(H')$ contains $p-1$ characters of weight q and of degree qv.

Let λ be an irreducible character of degree aq in $\mathcal{S}_1$. Then Lemma 4.5 [FT] yields that λ is induced by a linear character of some subgroup U of index a in S'. Define $\alpha = {1_U}^S - \lambda$. Since $U \triangleleft S'$ (as $S'/H \cong V$ is abelian), ${1_U}^{S'}$ is the regular representation of the group S'/U. Since $U = I(\theta)$ for some nonprincipal character θ with exactly a conjugates, $U \cap U^w = H$ for all $w \in Q^{*\sharp}$. If $|U : H| = b$, then $U/H \cong V_1 = I(\theta) \cap V$ is cyclic. Thus, UU^w/H is the set S_b/H of elements of order dividing b. It follows that S is Q^*-invariant. If a linear character ξ of S' has U in its kernel, $\ker \xi^w \supseteq U^w$. Thus, ξ^w has U in the kernel if and only if $\ker \xi \supseteq S_b$. Therefore, we can compute $(1_U)^S$. It is the sum of $\rho_{S/S'}$, irreducible characters induced by nonprincipal characters of S'/S_b with multiplicity q and $(a - a/b)$ other irreducible characters with multiplicity 1. Thus, it follows that $\|\alpha\|^2 = q + q^2((a/b)-1)/q + a - (a/b) + 1 = a + 1 + (q-1)a/b$.

The remaining portion of the proof is the same as that of Lemma 30.8 [FT]. Q.E.D.

Lemma 17.9. *S is of type II.*

Lemma 17.10. *If $\mathcal{S}$ contains an irreducible character of degree aq, then Hypothesis* 12.4 *is satisfied with $M = S$, $X = A(S)$, $H = S_F = P$,* *and $d = a$.*

Lemma 17.11. *If $\mathcal{S}$ contains an irreducible character of degree aq, then $|H\colon H'| \leq 4a^2q^2 + 1$.*

Proof. We need only to check that the present $\mathcal{S}(H')$ is the same as $\mathcal{S}(H')$ in Theorem 12.5. Suppose that $\lambda \in \mathcal{S}(H')$ in the sense of Theorem 12.5. Then, λ has the same degree as the character μ in $\mathcal{S}$ that has H' in the kernel. By the definition of $\mathcal{S}$, λ is induced by an irreducible character λ_1 of S'. Similarly, μ is induced by an irreducible character μ_1 of S'. Since $H' \subseteq \ker\mu$, the restriction of μ_1 on H is a direct sum of irreducible characters of degree 1. Since S'/H is abelian, Lemma 4.5 [FT] yields that $\mu_1(1)$ is prime to $|H|$. Note that H is a Hall subgroup of S. Since $\lambda_1(1) = \mu_1(1)$, the degree of λ_1 is prime to $|H|$. Therefore, the irreducible constituents of the restriction of λ_1 to H are linear. It follows that $H' \subseteq \ker\lambda$. Now, Theorem 12.5 yields Lemma 17.11 because $\mathcal{S}$ is not coherent. Q.E.D.

Lemma 17.12. *For $1 \leq i \leq t$, $(a, p_i - 1) = 1$ and P_iV/C_i is a Frobenius group.*

Lemma 17.13. *The group H is a nonabelian 3-group with $H' = \Phi(H)$. There is an irreducible character of degree aq in $\mathcal{S}$ and $a < 3^{q/2}$.*

Proof. By Lemma 17.8, $H' \neq 1$ so H is nonabelian. Choose the notation that $P_1' \neq 1$. Let

$$P_1 = P_{11} \supset P_{12} \supset \cdots \supset P_{1n} = P' \supset P_{1\ n+1} = P_0$$

be a part of a chief series of S. Then, P_1/P_o is a nilpotent group of class 2. Lemma 17.9 yields that S is of type II. Hence, by (IIv), $C_H(V) = 1$. It follows from Theorem 3.10 [BG] that Q^* centralizes some nonidentity in each chief factor. Since $C_H(Q^*)$ is cyclic, P_1/P_0 has exponent p^n. The mapping $y \to y^{p^{n-1}}$ induces a V-homomorphism of $P_1/\Phi(P_1)$ into P_1'/P_0. Therefore, the minimal polynomial of the generator x of U/C_1 on $P_1/\Phi(P_1)$ is the same as that on P_1'/P_0. By Lemma 6.2 [FT], we have $q > 3$ and $a < 3^{q/2}$.

If $\mathcal{S}$ contains no irreducible character of degree aq, Lemma 17.6 yields $H = P_1$ and $a = ({p_1}^q - 1)/(p_1 - 1)$. Hence,

$$3^{q-1} \leq {p_1}^{q-1} < a < 3^{q/2}.$$

This contradiction proves that there is an irreducible character of degree aq in $\mathcal{S}$.

Let $|P_1 : P_1'| = p_1{}^{mq}$. Then, by Lemma 17.11,

$$p_1{}^{mq}\Pi_{i>1}p_i{}^q \leq |H : H'| \leq 4a^2q^2 + 1 < 4 \cdot 3^q q^2 + 1.$$

By (5.9) [FT], we have $m = 1$, $t = 1$, and $p_1{}^q < 4 \cdot 3^q q^2 + 1$. Thus, p_1 is small. Eventually, we have $p_1 = 3$ (cf. page 960 [FT]). Hence, H is a 3-group because $t = 1$. Since $m = 1$, we have $\Phi(H) = H'$. Q.E.D.

Theorem 17.14. *Let S be a subgroup of type II, III, or IV. Let a be the exponent of the group V/V', and let T be the element of $\mathcal{M}$ paired with S in Theorem I. Then, the family $\mathcal{S}$ of characters is coherent except possibly if S is of type II, H is a nonabelian 3-group, HV/C is a Frobenius group with Frobenius kernel HC/C, $a < 3^{q/2}$, $|H : H'| = 3^q$, and T is of type V.*

§18. Characters of Subgroups of Type V

In this section let $T = T'W_2$ be a subgroup of type V. Let S be the subgroup in $\mathcal{M}$ which satisfies the conditions of Theorem I. By (d), S is of type II. We use the notation introduced at the beginning of §16.

Let $\mathcal{T}$ be the set of all characters of T which are induced by nonprincipal irreducible characters of T'. For $0 \leq i \leq q-1$, $0 \leq j \leq w_2 - 1$, let η_{ij} be the generalized characters of G associated with ω_{ij} of W and let ν_{ij} be the characters of T defined in Lemma 13.4. By Lemma 13.5, T' has exactly q irreducible characters which induce characters of weight w_2. Denote them $\nu_0 = 1_{T'}, \nu_1, \ldots, \nu_{q-1}$. Then, $\zeta_i = \nu_i{}^T$ has weight w_2. Since q is a prime, the characters $\nu_1, \ldots, \nu_{q-1}$ are algebraically conjugate. Therefore, $\nu_i(1) = \nu_1(1)$ for $1 \leq i \leq q-1$.

We prove a lemma.

Lemma P. *If λ is an irreducible character of $\mathcal{T}$, then λ^τ is defined and λ^τ is not equal to $\pm\eta_{st}$ for any s and t.*

Proof. If $\lambda \in \mathcal{T}$, then $\overline{\lambda}$ is an irreducible character in $\mathcal{T}$ and $\overline{\lambda} \neq \lambda$. Then, $\{\lambda, \overline{\lambda}\}$ is coherent and λ^τ is (not uniquely) defined by $(\lambda - \overline{\lambda})^\tau = \lambda^\tau - \overline{\lambda}^\tau$. Suppose that $\lambda^\tau = \pm\eta_{st}$. Then, for an element $x \in \widehat{W}$, we have

$$\lambda^\tau(x) = \pm\eta_{st}(x) = \pm\omega_{st}(x).$$

Since λ vanishes on $\widehat{W}$, $\lambda^\tau - \overline{\lambda}^\tau = (\lambda - \overline{\lambda})^\tau$ vanishes on x. Thus, we get that $\overline{\lambda}^\tau(x) = \pm\omega_{st}(x) \neq 0$. By Lemma 13.1, $\overline{\lambda}^\tau$ is one of $\pm\eta_{ij}$; in fact, $\omega_{st}(x) = \omega_{ij}(x)$ on $x \in \widehat{W}$ implies that $\overline{\lambda}^\tau = \pm\eta_{st} = \lambda^\tau$. This contradicts the inequality $\lambda \neq \overline{\lambda}$. Q.E.D.

Lemma 18.1. *The family $\mathcal{S}(H')$ contains an irreducible character of S except possibly if w_2 is a prime and $S' = HV$ is a Frobenius group with Frobenius kernel H.*

Proof. We can apply Lemma 17.6. If $\mathcal{S}(H')$ contains no irreducible character, then $H = P_1$ is a p_1-group, $H' = \Phi(H)$, $v = a = ({p_1}^q - 1)/(p_1 - 1)$ and $c_1 = c = 1$. Suppose that H is nonabelian. Choose a chief factor P_1'/P_0 of S. Then, $P_1'/P_0 \subseteq Z(P_1/P_0)$ and it is an elementary abelian. As in the proof of Lemma 17.13, Lemma 6.2 [FT] yields $a < 3^{q/2}$. Since $a = ({p_1}^q - 1)/(p_1 - 1)$, we have a contradiction $3^{q-1} < 3^{q/2}$. Therefore, H is abelian. It follows from $H' = \Phi(H)$ that H is elementary abelian. On each chief factor in H, Q^* has a nontrivial centralizer. Since $C_H(Q^*) = W_2$ is cyclic, $w_2 = |W_2|$ is a prime and VQ^* acts irreducibly on H. Thus, HV is a Frobenius group with Frobenius kernel H.

Q.E.D.

Lemma 18.2. *Let $a_{ij} = ((\nu_1(1)\zeta_0 - \zeta_i)^\tau, \eta_{0j})$. Then $a_{ij} \neq 0$ for $1 \leq i \leq q-1$, $0 \leq j \leq w_2 - 1$.*

Proof. Let $M \in \mathcal{M}$ be a supporting subgroup of T and let $N = M_F$. By (F*iii*), M is a group of type I. Let $E = M \cap T$. Then (F*ii*)(b) yields that E is a complement of N in M. We prove the following lemma.

The elements of $A(M)$ are $\pi(W_2)'$-elements.

Proof. Since T is of type V, we have $A(T) = T'$. Take an element $x \neq 1$ of $C_{T'}(W_2) = Q^*$. Then, by (F*ii*)(c), we have $(|N|, |C_T(x)|) = 1$. It follows that $(|N|, |W_2|) = 1$. Suppose that there is an element of $A(M)$ of order r for some prime r in $\pi(W_2)$. Since N is an r'-group, there is a subgroup R of order r in E with $C_N(R) \neq 1$. By replacing M by conjugate, we may assume $R \subseteq W_2$ because W_2 is a cyclic Hall subgroup of T. By Theorem 8.7 (d), $N_T(R) = Q^* \times W_2$ and it is cyclic. By Lemma 6.1 (d) and Theorem 6.5 (b), E has abelian Sylow subgroups. It follows that E has cyclic Sylow r-subgroups. Then, $r \in \tau_1(M) \cup \tau_3(M)$. Since M is a ϖ-subgroup by (F*ii*), $C_N(R) \neq 1$ implies that $r \in \kappa(M)$. This contradicts Proposition 10.1 (a). Q.E.D.

We claim that $\Theta = \eta_{0j}$ satisfies the property that Θ is constant on the cosets of N which lie in $M - N$. By Lemma 14.1, we need to check that Θ is orthogonal to the elements of $T(\alpha)$ for every nonprincipal irreducible character α of N. Take $\theta_1, \theta_2 \in S(\alpha)$. Since M is of type I, θ_i are irreducible characters of M which vanish outside $A(M)$ and $\theta_1 = \theta_2$ on N by Lemmas 4.3 and 4.5 [FT]. Thus, $\theta_1 - \theta_2$ vanishes outside $A(M) - N$. Since M is of type I, $A(M) = A_0(M)$ and $A(M) - N$ is a TI-set of G with normalizer M by (F*ii*)(d).

Thus, $(\theta_1 - \theta_2)^G$ is the difference of two irreducible characters of G. Suppose that $(\Theta, (\theta_1 - \theta_2)^G) \neq 0$. Then, $\Theta = \eta_{0j}$ is involved in $\Psi = (\theta_1 - \theta_2)^G$. The virtual character Ψ vanishes outside $\mathcal{C}_G(A(M))$. Since elements of $A(M)$ are $\pi(W_2)'$- elements by the lemma, there is a Galois automorphism of $\mathbb{Q}_{|G|}$ that leaves $(\theta_1 - \theta_2)^G$ invariant but moves η_{0j} to η_{0k} with $k \neq j$. Then, η_{0k} is involved in Ψ with the same multiplicity. This is a contradiction because Ψ is the difference of two characters. Hence, η_{0j} is constant on the sets of the form $A(x)$ for $x \in D^*$.

Lemma 11.4 yields now

$$(\nu_1(1)\zeta_0 - \zeta_i, (\eta_{0j})_T) = ((\nu_1(1)\zeta_0 - \zeta_i)^\tau, \eta_{0j}) = a_{ij}.$$

The rest of the proof is the same as that of Lemma 31.2 [FT]. Q.E.D.

From now on, the lemmas of this section will be proved under the assumption that $\mathcal{T}$ is not coherent, and we will derive a contradiction from this hypothesis.

By Corollary 9.6, we have $Q^* \subseteq T''$. Then, T/T'' is a Frobenius group with Frobenius kernel T'/T''. We check that Hypothesis 12.4 is satisfied for S, S', $\mathcal{T}$ in place of M, H, $\mathcal{S}$ with $d = 1$. Since T/T'' is a Frobenius group, there is an irreducible character of degree $w_2 = |T:T'|$. The last condition of Hypothesis 12.4 holds by Lemmas 13.5 and 13.7. If $H_1 = T''$, then $\mathcal{S}(H_1)$ in Theorem 12.5 is the set of irreducible characters of T/T'' which do not have T'/T'' in their kernel. Since T'/T'' is abelian, this family is coherent. Then, Theorem 12.5 yields that $\mathcal{T}$ is coherent if

$$|T':T''| > 4|T:T'|^2 + 1.$$

Since we are assuming that $\mathcal{T}$ is not coherent, we have

$$|T':T''| \leq 4w_2^2 + 1.$$

This implies that W_2 acts on T'/T'' irreducibly. It follows that $T' = Q$ is a q-group for the prime $q = |Q^*|$. Define

$$|Q : Q'| = q^b \quad \text{and} \quad |T : Q| = w_2 = e.$$

Lemma 18.3. *Suppose that $\mathcal{T}$ is not coherent and $|Q : Q'| = q^b$ with $b = 2c$ an even number. Then, $|T:Q| = e$ is not a power of any prime.*

Proof. This is Lemma 31.3 [FT]. We will paraphrase a part of their proof. Suppose that $e = p^h$ for some prime p. Since $\mathcal{T}$ is not coherent, Lemma 11.5 [FT] yields that $q^c+1=2p^h$, q^c is the degree of any nonlinear irreducible characters of $Q/[Q,Q']$, and if Q_1 is a normal subgroup of T such that $Q_1 \subseteq Q'$ and $Q_1 \neq Q'$, then T/Q_1 is not a Frobenius group. Note that $Q'/[Q,Q']$ is contained in the center of $Q/[Q,Q']$. Therefore, Lemma 4.1 [FT] yields that the degree of any irreducible character of $Q/[Q,Q']$ is at most q^c.

Since $\mathcal{T}$ is not coherent, Q is nonabelian. So, $Q' \neq 1$. Let Q_1 be a normal subgroup of T such that $Q_1 \subseteq Q'$ and Q'/Q_1 is a chief factor of T. Then, $[Q,Q'] \subseteq Q_1$. Since $Q_1 \neq Q'$, the group T/Q_1 is not a Frobenius group. Then, some nonidentity element of W_2 has a nontrivial centralizer in Q_1. By Proposition 8.2, W_2 acts in a prime manner on Q. Thus, we have $Q^* \nsubseteq Q_1$. Since $[Q,Q'] \subseteq Q_1$, Q^*Q_1 is a normal subgroup of Q. Clearly, W_2 normalizes Q^*Q_1. Therefore, Q^*Q_1 is a normal subgroup of T. Since Q'/Q_1 is a chief factor of T, we have $Q^*Q_1 = Q'$. Then, $|Q' : Q_1| = |Q^*| = q$. Any nonlinear irreducible representation of Q/Q_1 has degree q^c because $Q_1 \supseteq [Q,Q']$, and it represents the subgroup Q'/Q_1 (in the center of Q/Q_1) by scalar matrices. Since each coset of Q_1 in Q' contains an element of Q^*, any nonlinear irreducible character of Q/Q_1 is W_2-invariant. Thus, there are $q-1$ nonlinear irreducible characters $\nu_1,\ldots,\nu_{q-1}$ that induce reducible characters of T. Let $\zeta_i = \nu_i{}^T$ for $1 \le i \le q-1$. These characters are algebraically conjugate.

Since $|Q:Q'| = q^b$ with $b=2c$, $\mathcal{T}$ contains $(q^b-1)/e = 2(q^c-1)$ irreducible characters of degree e. Let $\{\lambda_i\}$ be these irreducible characters of degree e. Since $Q = A(T)$, $\{\lambda_i\}$ is coherent. Thus, the set of virtual characters $\{\lambda_i{}^\tau\}$ of weight one is defined by Lemma 10.1 [FT]. None of these $\lambda_i{}^\tau$ is equal to $\pm\eta_{st}$.

Define $\alpha = \zeta_0 - \lambda_1$ and $\beta = q^c\lambda_1 - \zeta_1$. Consider the decomposition of α^τ and β^τ as in the proof of Lemma 31.3 [FT]. Then,

$$\beta^\tau = q^c\lambda_1{}^\tau - x\sum_i \lambda_i{}^\tau + \varDelta$$

for some integer x and $(\lambda_i{}^\tau, \varDelta) = 0$ for all i. If we write

$$\varDelta = \sum a_{ij}\eta_{ij} + \varDelta_0$$

where $\varDelta_0$ does not involve any η_{ij}, Lemma 13.2 yields

$$a_{00} - a_{i0} - a_{0j} + a_{ij} = 0$$

because β^τ vanishes on $\widehat{W}$. By Lemma 11.3, $(\beta^\tau, 1_G) = (\beta, 1_T) = 0$ so $a_{00} = 0$.

The set $\{\zeta_s\}$, $1 \le s \le q-1$, is coherent by Lemma 13.7 with ${\zeta_s}^\tau = \varepsilon \sum_j \eta_{sj}$. Then, by Lemma 11.4,

$$(\Delta, {\zeta_s}^\tau - {\zeta_1}^\tau) = (\beta^\tau, {\zeta_s}^\tau - {\zeta_1}^\tau) = (\beta, \zeta_s - \zeta_1) = e.$$

It follows that $a_{s0} - a_{10} = \pm 1$ with the sign independent of s. With $a = a_{20}$, we have
(18.1)
$$(a\pm1)^2+(q-2)a^2+\sum_j {a_{0j}}^2+\sum_j\{(a\pm1+a_{0j})^2+(q-2)(a+a_{0j})^2\} \le \|\Delta\|^2.$$

Let k be the contribution from the third term. Since each pair of complex conjugate characters contributes an even integer to the sum, k is even. The terms in the last sum and the first two terms contribute at least one, so

$$k + e \le \|\Delta\|^2.$$

By definition, $(q^c\zeta_0 - \zeta_1)^\tau = q^c\alpha^\tau + \beta^\tau$. Lemma 18.2 implies that for any value of j $(1 \le j \le e-1)$

$$(\alpha^\tau, \eta_{oj}) \ne 0 \quad \text{or} \quad (\beta^\tau, \eta_{0j}) \ne 0.$$

Since $\|\beta^\tau\|^2 = q^{2c} + e$ by Lemma 11.4, we have

$$(q^c - x)^2 + x^2(2q^c - 3) + k + e \le q^{2c} + e, \quad x^2(q^c-1) - xq^c \le 0.$$

Therefore, $0 \le x \le q^c/(q^c-1) < 2$. Thus, $x = 0$ or $x = 1$. Suppose that $x \ne 0$. Then, $x = 1$ and $\|\Delta\|^2 = e + 2$. It follows that $k \le 2$. If $k = 0$, we get a contradiction as in [FT]. Assume that $k = 2$. Then, $a_{0r} = a_{0s} = \pm 1$ for exactly two r, s and the remaining a_{0j} are zero. The values taken by β^τ are in the field $\mathbb{Q}_{|Q|}$ by Lemma 11.1, while the values taken by η_{0j} are in the field $\mathbb{Q}_e$ by Lemma 13.1. Then, η_{0r} has at least $p-1$ algebraic conjugates η_{0j} with $a_{0j} = a_{0r}$. It follows that $p - 1 = 2$. Thus, $p = 3$ and $q \ne 3$.

Since $\|\Delta\|^2 = e + 2 = e + k$, the contribution from each term of the last sum in (18.1) is exactly one. Since $q - 2 > 1$, we have $a + a_{0j} = 0$ for each j with $1 \le j \le e-1$. Since the first two terms of (18.1) also contribute 1, we have $a = 0$. This contradicts $a_{0r} = a_{0s} = \pm 1$.

Therefore, $x = 0$ and we have

$$\beta^\tau = q^c {\lambda_1}^\tau + \Delta$$

with $\|\Delta\|^2 = e$. It follows that $k = 0$ and $a_{0j} = 0$ for $1 \le j \le e-1$. Then, (18.1) reads

$$e((a\pm1)^2 + (q-2)a^2) \le e.$$

Hence, $a = 0$ or $a \pm 1 = 0$ and $q = 3$. If $a = 0$, then $a_{ij} = 0$ for all $i \geq 2$ and $a_{1j} = a_{10} = \pm 1$. Thus,

$$\beta^\tau = q^c \lambda_1{}^\tau \pm \zeta_1{}^\tau.$$

If $a \pm 1 = 0$ and $q = 3$, $a_{10} = 0$ and $a_{20} = \pm 1$. Hence, $a_{1j} = 0$ and $a_{2j} = a_{20} = \pm 1$. Thus, $\beta^\tau = q^c \lambda_1{}^\tau \pm \zeta_2{}^\tau$. Since $q = 3$, we have only two characters ζ_1 and ζ_2. We see that the union of characters $\{\lambda_i\}$ and $\{\zeta_s\}$ is coherent. This set is precisely $\mathcal{T}(Q_1)$, the set of characters of $\mathcal{T}$ which are induced by characters of Q/Q_1. Thus, $\mathcal{T}(Q_1)$ is coherent. The index $|Q : Q_1|$ is q^{2c+1} and $q^{2c+1} > 4e^2 + 1$ because $e = (q^c + 1)/2 \geq 2$.

We check that Hypothesis 12.4 is satisfied with $d = 1$. We wish to apply Theorem 12.5. The only point we need to worry about is the definition of $\mathcal{T}(Q_1)$. Thus, suppose that μ is a character of $\mathcal{T}$ that is equivalent to $\tau \in \mathcal{T}$ that has Q_1 in its kernel. Then, τ is either an irreducible character of degree e or a character of degree q^c and of weight e. Our set $\mathcal{T}(Q_1)$ contains all the irreducible characters of degree e and all the reducible ones of degree $q^c e$ because there are only $q - 1$ such characters. Thus, $\mu \in \mathcal{T}(Q_1)$. Theorem 12.5 yields that $\mathcal{T}$ is coherent, contrary to the assumption. Q.E.D.

Lemma 18.4. *The family $\mathcal{S}$ for the group S is coherent.*

This follows from Theorem 17.14, Lemma 18.3, and Lemma 11.6 [FT] as shown in [FT]. Q.E.D.

We use the following notation. Let

$$1 = q^{f_0} < q^{f_1} < \cdots$$

be the set of degrees of irreducible characters of Q and

$$\nu_1(1) = q^{f_n}.$$

Since $Q^* \subseteq Q'$ by Theorem C(3), the principal character of Q is the only linear character of Q that is W_2-invariant. Thus, $\nu_1(1) > 1$, i.e. $n > 0$. For $0 \leq i \leq n-1$, let λ_i be an irreducible character of T with $\lambda_i(1) = eq^{f_i}$. Let $\mathcal{S}_i$ be the set of irreducible characters of T which are induced by irreducible characters of Q with degree q^{f_i}. Define j_s inductively as follows. Let $j_0 = 0$. Define j_s to be the largest integer not exceeding $n+1$ such that

$$\mathcal{T}_{s-1} = \bigcup_{i=j_{s-1}}^{j_s - 1} \mathcal{S}_i$$

is coherent. Let Q_0 be the normal closure of Q^* in T. Let

$$1 = q^{g_0} < q^{g_1} < \cdots < q^{g_m}$$

be all the degrees of irreducible characters of Q/Q_0. For any j with $0 \le j \le m$, let θ_j be an irreducible character of T/Q_0 of degree eq^{g_j}. Since T/Q_0 is a Frobenius group, any nonprincipal irreducible character of Q/Q_0 induces an irreducible character of T/Q_0. Define

$$\begin{aligned} \alpha &= \zeta_0 - \lambda_0, \\ \beta_i &= q^{f_i - f_{i-1}}\lambda_{i-1} - \lambda_i \qquad (1 \le i \le n-1), \\ \gamma_j &= q^{g_j - g_{j-1}}\theta_{j-1} - \theta_j \qquad (1 \le j \le m). \end{aligned}$$

Lemma 18.5. *With the notation introduced above, we have*

$$\begin{aligned} ({\beta_i}^\tau, \eta_{0t}) = 0 \quad &\textit{for} \quad 0 \le t \le e-1, 1 \le i \le n-1 \\ ({\gamma_j}^\tau, \eta_{0t}) = 0 \quad &\textit{for} \quad 0 \le t \le e-1, 1 \le j \le m. \end{aligned}$$

Furthermore, if e is a prime, then one of the following possibilities occurs:

$$\alpha^\tau = 1_G - {\lambda_0}^\tau + \sum_{t=1}^{e-1} \eta_{0t},$$

$$\alpha^\tau = 1_G + \overline{\lambda_0}^\tau + \sum_{t=1}^{e-1} \eta_{0t} \quad \textit{and} \quad 2e+1 = |Q : Q'|,$$

$$\alpha^\tau = 1_G + \sum_{s=1}^{q-1} \eta_{s0} + \Gamma$$

with $(\Gamma, \eta_{st}) = 0$ *for* $0 \le s \le q-1$, $0 \le t \le e-1$.

Proof. Write

$$\alpha^\tau = \Gamma_{00} + \Delta_{00}, \quad {\beta_i}^\tau = \Gamma_{i0} + \Delta_{i0}, \quad {\gamma_j}^\tau = \Gamma_{0j} + \Delta_{0j}$$

where Δ_{ij} is a linear combination of the generalized characters η_{st} and Γ_{ij} is orthogonal to each of these η_{st}. Since α^τ, ${\beta_i}^\tau$, and ${\gamma_j}^\tau$ vanish on $\widehat{W}$, Lemma 13.2 yields that $\Delta_{ij} = \sum a_{st}\eta_{st}$ with a_{st} (depending on i and j) satisfying

$$a_{00} - a_{s0} - a_{0t} + a_{st} = 0$$

for all s and t. For $1 \le s \le q-1$, $(\zeta_s - \zeta_1)^\tau$ is orthogonal to α^τ, β^τ, and γ^τ. Since ${\zeta_s}^\tau = \varepsilon \sum_j \eta_{sj}$ for $s > 1$, we have

$$a_{s0} = a_{10} \quad \text{for} \quad 1 \le s \le q-1.$$

Consider β_i. Suppose that $\lambda_{i-1} \in \mathcal{T}_s$ and write

$$\beta_i{}^\tau = \Delta + \Delta_1$$

where $\Delta_1 \in I(\mathcal{T}_s{}^\tau)$ and Δ is orthogonal to $I(\mathcal{T}_s{}^\tau)$. The Lemma at the beginning of Section 18 yields $\{\pm\eta_{st}\} \cap \mathcal{T}_s{}^\tau = \emptyset$. Thus, Δ_{i0} is a partial sum of Δ. By Theorem 12.1 [FT],

$$\|\Delta\|^2 \leq e + 1.$$

Since $(\beta_i{}^\tau, (\lambda_i - \overline{\lambda}_i)^\tau) \neq 0$, $\beta_i{}^\tau$ involves either $\lambda_i{}^\tau$ or $\overline{\lambda}_i{}^\tau$. If $\lambda_i \in \mathcal{T}_s$, the coherence of $\mathcal{T}_s$ yields that $\Delta = 0$. If $\lambda_i \notin \mathcal{T}_s$, then $\lambda_i{}^\tau$ (or $\overline{\lambda}_i{}^\tau$) is involved in Δ. Since $\lambda_i{}^\tau \neq \pm\eta_{st}$, we have

$$\|\Delta_{i0}\|^2 \leq e.$$

We can prove $a = 0$ as in Lemma 31.5 [FT]. Hence,

$$\Delta_{i0} = \sum_{t=1}^{e-1} a_{0t} \sum_{s=0}^{q-1} \eta_{st}.$$

By Lemma 11.1, the virtual characters of $I_0(\mathcal{T})^\tau$ take nonzero values only at q-singular elements. On the other hand, the virtual characters of $I_0(\mathcal{S})^\tau$ vanish on q-singular elements by Lemma 11.1 and (F*ii*)(c). Thus, $I_0(\mathcal{T})^\tau$ is orthogonal to $I_0(\mathcal{S})^\tau$. Since $\mathcal{S}$ is coherent by Lemma 18.4, we have $(\xi_k(1)\xi_r{}^\tau - \xi_r(1)\xi_k{}^\tau, \Delta_{i0}) = (\xi_k(1)\xi_r{}^\tau - \xi_r(1)\xi_k{}^\tau, \beta_i{}^\tau) = 0$. On the other hand, $(\xi_k{}^\tau, \Delta_{i0}) = \pm a_{0k}q$. Hence,

$$\xi_k(1)a_{0r} = \xi_r(1)a_{0k}.$$

Suppose that $a_{0t} \neq 0$ for some t. Then, $a_{0k} \neq 0$ for all k. Hence, $\|\Delta_{i0}\|^2 \geq q(e-1)$. This contradicts $\|\Delta_{i0}\|^2 \leq e$. Therefore, $\Delta_{i0} = 0$. The case for γ_j is similar.

The remainder of the proof is the same as the proof of Lemma 31.5 [FT]. Q.E.D.

We continue to use the notation introduced just before Lemma 18.5.

Lemma 18.6. *With the notation of the preceding lemma, let* $\lambda = \lambda_{n-1}$ *and* $\beta = q^{f_n - f_{n-1}}\lambda - \zeta_1$. *Then* $(\beta^\tau, \eta_{0t}) = 0$ *for* $0 \leq t \leq e-1$.

Proof. Let $\mathcal{T}_b$ be the coherent set that contains $\mathcal{S}_{n-1}$. If $\zeta_1 \in \mathcal{T}_b$, then $\beta^\tau \in I(\mathcal{T}_b{}^\tau)$ and $(\beta^\tau, \eta_{0t}) = 0$. If $\zeta_1 \notin \mathcal{T}_b$, we apply Theorem 12.1 [FT]. The proof is the same as that of Lemma 31.6 [FT]. Q.E.D.

Theorem 18.7. *The set $\mathfrak{T}$ is coherent.*

Proof. Suppose that $\mathfrak{T}$ is not coherent and use the notation introduced in Lemmas 18.5 and 18.6. In particular, α, β_i, γ_j, λ_i, and θ_j have the same meaning as in Lemmas 18.5 and 18.6. We may choose the notation $\lambda_0 = \theta_0$. We have

$$(q^{f_n}\zeta_0 - \zeta_1)^\tau = q^{f_n}\alpha^\tau + \sum_i q^{f_n - f_i}\beta_i{}^\tau.$$

By Lemma 18.2, $((q^{f_n}\zeta_0 - \zeta_1)^\tau, \eta_{0j}) \neq 0$. Since Lemmas 18.5 and 18.6 yield $(\beta_i{}^\tau, \eta_{0t}) = 0$ for all i with $1 \leq i \leq n$, we have $(\alpha^\tau, \eta_{0t}) \neq 0$ for $0 \leq t \leq e-1$. Thus, if $(\alpha^\tau, \eta_{0t}) = a_t$, then $a_t \neq 0$ and $\sum a_t{}^2 \leq \|\alpha^\tau\|^2 = e+1$. Therefore, $a_t = 1$ or -1, and α^τ involves exactly one more irreducible character with multiplicity 1 or -1. Since

$$(\alpha^\tau, (\lambda_0 - \overline{\lambda}_0)^\tau) = -1,$$

the extra character is either $\pm\lambda_0{}^\tau$ or $\pm\overline{\lambda}_0{}^\tau$. In the latter case, we have $|Q : Q'| = 2e+1$ and there are exactly 2 irreducible characters of T with degree e. We may choose the notation that

$$\alpha^\tau = 1_G - \lambda_0{}^\tau + \sum a_t\eta_{0t} \qquad (a_t = 1 \text{ or } -1). \tag{18.2}$$

Lemma 18.5 yields $(\gamma_s{}^\tau, \eta_{0t}) = 0$ for $1 \leq s \leq m$, $0 \leq t \leq e-1$. Since

$$(q^{g_j}\theta_0 - \theta_j)^\tau = \sum_{s=1}^{j} q^{g_j - g_s}\gamma_s,$$

we have

$$((q^{g_j}\theta_0 - \theta_j)^\tau, \alpha^\tau) = ((q^{g_j}\theta_0 - \theta_j)^\tau, -\lambda_0{}^\tau).$$

The left side is equal to $(q^{g_j}\theta_0 - \theta_j, \alpha) = -q^{g_j}$ by Lemma 11.4 (and the choice $\theta_0 = \lambda_0$). Since $\|(q^{g_j}\theta_0 - \theta_j)^\tau\|^2 = q^{2g_j} + 1$ and $((q^{g_j}\theta_0 - \theta_j)^\tau, (\theta_j - \overline{\theta}_j)^\tau) = -1$, we have

$$(q^{g_j}\theta_0 - \theta_j)^\tau = q^{g_j}\theta_0{}^\tau - \theta_j{}^\tau. \tag{18.3}$$

If there are only two irreducible characters of degree eq^{g_j}, there is an ambiguity in the definition of $\theta_j{}^\tau$. But, we can take a consistent notation. Let Q_0 be the normal closure of Q^* in T as defined before Lemma 18.5. Let $\mathfrak{T}(Q_0)$ be the set of irreducible characters of $\mathfrak{T}$ having the degrees eq^{g_j} with $0 \leq j \leq m$. Then, (18.3) implies that $\mathfrak{T}(Q_0)$ is coherent.

Consider $(q^{f_n}\lambda_0 - \zeta_1)^\tau = \sum_{i=1}^{n} q^{f_n - f_i}\beta_i$. By (18.2) together with Lemmas 18.5 and 18.6,

$$((q^{f_n}\lambda_0 - \zeta_1)^\tau, \alpha^\tau) = ((q^{f_n}\lambda_0 - \zeta_1)^\tau, -\lambda_0{}^\tau).$$

Lemma 11.4 yields that the left side is equal to $-q^{2f_n}$. Since $\|(q^{f_n}\lambda_0 - \zeta_1)^\tau\|^2 = q^{2f_n} + e$, we have

$$(q^{f_n}\lambda_0 - \zeta_1)^\tau = q^{f_n}\lambda_0{}^\tau + \Delta$$

with $\|\Delta\|^2 = e$. The set $\{\zeta_s\}$ of virtual characters ζ_s is subcoherent by Lemma 13.7. Hence, the definition of subcoherent set yields that $\Delta = \pm\zeta_s{}^\tau$. In fact, $\Delta = -\zeta_1{}^\tau$ except possibly when $q - 1 = 2$. In the exceptional case, there are exactly two virtual characters of weight e. We can choose the notation

$$(q^{f_n}\lambda_0 - \zeta_1)^\tau = q^{f_n}\lambda_0{}^\tau - \zeta_1{}^\tau. \tag{18.4}$$

Let Q_1 be a normal subgroup of T such that $Q_1 \subseteq Q_0$ and Q_0/Q_1 is a chief factor of T. It follows from the definition of Q_0 that $Q^* \not\subseteq Q_1$. Then, Q^*Q_1 is a normal subgroup of T and $Q^*Q_1 = Q_0$ (cf. the second paragraph of the proof of Lemma 18.3). Thus, $|Q_0 : Q_1| = q$.

Since $\mathcal{T}$ is not coherent and $\mathcal{T}(Q_0)$ is coherent, Theorem 12.5 yields that

$$|Q : Q_0| \le 4e^2 + 1.$$

Hence, Q/Q_0 has no proper W_2-invariant subgroup. Since T/Q_0 is a Frobenius group, this implies that $\Phi(Q) \subseteq Q_0$. On the other hand, $Q^* \subseteq Q'$ by Theorem C (3). Therefore, $Q_0 \subseteq Q'$. Thus, $\Phi(Q) = Q_0 = Q'$. The subgroup Q_1 satisfies $|Q_0 : Q_1| = q$. Hence, $Z(Q/Q_1) = Q_0/Q_1$ and Q/Q_1 is an extraspecial q-group. Thus, $|Q : Q'| = q^{2c}$ for some integer c. Define

$$\mathcal{T}(Q_1) = \mathcal{T}(Q_0) \cup \{\zeta_i \mid 1 \le i \le q - 1\}.$$

Then, $\mathcal{T}(Q_1)$ consists of all characters in $\mathcal{T}$ having the same weight and degree as some character in $\mathcal{T}$ which has Q_1 in its kernel. By (18.4), $\mathcal{T}(Q_1)$ is coherent. Since $\mathcal{T}$ is not coherent, Theorem 12.5 yields

$$q^{2c+1} = |Q : Q_1| \le 4e^2 + 1.$$

By Theorem 2.5 [BG], e divides $q^c + 1$ or $q^c - 1$. Since e is odd, we have $2e \le q^c + 1$. Then,

$$q^{2c+1} \le 4e^2 + 1 \le (q^c + 1)^2 + 1 < 2q^{2c}.$$

This contradiction proves Theorem 18.7. Q.E.D.

Corollary 18.8. $\alpha^\tau = 1_G - \lambda_0{}^\tau + \sum_{t=1}^{e-1} \eta_{0t}$.

Proof. Let $a_t = (\alpha^\tau, \eta_{0t})$. Since $\mathfrak{T}$ is coherent by Theorem 18.7, we have

$$(18.5) \qquad \begin{aligned}(\nu_1(1)\zeta_0 - \zeta_1)^\tau &= \nu_1(1)\alpha^\tau + (\nu_1(1)\lambda_0 - \zeta_1)^\tau \\ &= \nu_1(1)\alpha^\tau + \nu_1(1)\lambda_0{}^\tau - \zeta_1{}^\tau.\end{aligned}$$

By the Lemma at the beginning of this section, $(\lambda_0{}^\tau, \eta_{0t}) = 0$. Also, $(\zeta_1{}^\tau, \eta_{0t}) = 0$. This follows from Lemma 13.1 if $q > 3$ because $\zeta_1{}^\tau = \pm\sum_j \eta_{1j}$. If $q = 3$, $\zeta_1{}^\tau$ is not uniquely determined; however, $\zeta_1{}^\tau$ is either $\pm\sum_j \eta_{1j}$ or $\pm\sum_j \eta_{2j}$. Thus, we have $(\zeta_1{}^\tau, \eta_{0t}) = 0$. Lemma 18.2 and (18.5) yield

$$0 \neq ((\nu_1(1)\zeta_0 - \zeta_1)^\tau, \eta_{0t}) = \nu_1(1)a_t.$$

Since $|\mathfrak{T}| > 2$, α^τ involves $-\lambda_0{}^\tau$. Since $\lambda_0{}^\tau$ is not one of $\pm\eta_{st}$, we have

$$\alpha^\tau = 1_G - \lambda_0{}^\tau + \sum a_t\eta_{0t}.$$

It follows from $\|\alpha^\tau\|^2 = e+1$ that $a_t = 1$ or -1 for each t. By Lemma 13.1, $\lambda_0{}^\tau$ vanishes on $\widehat{W}$. The same holds for α^τ. By Lemmas 13.1 and 13.2, we have $a_t = 1$ for $0 \le t \le e-1$. Q.E.D.

Corollary 18.9. *The group S' is a Frobenius group and the number w_2 is prime.*

Proof. Suppose that Corollary 18.9 is false. By Lemma 31.1, $\mathcal{S}(H')$ contains an irreducible character θ. Consider the group S/H'. Let $E = Q^*V$ be a complement of H in S. Since S is of type II, E is a Frobenius group with Frobenius kernel V and $C_H(V) = 1$ (cf. (II*iv*) and the modified (II*v*)). By Theorem 3.10 [BG], Q^* centralizes a nonidentity element of H/H'. Thus, $\mathcal{S}(H')$ contains one of the reducible characters. Hence, we can take $\xi_i \in \mathcal{S}(H')$. Note that $\mathcal{S}(H')$ is coherent. This is clear if $\mathcal{S}$ is coherent. If $\mathcal{S}$ is not coherent, Lemma 17.8 yields that $\mathcal{S}(H')$ is coherent. Hence, $\mathcal{S}(H')$ is coherent always. If we define $\beta = \theta(1)\xi_1 - \xi_1(1)\theta$, $\beta \in I_0(\mathcal{S}(H'))$ and $\beta^\tau = \theta(1)\xi_1{}^\tau - \xi_1(1)\theta^\tau$.

Let α be the element of $I_0(\mathfrak{T})$ defined in Corollary 18.8. We prove that α^τ is orthogonal to β^τ. By Lemma 11.1, α^τ vanishes on elements not conjugate to an element of $A(x)$ for any $x \in T'^\sharp$. Suppose that $g = xy = yx \in A(x)$ and $\alpha^\tau(g) \neq 0$. We claim that $\beta^\tau(g) = 0$. Suppose $\beta^\tau(g) \neq 0$. By Lemma 11.1 applied to S, g is conjugate to an element of S or one of the supporting subgroups of S. Since T is of type V, T is not conjugate to any supporting subgroup by (F*ii*). If M is a supporting subgroup of S, then $\sigma(M) \cap \sigma(T) = \sigma(S) \cap \sigma(T) = \emptyset$ by Theorem 7.9. Since $g = xy$ is conjugate to an element of S or a supporting subgroup, the element x is conjugate to an element of S. Since $\beta^{\tau}(g) \neq 0$, x is conjugate to an element of $A(S) - H$. It follows that $(|C_G(x)|, |H|) \neq 1$. This implies that S is conjugate to a supporting subgroup of T. Let S^h be a conjugate of S that contains $C_G(x)$. Then, by (F*ii*), $S^h \cap T$ is a complement of H^h that contains $C_T(x)$. Since x is conjugate to an element of $A(S)$, the order of x is prime to q. On the other hand, $Q^* \subseteq C_T(x)$ because T' is nilpotent. This contradicts the structure of $S^h \cap T$ being a Frobenius group with Frobenius complement of order q. Thus, $(\alpha^{\tau}, \beta^{\tau}) = 0$. In fact, the above argument proves that any element of $I_0(\mathcal{T})$ is orthogonal to every element of $I_0(\mathcal{S})$. We compute $(\alpha^{\tau}, \beta^{\tau})$ using Corollary 18.8. We have

$$(\alpha^{\tau}, \beta^{\tau}) = (1_G - {\lambda_0}^{\tau} + \sum \eta_{0t}, \theta(1){\xi_1}^{\tau} - \xi_1(1)\theta^{\tau}).$$

Note that ${\lambda_0}^{\tau} \neq \theta^{\tau}$. This follows from $((\lambda_0 - \overline{\lambda}_0)^{\tau}, (\theta - \overline{\theta})^{\tau}) = 0$. Since ${\xi_1}^{\tau} = \varepsilon \sum_i \eta_{i1}$ (or $\pm \sum_i \eta_{i2}$), we have

$$(\alpha^{\tau}, \beta^{\tau}) = (\sum \eta_{0t}, \theta(1)\varepsilon \sum \eta_{i1}) = \varepsilon\theta(1).$$

This contradicts $(\alpha^{\tau}, \beta^{\tau}) = 0$. Q.E.D.

Theorem 18.10. *No element of $\mathcal{M}$ is of type V.*

Proof. We will paraphrase the proof of Theorem 32.1 [FT]. Suppose that $\mathcal{M}$ contains a subgroup T of type V. For $M = T$, we use the notation introduced at the beginning of Chapter II. Thus, D^* and $A(x)$ for $x \in D^*$ have the same meaning as defined there. We denote by S the subgroup of type II defined in Theorem I. The subgroup $H = S_F$ is a TI-set by (T7). In addition, the following notation is used: $T = T'W_2$, $S = S'Q^*$, $W = Q^* \times W_2$, $|W_2| = w_2 = e$, $|Q^*| = q$ and $|S' : H| = v$. Let $\mathcal{T}$ be the set of characters of T introduced at the beginning of this section. Then, by Theorem 18.7, $\mathcal{T}$ is coherent. Let

$$\mathcal{T}^* = \{\mathcal{T}, \zeta_0\}.$$

Corollary 18.8 yields that $\mathcal{T}^*$ is coherent if we define

$${\zeta_0}^{\tau} = 1_G + \sum_{t=1}^{e-1} \eta_{0t}.$$

The family $\mathcal{T}^*$ consists of irreducible characters of T and q reducible characters $\zeta_0, \zeta_1, \ldots, \zeta_{q-1}$ of weight e. There are irreducible characters ν_{ij} of T such that

$$\xi_i = \sum_{j=0}^{e-1} \nu_{ij} \quad \text{with} \quad (\nu_{ij})_{T'} = (\nu_{i0})_{T'}.$$

There is an irreducible character λ of degree e in $\mathcal{T}$. Lemma 14.4 applied with T and $\mathcal{T}^*$ in place of M and $\mathcal{S}$ yields

$$\lambda^\tau(x) = \lambda(x) + s\gamma(x) \quad \text{for } x \in T'^\sharp$$

where γ is orthogonal to every element of $\mathcal{T}^*$. Since the irreducible characters of T are $\{\nu_{ij}\}$ and the characters in $\mathcal{T}$, we have

$$\gamma = \sum a_{ij}\nu_{ij}.$$

Since $(\gamma, \zeta_i) = 0$ for $0 \le i \le q-1$, we have $\sum_j a_{ij} = 0$ for $0 \le i \le q-1$. It follows that

$$\gamma_{T'} = \sum_{i,j} a_{ij}(\nu_{ij})_{T'} = \sum_i \left(\sum_j a_{ij}\right)(\nu_{i0})_{T'} = 0.$$

This proves that $\lambda^\tau(x) = \lambda(x)$ for $x \in T'^\sharp$. By Lemma 14.3, λ^τ is constant on the set of the form $A(x)$ for $x \in D^*$. Hence, Lemma 11.5 yields

$$\text{(18.6)} \qquad \frac{1}{|G|}\sum_{x \in G_1} |\lambda^\tau(x)|^2 = \frac{1}{|T|}\sum_{x \in T'^\sharp} |\lambda(x)|^2 = 1 - \frac{e}{|T'|}.$$

Let G_1 be the set of elements of G which are conjugate to some element of $A(x)$ for $x \in D^*$. By Lemma 11.5 with Θ replaced by 1_G, we have

$$\frac{|G_1|}{|G|} = \frac{1}{e}\left(1 - \frac{1}{|T'|}\right).$$

Define $G_2 = \mathcal{C}_G(\widehat{W})$. By Theorem 8.7 (e),

$$\frac{|G_2|}{|G|} = 1 - \frac{1}{e} - \frac{1}{q} + \frac{1}{eq}.$$

Let G_3 be the set of elements of G which are conjugate to some elements of $H^\sharp$. Since H is a TI-set, we have

$$\frac{|G_3|}{|G|} = \frac{1}{qv|H|}(|H|-1).$$

These sets G_1, G_2, and G_3 are disjoint. Let G_0 be the complement of the union $G_1 \cup G_2 \cup G_3$. Then,

$$\begin{aligned}\frac{|G_0|}{|G|} &= 1 - \frac{1}{e}\left(1 - \frac{1}{|T'|}\right) - \left(1 - \frac{1}{e} - \frac{1}{q} + \frac{1}{eq}\right) - \frac{1}{qv} + \frac{1}{qv|H|} \\ &> \frac{1}{q} - \frac{1}{eq} - \frac{1}{qv} \geq \frac{1}{3q}\end{aligned} \tag{18.7}$$

because $e \geq 3$ and $v \geq 3$. By (18.6), we have

$$\frac{1}{|G|}\sum_{x \in G_0} |\lambda^\tau(x)|^2 \leq 1 - (1 - \frac{e}{|T'|}) = \frac{e}{|T'|}.$$

By Corollary 18.9, e is a prime and S' is a Frobenius group. Hence, $\eta_{01}, \ldots, \eta_{0e-1}$ are algebraically conjugate characters with values in $\mathbb{Q}_e$. Since S' is a Frobenius group, every element whose order is divisible by e lies in $G_2 \cup G_3$. Thus, η_{0t} take the same integral value on G_0. Since $(\zeta_0 - \lambda)^\tau$ vanishes off G_1,

$$\lambda^\tau(x) = 1 + (q-1)\eta_{01}(x) \quad \text{for} \quad x \in G_0.$$

In particular, $\lambda^\tau(x)$ is an odd integer so

$$|\lambda^\tau(x)|^2 \geq 1 \quad \text{for} \quad x \in G_0.$$

Thus,

$$\frac{|G_0|}{|G|} \leq \frac{1}{|G|}\sum_{x \in G_0} |\lambda^\tau(x)|^2 \leq \frac{e}{|T'|}.$$

Therefore, $|T'| < 3qe$ by (18.7). Theorem C (3) implies $Q^* \subseteq T''$. Hence, we get $|T' \colon T''| < 3e$. Since T/T'' is a Frobenius group, $|T' \colon T''| \geq 2e+1$. Thus,

$$|T''|(2e+1) \leq |T'| < 3qe, \quad q \leq |T''| < 3eq/(2e+1) < 2q.$$

It follows that $|T''| = q$ and W_2 acts irreducibly on T'/T''. This implies that T' is an extraspecial q-group. If $|T'/T''| = q^{2c}$, then Theorem 2.5 [BG] yields that e divides $q^c + 1$ or $q^c - 1$. Hence, $e \leq (q^c + 1)/2$ and

$$q^{2c} = |T' \colon T''| < 3e \leq 3(q^c + 1)/2 \leq 2q^c.$$

This contradiction proves Theorem 18.10.

Corollary 18.11. *Let S be a subgroup of type II, III, or IV in $\mathcal{M}$. Then, the family $\mathcal{S}$ is coherent.*

This follows from Theorems 17.14 and 18.10.

§19. Subgroups of Type I

We remark that a subgroup $M \in \mathcal{M}$ of type I is a Frobenius group if and only if $\tau_2(M)$ is empty. This is easy to see. If $\tau_2(M) = \emptyset$, then the complement E in (I*iii*) is a Z-group and the only subgroup of E with the same exponent as E is E. Thus, $E_0 = E$ and by (I*iii*), M is a Frobenius group. Suppose conversely that M is a Frobenius group. Then, $H = M_F$ is the nilpotent normal subgroup of maximal order in M. Then, H is the Frobenius kernel of the Frobenius group M. It follows from the property of a Frobenius complement that all Sylow subgroups of M/H are cyclic. This implies $\tau_2(M) = \emptyset$ (cf. the notation of §6).

Theorem 19.1. *Every subgroup of type I is a Frobenius group.*

The proof is by contradiction. Suppose that $\mathcal{M}$ has a subgroup of type I that is not a Frobenius group. The following notation will be used. Let ρ be the set of primes defined as follows: $p_i \in \rho$ if and only if $\mathcal{M}$ has a subgroup M_i of type I such that $p_i \in \tau_2(M_i)$. By Lemma H, the groups M_i are ϖ-groups; in particular, $p_i \in \varpi$. (The set ρ is denoted σ in [FT]; I have chosen this notation because σ has a different meaning in [BG] and we have been using σ in the sense of [BG].) The smallest prime in ρ will be denoted $p = p_k$. Let $M = M_k$, $K = M_F$, $P_0 \in Syl_p(M)$, $P \in Syl_p(G)$ such that $P_0 \subseteq P$, $A = \Omega_1(P_0)$, and

$$L \in \mathcal{M}(N_G(A)).$$

If L is of type I, let $U = L_F$ and choose a complement E of U in L. If L is not of type I, then L is of type II, III, or IV by Theorem 18.10. In this case, let $H = L_F$, $\mathfrak{U}$ a complement of H in L', and W_1 a complement of L' in L with $W_1 \subseteq N_L(\mathfrak{U})$. The order $|W_1|$ is a prime by (T7). Note the particular usage of the symbol $\mathfrak{U}$.

Let $\mathcal{L}$ be the set of characters of L defined as follows: If L is of type I, $\mathcal{L}$ is the set of all irreducible characters of L which do not have U in their kernel. If L is of type II, III, or IV, then $\mathcal{L}$ is the set of characters of L each of which is induced by a nonprincipal irreducible character of L' that vanishes outside $A(L)$. Thus, if L is of type I, $\mathcal{L}$ is the set of characters studied in §15. If L is of type III or IV, then $\mathcal{L}$ corresponds to the set $\mathcal{S}_0 \cup \mathcal{S}$ in §16.

Lemma 19.2. *The subgroup L is not of type II; it is either a Frobenius group with cyclic Frobenius complement or of type III or IV. There is no subset of L that is a TI-set of G and contains A. The group P is either an abelian group of rank 2 or the center $Z(P)$ is cyclic, and we can take $P \subseteq U$. Furthermore, L is the unique subgroup of $\mathcal{M}$ that contains $N_G(A)$.*

Proof. Since $p \in \tau_2(M)$, $A \in \mathcal{E}_p^2(M)$ and P_0 is an abelian group of rank 2 by Theorem 6.5 (b). We have $C_G(A) \subseteq M$ by Proposition 6.4 (a). It follows that either $P = P_0$ or $Z(P)$ is cyclic.

By Lemma 6.2 applied with A and L in place of X and M^*, we have that $p \in \sigma(L) \cup \tau_2(L)$. If $p \in \tau_2(L)$, Theorem 6.5 (b) applied with L in place of M yields $N_G(A) \not\subseteq L$, contradicting the definition of L. Thus, $p \in \sigma(L)$ and $A \subseteq L_\sigma$. In fact, $p \in \sigma_0(L)$ as $p \in \varpi$.

We have $A \in \mathcal{E}_p^2(M)$ by Theorem 6.5 (b). Since A normalizes K, some element $a \neq 1$ of A commutes with some element $y \neq 1$ of K by Proposition 1.16 [BG]. But, $K \cap L = 1$ by Theorem 6.5 (e). Thus, A is not contained in any subset of L that is a TI-set in G.

Suppose that L is of type II. Then, by Theorem 9.7 (a) with L in place of M, $L_\sigma = H \subseteq F(L)$ and $F(L)$ is a TI-set of G. Since $A \subseteq L_\sigma$, this contradicts what we proved in the preceding paragraph. Thus, L is not of type II.

If L is of type III or IV, $F(L)$ is a TI-set in G by (T7). Therefore, $A \not\subseteq F(L)$ but $A \subseteq L_\sigma = L'$. Thus, $p \in \pi(U)$. Since U is a Hall subgroup of L_σ, U contains a Sylow p-subgroup of G. We can choose $P \subseteq$ U.

Suppose that L is of type I. Then, $L_{\sigma_0} = U$. Since U is a Hall subgroup of G, we have $P \subseteq$ U. In fact, since U is nilpotent, P is a normal subgroup of L. We will prove that L is a Frobenius group. Suppose that L is not a Frobenius group. Then, $\tau_2(L)$ is not empty. Let $q \in \tau_2(L)$ and take $Q \in \mathcal{E}_q^2(L)$. Then, $q \in \rho$. It follows that $p < q$. By Theorem 6.5 (d), we have $C_U(Q) = 1$. Thus, Q acts on $\Omega_1(Z(P))$ nontrivially. Since $r(Z(P)) \leq 2$, we have $q < p$, contradicting the minimal nature of p. This proves that L is a Frobenius group.

The subgroup E is a Frobenius complement of L. Hence, E acts on $\Omega_1(Z(P))$ faithfully. If $Z(P)$ is cyclic, E is abelian. If P is abelian, $\Omega_1(P) = A$. Theorem 2.6 [BG] yields that E is abelian. A Frobenius complement is cyclic if it is abelian. Therefore, E is cyclic.

It remains to prove that $\mathcal{M}(N_G(A)) = \{L\}$. Suppose that P is nonabelian. Choose a subgroup P_1 such that $P_0 \subseteq P_1 \subseteq P$ with $|P_1 : P_0| = p$. Since $C_G(A) \subseteq M$ by Proposition 6.4 (a), P_1 is nonabelian. We have $P_1 \subseteq N_G(A)$ as $A = \Omega_1(P_0)$. By Theorem 6.13, $P_1 \in \mathcal{U}$. Hence, $N_G(A) \in \mathcal{U}$. Assume that P is abelian. Then, $P_0 = P$ and

$N_G(P) \subseteq N_G(A)$. Suppose that $L_0 \in \mathcal{M}(N_G(A))$. Then, $p \in \sigma(L_0) \cap \sigma(L)$. Theorem 7.9 yields that L_0 is conjugate to L: $L_0 = g^{-1}Lg$ for some $g \in G$. Then, $P, g^{-1}Pg \in Syl_p(L_0)$; hence, $P = h^{-1}g^{-1}Pgh$ for some $h \in L_0$. Thus, $L_0 = x^{-1}Lx$ with $x = gh \in N_G(P) \subseteq N_G(A) \subseteq L$. Therefore, $L_0 = L$. This proves the uniqueness of L. Q.E.D.

Lemma 19.3. *There exists an irreducible character $\lambda \in \mathcal{L}$ that does not have P in its kernel such that $\lambda(1)$ divides $p-1$ or $p+1$. The group L/L' is a cyclic group of order e with e dividing $p-1$ or $p+1$.*

Proof. Suppose that L is of type III or IV. Then, L/H is a Frobenius group with Frobenius kernel isomorphic to $\mathfrak{U}$. Since $P \subseteq U$, L/H has an irreducible character λ of degree w_1 that does not have P in its kernel. Since L/H is a Frobenius group, W_1 acts faithfully on $\Omega_1(Z(P))$. It follows that $w_1 \mid p^2-1$. Since w_1 is a prime by (T7), $\lambda(1) = w_1$ divides $p-1$ or $p+1$.

Suppose that L is of type I. Then, by Lemma 19.2, L is a Frobenius group with Frobenius kernel $\mathfrak{U}$ and Frobenius complement E that is cyclic. Thus, there is an irreducible character λ of L of degree $e = |E|$ that does not have P in its kernel. We need to prove that e divides either $p-1$ or $p+1$. As before, E acts faithfully on $\Omega_1(Z(P))$. Thus, if $Z(P)$ is cyclic, e divides $p-1$. If P is abelian, $e \mid p^2-1$. Suppose that e has prime divisors q_1 and q_2 such that

$$q_1 \mid p-1 \quad \text{and} \quad q_2 \mid p+1.$$

We will derive a contradiction. Let Q_i be a subgroup of E of order q_i. Then, Q_2 acts regularly on $\mathcal{E}_p^1(A)$, while Q_1 has at least two fixed points on $\mathcal{E}_p^1(A)$. Since Q_1Q_2 is abelian, Q_2 moves a Q_1-invariant subgroup to a Q_1-invariant subgroup. Thus, there are at least 3 Q_1-invariant subgroups of order p in A. It follows that Q_1 acts on A as a scalar, i.e. Q_1 does not centralize A but every subgroup of order p in A is Q_1-invariant. By Proposition 6.4 (b), there is $A_0 \in \mathcal{E}^1(A)$ such that $N_G(A_0) \subseteq M$. This implies that $Q_1 \subseteq M$. Then, $Q_1 \subseteq N_M(A)$. By Corollary 6.6 (b), $C = N_M(A)$ is a complement of K in M. The structure of M as a group of type I yields that there is a subgroup C_0 of C with the same exponent as C such that C_0 is a Frobenius complement of the Frobenius group KC_0. We are in the situation that $P = P_0$ is abelian. Then, Lemma 6.8 (a) yields that $q_1 \notin \tau_2(M)$ so C has a cyclic Sylow q_1-subgroup. We may take C_0 such that $Q_1 \subseteq C_0$. Then, $AQ_1 \cap C_0$ has order pq_1 and it is not cyclic. This contradicts the structure of a Frobenius complement. Thus, we have $e \mid p-1$ or $e \mid p+1$. Q.E.D.

Lemma 19.4. *The family $\mathfrak{L}$ is coherent. Let λ be the character defined in Lemma* 19.3. *Then, $\lambda^\tau(x) = \lambda(x)$ for $x \in A(L)^\sharp$.*

Proof. Let $e = |L:L'|$. We prove that the set $\mathfrak{L}$ of characters is coherent. Suppose that L is of type I. By Lemma 19.2, L is a Frobenius group with Frobenius kernel $\mathfrak{U}$ and L/U is a cyclic group of order e. By (Iv) for the group of type I, L satisfies one of the three conditions (a), (b), or (c) (cf. [BG], p.128). Since $P \subseteq U$, U is not a TI-set of L. Thus, the condition (a) does not nold. If L satisfies (b), U is abelian and $\mathfrak{L}$ is coherent. If L satisfies (c), then the exponent of L/U divides $p-1$. Hence, e divides $p-1$ and

$$|U:U'| \geq p^2 > 4e^2+1.$$

Since $\mathfrak{L}(U')$ is coherent, Theorem 12.5 yields that $\mathfrak{L}$ is coherent.

Suppose that L is of type III or IV. Then, L/H is a Frobenius group with Frobenius kernel $UH/H \cong \mathfrak{U}$. If $\mathfrak{L}(H)$ is not coherent, Lemma 12.6 yields that U is a nonabelian p-group with

$$|U:U'| \leq 4e^2+1.$$

Thus, $P = U$ is nonabelian. By Lemma 19.2, the center of P is cyclic. Then, Lemma 19.3 yields $e \mid p-1$. This is a contradiction because

$$p^2 \leq |U:U'| \leq (p-1)^2+1 < p^2.$$

It follows that $\mathfrak{L}(H)$ is coherent. By Theorem 16.1 (b), $\mathfrak{L}$ is coherent if L is of type IV.

Suppose that L is of type III and $\mathfrak{L}$ is not coherent. Then, L/H is a Frobenius group with abelian Frobenius kernel which is isomorphic to U. Let $\mathfrak{L}_0$ be the set of characters of L which are induced by nonprincipal irreducible characters of L'/H. Then, $\mathfrak{L}_0$ is coherent and $|\mathfrak{L}_0| = (u-1)/e$. If $\mathfrak{L}_1 = \mathfrak{L} - \mathfrak{L}_0$, then by Corollary 18.11, $\mathfrak{L}_1$ is coherent. Since we assumed that $\mathfrak{L}$ is not coherent, Theorem 16.1 (a) yields that H is an elementary abelian group of order r^e for some prime r, $C_U(H) = 1$, and UW_1 acts irreducibly on H. Since $A \subseteq \mathcal{E}_p^2(U)$, some nonidentity element of A lies in the inertia group of a nonprincipal linear character of H. As we can see from the proof of Lemma 13.8, there is an irreducible character $\mu \in \mathfrak{L}_1$ of degree de with $d \leq (u/p)$ where $u = |U|$. Let λ be a character of $\mathfrak{L}_0$. Then, λ is an irreducible character of degree e. Consider

$$\alpha = \xi_0 - \lambda \quad \text{and} \quad \beta = d\lambda - \mu,$$

where $\xi_0 = (1_{L'})^L$. If $\lambda_1, \lambda_2 \in \mathfrak{L}_0$ are distinct from λ,

$$(\beta^\tau, (\lambda_1 - \lambda_2)^\tau) = 0 \quad \text{and} \quad (\beta^\tau, (\lambda - \lambda_2)^\tau) = d$$

by Lemma 11.4. Therefore, we have

$$\beta^\tau = d\lambda^\tau - x \sum_{\nu \in \mathfrak{L}_0} \nu^\tau - \mu^\tau + \Delta_1 \tag{19.1}$$

where $(\Delta_1, \nu^\tau) = 0$ for every $\nu \in \mathfrak{L}_0$. Lemma 11.4 yields

$$\|\beta^\tau\|^2 = \|\beta\|^2 = d^2 + 1.$$

Hence, we have $(d-x)^2 + x^2(((u-1)/e) - 1) + 1 \leq d^2 + 1$, or

$$x^2(u-1)/e \leq 2dx. \tag{19.2}$$

By Lemma 19.3, e divides $p-1$ or $p+1$. If $2e \neq p+1$, then $2e \leq p-1$. It follows from (19.2) that

$$0 \leq x \leq 2ed/(u-1) \leq (p-1)d/(u-1) < 1$$

because $pd \leq u$ and $d > 1$. The above inequality yields $x = 0$ and $\mathfrak{L}$ is coherent. Therefore, we have $2e = p+1$. Since $2ed/(u-1) < 2$, we have $x = 1$.

Consider $\alpha^\tau = (\xi_0 - \lambda)^\tau$. If we define $\alpha^\tau = 1_G + \Delta - \lambda^\tau$, then $(\Delta, \nu^\tau) = 0$ for every $\nu \in \mathfrak{L}_0$, and $\|\Delta\|^2 = e - 1$ (cf. the proof of Lemma 12.7). We will show that

$$\Delta = \sum_{i=1}^{e-1} \eta_{i0}.$$

There is a long detour. The set $\mathfrak{L}_1$ contains $|W_2| = r$ reducible characters $\xi_1, \ldots, \xi_{r-1}$. We will show that $\xi_k(1) = ue$ for $k > 0$. Let θ be an irreducible character of minimal degree in $\mathfrak{L}_1$ with $\theta(1) = d_1 e$. Then, θ_H contains a nonprincipal linear character η of H and $I(\eta) \cap U \neq 1$. Take a prime $q \in \pi(I(\eta) \cap U)$. Since U is abelian and $A \subseteq \mathrm{U}$, we have $U \subseteq C_G(A) \subseteq M$ by Proposition 6.4 (a). In fact, U is contained in the complement $N_M(A)$ of K in M (Corollary 6.6 (b)). Suppose that $q \neq p$. If $q \in \tau_2(M)$, then $q > p$ by the minimal choice of p in the set ρ. If $q \notin \tau_2(M)$, then U has a cyclic Sylow q-subgroup. Since W_1 acts regularly on U, we have $q \equiv 1 \pmod{e}$. Since $e = (p+1)/2$, we have $q \geq p+2$. If $\beta_1 = d_1\lambda - \theta$,

$${\beta_1}^\tau = d_1\lambda^\tau - x_1 \sum \nu^\tau - \theta^\tau + \Delta_2$$

with $x_1^2(u-1) \leq 2ed_1x_1$. Since $d_1 \leq u/q$, the inequality $x_1 \neq 0$ yields $q(u-1) \leq (p+1)u$. Since $q \geq p+2$,

$$\frac{p+2}{p+1} \leq \frac{u}{u-1}.$$

This implies $p+1 \geq u-1 \geq p^2-1$. This contradiction proves that $x_1 = 0$ and $\mathfrak{L}$ is coherent. Thus,

$$\pi(I(\eta) \cap U) = \{p\}.$$

If $|I(\eta) \cap U| > p$, then we have an irreducible character of degree $d_2 e$ with $d_2 \leq u/p^2$. Then, a similar argument yields

$$p(u-1) \leq (p+1)u/p < 2u.$$

This is impossible. Thus, the degree of an irreducible character in $\mathfrak{L}_1$ is either ue/p or ue. For a nonprincipal linear character η of H, the index $|I(\eta) : H|$ cannot be equal to pe because W_1 does not normalize any subgroup of order p as $|W_1| = e = (p+1)/2$. Hence, the degree $\xi_k(1)$ of the reducible character ξ_k $(k > 0)$ is ue.

As remarked before, H is an abelian r-group for some prime r. We will show that $r \geq 2e$. We have seen that there is a subgroup B of order p in A such that $C_H(B) \neq 1$. Let $H_1 = C_H(B)$. Then, $H = H_1 \times H_2$ with $H_2 = [H, B]$ by Proposition 1.6 [BG]. Since U normalizes B, U acts on H_1 and H_2. If η is a nonprincipal linear character of H/H_2, then $I(\eta) \cap U = B$ as we have shown. It follows that the group U/B acts regularly on H_1. This implies that U/B is cyclic. Hence, $U = B \times C$ with C being cyclic. We will show that C can be chosen in such a way that C acts regularly on H.

The group UW_1 acts on H irreducibly by Theorem 16.1 (a). Since $N_H(A) = C_H(A)$ is W_1-invariant, we have

$$N_L(A) \cap H = N_H(A) = 1.$$

Note that $C_U(H) = 1$ by Theorem 16.1(a). It follows that $N_G(A) = N_L(A) = UW_1$. Since $U \subseteq C_G(A) \subseteq M$ but $N_G(A) \not\subseteq M$, we have $M \cap L = U$. The elementary abelian group A acts on K. Therefore, $C_K(A_1) \neq 1$ for some $A_1 \in \mathcal{E}^1(A)$. Since $C_K(A_1) \not\subseteq L$, M is one of the supporting subgroups of the F-set $A(L) = L'$. By (F*ii*), $C_G(A_1) \subseteq M$. Since $H \cap M = H \cap U = 1$, we have

$$C_H(A_1) = 1.$$

We can take $C \supseteq A_1$. Then, for any subgroup C_1 of prime order in C, $C_H(C_1) = 1$. Therefore, C acts regularly on H. It follows that $r^e \equiv 1 \pmod{u/p}$. If $|H_1| = r^m$, then $m < e$ and $r^m \equiv 1 \pmod{u/p}$. Since e is a prime, we have $r \equiv 1 \pmod{u/p}$. The prime p divides u/p. This implies

$$r - 1 \geq p \quad \text{or} \quad r \geq p + 1 = 2e.$$

Lemma 13.4 yields that $\xi_k = \sum \mu_{ik}$ where $\{\mu_{ik}\}$ is the set of irreducible characters associated with the selfnormalizing cyclic subgroup $W = W_1 \times W_2$. By the definition of the characters μ_{ik}, there is a sign ε that is independent of i such that $\mu_{ik}(x) = \varepsilon\omega_{ik}(x)$ for all $x \in W - W_2$. We claim that $\varepsilon = 1$. Consider the restriction $(\mu_{0k})_{W_1}$. Then, $(\mu_{0k})_{W_1} - \varepsilon 1_{W_1}$ vanishes on $W_1^\sharp$, so it is a multiple of the regular representation of W_1. Therefore, $\mu_{0k}(1) \equiv \varepsilon \pmod{e}$. Since $\mu_{0k}(1) = u$, we have $\varepsilon = 1$.

The group W_2 is of order r. Since r is a prime, the characters $\mu_{i1}, \mu_{i2}, \ldots, \mu_{ir-1}$ are r-conjugate; so are $\eta_{i1}, \ldots, \eta_{ir-1}$. Recall the definition of Δ. It is defined

$$(\xi_0 - \lambda)^\tau = 1_G + \Delta - \lambda^\tau.$$

The weight of Δ is $e - 1$ and $(1_G, \Delta) = (\nu^\tau, \Delta) = 0$ for every $\nu \in \mathfrak{L}_0$. We claim that Δ is r-rational and $(\Delta, \eta_{ik}) = 0$ if $k > 0$. Let ν be an irreducible character of $\mathfrak{L}_0$ different from λ. Then, α^τ as well as $(\lambda - \nu)^\tau$ are r-rational by Lemma 11.1. The proof of Lemma 12.1 shows that λ^τ is r-rational. Therefore, Δ is r-rational. Suppose that $(\Delta, \eta_{ik}) = a_k \neq 0$ for some i and $k > 0$. Since $\eta_{i1}, \ldots, \eta_{ir-1}$ are r-conjugate, we have $(\Delta, \eta_{it}) = a_k \neq 0$ for every $t > 0$. Thus, Δ involves $a_k \sum_t \eta_{it}$ and

$$\|\Delta\|^2 \geq r - 1.$$

Since $r - 1 \geq 2e - 1$, we have a contradiction that

$$e - 1 = \|\Delta\|^2 \geq 2e - 1.$$

Thus, $(\Delta, \eta_{ik}) = 0$ if $k > 0$.

Finally, we will prove that $\Delta = \sum_i \eta_{i0}$. For a fixed $k > 0$, consider

$$\gamma_i = \mu_{i0} - \mu_{ik} + \Sigma_0$$

where Σ_0 is the sum of irreducible characters of $\mathfrak{L}_0$. There are $(u-1)/e$ characters of degree e in $\mathfrak{L}_0$ and for $x \in {W_1}^\sharp$,

$$\mu_{ik}(x) = \omega_{ik}(x) = \omega_{i0}(x) = \mu_{i0}(x).$$

Since $\mu_{ik}(1) = u$ and $\mu_{i0}(1) = 1$, we have $\gamma_i \in I_0(A_0(L))$. Since

$$(\gamma_0 - \gamma_i)^\tau = (1_L - \mu_{0k} - \mu_{i0} + \mu_{ik})^\tau = 1_G - \eta_{0k} - \eta_{i0} + \eta_{ik},$$

we have ${\gamma_i}^\tau = \eta_{i0} - \eta_{ik} + \Gamma$ where Γ is independent of i. We will compute $({\gamma_0}^\tau, \alpha^\tau)$ using Lemma 11.4. Here, $\alpha = \xi_0 - \lambda$, so $\alpha^\tau = 1_G + \Delta - \lambda^\tau$. Since $\xi_0 = \sum_i \mu_{i0}$, we have

$$({\gamma_0}^\tau, \alpha^\tau) = (\gamma_0, \alpha) = 0.$$

Similarly, $({\gamma_i}^\tau, \alpha^\tau) = (\gamma_i, \alpha) = 0$. Thus, if $i > 0$,

$$\begin{aligned} 0 &= ({\gamma_0}^\tau, \alpha^\tau) = 1 + (\Gamma, \Delta) - (\Gamma, \lambda^\tau) \\ &= ({\gamma_i}^\tau, \alpha^\tau) = (\eta_{i0}, \Delta) + (\Gamma, \Delta) - (\Gamma, \lambda^\tau). \end{aligned}$$

Therefore, $(\eta_{i0}, \Delta) = 1$. We have used the lemma that $\lambda^\tau \neq \pm\eta_{st}$ for any s and t. We have

$$\Delta = \sum_{i=0}^{e-1} \eta_{i0}. \tag{19.3}$$

Clearly, Δ is a real-valued character. By the definition of β, we have $\overline{\beta} = d\overline{\lambda} - \overline{\mu}$. Then,

$$\beta - \overline{\beta} = d(\lambda - \overline{\lambda}) - (\mu - \overline{\mu}).$$

Since $\beta - \overline{\beta}$, $\lambda - \overline{\lambda}$, and $\mu - \overline{\mu} \in I_0(A(L))$, we have

$$(\beta - \overline{\beta})^\tau = d(\lambda - \overline{\lambda})^\tau - (\mu - \overline{\mu})^\tau = d(\lambda^\tau - \overline{\lambda}^\tau) - (\mu^\tau - \overline{\mu}^\tau).$$

On the other hand, we can compute $(\beta - \overline{\beta})^\tau = \beta^\tau - \overline{\beta}^\tau$ using (19.1). Since $\sum \nu^\tau$ is real, we have

$$\beta^\tau - \overline{\beta}^\tau = d(\lambda^\tau - \overline{\lambda}^\tau) - (\mu^\tau - \overline{\mu}^\tau) + \Delta_1 - \overline{\Delta}_1.$$

Therefore, $\Delta_1 = \overline{\Delta}_1$ is a real-valued virtual character. It follows from (19.3) that (Δ, Δ_1) is an even integer. We will contradict this by showing $(\Delta, \Delta_1) = -1$.

Compute $(\alpha^\tau, \beta^\tau)$ in two ways. Lemma 11.4 yields

$$(\alpha^\tau, \beta^\tau) = (\alpha, \beta) = -d.$$

By (19.1), we have

$$(\alpha^\tau, \beta^\tau) = (\Delta, \Delta_1) - (d-1) = (\Delta, \Delta_1) - d + 1.$$

Thus, $(\Delta, \Delta_1) = -1$. This contradiction proves that $\mathcal{L}$ is coherent in all cases.

We can apply Lemma 12.7 for M, H, H_1, h and $\mathcal{S}$ replaced by L, L', L'', p^2 and $\mathcal{L}$. Since $P \subseteq U$ and U is nilpotent, we have $|L' : L''| \geq p^2$. By Lemma 19.3, $e \leq (p+1)/2$. This implies $p^2 - 1 > e(e+1)$. If we define Δ by $(\xi_0 - \lambda)^\tau = 1_G + \Delta - \lambda^\tau$ and

$${\xi_0}^\tau = 1 + \Delta,$$

the set $\{\mathcal{L}, \xi_0\}$ is coherent by Lemma 12.7.

For $x \in A(L)^\sharp$, Lemma 14.4 yields that

$$\lambda^\tau(x) = \lambda(x) + s\gamma(x)$$

where s is a rational number and γ is a virtual character that is orthogonal to every element of $\mathcal{L}^* = \{\mathcal{L}, \xi_0\}$. If L is of type I, $\mathcal{L}$ consists of all nonlinear irreducible characters of L. Thus, $\gamma = \sum a_i \lambda_i$ where λ_i are linear characters of L/L'. Since $\xi_0 = \sum \lambda_i$, $(\gamma, \xi_0) = 0$ means $\sum a_i = 0$. Thus, for an element x of L', we have $\gamma(x) = 0$. This proves $\lambda^\tau(x) = \lambda(x)$ in this case.

If L is not of type I, then $\mathcal{L}^*$ consists of irreducible characters induced by characters of L' and ξ_k for $0 \le k \le w_2 - 1$. Thus, $\gamma = \sum a_{st}\mu_{st}$ with $(\gamma, \xi_k) = 0$ for all k. Then, for each k, $\sum_s a_{sk} = 0$. Since $(\mu_{sk})_{L'} = (\mu_{tk})_{L'}$ by Lemma 13.4, $\gamma(x) = 0$ for $x \in L'$. This proves that $\lambda^\tau(x) = \lambda(x)$ for $x \in A(L)^\sharp$. Q.E.D.

The next lemma is stated in [FT], p. 980, without proof.

Lemma P. *Let $M, L \in \mathcal{M}$. If M and L are not conjugate, no subgroup of $\mathcal{M}$ can serve as a supporting subgroup of $A(M)$ and at the same time of $A(L)$.*

Proof. Suppose that $N \in \mathcal{M}$ is a supporting subgroup of $A(M)$. Then, there is an element $x \in A(M)$ such that $C_G(x) \not\subseteq M$ and $C_G(x) \subseteq N$. By Theorem II, $x \in {M_{\sigma_0}}^\sharp$ and $M \cap N$ is a complement of N_σ in N. By Theorem 8.4, $\pi(\langle x \rangle) \subseteq \tau_2(N)$. Similarly, if N is a supporting subgroup of $A(L)$, there is an element $y \in {L_{\sigma_0}}^\sharp$ such that $C_G(y) \not\subseteq L$, $C_G(y) \subseteq N$, $\pi(\langle y \rangle) \subseteq \tau_2(N)$, and $L \cap N$ is a complement of N_σ in N. Since N_σ is a Hall normal subgroup of N, $L \cap N$ is conjugate to $M \cap N$ in N. Let $M \cap N = (L \cap N)^g$ for $g \in N$ and let $x' = y^g$. Then, $x, x' \in M \cap N$.

Take $p \in \tau_2(N)$ and suppose that G has a nonabelian Sylow p-subgroup. Then, by Theorem 6.7 (a), $\tau_2(N) = \{p\}$. Therefore, both x and y are p-elements and $\sigma(M) \cap \sigma(L) \neq \emptyset$. Since M is not conjugate to L, this contradicts Theorem 7.9. Hence, G has an abelian Sylow subgroup for every prime in $\tau_2(N)$. By Lemma 6.8 (a), a Hall $\tau_2(N)$-subgroup E_2 of $M \cap N$ is a normal abelian subgroup of $M \cap N$. Since $x, x' \in E_2$, they commute. The element x' is a $\sigma_0(L)$-element. Hence, x' is a $\sigma(M)'$-element by Theorem 7.9. By Corollary 8.3, we have either (1) $\pi(\langle x' \rangle) \subseteq \kappa(M)$ and $C_G(x) \subseteq M$, or (2) $\pi(\langle x' \rangle) \subseteq \tau_2(M)$ and $\mathcal{M}(C_G(x')) = \{M\}$. Since $C_G(x) \not\subseteq M$ and $C_G(x') \subseteq N$, neither case holds. This contradiction proves Lemma P. Q.E.D.

Lemma Q. *For each $M \in \mathcal{M}$, let $G_0(M)$ be the territory of M. Let $L \in \mathcal{M}$ and assume that L is not conjugate to M. Then,*

$$G_0(M) \cap G_0(L) = \emptyset$$

unless either M is conjugate to a supporting subgroup for $A(L)$ or L is conjugate to a supporting subgroup for $A(M)$.

Proof. The elements of $A(x)$ are of the form hx where $h \in C_H(x)$ and the order of x is prime to the order of h. The subgroup H is a supporting subgroup for $A(M)$; thus, $H = H_i = (M_i)_\sigma = (M_i)_F$ for some $M_i \in \mathcal{M}$. Suppose that $G_0(M) \cap G_0(L) \neq \emptyset$ and

$$g^{-1}(hx)g = ky$$

where k is an element of a supporting subgroup K of $A(L)$ and $ky = yk$ for some $y \in A(L)$.

Suppose that $h \neq 1$. Then, $C_G(x) \not\subseteq M$ and $C_G(x) \subseteq M_i$. By Theorem II, this implies $x \in {M_{\sigma_0}}^\sharp$ and $y \in {N_\sigma}^\sharp$. Note that any supporting subgroup is a ϖ-group. If $k \neq 1$, we have $y \in {L_{\sigma_0}}^\sharp$. Since $\pi(\langle y\rangle) \subseteq \pi(\langle h\rangle) \cup \pi(\langle x\rangle)$,

$$\sigma(L) \cap \sigma(M_i) \neq \emptyset \quad \text{or} \quad \sigma(L) \cap \sigma(M) \neq \emptyset.$$

By Theorem 7.9, L is conjugate to M_i that is a supporting subgroup for $A(M)$. Suppose that $k = 1$. If $\pi(\langle y\rangle) \cap \sigma(L) \neq \emptyset$, then the preceding argument shows that L is conjugate to M_i. Assume that $\pi(\langle y\rangle) \cap \sigma(L) = \emptyset$. Then, y is an $\sigma(L)'$-element of $A(L)$. Theorem II yields that $C_G(y) \subseteq L$. Since $g^{-1}(hx)g = y$, we have

$$C_G(y) \subseteq C_G(g^{-1}xg) = g^{-1}C_G(x)g \subseteq (M_i)^g.$$

By the definition of $A(L)$, y commutes with an element z of ${L_\sigma}^\sharp$. Since y is a ϖ-element, we have $z \in {L_{\sigma_0}}^\sharp$. Corollary 8.3 yields that either $\pi(\langle y\rangle) \subseteq \kappa(L)$ or $\mathcal{M}(C_G(y)) = \{L\}$. The definition of $A(L)$ yields that nonidentity elements of Hall $\kappa(L)$-subgroups are excluded from $A(L)$. Thus, the first possibility does not occur.

Therefore, we have $\mathcal{M}(C_G(y)) = \{L\}$. It follows from $C_G(y) \subseteq (M_i)^g$ that $L = {M_i}^g$.

If $k \neq 1$, a similar proof shows that M is conjugate to a supporting subgroup for $A(L)$. Suppose that $h = 1 = k$. Suppose that $\pi(\langle x\rangle) \subseteq \sigma_0(M)$. Then, $C_G(x)$ is contained in either M or a conjugate of a supporting subgroup. Since L is not conjugate to M, Theorem 7.9 yields that $\pi(\langle y\rangle) \cap \sigma(L) = \emptyset$. The argument of the preceding paragraph proves that $\mathcal{M}(C_G(y)) = \{L\}$. Since y is conjugate to x, we conclude that L is conjugate to a supporting subgroup of $A(M)$.

Suppose that $\pi(\langle x \rangle) \not\subseteq \sigma_0(M)$. Note that x centralizes some non-identity element of M_F. Since M_F is a ϖ-group, x is a ϖ-element. Since $\pi(\langle x \rangle) \not\subseteq \sigma_0(M)$, there is a Hall subgroup $\langle x' \rangle$ of $\langle x \rangle$ such that x' is a $\sigma(M)'$-element and x' commutes with an element u of ${M_{\sigma_0}}^{\sharp}$. As before, Corollary 8.3 yields that $\mathcal{M}(C_G(x')) = \{M\}$. Since $x^g = y$, the element $(x')^g = y'$ is a power of y. Thus, $y' \in A(L)$. It follows that $C_G(y')$ is contained in either L or a conjugate of a supporting subgroup of $A(L)$. Since

$$\mathcal{M}(C_G(y')) = \mathcal{M}(C_G(x'))^g = \{M^g\},$$

M is conjugate to a supporting subgroup for $A(L)$. Q.E.D.

Lemma 19.5. *Let λ be the irreducible character in $\mathcal{L}$ defined in Lemma 19.3. Then, λ^τ is conformal relative to $A(M)$ and*

$$\frac{1}{|M|} \sum_{x \in K^\sharp} |\lambda^\tau(x)|^2 < \frac{\lambda(1)^2}{|L|}.$$

Proof. We will prove that λ^τ is conformal relative to M. Let N be a supporting subgroup of $A(M)$. Since M is of type I but not a Frobenius group, Theorem II yields that N is of type I. Let $\Theta = \lambda^\tau$.

By Lemma 14.1, it suffices to check that Θ is orthogonal to every virtual character of the form $(\theta_1 - \theta_2)^G$ with $\theta_1, \theta_2 \in S(\alpha)$ for $\alpha \neq 1_H$, $\alpha \in \mathrm{Irr}(H)$. For the notation, see Lemma 14.1. Since N is of type I, θ_1 and θ_2 are irreducible characters of N and $\theta_1 - \theta_2$ vanishes outside $A(N) - H$. By (F*ii*)(d), $A(N) - H$ is a TI-set. This implies that $(\theta_1 - \theta_2)^G$ is a difference of two irreducible characters of G. Let

$$(\theta_1 - \theta_2)^G = \Theta_1 - \Theta_2.$$

If $\lambda^\tau = \Theta$ is not orthogonal to $(\theta_1 - \theta_2)^G$, then Θ must be either Θ_1 or Θ_2. The virtual character $\Theta_1 - \Theta_2$ vanishes outside the territory $G_0(N)$ of N. Lemma 19.2 yields that L is either a Frobenius group or of type III or IV. Thus, by (F*ii*)(d) or (F*iii*), L is not conjugate to any supporting subgroup for A(N). Since N is not a Frobenius group by (F*ii*)(d), N is not conjugate to L. By definition, N is a supporting subgroup for $A(M)$. Hence, by Lemma P, N is not conjugate to any supporting subgroup for $A(L)$. By Lemma Q, the territory of L is disjoint from that of N. Since $\mathcal{L}$ is coherent by Lemma 19.4, $\Theta - \overline{\Theta}$ vanishes outside of $G_0(L)$. Then, we have

$$((\theta_1 - \theta_2)^G, \Theta - \overline{\Theta}) = 0$$

because $G_0(N) \cap G_0(L) = \emptyset$. Thus, $(\theta_1 - \theta_2)^G$ contains Θ and $\overline{\Theta}$ with the same multiplicity. Since $\Theta \neq \overline{\Theta}$, this is a contradiction and proves that λ^τ is conformal relative to M.

We can apply Lemmas 12.5 and 12.6 to $\Theta(x) = |\lambda^\tau(x)|^2$. Let $G_1(M)$ be the proper territory of M. Then, $G_1(M)$ is the set of elements of G which are conjugate to some element of $A(x)$ with $x \in K^\sharp$. Since L is not conjugate to M, $\sigma(L) \cap \sigma(M) = \emptyset$ by Theorem 7.9. It follows that $G_1(M)$ is disjoint from $G_0(L)$. Lemma 12.6 yields

$$\frac{1}{|M|} \sum_{x \in K^\sharp} |\lambda^\tau(x)|^2 = \frac{1}{|G|} \sum_{x \in G_1(M)} |\lambda^\tau(x)|^2.$$

Since $G_1(M) \cap G_0(L) = \emptyset$, the orthogonality relation yields

$$\frac{1}{|G|} \sum_{x \in G_1(M)} |\lambda^\tau(x)|^2 < 1 - \frac{1}{|G|} \sum_{x \in G_0(L)} |\lambda^\tau(x)|^2.$$

Then, Lemmas 12.5 and 19.4 yield

$$\frac{1}{|M|} \sum_{x \in K^\sharp} |\lambda^\tau(x)|^2 < 1 - \frac{1}{|L|} \sum_{x \in (L')^\sharp} |\lambda(x)|^2.$$

The right side is equal to $\lambda(1)^2/|L|$ because λ vanishes outside L'.

Q.E.D.

Lemma 19.6. *Let $F = M \cap L$. Then, F is a complement of K in M. There is an element z of $A \cap Z(F)^\sharp$ such that $C_K(z) \not\subseteq K'$.*

Proof. We have $A \subseteq M \cap L$ and some nonidentity element of A has a nontrivial centralizer in K. Thus, M is a supporting subgroup for $A(L)$. By (F*ii*), $M \cap L = F$ is a complement of K in M.

Since M is of type I, F contains a subgroup F_0 of the same exponent as F that acts regularly on K. It follows that any subgroup of $\mathcal{E}^1(F_0)$ lies in the center $Z(F_0)$. Therefore, there is no Frobenius group that contains A. Note that A is the set of elements of order p in F by Corollary 6.6 (a) and Theorem 6.5 (b).

If L is of type I, L is a Frobenius group by Lemma 19.2. Since F is not a Frobenius group as shown in the preceding paragraph, we have $F \subseteq U$. Therefore, F is nilpotent. By (I*iv*) for M, every Sylow subgroup of F is abelian. Hence, F is abelian. The group $A \in \mathcal{E}_p^2(F)$ acts on K/K'. By Proposition 1.16 [BG], there is an element $z \in A^\sharp$ such that $C_{K/K'}(z) \neq 1$. Proposition 1.5 [BG] shows that $C_K(z) \not\subseteq K'$. This proves Lemma 19.6 if L is of type I.

Suppose that L is of type III or IV. We may assume $P \subseteq U$. If $F \not\subseteq L'$, we may choose $W_1 \subseteq F$. Then, $\langle A, W_1 \rangle$ is a Frobenius group in F. This does not occur. Therefore, $F \subseteq L'$. Let $F_1 = F \cap H$. Then, F_1 is a normal subgroup of F. We may assume that $F = F_1(F \cap U)$ by replacing U by a conjugate if necessary. Since U is nilpotent by (T2), $F \cap U$ is abelian. The subgroup A lies in $F \cap U$ and $A \lhd F$ by Corollary 6.6 (a). Therefore, $[F_1, A] = 1$ and $A \subseteq Z(F)$. Then, Lemma 19.6 holds as before. Q.E.D.

Lemma 19.7. *Let* M *be the set of all irreducible characters of* M *which do not have* K *in their kernel. Let* λ *be the character defined in Lemma* 19.3. *If* M *is coherent, then* λ^τ *is constant on* $K^\sharp$.

Proof. Let a be the least common multiple of the orders of all the elements of $A(L)$. By Lemma 19.2, we have $A(L) = L' = L_\sigma$. Since M is not conjugate to L, Theorem 7.9 yields that $\sigma(L) \cap \sigma(M) = \emptyset$. Thus, $(a, |K|) = 1$. Since $\mathfrak{L}$ is coherent by Lemma 19.4, we can apply Lemma 12.1 to conclude that the values taken by λ^τ lie in the field $\mathbb{Q}_a$. Lemma 19.5 yields that λ^τ is conformal relative to $A(M)$. Assume that M is coherent. We will show that λ^τ is orthogonal to every element of M^τ. Let α be a character of M. Then, α_K is not rational as $\overline{\alpha}_K \neq \alpha_K$. Since $(a, |K|) = 1$, there is a Galois automorphism that sends α_K to $\overline{\alpha}_K$ and induces the identity on $\mathbb{Q}_a$. This yields that $\lambda^\tau \neq \alpha^\tau$. Lemma 14.4 yields that there is a pair (r, β) of a rational number r and a virtual character β of M such that β is orthogonal to every element of M and $\lambda^\tau(x) = r\beta(x)$ for $x \in A(M)^\sharp$. Then, β is a linear combination of irreducible characters of M/K. Thus, $\lambda^\tau(x) = r\beta(x)$ for $x \in K^\sharp$ and λ^τ is constant on $K^\sharp$. Q.E.D.

Proof of Theorem 19.1. For some element x of $A^\sharp$, $C_K(x) \neq 1$. Take $y \in C_K(x)^\sharp$. Since M is a supporting subgroup for $A(L)$, Lemma 14.3 yields that λ^τ is conformal relative to $A(L)$. Thus, λ^τ is constant on the annex $A(x)$. It follows that

$$\lambda^\tau(xy) = \lambda^\tau(x) = \lambda(x).$$

The last equality comes from Lemma 19.4. Let $\mathbb{Q}_0$ be the field of primitive $|G|$th roots of unity and let $\mathfrak{P}$ be the prime ideal dividing p in the ring of integers in $\mathbb{Q}_0$. By Lemma 4.2 [FT], we have

$$\lambda^\tau(y) \equiv \lambda^\tau(xy) = \lambda(x) \equiv \lambda(1) \pmod{\mathfrak{P}}.$$

The values taken by λ^τ lie in $\mathbb{Q}_a$ where a is the exponent of L'. Therefore,

$\lambda^\tau(y)$ is a rational number, so we have

$$\lambda^\tau(y) \equiv \lambda(1) \pmod{p}.$$

By Lemma 19.3, $\lambda(1)$ divides $p+1$ or $p-1$. This yields that $\lambda(1) \leq (p+1)/2$ and

$$|\lambda^\tau(y)| \geq p - \lambda(1) \geq \lambda(1) - 1. \tag{19.4}$$

This inequality holds whenever $y \neq 1$ commutes with an element $x \in A^\sharp$. As before, let F be a complement of K in M. Lemma 19.6 yields that there is an element $z \in A^\sharp \cap Z(F)$ such that $C_K(z) \nsubseteq K'$. If $C_K(z) = K$, then (19.4) holds for every $y \in K^\sharp$. If $C_K(z) \neq K$, then Theorem 15.2 yields that $\mathcal{M}$ is coherent. By Lemma 19.7, λ^τ is constant on $K^\sharp$. Since (19.4) holds for at least one element of $K^\sharp$, it holds for every $y \in K^\sharp$ because λ^τ is constant on $K^\sharp$.

Let $e = \lambda(1)$. Then, Lemma 19.5 yields that

$$\frac{1}{|M|}(|K|-1)(e-1)^2 \leq \frac{1}{|M|}\sum_{x \in K^\sharp} |\lambda^\tau(x)|^2 < \frac{e^2}{|L|}.$$

Since $|M| = |K||M \cap L|$, we have

$$\frac{(|K|-1)}{|K|}\left(\frac{e-1}{e}\right)^2 < \frac{|M \cap L|}{|L|} \leq \frac{1}{3}.$$

Since $(e-1)/e \geq 2/3$, $|K| < 4$ and $|K| = 3$. The subgroup K is a Hall subgroup of G with $|N_G(K)|$ odd. Then, G is not simple. This contradicts the assumption. Thus, Theorem 19.1 holds. Q.E.D.

Theorem 19.8. *If there is no subgroup of type II, then G contains a nilpotent Hall ϖ-subgroup that is isolated.*

Proof. By Theorem I, all $M \in \mathcal{M}$ are of type I. By Theorem 19.1, they are Frobenius groups. It follows from (Fii)(d) that no supporting subgroup of type I is a Frobenius group. Thus, if $M \in \mathcal{M}$, there is no supporting subgroup for $A(M)$. Therefore, if $H = M_F$, then $H = M_{\sigma_0}$ and, for every $x \in H^\sharp$, $C_G(x) \subseteq M$. Since M is a Frobenius group, we have $C_G(x) \subseteq H$.

Take a prime $p \in \varpi$, $P \in Syl_p(G)$, and $M \in \mathcal{M}(N_G(P))$. Then, M is of type I. Therefore, M is a Frobenius group with Frobenius kernel $H = M_{\sigma_0} = M_F$ and $P \subseteq H$. Thus, H is a nilpotent ϖ-subgroup having the property that $C_G(x) \subseteq H$ for every $x \in H^\sharp$. We will show that H is a Hall ϖ-subgroup of G that is isolated.

Take a prime q in ϖ and suppose that pq is an edge of the prime graph of G. Then, there is a pair (x, y) of elements x and y such that $x \in P^\sharp$ and y is an element of $C_G(x)^\sharp$ of order q. Let $Q \in Syl_q(G)$ such that $y \in Q$ and let $z \in Z(Q)^\sharp$. Then, starting from $x \in P^\sharp$ we have in succession $y \in H$, $z \in H$, and $Q \subseteq H$. Repeating this argument, we conclude that if $r \in \varpi$, then H contains a Sylow r-subgroup of G. Therefore, H is a Hall ϖ-subgroup of G. It is nilpotent and isolated.

Q.E.D.

§20. The Pair of Subgroups S and T

In this section, we will assume that there is a subgroup in $\mathcal{M}$ that is not of type I. Theorem I yields that there is a pair of subgroups S and T which satisfy the conditions (a)—(e) of Theorem I. By Theorem 18.10, each of them is of type II, III, or IV. Throughout this section, we follow the notation of Section 34 of [FT]. Thus, p and q are distinct primes in ϖ such that

$$W = P^*Q^*, \;\; S = S'Q^*, \;\; T = T'P^*, \;\; |P^*| = p, \;\; \text{and} \;\; |Q^*| = q.$$

Let $P \in Syl_p(S)$ and $Q \in Syl_q(T)$. By Theorem C(2), $P^* \subseteq S_F$. Therefore, $P \subseteq S_F$ and P is a normal subgroup of S. It follows that $P^* \subseteq P$. Similarly, $Q^* \subseteq Q \lhd T$.

Let U be a Q^*-invariant complement of P in S'. Then, UQ^* is a complement of P in S. Let

$$C = C_{\mathrm{U}}(P).$$

Then, $C \lhd U$. Since $P^* \subseteq P$, we have $P^* \cap U = 1$. Proposition 8.2 (b) yields that Q^* acts regularly on U. Thus, the group UQ^* is a Frobenius group with Frobenius kernel U. Then, the prime q does not divide the order of U. Thus,

$$Q^* \in Syl_q(S).$$

Also, U is nilpotent. Since $C \subseteq U$, C is nilpotent; so is $PC = P \times C$. It follows that $PC \subseteq F(S)$. Clearly, we have $F(S) = P \times (F(S) \cap U) \subseteq PC$. Therefore,

$$F(S) = P \times C = PC.$$

By (T3), $S'' \subseteq F(S) \subseteq S'$. It follows that $S'/PC \cong U/C$ is abelian.

Similarly, let V be a P^*-invariant complement of Q in S'. Then, VP^* is a complement of Q in T and VP^* is a Frobenius group with Frobenius kernel V. Also, $P^* \in Syl_p(T)$. Let

$$D = C_V(Q).$$

Then, $D \lhd V$, $QD = F(T)$ and $T'/QD \cong V/D$ is abelian. Note that $A(S)$ is a TI-set of G with normalizer S. This is proved as follows. If $A(S)$ is not a TI-set, there is an element $x \in A(S)^\sharp$ such that $C_G(x) \not\subseteq S$. Then, $C_G(x)$ is contained in a conjugate of a supporting subgroup M_i by (F*ii*)(e). Since S is not of type I, M_i is of type I by (F*iii*). Then, by Theorem 19.1, M_i is a Frobenius group. But, none of the supporting subgroups can be a Frobenius group by (F*ii*)(d). Thus, $A(S)$ is a TI-set of G.

Similarly, $A(T)$ is a TI-set.

Let $\mathcal{S}$ be the set of characters of S which are induced by irreducible characters of S' not having P in their kernel. Since $P \subseteq S_F$, this set $\mathcal{S}$ is a part of the set of characters considered in §16 for subgroups of type II, III, or IV. Hence, Corollary 18.11 yields that the set $\mathcal{S}$ defined here is coherent. Let $\mathcal{T}$ be the set of characters of T induced by irreducible characters of T' which do not have Q in their kernel. Then, $\mathcal{T}$ is also coherent.

Let η_{ij} be the virtual characters of weight 1 associated with the self-normalizing cyclic group $W = P^*Q^*$. We use the notation of §13 and

$$\eta_{ij}(x) = \omega_{ij}(x) \quad \text{for} \quad x \in \widehat{W}.$$

Let μ_{ij} be the set of irreducible characters of S defined in Lemma 13.4. Then, $\mu_{ij}(x) = \varepsilon_j\omega_{ij}(x)$ for $x \in \widehat{W}$ with $\varepsilon_j = 1$ or -1. Let

$$\xi_k = \sum_{i=0}^{q-1} \mu_{ik}.$$

Similarly, let ν_{ij} be the set of irreducible characters of T defined in Lemma 13.4. Thus, $\nu_{ij}(x) = \pm\omega_{ij}(x)$ for $x \in \widehat{W}$, where the sign is independent of j. Let

$$\zeta_i = \sum_{j=0}^{p-1} \nu_{ij}.$$

By Lemma 13.5, characters of $\mathcal{S}$ (or $\mathcal{T}$) are either irreducible or one of the characters ξ_j $(0 \le j \le p-1)$ (or ζ_i $(0 \le i \le q-1)$).

We use the following notation:

$$|C| = c, \ |D| = d, \ |U\!:\!C| = u, \ |V\!:\!D| = v, \quad \text{and} \quad |G| = g.$$

For the following lemmas in this section, we maintain the symmetry between S and T. So, the results proved for S hold for T as well.

Lemma 20.1. *There is a normal subgroup P_0 of S such that $P_0 \subseteq P$, P/P_0 is an elementary abelian group of order p^q, and the group UQ^* acts irreducibly on P/P_0. Either U/C is a cyclic group with u dividing $(p^q-1)/(p-1)$ that acts irreducibly and regularly on P/P_0, or U/C is a product of at most $q-1$ cyclic groups with u dividing $(p-1)^{q-1}$. For $1 \le j \le p-1$, ξ_j is induced by a linear character of PC and $\xi_j(1) = uq$. Either PU is a Frobenius group with Frobenius kernel P such that $|P| = p^q$ and $u = (p^q-1)/(p-1)$, or $\mathcal{S}$ contains an irreducible character of degree uq that is induced by a linear character of PC.*

This is Lemma 34.1 [FT]. Some additional remarks included in Lemma 20.1 are really proved there. Q.E.D.

Lemma 20.2. *Either PU is a Frobenius group with Frobenius kernel P with $|P| = p^q$ and $u = (p^q-1)/(p-1)$, or QV is a Frobenius group with Frobenius kernel Q with $|Q| = q^p$ and $v = (q^p-1)/(q-1)$.*

Proof. This is Lemma 34.2 [FT]. We paraphrase their proof. Suppose that the result is false. Then, Lemma 20.1 yields that $\mathcal{S}$ contains an irreducible character λ of degree uq that is induced by a linear character of PC and $\mathcal{T}$ contains an irreducible character θ of degree vp that is induced by a linear character of QD. Define

$$\alpha = \lambda - \xi_1 \quad \text{and} \quad \beta = \theta - \zeta_1.$$

Then, α^τ takes nonzero values only on conjugates of $(PC)^\sharp$. Since $PC = F(S)$, α^τ is nonzero only at $\sigma(S)$-elements. Similarly, β^τ is nonzero only at $\sigma(T)$-elements. Since S is not conjugate to T, Theorem 7.9 yields that $\sigma(S) \cap \sigma(T) = \emptyset$; hence, $(\alpha^\tau, \beta^\tau) = 0$. Similarly,

$$((\lambda - \overline{\lambda})^\tau, (\beta - \overline{\beta})^\tau) = 0.$$

This implies $\lambda^\tau \neq \theta^\tau$ since $\lambda \neq \overline{\lambda}$.

By Lemma 13.7, $\xi_1^\tau = \pm \sum_{i=0}^{q-1} \eta_{i1}$ and $\zeta_1^\tau = \pm \sum_j \eta_{1j}$. By Lemma O, we have $\lambda^\tau \neq \pm\eta_{st} \neq \theta^\tau$. Thus,

$$(\alpha^\tau, \beta^\tau) = (\lambda^\tau - \xi_1^\tau, \theta^\tau - \zeta_1^\tau) = (\pm \sum_i \eta_{i1}, \pm \sum_j \eta_{1j}) = \pm 1.$$

This contradicts $(\alpha^\tau, \beta^\tau) = 0$. Q.E.D.

Lemma 20.3. *For $1 \le j \le p-1$,*

$$\sum_{x \in (PC)^\sharp} |\eta_{0j}(x)|^2 \geq uc|P| - u^2.$$

Proof. We paraphrase the proof of Lemma 34.3 [FT]. The set of irreducible characters of S consists of $\{\mu_{ij}\}$, $0 \le i \le q-1$, $0 \le j \le p-1$, the set $\operatorname{Irr} \mathcal{S}$ of irreducible characters in $\mathcal{S}$, and the set $\operatorname{Irr}(S/P)$. By Lemma 13.7, $\xi_t^{\tau} = \varepsilon_t \sum_i \eta_{it}$. Write the restriction $(\eta_{0t})_S$ as a linear combination of irreducible characters of S as follows:

$$(\eta_{0t})_S = \varepsilon\mu_{0t} + \sum_{s,t>0} c_{st}\mu_{st} + \sum_{\lambda\in \operatorname{Irr}\mathcal{S}} a_\lambda \lambda + \Delta \tag{20.1}$$

where $\varepsilon = \varepsilon_j$ and Δ is a character of S/P. Since $\mathcal{S}$ is coherent, Lemmas M and 11.4 yield

$$(\alpha^\tau, \eta_{0j}) = (\alpha, (\eta_{0j})_S) \tag{20.2}$$

for every $\alpha \in I_0(\mathcal{S})$. Take j and k with $1 \le j, k \le p-1$ and let $\alpha = \xi_j - \xi_k$. Note that $\xi_j(1) = uq = \xi_k(1)$, so $\alpha \in I_0(\mathcal{S})$. Then, (20.1) and (20.2) yield

$$\sum_{s=0}^{q-1} c_{sj} = \sum_{s=0}^{q-1} c_{sk}$$

including $k = t$. For each k, $(\mu_{ik})_{S'}$ is independent of i by Lemma 13.4 and $(\mu_{ik})_{S'} = \psi_k$ is an irreducible character of degree u of S'. Then, we have

$$\left(\sum_{s,t>0} c_{st}\mu_{st}\right)_{S'} = a\sum_{k=1}^{p-1} \psi_k(1)\psi_k$$

with $a\psi_k(1) = au = \sum_{s=0}^{q-1} c_{sk}$. Thus, a is a rational number such that au is an integer. If $\operatorname{Irr}\mathcal{S}$ is not empty, take $\lambda \in \operatorname{Irr}\mathcal{S}$. Then, $\lambda(1)$ is divisible by q because λ is induced by an irreducible character θ of S'. Let $\alpha = \theta(1)\xi_k - u\lambda$. Since $\xi_k(1) = uq$, we have $\alpha \in I_0(\mathcal{S})$. Then, (20.1) and (20.2), together with Lemma N, yield

$$\theta(1)\sum_{s=0}^{q-1} c_{sk} = ua_\lambda \quad \text{or} \quad a_\lambda = a\theta(1).$$

Therefore,

$$\left(\sum_{\lambda} a_\lambda\lambda\right)_{S'} = a\sum_{\lambda}\theta(1)(\theta_1 + \cdots + \theta_q)$$

where $\theta_1, \ldots, \theta_q$ are components of $\lambda_{S'}$. It follows that

$$\left(\sum_{s,t>0} c_{st}\mu_{st} + \sum_{\lambda} a_\lambda\lambda\right)_{S'} = a\rho_1$$

where ρ_1 is the portion of the regular representation of S' on the set of irreducible characters which do not have P in their kernel. Let ρ be the regular representation of S' and write $\rho = \rho_1 + \rho_2$. Then, ρ_2 is the regular representation of S'/P. If x is a nonidentity element of S', then

$$0 = \rho(x) = \rho_1(x) + \rho_2(x).$$

Let $\beta = -a\rho_2 + \Delta_{S'}$. Then, β is a linear combination of irreducible characters of S'/P with rational coefficients. It follows that for $x \in (S')^{\sharp}$,

$$\eta_{0t}(x) = \varepsilon\psi_t(x) + \beta(x).$$

Since $\rho_2(1) = |S'/P| = cu$, $\beta(1) = -acu + \Delta_{S'}(1)$ is an integer because $\Delta_{S'}$ is a character and au is an integer. The remainder of the proof is the same as the proof of Lemma 34.3 [FT]. We have

$$\begin{aligned}\sum_{x\in(PC)^{\sharp}} |\eta_{0t}(x)|^2 &= \sum(\varepsilon\psi_t(x) + \beta(x))(\varepsilon\overline{\psi}_t(x) + \overline{\beta}(x)) \\ &= \sum |\psi_t(x)|^2 + \varepsilon\sum(\psi_t(x)\overline{\beta}(x) \\ &\qquad + \overline{\psi}_t(x)\beta(x)) + \sum |\beta(x)|^2.\end{aligned}$$

Since ψ_t is an irreducible character that vanishes outside PC, the first term is $uc|P| - u^2$. Since β is a sum of irreducible characters of S'/P, the second sum is equal to $-2\varepsilon u\beta(1)$. The values of β are constant on each coset of P. Thus, the third sum is

$$|P|\sum_{x\in U} |\beta(x)|^2 - \beta(1)^2.$$

Lemma 20.1 yields that u divides either $(p^q - 1)/(p-1)$ or $(p-1)^{q-1}$, and $|P| \geq p^q$. Hence, we have

$$|P| \geq 2u + 1$$

and $|P|\beta(1)^2 - \beta(1)^2 - 2eu\beta(1) \geq 2u(\beta(1)^2 - \varepsilon\beta(1)) \geq 0$ because $\beta(1)$ is an integer. This proves Lemma 20.3. Q.E.D.

Lemma 20.4. *For $1 \leq i \leq q-1$,*

$$\sum_{x\in PC-C} |\eta_{i0}(x)|^2 \geq (|P| - 1)c.$$

Proof. We use the same method as in the proof of Lemma 20.3. Since η_{i0} is orthogonal to every character of $\mathcal{S}^{\tau}$, we have

$$(\eta_{i0})_{S'} = a\rho_1 + \gamma$$

where au is an integer, ρ_1 is the portion of the regular representation ρ with $\rho - \rho_1$ the regular representation of S'/P, and γ is a character of S'/P. Then, ρ_1 vanishes outside P, ρ_1 takes the value $-uc$ on $P-1$, and $\rho_1(1) = (|P|-1)uc$. Let $\delta = (\eta_{i0})_{\mathrm{U}}$ and $y \in P^{*\sharp}$. Since γ is a character of S'/P, $(\eta_{i0})_{S'}$ takes a constant value on each coset of P except at the identity. Thus,

(20.3)

$$\begin{aligned}\sum_{x\in PC-C} |\eta_{i0}(x)|^2 &= (|P|-1)\left(\sum_{x\in U^\sharp} |\delta(x)|^2 + |\eta_{i0}(y)|^2\right) \\ &= (|P|-1)(c\|\delta\|^2 - |\delta(1)|^2 + |\eta_{i0}(y)|^2).\end{aligned}$$

Clearly, $\|\delta\|^2$ is a nonzero integer. Let $z \in Q^{*\sharp}$, and let $\mathfrak{Q}$ be a prime ideal dividing q in the ring of algebraic integers of $\mathbb{Q}_{pq}$. Then, $\eta_{i0}(y) \equiv \eta_{i0}(yz) = \omega_{i0}(yz) \equiv \omega_{i0}(y) = 1 \pmod{\mathfrak{Q}}$. Thus, the left side of (20.3) is positive. It suffices to show that $|\delta(1)|^2 - |\eta_{i0}(y)|^2$ is an integral multiple of c. We have

$$\eta_{i0}(y) = a\rho_1(y) + \gamma(y) = -auc + \gamma(1),$$

$$\delta(1) = a\rho_1(1) + \gamma(1) = a(|P|-1)uc + \gamma(1).$$

Hence,

$$|\delta(1)|^2 - |\eta_{i0}(y)|^2 = (a(|P|-2)uc + 2\gamma(1))a|P|uc.$$

Since au is an integer, this is an integral multiple of c. Q.E.D.

Lemma 20.5. *Suppose that $\mathcal{S}$ contains an irreducible character λ of degree uq which is induced by a character of PC. Then,*

$$\sum_{x\in(PC)^\sharp} |\lambda^\tau(x)|^2 > uqc|P| - (uq)^2 - 2uq^2.$$

Proof. We have

$$(\lambda^\tau)_{S'} = \lambda_{S'} + a\rho_1 + \alpha$$

where au is an integer, ρ_1 is the portion of the regular representation ρ of S', $\rho = \rho_1 + \rho_2$ with ρ_2 the regular representation of S'/P, and α is a character of S'/P. Let

$$\beta = -a\rho_2 + \alpha.$$

Then for $x \in (S')^\sharp$, $\lambda^\tau(x) = \lambda(x) + \beta(x)$. The value of $\beta(x)$ is constant on each coset of P except at the identity. The proof of Lemma 34.5 [FT] may be applied. We have

(20.4)
$$\begin{aligned}\sum_{x\in(PC)^\sharp} |\lambda^\tau(x)|^2 &= \sum(\lambda(x)+\beta(x))(\overline{\lambda}(x)+\overline{\beta}(x))\\ &= \sum|\lambda(x)|^2 + \sum(\lambda(x)\overline{\beta}(x)+\beta(x)\overline{\lambda}(x)) + \sum|\beta(x)|^2.\end{aligned}$$

Since $\lambda \in \operatorname{Irr} S$ with $\lambda(1) = uq$, the first sum is $uqc|P| - (uq)^2$. None of the irreducible components of $\lambda_{S'}$ has P in its kernel. Hence, the second sum is $-2\lambda(1)\beta(1)$. Since β is constant on each coset of P,

$$\sum|\beta(x)|^2 = |P|\sum_{x\in U}|\beta(x)|^2 - |\beta(1)|^2.$$

Suppose $|\beta(1)| < q$. Then, $2\lambda(1)|\beta(1)| < 2uq^2$. The result follows from (20.4). On the other hand, if $|\beta(1)| \geq q$, then $2\lambda(1)|\beta(1)| \leq 2u|\beta(1)|^2 \leq (|P|-1)|\beta(1)|^2$. The result follows from (20.4) again. Q.E.D.

Lemma 20.6. *Let G_0 be the set of elements of G which are not conjugate to any element of PC, Q, or $\widehat{W}$. Suppose that $\mathcal{S}$ contains an irreducible character λ of degree uq. Define*

$$\begin{aligned}A_1 &= \{x\in G_0 \mid \lambda^\tau(x)\neq 0\},\\ A_2 &= \{x\in G_0 \mid \eta_{10}(x)\neq 0\}, \text{ and}\\ A_3 &= \{x\in G_0 \mid \eta_{01}(x)\neq 0 \text{ and } \eta_{01}(x)\equiv 0 \pmod{(q-1)}\}.\end{aligned}$$

Then, $G_0 = A_1 \cup A_2 \cup A_3$.

Lemma 20.7. *The following statements hold.*

(*i*) *If $q \geq 5$, then P is an elementary abelian group of order p^q and $u/c > 9p^{q-1}/20q$.*
(*ii*) *If $p, q \geq 5$, then $c = 1$ and $u > (13/20)p^{q-1}/q$.*
(*iii*) *If $p = 3$ and $c \neq 1$, then $u = 121$, $q = 5$, and $c = 11$.*
(*iv*) *If $q = 3$, then $c = 1$ or $c = 7$. Furthermore $u > (p^2+p+1)/13$.*

(*v*) *If* $q = 3$, *then* P *is an elementary abelian* p*-group and* $|P| = p^q$ *or* $p = 7$, $c = 1$, *and* $|P| = 7^4$.
(*vi*) *If* $q = 3$ *and* $c = 7$, *then* $u > (p^2 + p + 1)/2$.

Lemma 20.8. *If* $q \geq 5$, *then* PU/C *is a Frobenius group and we also have that* u *divides* $(p^q - 1)/(p - 1)$.

Lemma 20.9. *If* $p, q \geq 5$, *then* $c = 1$, $|P| = p^q$, *and either* $u = (p^q - 1)/(p - 1)$ *or* $p \equiv 1 \pmod{q}$ *and* $uq = (p^q - 1)/(p - 1)$.

These lemmas are proved as in [FT], §34. In the proof the references to Lemma 34.*n* [FT] should be to Lemma 20.*n* of this paper.

§21. Four Propositions

We continue to use the notation introduced at the beginning of §20. Thus, S and T are subgroups in $\mathcal{M}$, and p and q are distinct primes such that $|W| = pq$. The purpose of this section is to prove that $c = d = 1$, $|P| = p^q$, $|Q| = q^p$, PU is a Frobenius group, and QV is a Frobenius group.

Suppose that both p and q are greater than 3. Then, Lemma 20.7 (*i*) and (*ii*) yield that P is an elementary abelian group of order p^q and $c = 1$. By symmetry, Q is an elementary abelian group of order q^p and $d = 1$. By Lemma 20.8, PU is a Frobenius group and u divides $(p^q - 1)/(p - 1)$. By symmetry, QV is a Frobenius group. Thus, the result holds if $p, q \geq 5$. We may assume that $q = 3$ from now on. We prove four propositions.

Proposition 21.1. *If* $q = 3$, *then* $c = 1$.

Proof. Suppose that $q = 3$ and $c \neq 1$. By Lemma 20.7 (*iv*) and (*vi*), we have $c = 7$ and

$$u > (p^2 + p + 1)/2.$$

By Lemma 20.1, u divides either $p^2 + p + 1$ or $(p - 1)^2$. It follows from the inequality that $u = p^2 + p + 1$. Lemma 20.7 (*v*) yields that P is an elementary abelian group of order p^3. Then, by Lemma 20.1, U/C is a cyclic group that acts irreducibly and regularly on P. Hence, the group S'/C is a Frobenius group with Frobenius kernel PC/C. The group PC is nilpotent; so is $\mathfrak{U}$. Since U/C is cyclic, $\mathfrak{U}$ is abelian. Since $p \in \varpi$, we have $7 \in \varpi$ and $\mathfrak{U}$ is a ϖ-group. Therefore, S is a ϖ-group of type II or III.

Suppose that S is of type II. Then, $S_\sigma = S_F$ (Proposition 10.1); it is either P or PC. Suppose that $S_\sigma = PC$. Then, $(u, 7) = 1$. Let $U = C \times R$ with a $7'$-group R and let $M \in \mathcal{M}(N_G(R))$. Then, by (II$iv$), $N_G(R) \not\subseteq S$. Hence, M is not conjugate to S. Since $UQ^* \subseteq M$, M is not q-closed. Hence, M is not conjugate to T either. By Theorem I, M is of type I. Then, by Theorem 19.1, M is a Frobenius group with Frobenius kernel M_σ. It follows that $U \subseteq M_\sigma \cap S_\sigma$. This contradicts Theorem 7.9. Therefore, we have $S_\sigma = P$.

Let $M \in \mathcal{M}(N_G(U))$. As before, M is a Frobenius group with Frobenius kernel M_σ. Let $H = M_\sigma$. Then, $M = N_G(H)$ and $Q^* \subseteq M$. It follows from the structure of a Frobenius complement that $|M : H| = 3$ or $3p$. By (IIv), $N_G(C) \subseteq S$. Then, $C_G(C)$ is of rank at most 2. Therefore, H contains a characteristic subgroup of order 7 or 7^2. Thus, if $|M : H| = 3p$, then p divides $7-1$ or $(7^2-1)(7^2-7)$. This is impossible as $p \neq 3, 7$. Hence, we have $|M : H| = 3$.

Let M be the set of irreducible characters of M that do not have H in their kernel. Since M is a Frobenius group with H as the Frobenius kernel, M is the set of nonlinear irreducible characters of M. If M is not coherent, then H is a nonabelian group of prime power order (a power of 7) such that $|H : H'| \le 4|M : H|^2 + 1 = 37$. This implies that H is cyclic and $H \subseteq N_G(C) \subseteq S$. This is not the case. Hence, M is coherent. Let θ be the character of M induced by the principal character of H. Then, by Lemma 12.7, $\mathsf{M}^* = \mathsf{M} \cup \{\theta\}$ is coherent. We can determine θ^τ as follows. Take an irreducible character λ of M with $\lambda(1) = 3$. Then, $(\theta - \lambda)^\tau$ vanishes outside the territory $G_0(M)$. We check that S and some of its conjugates are the only supporting subgroups for $A(M)$. We remarked that no group of type I can be a supporting subgroup because it is a Frobenius group. A similar reasoning applies to the group T because QV is a Frobenius group by Lemma 20.2. Thus, the territory $G_0(M)$ consists of elements conjugate to some element of $H^\sharp$ or $PC - P$. In particular, $(\theta - \lambda)^\tau$ vanishes on $\widehat{W}$. Therefore, by Lemma 13.1, we have

$$(\theta^\tau - \lambda^\tau, \eta_{00} - \eta_{i0} - \eta_{0j} + \eta_{ij}) = 0.$$

It follows that θ^τ is a virtual character of weight 3 that involves η_{00} and one of η_{i0}, η_{0j}, or η_{ij}. Clearly, θ^τ is rational. Since η_{0j} or η_{ij} $(j \neq 0)$ has $p - 1$ algebraic conjugates, θ^τ does not involve η_{0j} or η_{ij}. Hence, $\theta^\tau = 1 + \eta_{10} + \eta_{20}$.

Let $\lambda \in \mathsf{M}$ be the irreducible character of degree 3 as above. We claim that $\lambda^\tau(x) = \lambda(x)$ for $x \in H^\sharp$. Note that λ^τ is well-behaved relative to $A(M)$ by Lemma 14.3. Then, by Lemma 14.4, there is a virtual character γ of M such that γ is orthogonal to every $\mu \in \mathcal{M}^*$ and

$$\lambda^\tau(x) = \lambda(x) + r\gamma(x) \qquad (x \in H^\sharp)$$

with some rational number r. If μ is a nonlinear irreducible character of M, then $\mu \in \mathcal{M}^*$. Hence, $(\gamma, \mu) = 0$ by the property of γ. Thus, γ does not involve μ. Hence, γ is a sum of linear characters. Then, $(\gamma, \theta) = 0$ implies that γ vanishes on $H^\sharp$. Hence, $\lambda^\tau(x) = \lambda(x)$ for $x \in H^\sharp$.

Let $G_0 = G_0(M)$. Then, Lemma 11.5 applied to $|\lambda^\tau(x)|^2$ and 1_G yield (with $h = |H|$)

$$\frac{1}{g}\sum_{x\in G_0} |\lambda^\tau(x)|^2 = \frac{1}{|M|}\sum_{x\in H^\sharp} |\lambda^\tau(x)|^2 = \frac{1}{|M|}\sum_{x\in H^\sharp} |\lambda(x)|^2 = 1 - \frac{3}{h},$$

$$\frac{1}{g}|G_0| = \frac{1}{|M|}\sum_{x\in H^\sharp} 1 = \frac{h-1}{3h}.$$

Let G_1 be the set of elements of $G - G_0$ which are not conjugate to any element of $\widehat{W}$, $P^\sharp$, or $Q^\sharp$. On G_1, $(\theta - \lambda)^\tau$ vanishes. The virtual characters η_{10} and η_{20} are 3-conjugate. Therefore, they take the same value on G_1. Thus,

$$1 + 2\eta_{10}(x) - \lambda^\tau(x) = 0$$

for $x \in G_1$. This implies that $\lambda^\tau(x) \neq 0$ on G_1. Hence,

$$\begin{aligned} \frac{3}{h} \geq \frac{1}{g}\sum_{x\in G_1} |\lambda^\tau(x)|^2 &\geq \frac{1}{g}|G_1| \\ &\geq 1 - \frac{h-1}{3h} - (1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}) - \frac{|P|-1}{|S|} - \frac{|Q|-1}{|T|}, \end{aligned}$$

$$\frac{8}{3h} + \frac{1}{3cu} + \frac{1}{pv} \geq \frac{2}{3p} + \frac{1}{|S|} + \frac{1}{|T|} > \frac{2}{3p}.$$

We have $u = p^2 + p + 1 \geq 3p$, $v = (3^p - 1)/2 \geq 63$, and $h \geq cu$. Hence, the left side of the above inequality is at most $(8+1+1)/63p = 10/63p < 1/6p$. This is a contradiction.

Suppose that S is of type III. Then, $S' = S_\sigma = A(S)$. Since there is no supporting subgroup for $A(S)$, S_σ is a TI-set of G with normalizer S. Let $\mathcal{S}_1 = \mathcal{S}_0 \cup \mathcal{S}$ in the notation of §16. Then, $\mathcal{S}_1$ is the set of characters of S which are induced by nonprincipal irreducible characters of S'. Let ξ_0 be the character of S induced by the principal character of S'. By

Theorem 16.1 (a), $\mathcal{S}_1$ is coherent. As before, $\mathcal{S}_2 = \mathcal{S}_1 \cup \{\xi_0\}$ is coherent and

$$\xi_0{}^\tau = 1 + \eta_{10} + \eta_{20}.$$

Let λ be an irreducible character of degree 3 lying in $\mathcal{S}_0$. By Lemma 13.5, the characters of $\mathcal{S}_2$ are either irreducible or one of ξ_j for $0 \le j \le p-1$. Then, any virtual character of S that is orthogonal to all $\mu \in \mathcal{S}_2$ vanishes on $(S')^\sharp$. It follows from Lemma 14.4 that $\lambda^\tau(x) = \lambda(x)$ for $x \in (S')^\sharp$. Let G_0 be the set of elements of G which are conjugate to some element of $(S')^\sharp$. Since S' is a TI-set in G, we have

$$\frac{1}{g}\sum_{x\in G_0} |\lambda^\tau(x)|^2 = \frac{1}{|S|}\sum_{x\in (S')^\sharp} |\lambda^\tau(x)|^2 = \frac{1}{|S|}\sum |\lambda(x)|^2 = 1 - \frac{3}{|S'|}$$

and $|G_0|/g = (|S'|-1)/|S|$. Let G_1 be the set of elements of $G - G_0$ which are not conjugate to any element of $\widehat{W}$ or $Q^\sharp$. Since $(\xi_0 - \lambda)^\tau$ vanishes on G_1 and $\eta_{10} = \eta_{20}$ on G_1, we see that λ^τ does not vanish on G_1. Thus,

$$\frac{3}{|S'|} \ge \frac{1}{g}\sum_{x\in G_1} |\lambda^\tau(x)|^2 \ge \frac{1}{g}|G_1| \ge 1 - (1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}) - \frac{|S'|-1}{|S|} - \frac{|T'|-1}{|T|}.$$

Hence,

$$\frac{3}{|S'|} + \frac{1}{p|V|} \ge \frac{2}{3p} + \frac{1}{|S|} + \frac{1}{|T|} > \frac{2}{3p}.$$

This is a contradiction. Q.E.D.

Proposition 21.2. *Suppose that $q = 3$. Then $d = 1$.*

Proof. Suppose that $q = 3$ and $d \ne 1$. Then, by Lemma 20.7 (*iii*) with q, c, and S replaced by p, d, and T, we have $p = 5$, $d = 11$, and $v = 121 = (11)^2$. Since $v = (3^5 - 1)/2$, V/D is cyclic by Lemma 20.1. It follows that V is abelian. Thus, T is of type II or III.

Suppose that T is of type III. Let $\mathcal{V}$ be the set of characters of T which are induced by nonprincipal irreducible characters of T'. By Theorem 16.1, $\mathcal{V}$ is coherent. Let ζ_0 be the character induced by the principal character of T'. By Lemma 12.7, $\mathcal{V}^* = \mathcal{V} \cup \{\zeta_0\}$ is coherent. We will see what $\zeta_0{}^\tau$ is. Let $\lambda \in \mathcal{V}$ be a character of degree p and let

$$\alpha = \zeta_0 - \lambda.$$

We use the same method as in the proof of Lemma 19.4. The characters ν_{ij} associated to the cyclic subgroup W satisfy $\nu_{ij}(1) = v$ for $i > 0$

(Lemma 20.1). Since $v \equiv 1 \pmod p$, all signs attached to ν_{ij} are 1. For a fixed $i \neq 0$, let

$$\gamma_j = \nu_{0j} - \nu_{ij} + \delta \qquad (0 \leq j \leq 4)$$

where δ is a sum of characters of degree p in $\mathcal{V}$ such that $\delta(1) = v - 1$. For example, let δ be the sum of distinct characters of degree 5 which have QD in their kernel. (There are exactly $(v-1)/5$ such characters.) We have $\gamma_j \in I_0(A_0(T))$. Thus, $\gamma_j{}^\tau$ are defined. Since

$$(\gamma_0 - \gamma_j)^\tau = \eta_{00} - \eta_{i0} - \eta_{0j} + \eta_{ij},$$

we have $\gamma_j{}^\tau = \eta_{0j} - \eta_{ij} + \Delta$ with Δ independent of j. For each j with $0 \leq j \leq 4$, $(\gamma_j{}^\tau, \zeta_0{}^\tau) = (\gamma_j, \zeta_0) = 1$. We have

$$5 = \sum_j (\eta_{0j}, \zeta_0{}^\tau) + (\sum_j \eta_{ij}, \zeta_0{}^\tau) + 5(\Delta, \zeta_0{}^\tau).$$

Since $\zeta_i{}^\tau = \pm \sum_j \eta_{ij}$ and $(\zeta_i{}^\tau, \zeta_0{}^\tau) = 0$, we get

$$5 = 1 + \sum_{j>o} (\eta_{0j}, \zeta_0{}^\tau) + 5(\Delta, \zeta_0{}^\tau).$$

Therefore, $(\eta_{0j}, \zeta_0{}^\tau) \neq 0$ for some $j > 0$. The characters $\eta_{01}, \ldots, \eta_{04}$ are p-conjugate, while $\zeta_0{}^\tau$ is p-rational. Hence, $(\eta_{0j}, \zeta_0{}^\tau)$ is independent of j. Since $\zeta_0{}^\tau$ is of weight 5, we have $\zeta_0{}^\tau = 1_G \pm \sum_{j>0} \eta_{0j}$. Since $(\gamma_j{}^\tau, \zeta_0{}^\tau) = 1$,

$$\zeta_0{}^\tau = 1_G + \sum_{j>0} \eta_{0j}.$$

Since $\mathcal{V}^*$ consists of all the characters of T which are induced by irreducible characters of T', Lemma 14.4 yields that

$$\lambda^\tau(x) = \lambda(x) \quad \text{for} \quad x \in (T')^\sharp.$$

Since there is no supporting subgroup, $A(T) = T'$ is a TI-set of G. Let G_0 be the set of elements of G which are conjugate to some element of $(T')^\sharp$. Then,

$$\begin{aligned}\frac{1}{g} \sum_{x \in G_0} |\lambda^\tau(x)|^2 &= \frac{1}{|T|} \sum_{x \in (T')^\sharp} |\lambda^\tau(x)|^2 \\ &= \frac{1}{|T|} \sum_{x \in (T')^\sharp} |\lambda(x)|^2 = 1 - \frac{5}{|T'|}\end{aligned}$$

because λ vanishes on $T - T'$. We have $|G_0|/g = (|T'| - 1)/|T|$. Let G_1 be the set of elements of $G - G_0$ which are not conjugate to any element of $\widehat{W}$ or $P^\sharp$. Then if $y \in G_1$, then $\alpha^\tau(y) = 0$ and $\eta_{01}(y) = \eta_{0j}(y)$ for all $j > 0$. It follows that

$$1 + 4\eta_{01}(y) - \lambda^\tau(y) = 0.$$

This implies $\lambda^\tau(y) \neq 0$. Then,

$$\begin{aligned}\frac{5}{|T'|} &\geq \frac{1}{g}\sum_{x\in G_1} |\lambda^\tau(x)|^2 \geq \frac{1}{g}|G_1| \\ &\geq 1 - \frac{|G_0|}{g} - (1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}) - \frac{|P|-1}{|S|}.\end{aligned}$$

Thus,

$$\frac{5}{|T'|} + \frac{1}{3u} \geq \frac{4}{15} + \frac{1}{|S|} + \frac{1}{|T|} > \frac{4}{15}.$$

This is not the case. Therefore, T is not of type III.

Suppose that T is of type II. In this case, $11 \in \varpi$ so T is a ϖ-group. Take $M \in \mathcal{M}(N_G(V))$. Since M contains VP^* which is not p-closed, M is not conjugate to P. The prime 11 lies in $\sigma(T)'$. In fact, D centralizes Q; hence, $N_G(D) \subseteq T$ by (IIv). Since $N_G(V) \not\subseteq T$ by (IIiv), V is not cyclic. Thus, $11 \in \tau_2(T) \cap \sigma(M)$ by Lemma 6.11. It follows that M is not conjugate to S. By Theorem I, M is of type I. Hence, by Theorem 19.1, M is a Frobenius group with Frobenius kernel M_σ. Let $H = M_\sigma$. Then, $M = N_G(H)$. Since $N_G(D) \subseteq T$, we have $N_H(D) = T \cap H = V$. It follows that $|H|$ is a power of the prime 11 and $Z(H)$ is cyclic. Therefore, $|M/H| = e$ divides $11 - 1 = 10$. Since P^* is contained in $N_G(V)$, we have $e = 5$.

Let $\mathcal{V}$ be the set of irreducible characters of M which do not have H in their kernel. If $\mathcal{V}$ is not coherent, then $|H : H'| \leq 4e^2 + 1 = 101$. This implies that H is cyclic. Therefore, $\mathcal{V}$ is coherent. Let ζ_0 be the character of M induced by the principal character of H. As before, $\mathcal{V}^* = \mathcal{V} \cup \{\zeta_0\}$ is coherent. Let λ be an irreducible character of M of degree 5, and let

$$\alpha = \zeta_0 - \lambda.$$

Then, $\alpha \in I_0(A(M))$ and α vanishes on the conjugates of $\widehat{W}$. (The group T is a supporting subgroup of $A(M)$. But, the territory of $A(M)$ does not intersect with $\widehat{W}$.) It follows that

$$(\alpha^\tau, \eta_{00} - \eta_{i0} - \eta_{0j} + \eta_{ij}) = 0.$$

Since $\lambda^\tau \neq \pm\eta_{st}$ for any s, t by Lemma N, we have

$$(\zeta_0{}^\tau, \eta_{00} - \eta_{i0} - \eta_{0j} + \eta_{ij}) = 0.$$

Since $\alpha^\tau = \zeta_0{}^\tau - \lambda^\tau$ involves the principal character of G, $(\zeta_0{}^\tau, \eta_{00}) = 1$. We will show that $(\zeta_0{}^\tau, \eta_{0j}) \neq 0$. Suppose that $(\zeta_0{}^\tau, \eta_{0j}) = 0$. If $(\zeta_0{}^\tau, \eta_{ij}) \neq 0$, then

$$\zeta_0{}^\tau = 1_G + a\sum_{j>0} \eta_{ij}$$

because η_{ij} for $1 \leq j \leq 4$ are p-conjugate. Then,

$$5 = \|\zeta_0{}^\tau\|^2 = 1 + 4a^2.$$

On the other hand, we have $(\zeta_0{}^\tau, \zeta_i{}^\tau) = (\zeta_0, \zeta_i) = 0$ for $i > 0$. Since $\zeta_i{}^\tau = \pm\sum_j \eta_{ij}$, $(\zeta_0{}^\tau, \zeta_i{}^\tau) = \pm 4a$. This is a contradiction. Hence, $(\zeta_0{}^\tau, \eta_{ij}) = 0$ for $i, j > 0$. Then, $(\zeta_0{}^\tau, \zeta_i{}^\tau) = 0$ implies $(\zeta_0{}^\tau, \eta_{i0}) = 0$. This contradiction finally proves $(\zeta_0{}^\tau, \eta_{0j}) \neq 0$. Then,

$$\zeta_0{}^\tau = 1 + \sum_{j>0} \eta_{0j}.$$

Lemma 14.4 yields that

$$\lambda^\tau(x) = \lambda(x) \quad \text{for} \quad x \in H^\sharp.$$

By Lemma 14.3, λ^τ is well-behaved relative to $A(M)$. Hence, we can apply Lemma 11.5. Let G_0 be the territory of $A(M)$. Then,

$$\frac{1}{g}\sum_{x\in G_0} |\lambda^\tau(x)|^2 = \frac{1}{|M|}\sum_{x\in H^\sharp} |\lambda^\tau(x)|^2 = \frac{1}{|M|}\sum_{x\in M^\sharp} |\lambda(x)|^2 = 1 - \frac{e}{|H|}.$$

And

$$\frac{1}{g}|G_0| = \frac{1}{g}\sum_{x\in G_0} 1 = \frac{1}{|M|}\sum_{x\in H^\sharp} 1 = \frac{|H|-1}{|M|}.$$

Let G_1 be the set of elements of $G - G_0$ which are not conjugate to any element of $\widehat{W}$, $P^\sharp$, or $Q^\sharp$. Then, $\alpha^\tau(x) = 0$ for $x \in G_1$. Also, we have $\eta_{01}(x) = \eta_{0j}(x)$ for $j > 0$ and $x \in G_1$. It follows that

$$1 + 4\eta_{01}(x) - \lambda^\tau(x) = 0 \quad \text{for} \quad x \in G_1.$$

This implies that $\lambda^\tau(x) \neq 0$ on G_1. Therefore, we have

$$\frac{e}{|H|} \geq \frac{1}{g} \sum_{x \in G_1} |\alpha(x)|^2 \geq \frac{1}{g}|G_1|$$
$$\geq 1 - \frac{|H|-1}{|M|} - (1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}) - \frac{|P|-1}{|S|} - \frac{|Q|-1}{|T|},$$
$$\frac{e}{|H|} + \frac{1}{3u} + \frac{1}{5 \cdot 11^3} \geq \frac{4}{15} + \frac{1}{|M|} + \frac{1}{|S|} + \frac{1}{|T|} > \frac{4}{15}.$$

Since $u = 31$ and $|H| \geq 11^3$, this is a contradiction. Q.E.D.

Proposition 21.3. *Suppose that $q = 3$. Then, P is an elementary abelian group of order p^3.*

Proof. Suppose that Proposition 21.3 fails. Then, by Lemma 20.7 (*v*), we have $p = 7$ and $|P| = p^4$. The group P is elementary abelian. The group U is abelian and its order u divides either $(p^3-1)/(p-1)$ or $(p-1)^{q-1}$. Since $p = 7$ and $(u, 6) = 1$, Lemma 20.1 yields that U is a cyclic group of order dividing $p^2 + p + 1 = 57 = 3 \cdot 19$. Since $(u, 3) = 1$, we must have $u = 19$.

Lemma 20.1 yields a normal subgroup P_0 such that UQ^* acts irreducibly on P/P_0. Then, $|P_0| = 7$ and U centralizes P_0. There is a subgroup P_1 of order p^3 such that UQ^* acts irreducibly on P_1. Then,

$$P = P_0 \times P_1$$

and $P_0 = C_P(U)$. Since $C_P(U) \neq 1$, S is not of type II. Therefore, S is of type III. Since there is no supporting subgroup, $S' = A(S)$ is a TI-set.

Let $\mathcal{U}$ be the set of characters of S which are induced by nonprincipal irreducible characters of S'. By Theorem 16.1 (a), $\mathcal{U}$ is coherent. Let ξ_0 be the character of S that is induced by the principal character of S'. Then, by Lemma 12.7, $\mathcal{U}^* = \mathcal{U} \cup \{\xi_0\}$ is coherent. Let λ be an irreducible character of degree 3 lying in $\mathcal{U}$, and let

$$\alpha = \xi_0 - \lambda.$$

Then, α^τ vanishes on any conjugate of $\widehat{W}$. It follows that

$$(\alpha^\tau, 1_G - \eta_{i0} - \eta_{0j} + \eta_{ij}) = 0 \quad \text{for} \quad i, j > 0.$$

By Lemma N, λ^τ is orthogonal to every η_{st}. Hence,

$$({\xi_0}^\tau, 1_G - \eta_{i0} - \eta_{0j} + \eta_{ij}) = 0.$$

The virtual characters η_{0j} $(1 \le j \le 6)$ are p-conjugate, while ${\xi_0}^\tau$ is p-rational. Since $\|{\xi_0}^\tau\|^2 = 3$, ${\xi_0}^\tau$ does not involve η_{0j}. By the same reasoning, ${\xi_0}^\tau$ does not involve η_{ij}. It follows that

$$ {\xi_0}^\tau = 1_G + \eta_{10} + \eta_{20}. $$

We can argue as in the previous propositions. We have

$$ \lambda^\tau(x) = \lambda(x) \quad \text{for} \quad x \in (S')^\sharp. $$

Let G_0 be the set of elements of G which are conjugate to some element of $(S')^\sharp$. Since S' is a TI-set,

$$ \begin{aligned} \frac{1}{g} \sum_{x \in G_0} |\lambda^\tau(x)|^2 &= \frac{1}{|S|} \sum_{x \in (S')^\sharp} |\lambda^\tau(x)|^2 \\ &= \frac{1}{|S|} \sum_{x \in (S')^\sharp} |\lambda(x)|^2 = 1 - \frac{3}{|S'|} \end{aligned} $$

because λ vanishes on $S - S'$. Similarly,

$$ |G_0|/g = (|S'| - 1)/|S|. $$

Let G_1 be the set of elements of $G - G_0$ which are not conjugate to any element of $\widehat{W}$ or $Q^\sharp$. Then, α^τ vanishes on G_1, and $\eta_{10}(y) = \eta_{20}(y)$ for $y \in G_1$. Thus,

$$ 1 + 2\eta_{10}(y) - \lambda^\tau(y) = 0 \quad \text{for} \quad y \in G_1. $$

This implies $\lambda^\tau(y) \ne 0$ for $y \in G_1$. Then,

$$ \begin{aligned} \frac{3}{|S'|} &\ge \frac{1}{g} \sum_{x \in G_1} |\lambda^\tau(x)|^2 \ge \frac{1}{g}|G_1| \\ &\ge 1 - \frac{|S'| - 1}{|S|} - (1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}) - \frac{|Q| - 1}{|T|}. \end{aligned} $$

Then,

$$ \frac{3}{|S'|} + \frac{1}{7 \cdot v} \ge \frac{1}{p} - \frac{1}{pq} + \frac{1}{|S|} + \frac{1}{|T|} > \frac{2}{21}. $$

Since $|S'| = 7^4 \cdot 19$ and $v = 1093$, this is a contradiction. Q.E.D.

Proposition 21.4. *Suppose that $q = 3$. Then, U is a cyclic group of order dividing p^2+p+1 that acts on P irreducibly and regularly. The group PU is a Frobenius group. The group Q is also an elementary abelian group of order 3^p and QV is a Frobenius group with Frobenius kernel Q. The group V is a cyclic group of order dividing $(3^p-1)/2$.*

Proof. Since $q = 3$, we have $p \geq 5$. By Lemma 20.7 (i) with q and P replaced by p and Q, Q is an elementary abelian group of order 3^p. By Proposition 21.2, we have $d = 1$ and $D = 1$. Lemma 20.8 with q and P replaced by p and Q yields that QV is a Frobenius group with Frobenius kernel Q and $v = |V|$ divides $(3^p-1)/2$. By Lemma 20.1 for T, V is a cyclic group.

Suppose that Proposition 21.4 fails. Then, by Lemma 20.1, the group U is a product of at most 2 cyclic groups and u divides $[(p-1)/2]^2$. Since U is abelian, S is either of type II or type III.

Suppose that S is of type III. Let $\mathcal{U}$ be the set of characters of S which are induced by nonprincipal irreducible characters of S'. If $\mathcal{U}$ is coherent, we can apply the same argument as the one in the proof of Proposition 21.3. At the end, we get

$$\frac{3}{|S'|} + \frac{|Q|}{|T|} \geq \frac{2}{3p}.$$

Since $|S'| = p^3u > 15p$ and $|T| = v|Q|$ with

$$v = (3^p-1)/2 \geq 5p,$$

we have a contradiction

$$\frac{2}{5p} > \frac{3}{|S'|} + \frac{|Q|}{|T|} \geq \frac{2}{3p}.$$

We will prove that $\mathcal{U}$ is coherent. Since $U \cong S'/P$ is abelian, $\mathcal{U}$ contains $(u-1)/3$ irreducible characters of degree 3. By assumption, U is an abelian group of exponent dividing $(p-1)/2$. Therefore, there is a U-invariant subgroup P_1 of P with index $|P : P_1| = p$. Then, $U/C_U(P/P_1)$ is cyclic. Since $C_U(P/P_1)$ is contained in the inertia group of any linear character of P/P_1, there is a linear character θ of P such that $|S' : I(\theta)| \leq \exp U \leq (p-1)/2$. It follows that there is an irreducible character μ of $\mathcal{U}$ having the degree $3d$ with $1 < d \leq (p-1)/2$ (cf. §11, the proof of Lemma 4.5 [FT]). Let λ be an irreducible character of degree 3 lying in $\mathcal{U}$. Let

$$\alpha = \xi - \lambda \quad \text{and} \quad \beta = d\lambda - \mu$$

where ξ is the character of S induced by the principal character of S'. Then, $\alpha, \beta \in I_0(A(S))$ and α^τ, β^τ are defined. Since S is of type III, we have $A(S) = S'$. Then, for characters ν, ν' of degree 3 in $\mathfrak{U}$,

$$(\beta^\tau, (\nu - \nu')^\tau) = 0$$

if $\nu \neq \lambda \neq \nu'$; while $(\beta^\tau, (\nu - \lambda)^\tau) = -d$. It follows that

$$\beta^\tau = d\lambda^\tau - x \sum \nu^\tau - \mu^\tau + \Delta$$

for some integer x where the sum is over all irreducible characters ν of degree 3. If $x = 0$, $\mathfrak{U}$ is coherent. Suppose that $x \neq 0$. We have $\|\beta^\tau\|^2 = d^2 + 1$. It follows from this

$$x^2(u-1)/3 \leq 2dx. \tag{21.1}$$

Note that $\mathfrak{U}$ contains exactly $(u-1)/3$ irreducible characters of degree 3. The above inequality implies $x > 0$. Lemma 20.7(iv) yields $u > (p^2 + p + 1)/13$. Since $d \leq (p-1)/2$, (21.1) yields $p \leq 37$. By Lemma 20.1, u divides $(p-1)^2/4$. Thus,

$$\frac{p^2+p+1}{13} < u \leq \frac{(p-1)^2}{4}.$$

If $p - 1$ is divisible by 4 or 3, we can replace $(p-1)^2/4$ by $(p-1)^2/16$ to get a contradiction. Thus, $p \equiv -1 \pmod{12}$ and $p = 11$ or 23. If $p = 23$, u divides 11^2. If $u = 11$, then $u \not\equiv 1 \pmod 3$. This contradicts the fact that UQ^* is a Frobenius group. Thus, $u = 121$ and (21.1) yields $x < 1$. Therefore, we have $p = 11$, $u = 25$, $d = 5$, and (21.1) yields $0 < x < 2$. Hence, $x = 1$ and

$$\beta^\tau = d\lambda^\tau - \sum \nu^\tau - \mu^\tau + \Delta.$$

As before, Δ is a real-valued virtual character such that $(1_G, \Delta) = (\nu^\tau, \Delta) = 0$ for every $\nu \in \mathfrak{U}$ with $\nu(1) = 3$. We check that

$$\alpha^\tau = 1_G + \eta_{10} + \eta_{20} - \lambda^\tau.$$

Since $\overline{\eta_{20}} = \eta_{10}$, (α^τ, Δ) is an even integer. But, $(\alpha^\tau, \beta^\tau) = (\alpha, \beta) = -d$ by Lemma 11.4. Thus, we have

$$(\alpha^\tau, \beta^\tau) = -(d-1) + (\alpha^\tau, \Delta)$$

which implies $(\alpha^\tau, \Delta) = -1$. This contradiction proves that $\mathfrak{U}$ is coherent. Thus, Proposition 21.4 is proved if S is of type III.

Suppose that S is of type II. Let $M \in \mathcal{M}(N_G(U))$. Then, M is not conjugate to S or T. Therefore, by Theorem I, M is of type I. By Theorem 19.1, M is a Frobenius group with Frobenius kernel M_σ. We have

$$U \subseteq M_\sigma.$$

The group Q^* is contained in M. Hence, $|M\!:\!M_\sigma| = 3$ or $3p$.

By Lemma 20.1, u divides $(p-1)^2/4$. As before, $p-1$ is not divisible by 4 or 3. Hence, if $u \neq (p-1)^2/4$, then

$$\frac{p^2+p+1}{13} < u \leq \frac{(p-1)^2}{20}$$

which is a contradiction. It follows that $u = (p-1)^2/4$ and U is the direct product of two cyclic groups of order $(p-1)/2$. Thus, all Sylow subgroups of U are abelian of rank 2. Hence, we have $\pi(\mathsf{U}) = \tau_2(S)$. Take $r \in \pi(U)$ and let $A \in \mathcal{E}_p^2(\mathsf{U})$. Then, for some $B \in \mathcal{E}^1(A)$, $C_P(B) \neq 1$ by Proposition 1.16 [BG]. For B, $C_G(B) \subseteq S$ by (IIv). This implies that

$$Z(M_\sigma) \subseteq S \cap M_\sigma = \mathsf{U}.$$

Since M_σ is nilpotent, $\pi(Z(M_\sigma)) = \pi(M_\sigma)$. Thus,

$$\pi(U) \subseteq \pi(M_\sigma) = \pi(Z(M_\sigma)) \subseteq \pi(U).$$

Therefore, $\pi(U) = \pi(M_\sigma)$. Suppose that $|\pi(U)| > 1$ or some Sylow subgroup of U is a Sylow subgroup of G. Then, by Theorem 6.7, G has abelian Sylow r-subgroups for each $r \in \pi(U)$. By Lemma 6.8(b), U is a Hall $\tau_2(S)$-subgroup of G. It follows that $U = M_\sigma$. Since $N_G(U) \nsubseteq S$, we have $|M\!:\!M_\sigma| = 3p$. Since M is a Frobenius group,

$$|A| \equiv 1 \pmod{3p}$$

for each $A \in \mathcal{E}^2(U)$. Since $u = |U| = (p-1)^2/4$, we have $U = A$ and $u \equiv 1 \pmod{3p}$. (If $A \neq U$, $|U| \geq (3p+1)^2$ which is impossible.) Thus,

$$(p-1)^2 - 4 = 12kp$$

for some integer k. Hence, p divides 3. This contradicts the assumption $p > q = 3$. Therefore, $\pi(U) = \{r\}$ for a single prime r and G has a nonabelian Sylow r-subgroup. It follows from the structure of S that $P = S_\sigma$. Then, Theorem 6.7 yields that $C = C_A(P)$ has order p. This contradicts Proposition 21.2. Q.E.D.

Theorem. *Let G be a finite simple group and let ϖ be a connected component of the prime graph $\Gamma(G)$ such that $2 \notin \pi$. Then, we have one of the two cases:*

(1) *G contains a nilpotent Hall ϖ-subgroup H that is isolated in G, or*

(2) *$\varpi = \{p, q\}$ and there is a self-normalizing cyclic group W of order pq.*

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