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On a Backward Estimate for Solutions of Parabolic Differential Equations and its Application to Unique Continuation

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Abstract.

We prove a new backward estimate and a new strong unique continuation property for solutions $u \in \mathcal{C} = C^o((0,T);\ H^2(\mathbf{R}^n;\ e^{-\alpha|x|^2}\,dx)) \cap C^1((0,T);\ L^2(\mathbf{R}^n;\ e^{-\alpha|x|^2}\,dx))$ of parabolic differential equations $\frac{\partial u}{\partial t} = \Delta u + V(x,t)u$ under certain conditions on V, where $\alpha > 0$ is a fixed number.

§1. Main results

We consider the following parabolic differential equation:

$$\frac{\partial u}{\partial t} = \Delta u + V(x,t)u \quad in \quad \mathbf{R}^n \times (0,T), \tag{1.1}$$

where V is real-valued, $T > 0$, and $n \geq 3$. Let $\alpha > 0$ be a fixed number and let $w(x) = e^{-\alpha|x|^2}$. We denote by $L^2(\mathbf{R}^n; w(x)dx)$ the closure of $C_o^\infty(\mathbf{R}^n)$ under the norm $\|u\|_{L^2(w)} = (\int_{\mathbf{R}^n} |u(x)|^2 w(x)\,dx)^{1/2}$. We also denote by $H^2(\mathbf{R}^n; w(x)dx)$ the closure of $C_o^\infty(\mathbf{R}^n)$ under the norm $\|u\|_{H^2(w)} = \left(\sum_{0\leq|\beta|\leq 2} \|D^\beta u\|_{L^2(w)}^2\right)^{1/2}$, where $D^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, $\partial_j = \frac{\partial}{\partial x_j}$, $|\beta| = \sum_{j=1}^n \beta_j$ for $\beta = (\beta_1, \cdots, \beta_n)$. Put $\mathcal{C} = C^o((0,T); H^2(\mathbf{R}^n; w(x)\,dx)) \cap C^1((0,T); L^2(\mathbf{R}^n; w(x)\,dx))$. We say $u \in \mathcal{C}$ is a solution of (1.1) if u satisfies (1.1) in $L^2(\mathbf{R}^n; w(x)dx)$ for each $t \in (0,T)$.

For a point $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$ and $0 < R < \sqrt{t_o}$, we set $S_R(t_o) = \{z = (x,t) \in \mathbf{R}^n \times (0,T) \mid t = t_o - R^2\}$. By using the backward heat kernel $G_{z_o}(z) = \dfrac{1}{(4\pi(t_o - t))^{n/2}} \exp(-\dfrac{|x-x_o|^2}{4(t_o-t)})$ which is defined

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for $t < t_o$, we define the weighted L^2 norm $H_{z_o}(R;u)$ and the weighted energy $I_{z_o}(R;u)$ over $S_R(t_o)$ as follows:

$$H_{z_o}(R;u) = \frac{1}{2}\int_{S_R(t_o)} u^2 G_{z_o}\,dx,$$

$$I_{z_o}(R;u) = \frac{1}{2}R^2\int_{S_R(t_o)} (|\nabla u|^2 - Vu^2)G_{z_o}\,dx.$$

Under certain assumptions on V we shall study the behaviour of $H_{z_o}(R;u)$ and $I_{z_o}(R;u)$ as $R \to 0$ and prove a 'monotonicity formula' for the weighted energy $I_{z_o}(R;u)$ (Lemma 3.1) and a doubling property for $H_{z_o}(R;u)$ (Theorem 1.3).

To state our assumptions on V, we first recall the definitions of the Fefferman-Phong class F_t and the Kato class K_n. $V \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is said to be of the Kato class K_n if

$$\lim_{r\to 0}\eta^K(r;V) = 0, \quad \eta^K(r;V) = \sup_{x\in\mathbf{R}^n}\int_{B_r(x)} \frac{|V(y)|}{|x-y|^{n-2}}\,dy,$$

where $B_r(x) = \{y \in \mathbf{R}^n \| x-y| < r\}$ for $r > 0$. For $1 \le t \le n/2$, $V \in L^t_{\mathrm{loc}}(\mathbf{R}^n)$ is said to be of the Fefferman and Phong class F_t if

$$||V||_{F_t} = \sup_{x\in\mathbf{R}^n, r>0} r^2\Big(\frac{1}{|B_r(x)|}\int_{B_r(x)} |V|^t\,dy\Big)^{1/t} < +\infty.$$

We note that $F_{n/2} = L^{n/2}(\mathbf{R}^n) \subset F_t \subset F_s$ for $1 \le s \le t \le n/2$ and weak-$L^{n/2}(\mathbf{R}^n) \subset F_t$ for every $t \in [1, n/2)$; $V \in K_n$ implies $V \in F_1$; but $L^{n/2}(\mathbf{R}^n)$ and K_n are incomparable for $n \ge 3$.

For $1 < t \le n/2$, we define the function space Q_t by $Q_t = \{V = V_1 + V_2;\ V_1 \in K_n,\ V_2 \in F_t\}$ and for $V \in Q_t$ set

$$\|V\|_{Q_t} = \|V\|_{Q_t}^{R_o} = \inf_{V=V_1+V_2\in Q_t}\{\eta^K(R_o;V_1) + \|V_2\|_{F_t}\} \tag{1.2}$$

for $R_o > 0$. Throughout this paper we fix $R_o > 0$.

Definition 1.1. For $1 \le t \le n/2, p \ge 1$, we say V belongs to the class $Q_{t,p}(0,T)$, if V satisfies
(1) for each $t_o \in (0,T)$, there exist positive functions $W, U \ge 0$ and a compact set $K \subset \mathbf{R}^n$ such that

$$|V(x, t_o - s)| \le W(x,s) + U(s), \quad \mathrm{supp}_x W(\cdot, s) \subset K \tag{1.3}$$

for every $s \in (0, t_o)$,
(2) $|W(\cdot, s)|^p \in Q_t$ for every $s \in (0, t_o)$.

Now we state our assumptions for V.

Assumption (A). V satisfies the following conditions for $1 < t \leq n/2, p > 2$.
(i) $V \in Q_{t,p}(0,T)$ and $\tilde{V} = 2V + (x - x_o) \cdot \nabla_x V + 2(t - t_o)\partial_t V \in Q_{t,p}(0,T)$;
(ii) for the expression $|V| \leq W_1 + U_1$ and $|\tilde{V}| \leq W_2 + U_2$, put $f_j^{(t_o)}(s) = s^{2-4/p} \, \| |W_j(s^2)|^p \|_{Q_t}^{1/p} + s^2 U_j(s^2), j = 1, 2$. Then there exists $s_1 > 0$ such that

$$f_1^{(t_o)}(s) \to 0 \ (s \to 0), \quad \int_0^{s_1} \frac{f_2^{(t_o)}(s)}{s} \, ds < +\infty \tag{1.4}$$

for every $t_o \in (0,T)$.

Example 1.2. (1) If $V \in C^1(\mathbf{R}^n \times (0,T))$ and $V, (1 + |x|)|\nabla V|, |\partial_t V| \in L^\infty(\mathbf{R}^n \times (0,T))$ and have compact support for each $t \in (0,T)$, then V satisfies Assumption (A).
(2) Let $V(x,t) = V(x)$ be independent of time variable. If $|V|^p$ and $|\tilde{V}|^p$, $\tilde{V} = 2V + (x - x_o) \cdot \nabla_x V$, belong to the class Q_t for some $1 < t \leq n/2$ and $p > 2$ and have compact support, then V satisfies Assumption (A).

We state our main results.

Theorem 1.3 (Backward Estimate). *Suppose Assumption* (A). *Let $u \in \mathcal{C}$ be a solution of* (1.1). *Then for $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$, there exist constants R^* and $C_o > 0$ such that*

$$\int_{S_{2R}(t_o)} u^2 G_{z_o} \, dx \leq C_o \int_{S_R(t_o)} u^2 G_{z_o} \, dx \tag{1.5}$$

for every $0 < R < R^* (< \sqrt{t_o})$. Here C_o is a constant independent of R.

Theorem 1.3 implies

Theorem 1.4 (Unique Continuation). *Suppose Assumption* (A). *Let $u \in \mathcal{C}$ be a solution of* (1.1) *and let $0 \leq \gamma < 1$. If u satisfies, for some $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$ and for arbitraly $N > 0$,*

$$\int_{S_R(t_o) \cap \{|x - x_o| < R^\gamma\}} u^2 G_{z_o} \, dx = O(R^N) \quad as \quad R \to 0, \tag{1.6}$$

then $u(x,t) \equiv 0$ on $\mathbf{R}^n \times (t_o - (R^*)^2, t_o)$, where $R^* > 0$ is the number given in Theorem 1.3.

As a corollary of the proof of Theorem 1.3, we obtain backward uniqueness, if we assume

Assumption (A′). In addition to Assumption (A), V satisfies that the compact set K (associated with the definition $V \in Q_{t,p}(0,T)$) can be taken uniformly in $t_o \in (0,T)$ and $F_1(s) \equiv \sup_{t_o \in (0,T)} f_1^{(t_o)}(s) \to 0$ as $s \to 0$, where $f_1^{(t_o)}(s)$ is the function defined in Assumption (A) (ii).

Corollary 1.5 (Backward Uniqueness). *Suppose Assumption* (A′). *If the solution* $u \in \mathcal{C}$ *of* (1.1) *satisfies* $u(\cdot, t_o) \equiv 0$ *for some* $t_o \in (0,T)$, *then* $u(\cdot, t) \equiv 0$ *for every* $t \in (0, t_o)$.

We note that if the assumption (A′) is satisfied, then we can take $R^* = \min(1/\sqrt{8\alpha}, \sqrt{t_o}, R_*)$, R_* is independent of t_o. By this observation Corollary 1.5 follows easily (cf. [GL], [Ku]). As a direct consequence of Theorem 1.4 we have

Corollary 1.6 (Weak UCP). *Suppose the Assumption* (A). *If the solution* $u \in \mathcal{C}$ *of* (1.1) *vanishes in some open set* $\omega \subset \mathbf{R}^n \times (0.T)$, *then* u *vanishes in the horizontal component of* ω *in* $\mathbf{R}^n \times (0.T)$.

There are several results on backward uniqueness and unique continuation theorems (see e.g., [L], [M], [So], [SS], [LP]), but Theorem 1.3 is new even in the case $V \equiv 0$, and Theorem 1.4 yields the different type of strong unique continuation property for solutions of (1.1). Moreover, the method of this paper is different from the previous works. This work is a parabolic version of [Ku].

If $\Omega \subset \mathbf{R}^n$ is bounded, smooth, and convex, we can show the same results for solutions u of

$$(1.7)\quad \frac{\partial u}{\partial t} = \Delta u + V(x,t)u \quad in \quad \Omega \times (0,T), \quad u = 0 \quad on \quad \partial\Omega \times (0,T).$$

Recently we also proved similar results for weak solutions. However, we do not know whether the backward estimate of type (1.5) also holds or not for u satisfying (1.7) locally (that is, without boundary condition).

§2. Preliminaries

In this section we show an inequality which controls singularities of V in the proof of Theorems. Let $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$ and put

$$\Phi_{z_o}(R;u) = \frac{1}{2}R^2 \int_{S_R(t_o)} |\nabla u|^2 G_{z_o}\, dx, \quad N_{z_o}(R;u) = \frac{I_{z_o}(R;u)}{H_{z_o}(R;u)}.$$

Then we have

Lemma 2.1. *Suppose $V \in Q_{t,p}(0,T)$ with $1 < t \le n/2, p \ge 1$. Then there exists a constant $C > 0$ such that*

$$\int_{S_R(t_o)} |V|u^2 G_{z_o}\, dx \le U(R^2)H_{z_o}(R;u) + CR^{-4/p}\||W(R^2)|^p\|_{Q_t}^{1/p}(\Phi_{z_o}(R;u) + H_{z_o}(R;u)) \tag{2.1}$$

for every $\sqrt{t_o} > R > 0$ and $u \in C^o((0,T); C_o^\infty(\mathbf{R}^n))$, where U and W are functions associated with V by Definition 1.1.

As an easy consequence of Lemma 2.1 we have

Lemma 2.2. *Suppose that $V \in Q_{t,p}(0,T)$ with $1 < t \le n/2, p > 2$ and that $f^{(t_o)}(s) \equiv s^2U(s^2) + s^{2-4/p}\||W(s^2)|^p\|_{Q_t}^{1/p} \to 0$ as $s \to 0$. Then there exist $C > 0$ and sufficiently small R_* such that*

$$C^{-1}\Phi_{z_o}(R : u) \le I_{z_o}(R : u) \le C\Phi_{z_o}(R; u) \tag{2.2}$$

for every $0 < R < R_*$ satisfying $N_{z_o}(R; u) > 1$.

To prove Lemma 2.1, first we note that if $V \in K_n$,

$$\int_{\mathbf{R}^n} |V|u^2 \le C(n)\eta^K(r; V)(\int_{\mathbf{R}^n} |\nabla u|^2\, dx + \frac{1}{r^2}\int_{\mathbf{R}^n} u^2\, dx)$$

for every $r > 0$ and $u \in C_o^\infty(\mathbf{R}^n)$, and that if $V \in F_t$ with $1 < t \le n/2$,

$$\int_{\mathbf{R}^n} |V|u^2 \le C(n,t)\|V\|_{F_t} \int_{\mathbf{R}^n} |\nabla u|^2\, dx$$

for every $u \in C_o^\infty(\mathbf{R}^n)$ (see e.g. [F], [Si]). Hence if $V \in Q_t$ with $1 < t \le n/2$, we have

$$\int_{\mathbf{R}^n} |V|u^2 \le C(n,t,R_o)\|V\|_{Q_t}(\int_{\mathbf{R}^n} |\nabla u|^2\, dx + \int_{\mathbf{R}^n} u^2\, dx) \tag{2.3}$$

for every $u \in C_o^\infty(\mathbf{R}^n)$, where $R_o > 0$ is a fixed constant.

Proof of Lemma 2.1. Let $t_o \in (0,T)$. We use the notaion $S_R = S_R(t_o)$ and $G = G_{z_o}$ for the sake of simplicity. Since $V \in Q_{t,p}(0,T)$, by the definition there exist $W, U \ge 0$ and a compact set $K \subset \mathbf{R}^n$ such that $|V(x, t_o - R^2)| \le W(x, R^2) + U(R^2)$ with $\mathrm{supp}_x W(\cdot, R^2) \subset K$ for every

$0 < R^2 < t_o$. Let $\eta \in C_o^\infty(\mathbf{R}^n)$ satisfy $\eta(x) \equiv 1$ on K, $0 \le \eta(x) \le 1$, and $|\nabla\eta(x)| \le C$. By Hölder's inequality, we have

$$\begin{aligned}
&\int_{S_R} |W(x,R^2)|u^2 G\,dx \\
&\qquad \le \Big(\int_{S_R} |W(x,R^2)|^p u^2 G\,dx\Big)^{1/p}\Big(\int_{S_R} u^2 G\,dx\Big)^{1/q} \\
&\qquad = (2H(R;u))^{1/q}\Big(\int_{S_R} |W(x,R^2)|^p u^2 G\,dx\Big)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$. The inequality (2.3) yields

$$\begin{aligned}
&\int_{S_R} |W|^p u^2 G\,dx \\
&\qquad \le \int_{S_R} |W|^p (\eta u)^2 G\,dx \qquad (2.4)\\
&\qquad \le C(n,t)\||W|^p\|_{Q_t}\Big(\int_{S_R} |\nabla(\eta u G^{1/2})|^2\,dx + \int_{S_R} u^2 G\,dx\Big).
\end{aligned}$$

Since $|\nabla(\eta u G^{1/2})|^2 \le C(n)(u^2 + |\nabla u|^2)G + C(n,K)\frac{u^2}{R^4}G$, we obtain

$$\begin{aligned}
&\int_{S_R} |W|^p u^2 G\,dx \qquad (2.5)\\
&\quad \le C(n,t,K)\||W|^p\|_{Q_t}\Big(\big(1+\frac{1}{R^4}\big)H(R;u) + \int_{S_R} |\nabla u|^2 G\,dx\Big).
\end{aligned}$$

Hence it follows that

(2.6)

$$\begin{aligned}
&\int_{S_R} |V|u^2 G\,dx \\
&\qquad \le \int_{S_R} |W|u^2 G\,dx + U(R^2)\int_{S_R} u^2 G\,dx \\
&\qquad \le 2U(R^2)H(R;u) \\
&\qquad\qquad + C(n,t,p)\||W|^p\|_{Q_t}^{1/p} H(R;u)^{1/q}\left(\frac{H(R;u)}{R^4}+\frac{\Phi(R;u)}{R^2}\right)^{1/p} \\
&\qquad \le 2U(R^2)H(R;u) \\
&\qquad\qquad + C(n,t,p)R^{-4/p}\||W|^p\|_{Q_t}^{1/p}(H(R;u)+\Phi(R;u)).
\end{aligned}$$

Q.E.D.

Proof of Lemma 2.2. Note that $I(R;u) = \Phi(R;u) - \Psi(R;u)$ and that $N(R;u) > 1$ implies $H(R;u) < I(R;u)$ by definition. By Lemma 2.1 we have, for $R > 0$ satisfying $N(R;u) > 1$,

$$|\Psi(R;u)| \le Cf^{(t_o)}(R)\Phi(R;u). \tag{2.7}$$

Hence, by the assumption $f^{(t_o)}(s) \to 0$ as $s \to 0$, there exists $R_* > 0$ such that $Cf^{(t_o)}(R) < 1/2$ for every $0 < R < R_*$. Hence we obtain the desired estimate. Q.E.D.

§3. Proof of theorems

In this section we prove theorems. Suppose that V satisfies Assumption (A) and $u \in \mathcal{C}$ is a solution of (1.1) throughout this section. Without loss of generality, we may assume $z_o = (O,0)$ and consider (1.1) for $t < 0$. We write $S_R = S_R(0) = \{(x,t)|t = -R^2\}$, $G_o = G_{(O,0)}$, $H(R) = H_{(O,0)}(R;u)$, $I(R) = I_{(O,0)}(R;u)$ and $N(R) = N_{(O,0)}(R;u)$, and use the notation $P(u) = x\cdot\nabla u + 2t\partial_t u$. Let $R^* = \min(1/\sqrt{8\alpha}, R_*)$, where R_* is the number determined by Lemma 2.2 with respect to $t_o = 0$. Then we have

Lemma 3.1. *For* $0 < R < R^*$, $I(R)$ *is differentiable and satisfies*

$$\begin{aligned}
I'(R) &= \frac{1}{2R}\int_{S_R} P(u)^2 G_o\,dx \\
&\qquad - \frac{R}{2}\int_{S_R}(2V + x\cdot\nabla V + 2t\partial_t V)u^2 G_o\,dx.
\end{aligned}\tag{3.1}$$

If $u(x,t) \not\equiv 0$ on $\mathbf{R}^n \times (-(R^*)^2, 0)$, then it follows that $H(R) > 0$ for every $0 < R < R^*$. We note that this fact can be proved by the similar argument as in [GL;p264] (see also the proof of Theorem 1.5 in [Ku]). Therefore we may assume that $H(R) > 0$ for every $0 < R < R^*$, and hence $N(R)$ is also differentiable on $(0, R^*)$. Let $\tilde{V} = 2V + x\cdot\nabla V + 2t\partial_t V$ have the expression $|\tilde{V}| \le W_2 + U_2$ by the assumption $\tilde{V} \in Q_{t,p}(0,T)$. Then by Lemma 3.1 we obtain the following differential inequality for $N(R)$.

Lemma 3.2. *There exists $C > 0$ such that*

(3.2)
$$\frac{N'(R)}{N(R)} \ge -C(\frac{R^2U_2(R^2) + R^{2-4/p}\||W_2(R^2)|^p\|_{Q_t}^{1/p}}{R}) \equiv -C\frac{f_2^{(0)}(R)}{R}$$

for $0 < R < R^*$ satisfying $N(R) > 1$.

Proof of Lemma 3.1. We follow the computation of Struwe [St]. Let $u_R(x,t) = u(Rx, R^2t)$. Then we have $\Phi(R;u) = \Phi(1, u_R)$. If u is a solution of $\dfrac{\partial u}{\partial t} = \Delta u + V(x,t)u$, then u_R is a solution of $\dfrac{\partial u_R}{\partial t} = \Delta u_R + V_R(x,t)u_R$, where $V_R(x,t) = R^2V(Rx, R^2t)$. By noting $\nabla G_o = -(x/2R^2)G_o$ on S_R, we obtain

$$\begin{aligned}
\Phi'(R;u) &= \frac{d\Phi(1;u_R)}{dR} \\
&= \int_{S_1} \nabla u_R \cdot \nabla(\frac{du_R}{dR})G_o\, dx \\
&= -\int_{S_1} (\Delta u_R G_o + \nabla u_R \cdot \nabla G_o)\frac{du_R}{dR}\, dx \qquad (3.3)\\
&= \int_{S_R} \frac{P(u)}{R}(\frac{P(u)}{2} + R^2Vu)G_o\, dx \\
&= \frac{1}{2R}\int_{S_R} P(u)^2G_o\, dx + R\int_{S_R} P(u)VuG_o\, dx.
\end{aligned}$$

On the other hand, since $\Psi(R;u) = \frac{1}{2}\int_{S_R} Vu^2G_o\, dx = \frac{1}{2}\int_{S_1} V_Ru_R^2G_o\, dx$, we have

$$\begin{aligned}
\Psi'(R;u) &= \frac{1}{2}\int_{S_1} \frac{dV_R}{dR} u_R^2 G_o\,dx + \int_{S_1} V_R \frac{du_R}{dR} u_R G_o\,dx \\
&= \frac{R}{2}\int_{S_R} (2V + x\cdot\nabla V + 2t\partial_t V)u^2 G_o\,dx \\
&\quad + R\int_{S_R} P(u) V u G_o\,dx.
\end{aligned} \tag{3.4}$$

Combining (3.3) with (3.4) we complete the proof. Q.E.D.

Proof of Lemma 3.2. Since $H(R;u) = H(1;u_R)$, we have

$$H'(R) = H'(R;u) = \int_{S_1} u_R \frac{du_R}{dR} G_o\,dx = \frac{1}{R}\int_{S_R} uP(u) G_o\,dx. \tag{3.5}$$

On the other hand, multiplying uG_o to (1.1) and integrating over S_R, we obtain

$$\int_{S_R} u\partial_t u G_o\,dx = -\int_{S_R} |\nabla u|^2 G_o\,dx - \int_{S_R} u\nabla u\cdot\nabla G_o\,dx + \int_{S_R} V u^2 G_o\,dx.$$

Since $\nabla G_o = \frac{x}{2t} G_o$ on S_R, this implies

$$I(R) = \frac{1}{4}\int_{S_R} P(u) u G_o\,dx. \tag{3.6}$$

Hence we obtain $H'(R) = \frac{4}{R} I(R)$. Therefore, for $0 < R < R^*$, (3.1) and (3.7) yield

$$\begin{aligned}
\frac{N'(R)}{N(R)} &= \frac{I'(R)}{I(R)} - \frac{H'(R)}{H(R)} \\
&= \frac{\int_{S_R} P(u)^2 G_o\,dx}{2RI(R)} - \frac{4I(R)}{RH(R)} \\
&\quad - \frac{R}{2I(R)}\int_{S_R} (2V + x\cdot\nabla V + 2t\partial_t V)u^2 G_o\,dx.
\end{aligned} \tag{3.7}$$

By Schwarz's inequality,

$$\begin{aligned}
&\frac{\int_{S_R} P(u)^2 G_o\,dx}{2RI(R)} - \frac{4I(R)}{RH(R)} \\
&\quad = \frac{\int_{S_R} P(u)^2 G_o\,dx}{\frac{R}{2}\int_{S_R} P(u) u G_o\,dx} - \frac{\int_{S_R} P(u) u G_o\,dx}{\frac{R}{2}\int_{S_R} u^2 G_o\,dx} \geq 0.
\end{aligned} \tag{3.8}$$

Thus we arrive at

$$\frac{N'(R)}{N(R)} \geq -\frac{R}{2I(R)} \int_{S_R} (2V + x \cdot \nabla V + 2t\partial_t V) u^2 G_o \, dx \tag{3.9}$$

for $0 < R < R^*$. By Lemmas 2.1 and 2.2 we can conclude the desired esimate. Q.E.D.

Proof of Theorem 1.3. Note that the set $\{0 < R < R^* : N(R) > 1\}$ is open, because $N(R)$ is continuous. Hence there exist countable open disjoint intervals (R_j, R_{j+1}) such that $\{0 < R < R^* : N(R) > 1\} = \cup_{j=1}^{\infty}(R_j, R_{j+1})$. By Assumption (A) and Lemma 3.2, we have

$$\log(\frac{N(R_{j+1})}{N(R_j)}) \geq -C \int_0^{R^*} \frac{f_2(s)}{s} \, ds$$

for each $j = 1, 2, \cdots$. This implies

$$N(R) \leq \max(1, N(R^*)) \exp(-C \int_0^{R^*} \frac{f_2(s)}{s} \, ds) (\equiv N_o) \tag{3.10}$$

for $0 < R < R^*$. Since $H'(R) = (4/R)I(R)$, we obtain

$$H(2R) \leq H(R) \exp(4N_o \log 2), \ 0 < R < R^*. \tag{3.11}$$

This complete the proof of Theorem 1.3. Q.E.D.

Proof of Theorem 1.4. It is well-known that when the doubling estimate (1.5) in Theorem 1.3 holds, the condition that $H(R) = O(R^N)$ for every $N > 0$ as $R \to 0$ implies $H(R) \equiv 0$ for every $R \in (0, R^*)$ (see e.g.,[GL]). Hence it suffices to show $H(R) = O(R^N)$ for every $N > 0$. Let $0 \leq \gamma < 1$ and put

$$g(R) = \int_{S_R(t_o) \cap \{x; |x - x_o| \geq R^\gamma\}} u^2 G_{z_o} \, dx.$$

Then it is easy to see that there exists a constant M such that

$$g(R) \leq \frac{M}{R^n} \exp(-\frac{1}{8R^{2(1-\gamma)}}).$$

Actually we can take

$$M = \sup_{t \in [t_o - (R^*)^2, t_o]} \int_{\mathbf{R}^n} u^2(x,t) e^{-\frac{|x - x_o|^2}{8(R^*)^2}} \, dx < +\infty,$$

since $R^* \leq 1/\sqrt{8\alpha}$. Hence $g(R) = O(R^N)$ for every $N > 0$. By the assumption $f(R) = \int_{S_R(t_o)\cap\{x;|x-x_o|<R^\gamma\}} u^2 G_{z_o}\, dx = O(R^N)$, we can conclude that $H(R) = O(R^N)$ for every $N > 0$. Thus we complete the proof. Q.E.D.

References

[F] C.L. Fefferman, The uncertainty principle, Bull.A.M.S.(N.S.), **9** (1983), 129–206.

[GL] N. Garofalo and F.H. Lin, monotonicity properties of variational integrals. A_p weights and unique continuation, Indiana Univ. Math. J., **35** (1986), 245–268.

[Ku] K. Kurata, On unique continuation theorem for uniformly elliptic equations with strongly singular potentials, Gakushuin University Preprint Series No.2 (1991).

[LP] M. Lee and M.H. Protter, Unique Continuation for Parabolic Differential Equations and Inequalities, Duke Math. J., **28** (1961), 369–382.

[L] F.H. Lin, A uniqueness theorem for parabolic equations, Comm.Pure Appl.Math., **48** (1990), 127–136.

[M] K. Masuda, A note on the analyticity in time and the unique continuation property for solutions of diffusion equations, Proc. Japan Acad., **43** (1967), 420–422.

[SS] J.C. Saut and B. Scheurer, Unique Continuation for Some Evolution Equations, J. of Diff.Eq., **66** (1987), 118–139.

[Si] B. Simon, Schrödinger semigroups, Bull.Amer.Math.Soc., **7** (1982), 447–526.

[So] C.D. Sogge, A unique continuation theorem for second order parabolic differential operators, Ark. Mat., **28** (1990), 159–182.

[St] M.Struwe, On the evolution of harmonic maps in heigher dimensions, J. Diff.Geom., **28** (1988), 485–502.

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