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Sufficient Condition for Non-uniqueness of the Positive Cauchy Problem for Parabolic Equations

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Dedicated to Professor ShigeToshi Kuroda on the occasion of his 60th birthday

§1. Introduction

The purpose of this paper is to give a sufficient condition for non-uniqueness of non-negative solutions of the Cauchy problem

$$(1) \qquad (\partial_t - \Delta + V(x))u(x,t) = 0 \qquad \text{in} \quad R^n \times (0,\infty),$$
$$(2) \qquad u(x,0) = 0 \qquad \text{on} \quad R^n,$$

where V is a real-valued function in $L_{p,\mathrm{loc}}(R^n)$, $p > n/2$ for $n \geq 2$ and $p = 1$ for $n = 1$. We mean by a solution of (1)–(2) a function which belongs to

$$C^0(R^n \times [0,\infty)) \cap L_{2,\mathrm{loc}}([0,\infty); H^1_{\mathrm{loc}}(R^n_x))$$

and satisfies (1) and (2) in the weak sense and continuously, respectively (cf. [A]). We assume that

$$(3) \qquad |V(x) - W(|x|)| \leq C \qquad \text{on} \quad R^n$$

for some constant $C \geq 0$ and a measurable function W on $[0,\infty)$ with $\inf_{r\geq 0} W(r) > 0$. Our main result is the following

Theorem. *Suppose that*

$$(4) \qquad \int_1^\infty W(r)^{-1/2} dr < \infty.$$

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Then there exists a solution u of (1)–(2) *such that*

$$u(x,t) > 0 \qquad in \quad R^n \times (0,\infty). \tag{5}$$

The proof of this theorem is given in Section 2.

In [M1], among other things, we have shown that:

Under some additional conditions on W, nonnegative solutions of (1)–(2) *are not unique if and only if* (4) *holds.*

The aim of this paper is to establish a half of this result without the additional conditions on W.

§2. Proof

In this section we prove the Theorem. A main idea of the proof is to exploit a relative version (see Lemmas 3 $\sim$ 6 below) of methods developed in connection with non-conservation of probability (cf. [D] and [Kh]). The proof is divided into several lemmas.

First, without loss of generality, we may and will assume that $W \geq 1$.

Consider the initial value problem

$$-g'' - [(n-1)/r]g' + W(r)g = 0 \qquad \text{in} \quad (0,\infty), \tag{6}$$

$$g(r) = 1 + o(r^\alpha) \qquad \text{as} \quad r \to 0, \tag{7}$$

where $\alpha = 1$ for $n = 1$ and $\alpha = 0$ for $n > 1$. A solution of (6)–(7) means a function g in $C^0([0,\infty)) \cap C^1((0,\infty))$ such that its derivative g' is absolutely continuous on any compact subinterval of $(0,\infty)$, and g satisfies (6) and (7). Let us see that (6)–(7) has a unique solution when $n > 2$. (When $n = 2$, it can be shown similarly; and it is clear if $n = 1$.) Since $W \in L_{p,\mathrm{loc}}(R^n)$, $p > n/2$, we have by Hölder's inequality

$$r^{2-n}\int_0^r s^{n-1}W(s)ds \leq Cr^{2-n/p}\Big(\int_0^r W(s)^p s^{n-1}ds\Big)^{1/p} < \infty \tag{8}$$

for any $r > 0$, where C is a positive constant independent of r. Thus a solution g of (6)–(7) satisfies

$$\lim_{r\to 0} rg'(r) = 0, \tag{9}$$

$$g'(r) = \int_0^r (s/r)^{n-1}W(s)g(s)ds, \qquad r > 0. \tag{10}$$

Putting

$$K(r,s) = [(s^{2-n} - r^{2-n})/(n-2)]W(s)s^{n-1}, \tag{11}$$

we have

$$\begin{aligned} \int_0^r dt \int_0^t (s/t)^{n-1} W(s)ds &= \int_0^r K(r,s)ds \\ &\le Cr^{2-n/p}\Big(\int_0^r W(s)^p s^{n-1} ds\Big)^{1/p} < \infty \end{aligned} \tag{12}$$

for any $r > 0$, where C is a positive constant independent of r. Thus g satisfies the integral equation

$$g(r) = 1 + \int_0^r K(r,s)g(s)ds \tag{13}$$

on $[0,\infty)$. Conversely, a solution of (13) in $C^0([0,\infty))$ is also a solution of the initial value problem (6)–(7). Now, in view of (12), the iteration method shows that (13) has a unique solution on $[0,\delta]$ for a sufficiently small positive number δ. The obtained solution is also a unique solution of (6)–(7) with $(0,\infty)$ replaced by $(0,\delta)$. By extending it, we get a unique solution g of (6)–(7). Furthermore, we see that $g > 0$ and $g' > 0$ in $(0,\infty)$.

With $f(r) = r^{(n-1)/2}g(r)$ and $w(r) = W(r) + (n-1)(n-3)/4r^2$, we have

$$f'' = w(r)f \qquad \text{in} \quad (0,\infty), \tag{14}$$

$$f(r) = r^{(n-1)/2}[1 + o(r^\alpha)] \qquad \text{as} \quad r \to 0. \tag{15}$$

The following Lemmas 1 and 2 play a technically main part in removing the additional conditions on W mentioned in the Introduction.

Lemma 1. *$f, f' > 0$ in $(0,\infty)$, $\inf_{r>1} f'(r)/f(r) > 0$, and*

$$\int_1^\infty (f/f')dr < \infty. \tag{16}$$

Proof. We have only to show the second and third assertions. With $F = f'/f$, we have from (14)

$$F' + F^2 = w \tag{17}$$

Let $a(r)$ be the solution of the initial value problem

$$a'' = (1/4)a \quad \text{in} \quad (1,\infty), \qquad a(1) = f(1), \qquad a'(1) = f'(1).$$

With $A = a'/a$,

$$(F - A)' + (F + A)(F - A) = w - 1/4 \geq 0 \qquad \text{in} \quad (1,\infty),$$
$$(F - A)(1) = 0.$$

Thus $F \geq A$, and so $\inf_{r>1} F(r) > 0$. We next show (16) simplifying an argument in [KN, 4.2 and 4.3]. We claim that

$$1/F + (1/2)(1/F^2)' \leq 2/w^{1/2} \tag{18}$$

in $(1,\infty)$. By (17),

$$(1/w)(F'/F^2) + 1/w = 1/F^2.$$

If $F' \geq 0$, then $F \leq w^{1/2}$; and so

$$1/F = F[1/w + (1/w)(F'/F^2)] \leq 1/w^{1/2} + F'/F^3.$$

If $F' < 0$, then $1/F \leq 1/w^{1/2}$ and

$$(1/2)(1/F^2)' = -F'/F^3 = 1/F - w/F^3 < 1/w^{1/2}.$$

Thus we get (18). Hence

$$\int_1^R F^{-1} dr + \frac{1}{2}[F(R)^{-2} - F(1)^{-2}] \leq \int_1^R 2w^{-1/2} dr \leq \int_1^\infty 4W^{-1/2} dr.$$

This together with (4) implies (16). Q.E.D.

Let f_1 be the solution of (14)–(15) with w replaced by $w+1$. Then we have

Lemma 2. *The function f_1/f is increasing and* $0 < \lim_{r\to\infty} (f_1/f)(r) < \infty$.

Proof. With $v = f_1/f$, we have

$$f^{-2}(f^2 v')' = v \qquad \text{in} \quad (0,\infty), \tag{19}$$

$$v(r) = 1 + o(r^\alpha) \qquad \text{as} \quad r \to 0. \tag{20}$$

From (19)–(20) we get along the line in deriving (13) the equation

$$v(r) = 1 + \int_0^r [\int_s^r (f(s)/f(t))^2 dt] v(s) ds. \tag{21}$$

This implies that v is strictly increasing. Next, let us show the second assertion along the line given in [KN, 2.5]. With $u = \log(f_1/f)$ and $F = f'/f$, we have

$$u'' + (2F)u' + (u')^2 = 1. \tag{22}$$

This implies that $2u' \le 1/F - u''/F$. Thus, for any $R > 1$,

$$\begin{aligned} 2\int_1^R u' dr \le & \int_1^R (1/F) dr \\ & - u'(R)/F(R) + u'(1)/F(1) + \int_1^R (-F'/F^2) u' dr. \end{aligned}$$

Since $-F'/F^2 = 1 - w/F^2 < 1$ and $u' > 0$, we then have

$$2\int_1^R u' dr \le \int_1^R (1/F) dr + u'(1)/F(1) + \int_1^R u' dr.$$

Hence

$$u(R) \le \int_1^R (1/F) dr + u'(1)/F(1) + u(1).$$

This together with (16) implies that $\lim_{r\to\infty} f_1(r)/f(r) < \infty$.

Q.E.D.

Now put

$$H(x) = h(|x|) = (f_1/f)(|x|)[\lim_{s\to\infty} (f_1/f)(s)]^{-1}, \tag{23}$$

$$L = -g(|x|)^{-2} \sum_{j=1}^n (\partial/\partial x_j)(g(|x|)^2 \partial/\partial x_j), \tag{24}$$

where g is the solution of (6)–(7). Then we can easily obtain the following lemma.

Lemma 3. *H is a solution of the equation*

$$(L+1)H = 0 \qquad in \quad R^n \tag{25}$$

such that $0 < H < 1$ *and* $\lim_{|x|\to\infty} H(x) = 1$.

Let $G(x,y)$ be the minimal Green function for $(L+1, R^n)$ (cf. [M3]). Then we have

Lemma 4. $0 < \int_{R^n} G(x,y)dy \le 1 - H(x)$ *on* R^n.

Proof. Recall that $G = \lim_{R\to\infty} G_R$, where G_R is the Green function for $(L+1, B_R)$ with $B_R = \{x \in R^n; |x| < R)$. Put $U_R(x) = \int_{|y|<R} G_R(x,y)dy$. Then

$$(L+1)U_R = 1 \quad \text{in} \quad B_R, \qquad U_R = 0 \quad \text{on} \quad \partial B_R.$$

On the other hand,

$$(L+1)(1-H) = 1 \quad \text{in} \quad B_R, \qquad 1 - H > 0 \quad \text{on} \quad \partial B_R.$$

Thus the maximum principle shows that $U_R < 1 - H$ in B_R. But

$$\lim_{R\to\infty} U_R(x) = \int_{R^n} G(x,y)dy.$$

This proves the lemma. Q.E.D.

Since Lemma 4 implies that $[(L+1)^{-1}1](x) < 1$, we can now apply a criterion for non-conservation of probability (cf. [D, Lemma 2.1]), which goes back to Khas'minskii [Kh]. Let $K(x,y,t)$ be the smallest fundamental solution for $(\partial_t + L, R^n \times (0,\infty))$ (cf. [M1, M2]), and put

$$v(x,t) = \int_{R^n} K(x,y,t)dy. \tag{26}$$

Then we have

Lemma 5. $v(x,0) = 1$, *and*

$$(\partial_t + L)v = 0 \quad \text{and} \quad 0 < v < 1 \qquad \text{in} \quad R^n \times (0,\infty). \tag{27}$$

Proof. For self-containedness, we briefly show that $0 < v < 1$. The maximum principle for a parabolic equation on a cylinder together with the semigroup property of the smallest fundamental solution implies that either $v = 1$ or $0 < v < 1$ in $R^n \times (0,\infty)$. On the other hand, by Lemma 4,

$$\int_0^\infty e^{-t}v(x,t)dt = \int_{R^n} G(x,y)dy < 1 \qquad \text{on} \quad R^n.$$

Hence $0 < v < 1$. Q.E.D.

The final step of the proof is the following

Lemma 6. *There exists a solution u having the desired properties of the Theorem.*

Proof. With v being the function given by (26), put

$$w(x,t) = g(x)(1 - v(x,t)). \tag{28}$$

Then we see that $w(x,0) = 0$, and

$$(\partial_t - \Delta + W)w = 0 \quad \textit{and} \quad 0 < w(x,t) < g(x) \quad \text{in} \quad R^n \times (0,\infty). \tag{29}$$

For $R > 0$, let u_R be the solution of the mixed problem

$$(\partial_t - \Delta + V)u_R = 0 \text{ in } B_R \times (0,\infty), \ u_R = w \text{ on } \partial(B_R \times (0,\infty))$$

(cf. [A]). Since $W - C \leq V \leq W + C$ by (3), the comparison theorem shows that

$$e^{-Ct} \leq u_R(x,t)/w(x,t) \leq e^{Ct} \qquad \text{in} \quad B_R \times (0,\infty).$$

We see that for some sequence $R_j \to \infty$, u_{R_j} converges uniformly on each compact subset of $R^n \times [0,\infty)$ to a solution u of (1) satisfying

$$e^{-Ct} \leq u(x,t)/w(x,t) \leq e^{Ct} \qquad \text{in} \quad R^n \times (0,\infty). \tag{30}$$

This proves the lemma. Q.E.D.

Remark. We can also prove the Theorem by using Theorem 5.5 of [M1] after establishing Lemma 2; because Lemma 2 and (21) imply that

$$\int_1^\infty ds \int_s^\infty (s/t)^{n-1}(g(s)/g(t))^2 dt < \infty.$$

But the proof given in this paper is more direct than the one based on Theorem 5.5 of [M1].

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