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Approximation of Expectation of Diffusion Process and Mathematical Finance

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§1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{(B^1(t), \ldots, B^d(t)\ ;\ t \in [0,\infty)\}$ be a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose devivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that with probability one $X(t,x)$ is continuous in t and smooth in x.

In many fields, it is important to compute $E[f(X(T,x))]$ numerically, where f is a function defined in $\mathbf{R}^N$. Let $u(t,x) = E[f(X(t,x))]$, $t > 0$, $x \in \mathbf{R}^N$. Then u satisfies the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x), \\ u(0,x) = f(x). \end{cases}$$

Here $L = \frac{1}{2}\sum_{i=1}^N V_i^2 + V_0$. So to compute $E[f(X(T,x))]$ is the same to compute the solution $u(T,x)$ to PDE. However, in mathematical finance, if we think of the problem of pricing of Europian options, there are sometimes following difficulties.

(1) L can be degenerate. Moreover, L may not satisfy even the Hörmander condition.

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(2) f may not be continuously differentiable.

Bally and Talay [1] showed that under the Hörmander condition, Euler-Maruyama approximation gives a good approximation, even if the function f is only bounded measurable. In this paper, we introduce a new method to compute $E[f(X(T,x))]$ numerically. Our method works when the function f is Lipschitz continuous. Our main tools are Malliavin calculus and stochastic Taylor approximation based on Lie algebra. Such stochastic Taylor expansion was initiated by Ben Arous [2], and has been studied by many authors ([3], [8], [10], also see [9]).

§2. Notation and Results

Let $\mathcal{A} = \{\emptyset\} \cup \bigcup_{k=1}^{\infty}\{0,1,\dots,d\}^k$ and for $\alpha \in \mathcal{A}$, let $|\alpha| = 0$ if $\alpha = \emptyset$, let $|\alpha| = k$ if $\alpha = (\alpha^1,\dots,\alpha^k) \in \{0,1,\dots,d\}^k$, and let $\|\alpha\| = |\alpha| + \mathrm{card}\{1 \le i \le |\alpha| \; ; \alpha^i = 0\}$. For $\alpha, \beta \in \mathcal{A}$, we define $\alpha * \beta \in \mathcal{A}$ by $\alpha * \beta = (\alpha^1,\dots,\alpha^k,\beta^1,\dots,\beta^\ell)$ if $\alpha = (\alpha^1,\dots,\alpha^k) \in \{0,1,\dots,d\}^k$ and $\beta = (\beta^1,\dots,\beta^\ell) \in \{0,1,\dots,d\}^\ell$. Then $\mathcal{A}$ becomes a semigroup with respect the product $*$ with the identity $\emptyset$.

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ denote $\mathcal{A}\setminus\{\emptyset\}$ and $\mathcal{A}\setminus\{\emptyset,(0)\}$, respectively. Also, for each $m \ge 1$, $\mathcal{A}(m)$, let $\mathcal{A}_0(m)$ and $\mathcal{A}_1(m)$ denote $\{\alpha \in \mathcal{A} \; ; \|\alpha\| \le m\}$, $\{\alpha \in \mathcal{A}_0 \; ; \|\alpha\| \le m\}$ and $\{\alpha \in \mathcal{A}_1 \; ; \|\alpha\| \le m\}$ respectively.

Let $B^{\circ\alpha}(t)$, $t \in [0,\infty)$, $\alpha \in \mathcal{A}$, be inductively defined by

$$B^{\circ\emptyset} = 1, \quad B^{\circ(i)} = B^i(t), \quad i = 0,1,\dots,d$$

and

$$B^{\circ\alpha*(i)}(t) = \int_0^t B^{\circ\alpha}(s) \circ dB^i(s), \quad i = 0,1,\dots,d.$$

We define a vector field $V_{[\alpha]}$, $\alpha \in \mathcal{A}$, inductivel by

$$V_{[\emptyset]} = 0, \quad V_{[i]} = V_i, \quad i = 0,1,\dots,d$$
$$V_{[\alpha*(i)]} = [V_\alpha, V_i], \quad i = 0,1,\dots,d.$$

Now we assume the following throughout the paper.

(UFG) There is an integer ℓ such that for any $\alpha \in \mathcal{A}_1$, there are $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbf{R}^N)$, $\alpha \in \mathcal{A}_1$, $\beta \in \mathcal{A}_1(\ell)$, satisfying the following.

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(\ell)} \varphi_{\alpha,\beta} V_{[\beta]}.$$

Remark.

(1) Let us think of $C_b^\infty(\mathbf{R}^N)$-module $M = \sum_{\alpha\in\mathcal{A}_0} C_b^\infty(\mathbf{R}^N)V_{[\alpha]}$. Then the assumption (UFG) is equivalent to the assumption that M is finitely generated as a $C_b^\infty(\mathbf{R}^N)$-module.

(2) The following condition (UH) (Uniform Hörmander condition) implies the assumption (UFG).

(UH) There are an integer ℓ and a constant $c > 0$ such that

$$\sum_{\alpha\in\mathcal{A}_1(\ell)} (V_{[\alpha]}, \xi)^2 \geq c|\xi|^2, \quad \text{for all } x, \xi \in \mathbf{R}^N$$

Let V_α, $\alpha \in \mathcal{A}$, be differential operators given by

$$V_\alpha = \text{Identity}, \quad \text{if } \alpha = \emptyset,$$

and

$$V_\alpha = V_{\alpha_1} \cdots V_{\alpha_k}, \quad \text{if } \alpha = (\alpha_1, \dots, \alpha_k).$$

Let us define a semi-norm $\|\cdot\|_{V,n}$, $n \geq 1$, on $C_0^\infty(\mathbf{R}^N; \mathbf{R})$ by

$$\|f\|_{V,n} = \sum_{k=1}^{n} \sum_{\substack{\alpha_1,\dots,\alpha_k\in\mathcal{A}_1 \\ \|\alpha_1*\cdots*\alpha_k\|=n}} \|V_{[\alpha_1]}\cdots V_{[\alpha_k]}f\|_\infty.$$

Now let us define a semigroup of linear operators $\{P_t\}_{t\in[0,\infty)}$ by

$$(P_t f)(x) = E[f(X(t,x))], \quad t \in [0,\infty),\ f \in C_b^\infty(\mathbf{R}^N).$$

Then we can prove the following by using a similar argument in Kusuoka-Stroock [7] (also see [5] for the details).

Theorem 1. *For any $n, m \geq 0$ and $\alpha_1, \dots, \alpha_{n+m} \in \mathcal{A}_1$, there is a constant $C > 0$ such that*

$$\|V_{[\alpha_1]}\cdots V_{[\alpha_n]}P_t V_{[\alpha_{n+1}]}\cdots V_{[\alpha_{n+m}]}f\|_\infty \leq \frac{C}{t^{\|\alpha_1*\cdots*\alpha_{n+m}\|/2}}\|f\|_\infty, \quad f \in C_b^\infty(\mathbf{R}^N).$$

Corollary 2. *For any $n \geq 0$ and $\alpha_1, \dots, \alpha_n \in \mathcal{A}_1$, there is a constant $C > 0$ such that*

$$\|V_{[\alpha_1]}\cdots V_{[\alpha_n]}P_t f\|_\infty \leq \frac{Ct^{1/2}}{t^{\|\alpha_1*\cdots*\alpha_n\|/2}}\|\nabla f\|_\infty, \quad f \in C_b^\infty(\mathbf{R}^N).$$

Definition 3. We say that a family of random variables $\{Z_\alpha \ ; \alpha \in \mathcal{A}_0\}$ is m-moment similar, $m \geq 1$, if $Z_{(0)} = 1$,

$$E[|Z_\alpha|^n] < \infty \quad \text{for any } n \geq 1,\ \alpha \in \mathcal{A}_0,$$

and if

$$E[Z_{\alpha_1} \cdots Z_{\alpha_k}] = E[B^{\circ\alpha_1}(1) \cdots B^{\circ\alpha_k}(1)]$$

for any $k = 1, 2, \ldots, m$ and $\alpha_1, \ldots, \alpha_k \in \mathcal{A}_0$ with $\|\alpha_1\| + \cdots + \|\alpha_k\| \leq m$.

Let $H : \mathbf{R}^N \to \mathbf{R}^N$ be given by $H(x) = (x_1, x_2, \ldots, x_N)$, $x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N$.

Our main result is the following.

Theorem 4. *Let m be an integer and suppose that a family of random variables $\{Z_\alpha \ ; \alpha \in \mathcal{A}_0\}$ is m-moment similar. Let $Q_{(s)}$ be a Markov operator in $C_b(\mathbf{R}^N)$*

$$\begin{aligned} Q_{(s)} f(x) =& E\Big[f\Big(\sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{\alpha_1, \ldots, \alpha_k \in \mathcal{A}_0, \\ \|\alpha_1\| + \cdots + \|\alpha_k\| \leq m}} s^{(\|\alpha_1\| + \cdots + \|\alpha_k\|)/2} \\ & \times (P^0_{\alpha_1} \cdots P^0_{\alpha_k})(V_{[\alpha_1]} \cdots V_{[\alpha_k]} H)(x) \Big) \Big] \end{aligned}$$

for $f \in C_b(\mathbf{R}^N)$ and $x \in \mathbf{R}^N$. Here

$$P^0_\alpha = |\alpha|^{-1} \sum_{k=1}^{|\alpha|} \frac{(-1)^{k-1}}{k} \sum_{\beta_1 * \cdots * \beta_k = \alpha} Z_{\beta_1} \cdots Z_{\beta_k}.$$

Then for any $n \geq 1$ there is a constant $C > 0$ such that

$$\|P_s f - Q_{(s)} f(x)\|_\infty \leq C \Big(\sum_{k=m+1}^{n(m+1)} s^{k/2} \|f\|_{V,k} + s^{(m+1)/2} \|\nabla f\|_\infty \Big),$$
$$s \in (0, 1],\ f \in C_b^\infty(\mathbf{R}^N; \mathbf{R}).$$

Let $T > 0$ and $\gamma > 0$. Let $t_k = t_k^{(n)} = k^\gamma T / n^\gamma$, $n \geq 1$, $k = 0, 1, \ldots, n$, and let $s_k = s_k^{(n)} = t_k - t_{k-1}$, $k = 1, \ldots, n$. Then we have the following.

Theorem 5. *Let $m \geq 1$ and $Q_{(s)}$, $s > 0$ be as in Theorem 4. Then we have the following.*

For $\gamma \in (0, m-1)$, there is a constant $C > 0$ such that

$$\|P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f\|_\infty \leq C n^{-\gamma/2} \|\nabla f\|_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

For $\gamma = m-1$, there is a constant $C > 0$ such that

$$\|P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f\|_\infty \leq C n^{-(m-1)/2} \log(n+1) \|\nabla f\|_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

For $\gamma > m-1$, there is a constant $C > 0$ such that

$$\|P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f\|_\infty \leq C n^{-(m-1)/2} \|\nabla f\|_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

§3. Example of 5-momemt similar family

Let η_i, $i = 1, \ldots, d$ and η_{ij}, $1 \leq i < j \leq d$, are independent random variables such that

$$P(\eta_i = 0) = \frac{1}{2}, \quad P\left(\eta_i = \pm\sqrt{2 \pm \sqrt{2}}\right) = \frac{1}{8},$$

and

$$P(\eta_{ij} = \pm 1) = \frac{1}{2}.$$

Then we see that

$$E[\eta_i] = E[\eta_i^3] = 0, \quad E[\eta_i^2] = 1, \quad E[\eta_i^4] = 3,$$

and

$$E[\eta_{ij}] = 0, \quad E[\eta_{ij}^2] = 1.$$

Now let us define random variables $\{Z_\alpha\ ;\ \alpha \in \mathcal{A}_0\}$ as follows.

(1) The case where $\|\alpha\| = 1$.

$$Z_i = \eta_i, \quad i = 1, \ldots, d.$$

(2) The case where $\|\alpha\| = 2$.

$$Z_0 = 1,$$
$$Z_{ij} = \begin{cases} \frac{1}{2}(\eta_i\eta_j + \eta_{ij}), & 1 \le i < j \le d, \\ \frac{1}{2}(\eta_i\eta_j - \eta_{ji}), & 1 \le j < i \le d, \\ \frac{1}{2}\eta_i\eta_j, & 1 \le i = j \le d. \end{cases}$$

(3) The case where $\|\alpha\| = 3$.

$$Z_{i0} = Z_{0i} = \frac{1}{2}\eta_i, \quad Z_{iii} = \frac{1}{6}\eta_i^3, \quad 1 \le i \le d,$$
$$Z_{iij} = Z_{jii} = \frac{1}{4}\eta_i, \quad Z_{iji} = 0, \quad 1 \le i \ne j \le d,$$

and $Z_\alpha = 0$ in other cases.

(4) The case where $\|\alpha\| = 4$.

$$Z_\alpha = E[B^{\circ\alpha}],$$

that is

$$Z_{iijj} = \frac{1}{8}, \quad 1 \le i, j \le d,$$
$$Z_{0ii} = Z_{ii0} = \frac{1}{4}, \quad 1 \le i \le d,$$
$$Z_{00} = \frac{1}{2},$$

and $Z_\alpha = 0$ in the other case.

(5) The case where $\|\alpha\| \ge 5$.

$$Z_\alpha = 0.$$

Then the family of random variables $\{Z_\alpha \ ; \ \alpha \in \mathcal{A}_0\}$ is 5-moment similar.

§4. Preparation from Algebra

We say that a polynomial p of $x_\alpha, \alpha \in \mathcal{A}_0$, is m-homogeneous, $m \ge 0$, if

$$p(\varepsilon^{\|\alpha\|}x_\alpha, \alpha \in \mathcal{A}_0) = \varepsilon^m p(x_\alpha, \alpha \in \mathcal{A}_0), \quad \varepsilon > 0.$$

Let $\mathcal{U}$ be the free algebra generated by $\{v_0, v_1, \dots, v_d\}$ over $\mathbf{R}$. Then the algebra $\mathcal{U}$ can be extended to the algebra $\overline{\mathcal{U}}$ of formal power series in $\{v_0, v_1, \ldots, v_d\}$. We define $v^\alpha \in \mathcal{U}$, $\alpha \in \mathbf{A}$, by $v^\emptyset = 1$, and by $v^\alpha = v^{\alpha_1} \cdots v^{\alpha_k}$, if $\alpha = (\alpha^1, \ldots, \alpha^k)$. Then $\overline{\mathcal{U}}$ is the complete direct sum of the space $\mathbf{R}v^\alpha$, $\alpha \in \mathcal{A}$. We define convergence in $\overline{\mathcal{U}}$ by $\sum_{\alpha\in\mathcal{A}} a_{\alpha,n}v^\alpha \to \sum_{\alpha\in\mathcal{A}} a_\alpha v^\alpha$, $n \to \infty$, if $a_{\alpha,n} \to a_\alpha$ for any $\alpha \in \mathcal{A}$.

For $x, y \in \overline{\mathcal{U}}$, let $[xy] = xy - yx$. For $\alpha \in \mathbf{A}$, let $v^{[\alpha]} \in \mathcal{U}$ denote 0, if $\alpha = \emptyset$, v_i, if $\alpha = i \in \{0, 1, \ldots, d\}$, and $[\cdots[[v_{\alpha^1}v_{\alpha^2}]v_{\alpha^3}]\cdots, v_{\alpha^k}]$, if $\alpha = (\alpha^1, \ldots, \alpha^k)$ and $k \geq 2$. Let $\overline{\mathcal{U}}^{\mathcal{L}}$ be the closure of $\sum_{\alpha\in\mathcal{A}} \mathbf{R}v^{[\alpha]}$ in $\overline{\mathcal{U}}$. Then $\overline{\mathcal{U}}^{\mathcal{L}}$ is closed under Lie product $[\quad]$ (see Jacobson [4, p.168]).

We use the following two theorems (see Jacobson [J, pp.167–174]).

Theorem 6 (Friedrichs). *Let δ be a continuous homomorphism from $\overline{\mathcal{U}}$ into $\overline{\mathcal{U}} \otimes \overline{\mathcal{U}}$ determined by $\delta(1) = 1 \otimes 1$ and $\delta(v_i) = v_i \otimes 1 + 1 \otimes v_i$, $i = 0, 1, \ldots, d$. Then for $x \in \overline{\mathcal{U}}$, $x \in \overline{\mathcal{U}}^{\mathcal{L}}$ if and only if $\delta(x) = x \otimes 1 + 1 \otimes x$.*

Theorem 7. *Let σ be a linear continuous operator from $\overline{\mathcal{U}}$ into $\overline{\mathcal{U}}^{\mathcal{L}}$ given by $\sigma(v^\alpha) = |\alpha|^{-1} v^{[\alpha]}$, $\alpha \in \mathcal{A}$. Then the restriction of σ on $\overline{\mathcal{U}}^{\mathcal{L}}$ is identity.*

Let $\mathcal{B}_{\overline{\mathcal{U}}}$ be a Borel algebra over $\overline{\mathcal{U}}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. One can define $\overline{\mathcal{U}}$-valued random variables and their expectaions etc. naturally. Let $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ be a filtration satisfying a usual hypothesis, $(B^1(t), \ldots, B^d(t))$, $t \in [0, \infty)$, be a d-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion, and $B^0(t) = t$, $t \in [0, \infty)$. We say that $X(t)$ is a $\overline{\mathcal{U}}$-valued continuous semimartingale, if there are continuous semimartingales X_α, $\alpha \in \mathcal{A}$, such that $X(t) = \sum_{\alpha\in\mathbf{A}} X_\alpha(t)v^\alpha$. For $\overline{\mathcal{U}}$-valued continuous semimartingale $X(t)$, $Y(t)$, we can define $\overline{\mathcal{U}}$-valued continuous semimartingales $\int_0^t X(s) \circ dY(s)$ and $\int_0^t \circ dX(s)Y(s)$ by

$$\int_0^t X(s) \circ dY(s) = \sum_{\alpha,\beta\in\mathcal{A}} \Big(\int_0^t X_\alpha(s) \circ dY_\beta(s)\Big) v^\alpha v^\beta,$$

$$\int_0^t \circ dX(s)Y(s) = \sum_{\alpha,\beta\in\mathcal{A}} \Big(\int_0^t Y_\beta(s) \circ dX_\alpha(s)\Big) v^\alpha v^\beta,$$

where

$$X(t) = \sum_{\alpha\in\mathcal{A}} X_\alpha(t)v^\alpha, \quad Y(t) = \sum_{\beta\in\mathcal{A}} Y_\beta(t)v^\beta.$$

Then we have

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s) \circ dY(s) + \int_0^t \circ dX(s)Y(s).$$

Since $\mathbf{R}$ is regarded a vector subspace in $\overline{\mathcal{U}}$, we can define $\int_0^t X(s) \circ dB^i(s)$, $i = 0, 1, \dots, d$, naturally. We can similarly think of $\overline{\mathcal{U}} \otimes \overline{\mathcal{U}}$-valued semimartingales and stochastic calculus for them.

Now let us consider SDE on $\overline{\mathcal{U}}$

$$X(t) = 1 + \sum_{i=0}^{d} \int_0^t X(s) v_i \circ dB^i(s), \quad t \geq 0.$$

One can easily solve this SDE and obtain

$$X(t) = 1 + \sum_{\alpha \in \mathcal{A}_0} B^{\circ\alpha}(t) v^{\alpha}.$$

We also have the following.

Proposition 8. *Let p^0_α, $\alpha \in \mathcal{A}_0$, be $\|\alpha\|$-homogeneous polynomials in x_β, $\beta \in \mathcal{A}_0$, given by*

$$p^0_\alpha(x_\beta, \beta \in \mathcal{A}_0) = |\alpha|^{-1} \sum_{k=1}^{|\alpha|} \frac{(-1)^{k-1}}{k} \sum_{\substack{\beta_1, \dots, \beta_k \in \mathcal{A}_0 \\ \beta_1 * \cdots * \beta_k = \alpha}} x_{\beta_1} \cdots x_{\beta_k}.$$

Then

$$\log X(t) = \sum_{\alpha \in \mathcal{A}_0} p^0_\alpha(B^{\circ\beta}(t), \beta \in \mathcal{A}_0) v^{[\alpha]}.$$

In other words,

$$X(t) = 1 + \sum_{\alpha \in \mathcal{A}_0} B^{\circ\alpha}(t) v^{\alpha} = \exp\left(\sum_{\alpha \in \mathcal{A}_0} p^0_\alpha(B^{\circ\beta}(t), \beta \in \mathcal{A}_0) v^{[\alpha]} \right).$$

Proof. Note that

$$\delta(X(t)) = 1 \otimes 1 + \sum_{i=0}^{d} \int_0^t \delta(X(s))(v_i \otimes 1 + 1 \otimes v_i) \circ dB^i(s),$$

and

$$\begin{aligned} X(t) \otimes X(t) =& 1 \otimes 1 + \int_0^t \circ d(X(s) \otimes 1)(1 \otimes X(s)) \\ &+ \int_0^t (X(s) \otimes 1) \circ d(1 \otimes X(s)) \\ =& 1 \otimes 1 + \sum_{i=0}^{d} \int_0^t X(s) \otimes X(s)(v_i \otimes 1 + 1 \otimes v_i) \circ dB^i(s). \end{aligned}$$

Since one can easily see the uniqueness of such SDE on $\overline{\mathcal{U}} \otimes \overline{\mathcal{U}}$, we have

$$\delta(X(t)) = X(t) \otimes X(t).$$

For any $u \in \overline{\mathcal{U}}$ with of the form $u = \sum_{\alpha \in \mathcal{A}_0} a_\alpha v^\alpha$, we have

$$\exp(u) \otimes \exp(u) = \exp(u \otimes 1 + 1 \otimes u),$$

which implies

$$\log((1+u) \otimes (1+u)) = \log(1+u) \otimes 1 + 1 \otimes \log(1+u).$$

So we have

$$\delta(\log X(t)) = \log(\delta X(t)) = \log X(t) \otimes 1 + 1 \otimes \log X(t).$$

So by Theorem 6 we see that $\log X(t) \in \overline{\mathcal{U}}^{\mathcal{L}}$ *P-a.s.* On the other hand,

$$\log X(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \Bigg(\sum_{\alpha_1, \dots, \alpha_k \in \mathcal{A}_0} B^{\circ \alpha_1}(t) \cdots B^{\circ \alpha_k}(t) v^{\alpha_1 * \dots * \alpha_k} \Bigg).$$

So acting the linear operator σ in Theorem 7, we have our assertion. Q.E.D.

Proposition 9. *There are polynomials q^0_α, $\alpha \in \mathcal{A}_0$, in x_β, $\beta \in \mathcal{A}_0$, such that*

$$\log\Bigg(\exp(-x_0 v_0) \exp\Big(\sum_{\alpha \in \mathcal{A}_0} x_\alpha v^{[\alpha]} \Big) \Bigg) = \sum_{\alpha \in \mathcal{A}_1} q^0_\alpha (x_\beta,\, \beta \in \mathcal{A}_0) v^{[\alpha]}$$

for any $x_\beta \in \mathbf{R}$, $\beta \in \mathcal{A}_0$. Moreover, $q^0_0 = 0$ and q^0_α is $\|\alpha\|$-homogeneous for each $\alpha \in \mathcal{A}_1$.

Proof. Similarly to the proof of Proposition 8, we see that $\log\big(\exp(-x_0 v_0) \exp(\sum_{\alpha \in \mathcal{A}_0} x_\alpha v^{[\alpha]})\big) \in \overline{\mathcal{U}}^{\mathcal{L}}$. Since we have

$$\exp(-x_0 v_0) \exp\Bigg(\sum_{\alpha \in \mathcal{A}_0} x_\alpha v^{[\alpha]} \Bigg) = 1 + \sum_{\alpha \in \mathcal{A}_1} x_\alpha v^{[\alpha]}$$
$$+ \sum_{\ell + k \geq 2} \sum_{\alpha_1, \dots, \alpha_k \in \mathcal{A}_0} \frac{1}{\ell! k!} (-x_0)^\ell x_{\alpha_1} \cdots x_{\alpha_k} v_0^\ell v^{[\alpha_1]} \cdots v^{[\alpha_k]}.$$

Note that $v_0^\ell v^{[\alpha_1]} \cdots v^{[\alpha_k]} \in \mathcal{U}'_{2\ell + \|\alpha_1\| \cdots \|\alpha_k\|}$. So acting the linear operator σ in Theorem 7 again, we have our assertion. Q.E.D.

§5. Basic Estimates

For $n \geq 0$ let φ_n denote a map from $\overline{\mathcal{U}}$ into the space of differential operators in $\mathbf{R}^N$ of order n given by

$$\varphi_n\Big(\sum_{\alpha\in\mathcal{A}} a_\alpha v^\alpha\Big) = \sum_{\alpha\in\mathcal{A}(n)} a_\alpha V_\alpha, \quad a_\alpha \in \mathbf{R},\ \alpha \in \mathcal{A}.$$

Note that if $u \in \overline{\mathcal{U}}^{\mathcal{L}}$, then $\varphi_n(u)$ is a vector field.

First we observe the following.

Proposition 10. *For any $U \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$,*

$$\left\| f(\exp(U)(\,\cdot\,)) - \sum_{k=0}^{n} \frac{1}{k!} U^k f \right\|_\infty \leq \frac{1}{(n+1)!} \|U^{n+1} f\|_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$ and $n \geq 1$.

Proof. One can prove the following inductively.

$$f(\exp(tU)(x)) = \sum_{k=0}^{n} \frac{t^k}{k!}(U^k f)(x) + \int_0^t \frac{(t-s)^n}{n!}(U^{n+1}f)(\exp(sU)(x))\,ds.$$

Then we have our assertion. Q.E.D.

As corllaries of the above Proposition, we have the following.

Proposition 11. *For any $u = \sum_{\alpha\in\mathcal{A}_1} a_\alpha v^{[\alpha]} \in \overline{\mathcal{U}}^{\mathcal{L}}$, and $n \geq 1$ we have*

$$\|f(\exp(\varphi_n(u))(\,\cdot\,)) - (\varphi_n(\exp(u))f)(\,\cdot\,)\|_\infty \leq \sum_{k=n+1}^{n(n+1)} \max\{|a_\alpha|^{1/\|\alpha\|}\,;\, \alpha \in \mathcal{A}_1(n)\}^k \|f\|_{V,k}$$

for any $f \in C_b^\infty(\mathbf{R}^N)$.

Proposition 12. *For any $u = \sum_{\alpha\in\mathcal{A}_0} a_\alpha v^{[\alpha]} \in \overline{\mathcal{U}}^{\mathcal{L}}$, and $n \geq 1$ we have a constant C depending only on d and n such that*

$$\|f(\exp(\varphi_n(u))(\,\cdot\,)) - (\varphi_n(\exp(u))f)(\,\cdot\,)\|_\infty \leq C \sum_{k=n+1}^{n(n+1)} \max\{|a_\alpha|^{1/\|\alpha\|}\,;\, \alpha \in \mathcal{A}_0(n)\}^k \sum_{\alpha\in\mathcal{A},\,\|\alpha\|=k} \|V_\alpha f\|_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$.

Also, we have the following.

Proposition 13. *For any* $u^{(i)} = \sum_{\alpha \in \mathcal{A}_0} a_\alpha^{(i)} v^{[\alpha]} \in \overline{\mathcal{U}}^{\mathcal{L}}$, $i = 1, 2$, *and* $n \geq 1$, *we have a constant* C *depending only on* d *and* n *such that*

$$\|f(\exp(\varphi_n(u^{(1)}))(\exp(\varphi_n(u^{(2)}))(\cdot))) - (\varphi_n(\exp(u^{(2)})\exp(u^{(1)}))f)(\cdot)\|_\infty$$
$$\leq C \sum_{k=n+1}^{2n(n+1)} \max\{|a_\alpha^{(i)}|^{1/\|\alpha\|}\ ;\ \alpha \in \mathcal{A}_0(2n),\ i = 1, 2\}^k \sum_{\alpha \in \mathcal{A},\ \|\alpha\| = k} \|V_\alpha f\|_\infty$$

for any $f \in C_b^\infty(\mathbf{R}^N)$.

Proof. Note that

$$\begin{aligned}
&f(\exp(\varphi_n(u^{(1)}))(\exp(\varphi_n(u^{(2)}))(x))) - (\varphi_n(\exp(u^{(2)})\exp(u^{(1)}))f)(x) \\
&\quad = f(\exp(\varphi_n(u^{(1)}))(\exp(\varphi_n(u^{(2)}))(x))) \\
&\qquad - (\varphi_n(\exp(u^{(1)}))f)(\exp(\varphi_n(u^{(2)}))(x))) \\
&\qquad + (\varphi_n(\exp(u^{(1)}))f)(\exp(\varphi_n(u^{(2)}))(x))) \\
&\qquad - \varphi_n(\exp(u^{(2)}))(\varphi_n(\exp(u^{(1)}))f)(x)) \\
&\qquad + \varphi_n(\exp(u^{(2)}))(\varphi_n(\exp(u^{(1)}))f)(x)) \\
&\qquad - (\varphi_n(\exp(u^{(2)})\exp(u^{(1)}))f)(x).
\end{aligned}$$

Then we have our assertion from previous two propositions. Q.E.D.

§6. Moment Equivalent Families

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 14. We say that families of random variables $\{Z_\alpha\ ;\ \alpha \in \mathcal{A}_0\}$ and $\{Z'_\alpha\ ;\ \alpha \in \mathcal{A}_0\}$ are m-moment equivalent, $m \geq 1$, if

$$E[|Z_\alpha|^n] < \infty, \quad E[|Z'_\alpha|^n] < \infty, \quad \text{for any } n \geq 1 \text{ and } \alpha \in \mathcal{A}_0,$$

and

$$E[Z_{\alpha_1} \cdots Z_{\alpha_k}] = E[Z'_{\alpha_1} \cdots Z'_{\alpha_k}]$$

for any $k = 1, 2, \ldots, m$ and $\alpha_1, \ldots, \alpha_k \in \mathcal{A}_0$ with $\|\alpha_1\| + \cdots + \|\alpha_k\| \leq m$.

The main result in this section is the following.

Theorem 15. *Let $m \geq 1$. Let $\{Z_\alpha^{(1)} ; \alpha \in \mathcal{A}_0\}$ and $\{Z_\alpha^{(2)} ; \alpha \in \mathcal{A}_0\}$ are m-moment equivalent families of random variables such that $Z_{(0)}^{(1)} = Z_{(0)}^{(2)} = 1$. Let $Z^{(i)}(\varepsilon)$, $\varepsilon > 0$, be a $\overline{\mathcal{U}}^{\mathcal{L}}$-valued random variable given by $Z^{(i)}(\varepsilon) = \sum_{\alpha \in \mathcal{A}_0} \varepsilon^{\|\alpha\|} Z_\alpha^{(i)} v^{[\alpha]}$.*

Then for any $n \geq 1$, there is a constant $C > 0$ depending only on n and moments of $Z_\alpha^{(i)}$, $i = 1, 2$, $\alpha \in \mathcal{A}_0(n)$, such that

$$\sup_{x \in \mathbf{R}^N} \left| E[f(\exp(\varphi_n(Z^{(1)}(\varepsilon)))(x))] - E[f(\exp(\varphi_n(Z^{(2)}(\varepsilon)))(x))] \right|$$
$$\leq C \left(\sum_{k=m+1}^{n(m+1)} \varepsilon^k \|f\|_{V,k} + \varepsilon^{n+1} \|\nabla f\|_\infty \right), \quad \varepsilon \in (0, 1],\ f \in C_b^\infty(\mathbf{R}^N; \mathbf{R}).$$

To prove this theorem we need some preparations.

First we have the following combining Propositions 12 and 13.

Proposition 16. *Let $\{Z_\alpha ; \alpha \in \mathcal{A}_0\}$ is a family of random variables such that $Z_0 = 1$. Let $Z(\varepsilon) = \sum_{\alpha \in \mathcal{A}_0} \varepsilon^{\|\alpha\|} Z_\alpha^{(i)} v^{[\alpha]}$. Then for any $n \geq 1$ and $p \in [1, \infty)$, there is a constant $C > 0$ depending only on n, p, and moments of Z_α, $\alpha \in \mathcal{A}_0(n)$, such that*

$$\sup_{x \in \mathbf{R}^N} E\Bigg[\bigg| f(\exp(\varphi_n(Z(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)))$$
$$- f\bigg(\exp\bigg(\varphi_n \bigg(\sum_{\alpha \in \mathcal{A}_0} \varepsilon^{\|\alpha\|} q_\alpha^0 (Z_\beta,\ \beta \in \mathcal{A}_0) v^{[\alpha]} \bigg) \bigg)(x) \bigg) \bigg|^p \Bigg]^{1/p}$$
$$\leq C \sum_{\substack{\alpha \in \mathcal{A}_0 \\ n+1 \leq \|\alpha\| \leq 2n(n+1)}} \varepsilon^{\|\alpha\|} \|V_\alpha f\|_\infty, \quad \varepsilon \in (0, 1],\ f \in C_b^\infty(\mathbf{R}^N; \mathbf{R}).$$

Here polynomials q_α^0, $\alpha \in \mathcal{A}_1$, are as in Proposition 9.

As a corollary we have the following.

Corollary 17. *Let us assume the same as the previous proposition. Then for any $n \geq 1$ and $p \in [1, \infty)$, there is a constant $C > 0$*

depending only on n, p and moments of Z_α, $\alpha \in \mathcal{A}_0(n)$, such that

$$\sup_{x\in\mathbf{R}^N} E\Bigg[\Bigg|\exp(\varphi_n(Z(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)) - \exp\bigg(\varphi_n\Big(\sum_{\alpha\in\mathcal{A}_0} \varepsilon^{\|\alpha\|} q^0_\alpha(Z_\beta,\, \beta \in \mathcal{A}_0)v^{[\alpha]}\Big)(x)\bigg)\Bigg|^p\Bigg]^{1/p}$$

$$\leq C\varepsilon^{n+1} \sum_{\substack{\alpha\in\mathcal{A}_0 \\ n+1\leq\|\alpha\|\leq 2n(n+1)}} \|V_\alpha H\|_\infty, \quad \varepsilon \in (0,1].$$

Proof. Let $\psi \in C_b^\infty(\mathbf{R};\mathbf{R})$ such that $\psi(t) = t$, $|t| < 1$, and $0 \leq \psi'(t) \leq 1$, $t \in \mathbf{R}$. Let $f_{\ell,j} \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$, $\ell \geq 1$, $j = 1,\ldots,N$, be given by $f_{\ell,j}(x) = \ell\psi(\ell^{-1}x_j)$. Then we see that

$$\sup_{\ell\geq 1,\, j=1,\ldots,N} \|\nabla f_{\ell,j}\|_\infty < \infty,$$

and

$$\max_{j=1,\ldots,N} \|\nabla^k f_{\ell,j}\|_\infty \to 0, \ \ \ell \to \infty, \quad k \geq 2.$$

So we see that

$$\sup_{\ell\geq 1,\, j=1,\ldots,N} \|V_\alpha f_{\ell,j}\|_\infty < \infty, \quad \alpha \in \mathcal{A}_0.$$

Therefore applying the previous proposition for $f_{\ell,j}$ and letting $\ell \uparrow \infty$, we have our assetion. Q.E.D.

Similarly by using Proposition 12, we have the following.

Proposition 18. *Let us assume the same as the previous proposition. Then for any $n \geq 1$ and $p \in [1,\infty)$, there is a constant $C > 0$ depending only on n, p and moments of Z_α, $\alpha \in \mathcal{A}_0(n)$, such that*

$$\sup_{x\in\mathbf{R}^N} E[|\exp(\varphi_n(Z(\varepsilon))(x)) - (\varphi_n(\exp(Z(\varepsilon)))H)(x)|^p]^{1/p}$$

$$\leq C\varepsilon^{n+1} \sum_{\substack{\alpha\in\mathcal{A}_0 \\ n+1\leq\|\alpha\|\leq 2n(n+1)}} \|V_\alpha H\|_\infty, \quad \varepsilon \in (0,1].$$

Now let us prove Theorem 15. Note that

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^N}\left|E[f(\exp(\varphi_n(Z^{(1)}(\varepsilon)))(x))] - E[f(\exp(\varphi_n(Z^{(2)}(\varepsilon)))(x))]\right| \\
&\quad = \sup_{x\in\mathbf{R}^N}\Big|E[f(\exp(\varphi_n(Z^{(1)}(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)))] \\
&\qquad\qquad - E[f(\exp(\varphi_n(Z^{(2)}(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)))]\Big|
\end{aligned}$$

On the other hand, by Corollary 17, we have

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^N} E\Bigg[\Bigg|f(\exp(\varphi_n(Z^{(i)}(\varepsilon))(\exp(-\varepsilon^2 V_0)(x))) \\
&\qquad - f\Bigg(\exp\Bigg(\varphi_n\Big(\sum_{\alpha\in\mathcal{A}_0}\varepsilon^{\|\alpha\|}q^0_\alpha(Z^{(i)}_\beta,\,\beta\in\mathcal{A}_0)v^{[\alpha]}\Big)(x)\Bigg)\Bigg)\Bigg|\Bigg] \\
&\leq C\varepsilon^{n+1}\Bigg(\sum_{\substack{\alpha\in\mathcal{A}_0\\ n+1\leq\|\alpha\|\leq 2n(n+1)}}\|V_\alpha H\|_\infty\Bigg)\|\nabla f\|_\infty, \\
&\qquad\qquad\qquad\qquad \varepsilon\in(0,1],\ f\in C_b^\infty(\mathbf{R}^N;\mathbf{R}).
\end{aligned}$$

Also, since $q^0_{(0)}=0$, by Proposition 11 we have

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^N} E\Bigg[\Bigg|f\Bigg(\exp\Bigg(\varphi_n\Big(\sum_{\alpha\in\mathcal{A}_0}\varepsilon^{\|\alpha\|}q^0_\alpha(Z^{(i)}_\beta,\,\beta\in\mathcal{A}_0)v^{[\alpha]}\Big)(x)\Bigg)\Bigg) \\
&\qquad - E\Bigg[\Bigg(\varphi_n\Bigg(\exp\Big(\sum_{\alpha\in\mathcal{A}_0}\varepsilon^{\|\alpha\|}q^0_\alpha(Z^{(i)}_\beta,\,\beta\in\mathcal{A}_0)v^{[\alpha]}\Big)\Bigg)f\Bigg)(x)\Bigg|\Bigg] \\
&\leq C\varepsilon^{n+1}\Bigg(\sum_{k=n+1}^{n(n+1)}\|f\|_{V,k}\Bigg), \quad \varepsilon\in(0,1],\ f\in C_b^\infty(\mathbf{R}^N;\mathbf{R}).
\end{aligned}$$

Note that $\{q^0_\alpha(Z^{(i)}_\beta,\,\beta\in\mathcal{A}_0)\ ;\ \alpha\in\mathcal{A}_0\}$, $i=1,2$, are m-moment equivalent to each other, we see that

$$\begin{aligned}
&E\Bigg[\Bigg(\varphi_m\Bigg(\exp\Big(\sum_{\alpha\in\mathcal{A}_0}\varepsilon^{\|\alpha\|}q^0_\alpha(Z^{(1)}_\beta,\,\beta\in\mathcal{A}_0)v^{[\alpha]}\Big)\Bigg)f\Bigg)(x)\Bigg] \\
&\quad = E\Bigg[\Bigg(\varphi_m\Bigg(\exp\Big(\sum_{\alpha\in\mathcal{A}_0}\varepsilon^{\|\alpha\|}q^0_\alpha(Z^{(2)}_\beta,\,\beta\in\mathcal{A}_0)v^{[\alpha]}\Big)\Bigg)f\Bigg)(x)\Bigg].
\end{aligned}$$

Therefore we have our theorem.

§7. SDE

Let $X(t,x)$ be the solution of SDE (1). Also, let $\tilde{X}(t)$ be the solution to SDE (2) in $\overline{\mathcal{U}}$. Then we have the following.

Proposition 19. *For any $n \geq 1$, there is a constant C depending only on d and n such that*

$$\sup_{x \in \mathbf{R}^N} E\left[\left|f(X(t,x)) - (\varphi_n(\tilde{X}(t))f)(x)\right|^2\right]^{1/2}$$
$$\leq Ct^{(n+1)/2} \sum_{\substack{\alpha \in \mathcal{A} \\ \|\alpha\|=n+1,n+2}} \|V_\alpha f\|_\infty, \quad t \in (0,1],\ f \in C_b^\infty(\mathbf{R}^N;\mathbf{R}).$$

Proof. Note that

$$f(X(t,x)) = f(x) + \sum_{i=0}^d \int_0^t (V_i f)(X(t,x)) \circ dB^i(t).$$

So we have

$$f(X(t,x)) = \sum_{\alpha \in \mathcal{A}(n)} (V_\alpha f)(x) B^{\circ\alpha}(t) + R(t,x).$$

Here

$$R(t,x) = {\sum}' \int_0^t \circ dB^{\alpha^k}(s_k) \int_0^{s_k} \circ dB^{\alpha^{k-1}} \cdots$$
$$\cdots \int_0^{s_1} \circ dB^i(s_0)(V_i V_\alpha f)(X(s_0,x))$$

and $\sum'$ is the summation with respect to $\alpha = (\alpha^1, \ldots, \alpha^k) \in \mathcal{A}(n)$ and $i = 0, 1, \ldots, d$, with $\|(i) * \alpha\| \geq n+1$. Since

$$\int_0^t (V_i V_\alpha f)(X(s,x)) \circ dB^i(s)$$
$$= \int_0^t (V_i V_\alpha f)(X(s,x))\, dB^i(s) + (1 - \delta_{0,i})\frac{1}{2} \int_0^t (V_i^2 V_\alpha f)(X(s,x))\, ds.$$

we see that there is a constant $C(d,n)$ depending only on d and n such that

$$\sup_{x \in \mathbf{R}^N} E[|R(t,x)|^2]^{1/2}$$
$$\leq C(d,n)t^{(n+1)/2} \max\{\|V_\alpha f\|_\infty\ ;\ \alpha \in \mathcal{A},\ \|\alpha\| = n+1, n+2\}.$$

Since $X(t,\cdot):\mathbf{R}^N\to\mathbf{R}^N$ is a diffeomorphism, we can think of the push-forward $X(t)^*$. Then we have

$$X(t)^*=\text{Identity}+\sum_{i=0}^{d}\int_0^t X(s)^*V_i\circ dB^i(s)$$

as linear operators in $C^\infty(\mathbf{R}^N)$. So we have

$$\sum_{\alpha\in\mathcal{A}(n)}B^{\circ\alpha}(t)V_\alpha=\varphi_n(\tilde{X}(t)).$$

This proves our assrtion. Q.E.D.

Combining the previous proposition with Propositions 8 and 12, and applying the argument in Corollary 17, we have the following.

Proposition 20. *For any $n\geq 1$, there is a constant $C>0$ depending only on n and d such that*

$$\begin{aligned}\sup_{x\in\mathbf{R}^N}E\Bigg[\Bigg|&f(X(s,x))\\ &-f\Bigg(\exp\Bigg(\varphi_n\Bigg(\sum_{\alpha\in\mathcal{A}_0}s^{\|\alpha\|/2}p^0_\alpha(B^{\circ\beta}(1),\,\beta\in\mathcal{A}_0)v^{[\alpha]}\Bigg)\Bigg)(x)\Bigg)\Bigg|^2\Bigg]^{1/2}\\ \leq C\Bigg(&\sum_{\substack{\alpha\in\mathcal{A}\\ n+1\leq\|\alpha\|\leq n(n+2)}}s^{\|\alpha\|/2}\|V_\alpha f\|_\infty\Bigg),\\ &\qquad s\in(0,1],\ f\in C_b^\infty(\mathbf{R}^N;\mathbf{R}).\end{aligned}$$

In particular for any $n\geq 1$, there is a constant $C'>0$ depending only on n and d such that

$$\begin{aligned}\sup_{x\in\mathbf{R}^N}E\Bigg[\Bigg|&X(s,x)\\ &-\exp\Bigg(\varphi_n\Bigg(\sum_{\alpha\in\mathcal{A}_0}s^{\|\alpha\|/2}p^0_\alpha(B^{\circ\beta}(1),\,\beta\in\mathcal{A}_0)v^{\alpha}\Bigg)(x)\Bigg|^2\Bigg]^{1/2}\\ \leq C'&s^{(n+1)/2}\sum_{\substack{\alpha\in\mathcal{A}_0\\ n+1\leq\|\alpha\|\leq 2n(n+1)}}\|V_\alpha H\|_\infty,\quad s\in(0,1].\end{aligned}$$

Here p^0_α are polynomials in Proposition 8.

§8. Proof of Theorems

By Theorem 15 and Proposition 20, we have the following.

Theorem 21. *Let $m \geq 1$. Let $\{Z_\alpha\ ;\ \alpha \in \mathcal{A}_0\}$ be m-moment similar family of random variables. Then for any $n \geq 1$, there is a constant $C > 0$ depending only on n and moments of Z_α, $\alpha \in \mathcal{A}_0$ such that*

$$\begin{aligned}
\sup_{x\in\mathbf{R}^N} \Bigg| & E[f(X(s,x))] \\
& - E\Bigg[f\Bigg(\exp\Bigg(\varphi_n\Bigg(\sum_{\alpha\in\mathcal{A}_0} s^{\|\alpha\|/2} p^0_\alpha(Z_\beta,\ \beta\in\mathcal{A}_0) v^{[\alpha]}\Bigg)\Bigg)(x)\Bigg)\Bigg]\Bigg| \\
\leq C\Bigg(& \sum_{k=m+1}^{n(m+1)} s^{k/2}\|f\|_{V,k} \\
& + s^{(n+1)/2}\Bigg(\sum_{\substack{\alpha\in\mathcal{A}_0 \\ n+1\leq\|\alpha\|\leq 2n(n+1))}} \|V_\alpha H\|_\infty\Bigg)\|\nabla f\|_\infty\Bigg),
\end{aligned}$$

for $\varepsilon \in (0,1]$, $f \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$.

Now Theorem 4 is an easy consequence of Theorems 15, 21 and Proposition 18.

Now let us prove Theorem 5. By Theorem 4, Corollary 2 and the argument in [6], we have the following.

Proposition 22. *For any $a \geq 1$, there is a constant $C > 0$ such that*

$$\|P_{t+s}f - Q_{(s)}P_t f\|_\infty \leq \frac{Cs^{(m+1)/2}}{t^{m/2}}\|\nabla f\|_\infty$$

for any $s,t \in (0,a]$ and $f \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$ with $s \leq at$.

By this proposition, under the assumption in Theorem 5, we have

$$\begin{aligned}
&\|P_T f - Q_{(s_n)}Q_{(s_{n-1})}\cdots Q_{(s_1)}f\|_\infty \\
&\leq \sum_{k=1}^{n} \|Q_{(s_n)}\cdots Q_{(s_{k+1})}(P_{s_k} - Q_{(s_k)})P_{t_{k-1}}f\|_\infty \\
&\leq \sum_{k=2}^{n} \|P_{t_{k-1}+s_k}f - Q_{(s_k)}P_{t_{k-1}}f\|_\infty + \|P_{s_1}f - Q_{(s_1)}f\|_\infty.
\end{aligned}$$

It is easy to see that there is a constant $C > 0$ such that

$$\|P_s f - f\|_\infty \le C s^{1/2} \|\nabla f\|_\infty$$

and

$$\|Q_{(s)} f - f\|_\infty \le C s^{1/2} \|\nabla f\|_\infty$$

for any $s \in (0,1]$ and $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$. So we see that there is a constant $C > 0$ such that

$$\begin{aligned} &\|P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f\|_\infty \\ &\le C n^{-\gamma/2} \left(1 + \sum_{k=1}^{n-1} \frac{k^{(m+1)(\gamma-1)/2}}{k^{m\gamma/2}}\right) \|\nabla f\|_\infty \\ &= C n^{-\gamma/2} \left(1 + \sum_{k=1}^{n-1} k^{(\gamma-m-1)/2}\right) \|\nabla f\|_\infty. \end{aligned}$$

This implies our theorem.

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