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Einstein-Kähler Metrics on Minimal Varieties of General Type and an Inequality between Chern Numbers

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In this paper, we shall give a complete proof of the results which were announced in Part I in this volume. For earlier results on Einstein-Kähler metrics and tangent sheaves of minimal varieties, see the introduction of [Sg].

The contents of this paper are as follows. In Section 1 through 4, solving a degenerate Monge-Ampère equation, we shall construct a family of Einstein-Kähler metrics on the smooth part of minimal varieties of general type. In Section 5, we shall show a subsequence of this family of Einstein-Kähler metrics converges to an Einstein-Kähler metric, whose cohomology class corresponds properly to a negative constant multiple of the first Chern class of the variety. In Section 6, an inequality between Chern numbers for minimal varieties, so called Bogomolov-Gieseker type inequality, will be proved. In Sections 7 and 8, using the metric constructed in Section 5, we shall obtain a sufficient condition for the tangent sheave of certain varieties to be stable.

After the completion of this work, we were informed that S. Bando and R. Kobayashi obtained, simultaniously with ours, the same result as Theorem 5.6 by a heat equation method which is different from ours. We added Theorem 5.8 according to their suggestions via correspondences. I wish to express my hearty gratitude to them.

§1. A degenerate Monge-Ampère equation

Let M be an n dimensional compact projective algebraic manifold and

$$E = \{E_i\}_{i=1}^{N}$$

effective divisors on M. Assume that the following condition is satisfied.

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Condition 1.1. *There exist positive numbers a_i $(1 \leq i \leq N)$ so that $K_M - \sum_{i=1}^N a_i E_i$ is ample.*

Let $\mathbf{h}_i$ be a Hermitian fibre metric of the line bundle $[E_i]$. Then, from the condition 1.1, a C^∞ d-closed real (1.1)-form $\omega_{\mathbf{R}} \in 2\pi c_1(K_M)$ exists on M such that

$$\omega_{\mathbf{R}} - \sum_{i=1}^{N} a_i \sqrt{-1}\partial\bar{\partial}\log \mathbf{h}_i$$

is positive definite everywhere on M. From [Y-2], there exists a Kähler metric G on M such that $\sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} = \omega_{\mathbf{R}}$ and therefore

$$\omega_{\mathbf{a}} := \sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} - \textstyle\sum_{i=1}^N a_i\sqrt{-1}\partial\bar{\partial}\log\det\mathbf{h}_i$$

is positive definite everywhere on M. Let s_i be the non-zero holomorphic section of $[E_i]$ whose the divisor is just E_i. Now we consider the equation:

$$\det(\mathbf{G}(a)_{i\bar{j}} + u_{:i\bar{j}}) = \prod_{i=1}^{N} \| s_i \|_{\mathbf{h}_i}^{2a_i} e^u \det\mathbf{G} \tag{1.2}$$

where $\omega_{\mathbf{a}} = \sqrt{-1}\sum_{ij} G(a)_{i\bar{j}} dz_i \wedge d\bar{z}_j$ and where $u_{:i\bar{j}} := \partial_i\bar{\partial}_j u$. Let ϕ be a C^∞ solution of (1.2) on $M \setminus \cup_i E_i$. Then it is easy to see that $\tilde{\omega}_{\mathbf{a}} := \omega_{\mathbf{a}} + \sqrt{-1}\partial\bar{\partial}\phi$ is an Einstein-Kähler metric on $M \setminus \cup_i E_i$. In order to solve the equation (1.2), we consider the following perturbed equation:

$$\det(\mathbf{G}(a)_{i\bar{j}} + u_{:i\bar{j}}) = \prod_{i=1}^{N} (\| s_i \|_{\mathbf{h}_i}^{2} + \delta)^{a_i} e^u \det\mathbf{G} \tag{1.3}$$

It is known by Yau (for instance, see [Y-2] Theorem 4) that the equation (1.3) always has the unique C^∞ solution ϕ_δ for any positive number δ. In the following sections, we shall show that a subsequence of $\{\phi_\delta\}_{\delta>0}$ converges to a C^∞ function on $M \setminus \cup_i E_i$ with respect to certain norm as δ goes to 0.

§2. A uniform bound for ϕ_δ

All results are same as [Sg], but for the sake of completeness, we shall prove them again. Firstly, we shall obtain a lower bound for ϕ_δ. In what follows, C_i always denotes a positive number which depends only on $\{a_i\}$. Let fix a positive number δ and x_0 a point of M such that $\phi_\delta(x_0) = \inf_M \phi_\delta$. From the definition of ϕ_δ,

$$\begin{aligned}\det(\mathbf{G}(a)_{i\bar{j}})(x_0) &\leq \det(\mathbf{G}(a)_{i\bar{j}} + \phi_{\delta:i\bar{j}})(x_0) \\ &= \prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{a_i}(x_0) e^{\phi_\delta(x_0)} \det \mathbf{G}(x_0).\end{aligned}$$

Therefore, we see

$$\begin{aligned}\phi_\delta(x_0) &\geq \log\{\prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{-a_i}(x_0) \det(\mathbf{G}(a))(x_0) / \det \mathbf{G}(x_0) \\ &\geq -C_1.\end{aligned}$$

Hence we have the following lemma.

Lemma 2.1. *For any* $0 < \delta < 1$, *we have* $\inf_M \phi_\delta \geq -C_1$.

Next we shall obtain an upper bound for ϕ_δ. We set

$$\psi_\delta := \phi_\delta + \sum_{i=1}^{N} a_i \log \| s_i \|^2_{\mathbf{h}_i} . \tag{2.2}$$

Then ψ_δ is a C^∞ function on $M \setminus \cup_i E_i$ which is bounded above, and $\psi_\delta \mid \cup_i E_i = -\infty$. Since, on $M \setminus \cup_i E_i$,

$$\mathbf{G}(a)_{i\bar{j}} + \phi_{\delta:i\bar{j}} = \partial_i \bar{\partial}_j \log \det \mathbf{G} + \psi_{\delta:i\bar{j}},$$

we obtain

$$\det(\partial_i \bar{\partial}_j \log \det \mathbf{G} + \psi_{\delta:i\bar{j}}) = \prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{a_i} \| s_i \|^{-2a_i}_{\mathbf{h}_i} e^{\psi_\delta} \det \mathbf{G}.$$

Now fix $0 < \delta < 1$, and choose y_0 so that $\psi_\delta(y_0) = \sup_M \psi_\delta$. Since $\sqrt{-1}\partial\bar{\partial}\psi_\delta(y_0) \leq 0$, we obtain

$$\begin{aligned}&\det(\partial_i \bar{\partial}_j \log \det \mathbf{G})(y_0) \\ &\geq \det(\partial_i \bar{\partial}_j \log \det \mathbf{G} + \psi_{\delta:i\bar{j}})(y_0) \\ &= \prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{a_i} \| s_i \|^{-2a_i}_{\mathbf{h}_i} (y_0) e^{\psi_\delta(y_0)} \det \mathbf{G}(y_0).\end{aligned}$$

Hence $\sup_M \psi_\delta \leq C_2$ for a certain positive number C_2. Now, from (2.2), we have

Lemma 2.3. $\phi_\delta \leq C_2 - \sum_{i=1}^N a_i \log \| s_i \|^2_{\mathbf{h}_i}$ *for any* $0 < \delta < 1$.

Since $\log \| s_i \|^2_{\mathbf{h}_i} \in L^1(M, \mathbf{G}(a))$, and from (2.1) and (2.3), we have

$$\int_M | \phi_\delta | \, dV_{\mathbf{G}(a)} \leq C_3$$

for a certain positive number C_3. On the other hand, since $\omega_{\mathbf{G}(a)} + \sqrt{-1}\partial\bar{\partial}\phi_\delta$ is a Kähler form on M, we have $\Delta\phi_\delta + n > 0$, where Δ is the normalized Laplacian of $\mathbf{G}(a)$. Let $\mathbf{G}(x, y)$ be the Green's kernel function of Δ under the Neumann's condition, and set

$$\hat{\phi}_\delta := \phi_\delta - \frac{1}{Vol_{\mathbf{G}(a)}(M)} \int_M \phi_\delta dV_{\mathbf{G}(a)}.$$

Choose a positive number C_4 so that $\mathbf{G}(x, y) + C_4 > 0$ on $M \times M$. Then

$$\begin{aligned} \hat{\phi}_\delta(x) &= -\int_M \mathbf{G}(x, y)\Delta\hat{\phi}_\delta(y) dV_{\mathbf{G}(a)} \\ &= -\int_M \{\mathbf{G}(x, y) + C_4\}\Delta\hat{\phi}_\delta(y) dV_{\mathbf{G}(a)} \\ &\leq n \int_M \{\mathbf{G}(x, y) + C_4\} dV_{\mathbf{G}(a)}. \end{aligned}$$

Hence we obtain the following lemma.

Lemma 2.4. *There exists a positive number* C_5 *such that*

$$\sup_M \phi_\delta \leq C_5$$

for any $0 < \delta < 1$.

§3. The second order estimate of ϕ_δ

In this section, following Yau's argument [Y-2], we shall obtain the second order estimate of ϕ_δ. For simplicity, we use the following notations.

Notations 3.1.

$\mathbf{g}_{i\bar{j}} := \mathbf{G}(a)_{i\bar{j}}$,

$\mathbf{g}'_{i\bar{j}} := \mathbf{G}(a)_{i\bar{j}} + \phi_{\delta:i\bar{j}}$,

$F := \log\{\det \mathbf{G}_{i\bar{j}} / \det \mathbf{G}(a)_{i\bar{j}}\}$,

Δ : *the normalized Laplacian with respect to the metric* $\mathbf{g}$,

Δ' : *the normalized Laplacian with respect to the metric* $\mathbf{g}'$.

Using (3.1), the equation (1.3) can be written by

$$\det(\mathbf{g}_{i\bar{j}} + u_{:i\bar{j}}) = \prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{a_i} e^{u+F} \det \mathbf{g}_{i\bar{j}}. \tag{3.2}$$

Mimicking the computation of [Y-2 p.346–p.351], we have the following lemma.

Lemma 3.3. *Let fix $x \in M$, and $(z_1, \cdots, z_n)$ a complex normal coordinate at x with respect to the Kähler metric $\mathbf{g}$. $R_{a\bar{b}c\bar{d}}$ denotes the curvature tensor of $\mathbf{g}$ at x in terms of the coordinate $(z_1, \cdots, z_n)$. Choose a positive number C so that $C + \inf_{i \neq l} R_{i\bar{i}l\bar{l}} > 1$ on M. Then, at x, we have*

$$\begin{aligned}
&\Delta'(\exp\{-C\phi_\delta\}(n + \Delta\phi_\delta)) \\
&\geq \exp\{-C\phi_\delta\}[n + \Delta\phi_\delta] \\
&+ \exp\{-C\phi_\delta\}[\Delta(F + \sum_{i=1}^{N} a_i \log(\| s_i \|^2_{\mathbf{h}_i} + \delta))] \\
&- \exp\{-C\phi_\delta\}[n + n^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}}] \\
&- \exp\{-C\phi_\delta\}[Cn(n + \Delta\phi_\delta)] \\
&+ \exp\{-C\phi_\delta\}[n + \Delta\phi_\delta]^{1+\frac{1}{n-1}} \prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{\frac{-a_i}{n-1}} \\
&\times \exp\left\{\frac{-1}{n-1}(\phi_\delta + F)\right\}.
\end{aligned}$$

Now set $\eta_\delta := \exp\{-C\phi_\delta\}(n + \Delta\phi_\delta)$. From (3.3), we have

$$\begin{aligned}
&\Delta'\eta_\delta \geq (1 - Cn)\eta_\delta \\
&+ \exp\{-C\phi_\delta\}[\Delta(F + \sum_{i=1}^{N} a_i \log(\| s_i \|^2_{\mathbf{h}_i} + \delta))] \\
&- \exp\{-C\phi_\delta\}[n + n^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}}] \\
&+ \exp\left\{\frac{C-1}{n-1}\phi_\delta\right\} \prod_{i=1}^{N}(\| s_i \|^2_{\mathbf{h}_i} + \delta)^{\frac{-a_i}{n-1}} \exp\left\{\frac{-F}{n-1}\right\} \eta_\delta^{1+\frac{1}{n-1}}
\end{aligned}$$

Now, since there exists a positive number C_6 such that

$$\sum_{i=1}^{N} a_i \Delta \log(\| s_i \|^2_{\mathbf{h}_i} + \delta) \geq -C_6$$

for any $0 < \delta < 1$, using (2.1) and (2.4), we have

$$\Delta' \eta_\delta \geq (1 - Cn)\eta_\delta - C_7 + C_8 \eta_\delta^{1+\frac{1}{n-1}} \tag{3.4}$$

where C_i is a positive number independent of δ. Choose $z_0 \in M$ so that $\eta_\delta(z_0) = \sup_M \eta_\delta$. Then, from (3.4), there exists a positive number C_9 such that

$$\sup_M \eta_\delta \leq C_9$$

for any $0 < \delta < 1$. Then, from (2.4), we have $n + \Delta\phi_\delta \leq C_{10}$. Since $n + \Delta\phi_\delta > 0$, we obtain the following lemma.

Lemma 3.6. *$-n < \inf_M \Delta\phi_\delta \leq \sup_M \Delta\phi_\delta \leq C_{10}$ for any $0 < \delta < 1$.*

In what follows, we fix $x \in M$ and choose a complex normal coordinate $(z_1, \cdots, z_n)$ with respect to $\mathbf{g}$ so that

$$\phi_{\delta:i\bar{j}}(x) = \delta_{ij}\phi_{\delta:i\bar{i}}(x).$$

Since $\mathbf{g}'_{i\bar{i}}(x) > 0$, $1 + \phi_{\delta:i\bar{i}}(x) > 0$ for any i. On the other hand, since

$$\phi_{\delta:i\bar{i}} = \Delta\phi_\delta - \sum_{j\neq i} \phi_{\delta:j\bar{j}},$$

we have $\phi_{\delta:i\bar{i}}(x) \leq C_{11}$. In particular,

$$\mathbf{g}'_\delta := \mathbf{g}' \leq C_{12}\mathbf{g}$$

for any $0 < \delta < 1$. Next we shall obtain a lower bound for $\mathbf{g}'_\delta$. From (1.3) and (2.1), we have

$$\begin{aligned}[1 + \phi_{\delta:i\bar{i}}(x)]^{-1} &= \prod_{j\neq i}[1 + \phi_{\delta:j\bar{j}}]/\det \mathbf{g}'_\delta(x) \\ &\leq C_{13} \prod_{k=1}^{N} \| s_k \|^{-2a_k}_{\mathbf{h}_k},\end{aligned}$$

and therefore

$$1 + \phi_{\delta:i\bar{i}} \geq C_{14} \prod_{k=1}^{N} \| s_k \|_{\mathbf{h}_k}^{2a_k} .$$

Now, combining the Schauder estimates, we obtain the following proposition.

Proposition 3.7. *There exists a positive number C_{15} depending only on $\{a_i\}$ such that $\| \phi_\delta \|_{C^2(M)} \leq C_{15}$ and that*

$$C_{15}^{-1} \prod_{i=1}^{N} \| s_i \|_{\mathbf{h}_i}^{2a_k} \mathbf{g} \leq \mathbf{g}'_\delta \leq C_{15} \mathbf{g}.$$

Now use [G-T, Theorem 17.14] and regularity theorem for elliptic operators, we finally obtain the following theorem.

Theorem 3.8. *There exists a unique*

$$\phi \in C^\infty(M \setminus \cup_i E_i) \cap C^{2-\alpha}(M)$$

such that, on $M \setminus \cup_i E_i$, $\omega_{\mathbf{a}} + \sqrt{-1}\partial\bar{\partial}\phi$ is a Kähler form and that satisfies

$$\det(\mathbf{G}(a)_{i\bar{j}} + \phi_{:i\bar{j}}) = \prod_{i=1}^{N} \| s_i \|_{\mathbf{h}_i}^{2a_i} e^{\phi} \det \mathbf{G}.$$

Moreover $\{\phi_{:i\bar{j}}\}$ are bounded.

Proof. It is enough to show only uniqueness. Let ψ be another solution of the equation satisfying the conditions above. Then, by the similar argument of [Y-2, Theorem 6], one can show $\phi = \psi + C$ where C is a constant. Since

$$\begin{aligned}
&\prod_{i=1}^{N} \| s_i \|_{h_i}^{2a_i} e^{\psi} \det \mathbf{G} \\
&= \det(\mathbf{G}(a)_{i\bar{j}} + \psi_{:i\bar{j}}) \\
&= \det(\mathbf{G}(a)_{i\bar{j}} + \phi_{:i\bar{j}}) \prod_{i=1}^{N} \| s_i \|_{\mathbf{h}_i}^{2a_i} e^{\phi} \det \mathbf{G},
\end{aligned}$$

we obtain $e^C = 1$. Therefore $C = 0$. Q.E.D.

Corollary 3.9. *There exists a d-closed real positive current of type* (1.1)

$$\hat{\gamma} \in c_1(K_M - \sum_{i=1}^{N} a_i E_i)$$

such that $\hat{\gamma}\mid_{(M\setminus \cup E_i)}$ *is a* C^∞ *Einstein-Kähler metric and that* $\det \hat{\gamma}(x)$ *goes to 0 when* $x \to \cup_i E_i$.

§4. Certain examples

In this section, we shall give certain examples which satifies the condition (1.1). These examples are also given in [Sg], but for the sake of completeness, we shall give them again.

Example 4.1. Let X be a compact normal projective variety. We say X has *only canonical singularities* if X is $\mathbf{Q}$-Gorenstein and if there exists a resolution of singularities $\mu : Y \longrightarrow X$ such that

$$K_Y = \mu^* K_X + \sum_i a_i E_i$$

where a_i are nonnegative rational numbers and E_i vary all the exceptional prime divisor for μ. It is well-known that the definition above does not depend on a choice of desingularization of X. Let assume K_M is ample. We take Hironaka's resolution as μ. Then there exist positive rational numbers $\{c_i\}$ such that

$$\mu^* K_X - \sum_i c_i E_i$$

is ample. Then, by (3.9), we have a C^∞ Einstein-Kähler metric on X_{reg}.

Example 4.2. Let X be a compact nonsingular projective variety whose K_X is nef and big. Then, by [K-M-M, Corollary 3.5], there exists an effective divisor E_0 such that $K_X - aE_0$ is ample for any sufficiently small positive rational number a. Therefore, by (3.9) again, we obtain a C^∞ Einstein-Kähler metric on $X \setminus E_0$.

In what follows, we shall investigate the examples in more detail.

Lemma 4.3. *In the example* (4.1), *there exists Hermitian fibre metric* $\mathbf{h}_i$ *of* $[E_i]$ *and there exists a Kähler metric* $\mathbf{g}$ *on* Y *such that*

$$\sqrt{-1}\partial\bar{\partial} \log \det \mathbf{g} - \textstyle\sum_i (a_i + \delta c_i) \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i$$

is positive definite everywhere for any $0 < \delta < 1$.

Proof. Since K_X is ample, there exists a C^∞ d-closed real (1.1)-form $\omega \in c_1(\mu^* K_X)$ on Y with positive semidefinite everywhere. Since $\mu^* K_X - \sum_i c_i E_i$ is ample, we choose Hermitian fibre metrics $\mathbf{h}_i$ of $[E_i]$ such that $\omega - \sum_i c_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i$ is positive definite everywhere. Note that $K_Y = \mu^* K_X + \sum_i a_i E_i$, and there exists a Kähler metric $\mathbf{g}$ on Y with

$$\sqrt{-1}\partial\bar{\partial} \log \det \mathbf{g} = \omega + \textstyle\sum_i a_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i.$$

On the other hand, since ω is positive semidefinite,

$$\omega - \delta \sum_i c_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i = \delta(\omega - \textstyle\sum_i c_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i) + (1-\delta)\omega > 0,$$

for any $0 < \delta < 1$. Hence

$$\sqrt{-1}\partial\bar{\partial} \log \det \mathbf{g} - \textstyle\sum_i (a_i + \delta c_i) \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i > 0$$

on Y. Q.E.D.

Let X be a compact nonsingular projective variety whose canonical divisor K_X is nef and big. Then, due to Kawamata's Base Points Free Theorem [K-M-M, Theorem 3-1-1], there exists a C^∞ d-closed real (1.1)-form $\Omega \in c_1(K_X)$ which is positive semidefinite everywhere. Therefore the following lemma can be proved in the same way as (4.3).

Lemma 4.4. *In the example* (4.2), *we have a Kähler metric* $\mathbf{G}$ *on* X *and a Hermitian fibre metric* $\mathbf{h}_0$ *of* $[E_0]$ *which satisfies*

$$\sqrt{-1}\partial\bar{\partial} \log \det \mathbf{G} - a\sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_0 > 0$$

on X *for a sufficiently small positive number* a.

Example (4.1) (resp. (4.2)) says that, for any $0 < \delta < 1$ (resp. for any sufficiently small $a > 0$), there exists a C^∞ Einstein-Kähler metric on X_{reg}(resp. $X \setminus E_0$). In the next section, we shall show a subsequence of these Einstein-Kähler metrics also converges to an Einstein-Kähler metric as δ (resp. a) goes to 0.

§5. Convergence of Einstein-Kähler metrics

Let M be a nonsingular projective variety and $\{E_i\}$ effective divisors on M. In this section, we always assume the following condition.

Condition 5.1. *For each i, there exists a Hermitian fibre metric $\mathbf{h}_i$ of $[E_i]$ and also there exists a Kähler metric $\mathbf{G}$ on M such that, for certain nonnegative number α_i and for sufficiently small positive number δ_0,*

$$\sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} - \textstyle\sum_i(\alpha_i + \delta c_i)\sqrt{-1}\partial\bar{\partial}\log\mathbf{h}_i$$

is a Kähler form for any $0 < \delta < \delta_0$, where c_i is a positive number.

Note that the examples (4.1) and (4.2) satisfy (5.1) by (4.3) and (4.4) respectively. Now we set $a_i := \alpha_i + \frac{\delta_0 c_i}{2}$. Then, by (5.1),

$$\sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} - \textstyle\sum_i(a_i + \delta c_i)\sqrt{-1}\partial\bar{\partial}\log\mathbf{h}_i$$

is a Kähler form for any $\frac{-\delta_0}{2} < \delta < \frac{\delta_0}{2}$. We denote

$$\omega_a := \sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} - \textstyle\sum_i a_i\sqrt{-1}\partial\bar{\partial}\log\mathbf{h}_i$$

and

$$\omega_{\alpha+\epsilon} := \sqrt{-1}\partial\bar{\partial}\log\det\mathbf{G} - \textstyle\sum_i(\alpha_i + \epsilon c_i)\sqrt{-1}\partial\bar{\partial}\log\mathbf{h}_i.$$

For any $0 < \epsilon < \delta_0$, we consider the equation:

$$\det(\mathbf{G}(\alpha+\epsilon)_{i\bar{j}} + u_{i\bar{j}}) = \prod_i \| s_i \|_{\mathbf{h}_i}^{2(\alpha_i+\epsilon c_i)} e^u \det\mathbf{G}, \tag{5.2}$$

where $\omega_{\alpha+\epsilon} = \sqrt{-1}\sum_{ij}\mathbf{G}(\alpha+\epsilon)_{i\bar{j}}dz_i \wedge dz_{\bar{j}}$ and s_i is a holomorphic section of $[E_i]$ whose divisor is E_i. From (3.8), (5.2) has the unique solution $\phi_\epsilon \in C^\infty(M \setminus \cup_i E_i) \cap C^{2-\gamma}(M)$. In what follows, we shall show that a subsequence of $\{\phi_\epsilon\}$ converges to a C^∞function on $M \setminus \cup_i E_i$ as ϵ goes to 0. C_i always denotes a positive number depending only on $\{a_i\}$.

Lemma 5.3. *There exists a positive number C_{16} such that*

$$\phi_\epsilon \geq -C_{16} + \frac{\delta_0}{2}\sum_i c_i \log \| s_i \|_{\mathbf{h}_i}^2$$

for any $0 < \epsilon < \frac{\delta_0}{2}$.

Proof. We set $\psi_\epsilon := \phi_\epsilon - \frac{\delta_0}{2}\sum_i c_i \log \| s_i \|_{\mathbf{h}_i}^2$. Then ψ_ϵ is bounded below and $\psi_\epsilon \mid \cup_i E_i = +\infty$. Note that, on $M \setminus \cup_i E_i$,

$$\mathbf{G}(\alpha+\epsilon)_{i\bar{j}} + \phi_{\epsilon:i\bar{j}} = \mathbf{G}(a+\epsilon)_{i\bar{j}} + \psi_{:i\bar{j}}.$$

Now choose $x_1 \in M$ so that $\psi_\epsilon(x_1) = \inf_M \psi_\epsilon$. Since, for any $0 < \epsilon < \frac{\delta_0}{2}$, $\omega_{a+\epsilon}$ is a Kähler form, we have

$$\begin{aligned}
&\det \mathbf{G}(a+\epsilon)_{i\bar{j}}(x_1) \\
&\leq \det(\mathbf{G}(a+\epsilon)_{i\bar{j}} + \psi_{\epsilon:i\bar{j}})(x_1) \\
&= \prod_i \| s_i \|_{\mathbf{h}_i}^{2(a_i+\epsilon c_i)} (x_1) e^{\psi_\epsilon(x_1)} \det \mathbf{G}(x_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
\psi_\epsilon(x_1) &\geq \log\{\prod_i \| s_i \|_{\mathbf{h}_i}^{-2(a_i+\epsilon c_i)} (x_1) \det \mathbf{G}(a+\epsilon)_{i\bar{j}} / \det \mathbf{G}(x_1)\} \\
&\geq -C_{17}
\end{aligned}$$

for any $0 < \epsilon < \frac{\delta_0}{2}$. Q.E.D.

Lemma 5.4. *There exists a positive number C_{18} such that*

$$\sup_M \phi_\epsilon \leq C_{18} \quad \textit{for any} \quad 0 < \epsilon < \frac{\delta_0}{2}.$$

Proof. Let $\phi_{\epsilon,\delta}$ be the C^∞solution of the equation

$$\det(\mathbf{G}(\alpha+\epsilon)_{i\bar{j}} + u_{:i\bar{j}}) = \prod_i (\| s_i \|_{\mathbf{h}_i}^2 + \delta)^{\alpha_i+\epsilon c_i} e^u \det \mathbf{G}$$

and set $\eta_{\epsilon,\delta} := \phi_{\epsilon,\delta} + \sum(\alpha_i + \epsilon c_i) \log \| s_i \|_{\mathbf{h}_i}^2$. Then, by the same argument in Section 2, we have a positive number C_{19} such that $\sup_M \eta_{\epsilon,\delta} \leq C_{19}$ for any $0 < \epsilon < \frac{\delta_0}{2}$ and $0 < \delta < 1$. On the other hand, one can see that $\{\phi_{\epsilon,\delta}\}$ have a uniform lower bound so that

$$\phi_{\epsilon,\delta} \geq -C_{20} + \frac{\delta_0}{2} \sum c_i \log \| s_i \|_{\mathbf{h}_i}^2 .$$

Therefore there exists a positive number C_{21} such that $\int_M | \phi_{\epsilon,\delta} | dV_{\mathbf{G}(a)} \leq C_{21}$. Now we use the Grees's kernel of the metric $\mathbf{G}(a)$ and let $\delta \to 0$ to obtain the required estimate by the same argument as in Section 2. Q.E.D.

Lemma 5.5. *There exists a positive numbers C_{22} and C_{23} such* *that for any* $0 < \epsilon \ll \frac{\delta_0}{2}$,

$$\begin{aligned}
&C_{22}^{-1} \prod_i \| s_i \|_{\mathbf{h}_i}^{2(a_i+\epsilon c_i)+C_{23}\delta_0} \omega_{a+\epsilon} \\
&\leq \omega_{\alpha+\epsilon} + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon \\
&\leq C_{22} \prod_i \| s_i \|_{\mathbf{h}_i}^{-C_{23}\delta_0} \omega_{a+\epsilon}
\end{aligned}$$

on $M \setminus \cup_i E_i$.

Proof. As in (5.3), we set

$$\psi_\epsilon := \phi_\epsilon - \frac{\delta_0}{2} \sum_i c_i \log \| s_i \|_{\mathbf{h}_i}^2 .$$

Then, by (5.3), ψ_ϵ is bounded below, and satisfies

$$\det(\mathbf{G}(a+\epsilon)_{i\bar{j}} + \psi_{\epsilon:i\bar{j}}) = \prod_i \| s_i \|_{\mathbf{h}_i}^{2(a_i+\epsilon c_i)} e^{\psi_\epsilon + F_\epsilon} \det \mathbf{G}(a+\epsilon)_{i\bar{j}}$$

on $M \setminus \cup_i E_i$, where $F_\epsilon := \log[\det \mathbf{G} / \det \mathbf{G}(a+\epsilon)]$. Let Δ_ϵ be the normalized Laplacian with respect to the metric $\mathbf{G}(a+\epsilon)$ and $R_{a\bar{b}c\bar{d}}(x)$ be the curvature tensor of $\mathbf{G}(a)$ in terms of a complex normal coordinate with respect to $\mathbf{G}(a)$ at x. We choose a positive number $C > 1$ so that $C + \inf_{i \neq l} R_{i\bar{i}l\bar{l}}(x) > 1$ for any x. Since

$$\omega_{a+\epsilon} + \sqrt{-1}\partial\bar{\partial}\psi_\epsilon = \omega_{\alpha+\epsilon} + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon$$

is positive definite everywhere on $M \setminus \cup_i E_i$, taking trace with respect to $\mathbf{G}(a+\epsilon)$, we have $n + \Delta_\epsilon \psi_\epsilon > 0$ on $M \setminus \cup_i E_i$. On the other hand, note that

$$\begin{aligned}
&\exp\{-C\psi_\epsilon\}(n + \Delta_\epsilon \psi_\epsilon) \\
&= \prod \| s_i \|_{\mathbf{h}_i}^{C\delta_0 c_i} \exp\{-C\phi_\epsilon\}(n + \Delta_\epsilon \phi_\epsilon \\
&+ \frac{\delta_0}{2} \sum_i \mathrm{Tr}_{\mathbf{G}(a+\epsilon)}(c_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i).
\end{aligned}$$

Therefore, by (3.8), we have

$$\exp\{-C\psi_\epsilon\}(n + \Delta_\epsilon \psi_\epsilon) > 0$$

on $M \setminus \cup_i E_i$, and

$$\exp\{-C\psi_\epsilon\}(n + \Delta_\epsilon \psi_\epsilon) = 0$$

on $\cup_i E_i$. Using a complex normal coordinate at x with respect to $\mathbf{G}(a+\epsilon)$, by the same computation as (3.3), for $0 < \epsilon \ll \frac{\delta_0}{2}$ we obtain

$$\begin{aligned}
&\Delta'_\epsilon(\exp\{-C\psi_\epsilon\}(n+\Delta_\epsilon\psi_\epsilon)) \\
&\geq \exp\{-C\psi_\epsilon\}[n+\Delta_\epsilon\psi_\epsilon] \\
&+\exp\{-C\psi_\epsilon\}[\Delta_\epsilon(F_\epsilon+\sum_i(a_i+\epsilon c_i)\log \| s_i \|^2_{\mathbf{h}_i})] \\
&-\exp\{-C\psi_\epsilon\}[n+n^2\inf_{i\neq l} R(\epsilon)_{i\bar{i}l\bar{l}}] \\
&-\exp\{-C\psi_\epsilon\}[Cn(n+\Delta_\epsilon\psi_\epsilon)] \\
&+\exp\{-C\psi_\epsilon\}(n+\Delta_\epsilon\psi_\epsilon)^{1+\frac{1}{n-1}}\prod_i \| s_i \|_{\mathbf{h}_i}^{-\frac{2}{n-1}(a_i+\epsilon c_i)} \\
&\times \exp\left\{-\frac{1}{n-1}(\psi_\epsilon+F_\epsilon)\right\}
\end{aligned}$$

at x. Here $R(\epsilon)_{a\bar{b}c\bar{d}}$ is the curvature tensor of $\mathbf{G}(a+\epsilon)$ and Δ'_ϵ is the normalized Laplacian with respect to the metric $\omega_{a+\epsilon}+\sqrt{-1}\partial\bar{\partial}\psi_\epsilon$. We set

$$\eta_\epsilon := \exp\{-C\psi_\epsilon\}(n+\Delta_\epsilon\psi_\epsilon).$$

Then, by (5.3), we have

$$\Delta'_\epsilon\eta_\epsilon \geq (1-Cn)\eta_\epsilon - C_{24} + C_{25}\eta_\epsilon^{1+\frac{1}{n-1}}.$$

We choose $z_1 \in M \setminus \cup_i E_i$ to be $\eta_\epsilon(z_1) = \sup_M \eta_\epsilon$. Then, using the maximum principle, we obtain $\eta_\epsilon(z_1) \leq C_{26}$, and, by (5.4), it holds

$$n+\Delta_\epsilon\psi_\epsilon \leq C_{27}\prod_i \| s_i \|_{\mathbf{h}_i}^{-C\delta_0 c_i}.$$

Now, fix a point x of $M\setminus\cup_i E_i$ and choose a complex normal coordinate $(z_1,\cdots,z_n)$ at x with respect to the metric $\mathbf{G}(a+\epsilon)$ so that

$$\psi_{\epsilon:i\bar{j}}(x) = \delta_{ij}\psi_{\epsilon:i\bar{i}}(x),$$

the required estimate follows by the same argument as Section 3.

Q.E.D.

Let $\epsilon \to 0$, and we obtain the following theorem.

Theorem 5.6. *Under the condition* (5.1), *there exists a* $\phi \in C^\infty(M \setminus \cup_i E_i)$ *which is bounded above and satisfies the equation*

$$\det(\mathbf{G}(\alpha)_{i\bar{j}} + \phi_{:i\bar{j}}) = \prod_i \| s_i \|_{\mathbf{h}_i}^{2\alpha_i} e^{\phi} \det \mathbf{G}$$

on $M \setminus \cup_i E_i$. *Moreover there exists a d-closed positive* (1.1) *current* $\Omega \in c_1(K_M - \sum_i \alpha_i E_i)$ *such that* $\Omega |_{(M \setminus \cup_i E_i)} = \omega_\alpha + \sqrt{-1}\partial\bar{\partial}\phi$, *and in particular* Ω *is an Einstein-Kähler metric on* $M \setminus \cup_i E_i$.

Proof. By (3.9), for any $\epsilon > 0$, there exists a d-closed real positive (1.1)-current $\Omega_\epsilon \in c_1(K_M - \sum_i(\alpha_i + \epsilon c_i)E_i)$ such that, on $M \setminus \cup_i E_i$, Ω_ϵ is a C^∞ Einstein-Kähler metric $\omega_{\alpha+\epsilon} + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon$ where ϕ_ϵ is the solution of (5.2). Let ω_A be a Kähler form associated with an ample divisor A on M. We denote the mass norm with respect to ω_A by M_A. Since Ω_ϵ is positive,

$$M_A(\Omega_\epsilon) = < \Omega_\epsilon, \omega_A^{n-1} > = \{K_M - \sum_i (\alpha_i + \epsilon)E_i\} \cdot A^{n-1},$$

where $\cdot$ denotes the intersection product. Then, by compactness theorem of positive currents, there exists a d-closed real positive (1.1)-current Ω such that, if necessary taking a subsequence,

$$< \Omega, \varphi > = \lim_{\epsilon \to 0} < \Omega_\epsilon, \varphi >$$

for any C^∞ $(n-1.n-1)$-form φ on M. Therefore, in particular, for any d-closed real $(n-1.n-1)$-form ψ, we have

$$< \Omega, \psi > = \lim_{\epsilon \to 0} < \Omega_\epsilon, \psi > = [K_M - \sum_i \alpha_i E_i] \cdot [\psi],$$

where $[\psi]$ represents the cohomology class defined by ψ. Hence $\Omega \in c_1(K_M - \sum_i \alpha_i E_i)$. Let φ be a C^∞ real $(n-1.n-1)$-form with supp$\varphi \subset\subset M \setminus \cup_i E_i$. Then, if necessary taking a subsequence again, by Lebesgue's convergence theorem, we have

$$< \Omega, \varphi > = \lim_{\epsilon \to 0} < \Omega_\epsilon, \varphi > = < \omega_\alpha + \sqrt{-1}\partial\bar{\partial}\phi, \varphi > .$$

Therefore

$$\Omega |_{(M \setminus \cup_i E_i)} = (\omega_\alpha + \sqrt{-1}\partial\bar{\partial}\phi) |_{(M \setminus \cup_i E_i)} .$$

Q.E.D.

Bando-Kobayashi [B-K] pointed out that one can strengthen Theorem (5.6) in the following form. The proof of Theorem (5.7) is due to them.

Theorem 5.7. *Let ω_0 be a C^∞ Kähler form on M. Assume that a sequence of C^∞ Kähler forms*

$$\{\omega_k := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_k\}_k$$

converges a singular Kähler form ω which is C^∞ on $M \setminus \cup_i E_i$. Moreover assume that $\{\varphi_k\}$ satify the estimate

$$\delta \sum_i \log \| s_i \|^2_{h_i} - C_\delta \leq \varphi_k \leq C$$

for an arbitrary small positive number δ, where C_δ is a positive number which depends only on δ and where C is a positive number independent δ. Then ω is cohomologous to ω_0 as a current.

Proof. It is only sufficient to show that, for any $\epsilon > 0$, there exists an open neighborhood U of $\cup_i E_i$ such that

$$\int_U \omega_k \wedge \omega_0^{n-1} \leq \epsilon$$

for any k. We set

$$\theta := -\sum_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i$$

and

$$\varphi_{k,\delta} := \varphi_k - 2\delta \sum_i \log \| s_i \|^2_{\mathbf{h}_i} .$$

Let Ω be an open neighborhood of $\cup_i E_i$ with a smooth boundary. Then, by Poincare-Lelong's formula and Stokes' theorem, we get

$$\begin{aligned} \int_\Omega \omega_k \wedge \omega_0^{n-1} = \int_\Omega \omega_0^n + 2\delta \int_\Omega \theta \wedge \omega_0^{n-1} \\ + 4\pi\delta \sum_i \int_{E_i} \omega_0^{n-1} + \int_{\partial\Omega} \frac{\partial}{\partial n}(\varphi_{k,\delta}), \end{aligned}$$

where $\frac{\partial}{\partial n}$ is the outer normal of $\partial\Omega$ with respect to the metric ω_0. On the other hand, by the assumption, we have

$$-\delta \sum_i \log \| s_i \|^2_{\mathbf{h}_i} - C_\delta \leq \varphi_{k,\delta} \leq C - 2\delta \sum_i \log \| s_i \|^2_{\mathbf{h}_i} .$$

Therefore we can choose a sequence of positive numbers $\{A_k\}$ with $A_k \uparrow \infty$ as $k \uparrow \infty$ such that the boundary of the set

$$\Omega_k := \{x \in M \mid \varphi_{k,\delta}(x) \geq A_k\}$$

is smooth and that, for an arbitrary neighborhood U of $\cup_i E_i$, it holds $\Omega_k \subset\subset U$ for $k \gg 1$. Moreover we can assume that $\frac{\partial}{\partial n}\varphi_{k,\delta} \leq 0$ on $\partial\Omega_k$. Then, by (5.9), we have

$$\int_{\Omega_k} \omega_k \wedge \omega_0^{n-1} \leq \int_{\Omega_k} \omega_0^n + 2\delta \int_{\Omega_k} \theta \wedge \omega_0^{n-1} + 4\pi\delta \sum_i \int_{E_i} \omega_0^{n-1}.$$

Taking a positive number δ sufficiently small, the proof is completed.
Q.E.D.

§6. An inequality between Chern numbers

Let M be an n dimensional compact **Q**-Gorenstein projective variety and $\mu : N \longrightarrow M$ a birational morphism from an n dimensional nonsingular compact projective variety. Let E be a holomorphic vector bundle of rank r on N, and H an ample divisor on M. Assume that E is μ^*H-semistable, namely, for any coherent subsheaf $\mathcal{S}$ of E with positive rank, the inequality

$$\frac{1}{\mathrm{rk}(\mathcal{S})}\{c_1(\mathcal{S}) \cdot (\mu^*H)^{n-1}\} \leq \frac{1}{r}\{c_1(E) \cdot (\mu^*H)^{n-1}\}$$

holds. In this section, we shall prove the following theorem.

Theorem 6.1. *We have an inequality between Chern numbers of E;*

$$\{(r-1)c_1(E)^2 - 2rc_2(E)\} \cdot (\mu^*H)^{n-2} \leq 0.$$

Fact 6.2 ([E] Theorem 1.1). *Let X be an n dimensional compact minimal Kähler space and $f : Y \longrightarrow X$ a birational morphism from a compact Kähler manifold Y. Then the tangent bundle $\mathcal{T}_Y$ of Y is f^*K_X-semistable, where K_X is the canonical divisor of X. Moreover if f^*K_X is cohomologous to zero, $\mathcal{T}_N$ is $f^*\Phi_X$-semistable with respect to any Kähler form Φ_X on X.*

Combining (6.1), (6.2) and [K-M-M, Corollary 3-3-2], we obtain a generalization of Bogomolov's inequality.

Corollary 6.3. *Let M be an n dimensional compact minimal variety of general type, and $\mu : N \longrightarrow M$ an arbitrary resolution. Then we have*

$$\{(n-1)c_1(\mathcal{T}_N)^2 - 2nc_2(\mathcal{T}_N)\} \cdot (\mu^* K_M)^{n-2} \leq 0.$$

Proof. For a sufficiently large integer m, there exist an n dimensional compact projective variety Z with only canonical singularirties such that K_Z is ample, and a proper birational morphism $\Phi : M \longrightarrow Z$ defined by the linear system $| mK_M |$, which satisfies $K_M = \Phi^*(K_Z)$. We consider the following commutative diagram.

$$\begin{tikzcd} N \arrow[r, "\text{identity}"] \arrow[d, "\mu"'] & N \arrow[d, "\Psi"'] \\ M \arrow[r, "\Phi"] & Z \end{tikzcd}$$

By definition of minimal projective variety, we have

$$K_N = \mu^* K_M + \sum a_i E_i$$

for non-negative rational numbers a_i, where E_i is exceptional for μ. Therefore

$$K_N = \Psi^* K_M + \sum a_i E_i,$$

and by (6.2), $\mathcal{T}_N$ is $\Psi^*(K_Z)$-semistable. Hence, by (6.1), we obtain

$$\{(n-1)c_1(\mathcal{T}_N)^2 - 2nc_2(\mathcal{T}_N)\} \cdot (\Psi^* K_Z)^{n-2} \leq 0.$$

Note that $\Psi^*(K_Z) = \mu^*(K_M)$, and we finish the proof. Q.E.D.

(6.3) and [Mi-2, Theorem 6.6] imply the following results.

Corollary 6.4. *Let M be an n dimensional compact minimal variety smooth in codimension 2. Then we have the same inequality between chern numbers of N as in* (6.3).

Proof. Let

$$\nu(\mu^* K_M) := \max\{e \mid (\mu^* K_M)^e \not\equiv 0\},$$

where $\equiv$ denotes numerical equivalence. If $\nu(\mu^* K_M) < n-2$, the required inequality obviously holds. If $\nu(\mu^* K_M) = n$, we have proved in

(6.3). So we investigate when $\nu(\mu^* K_M) = n-2$ or $n-1$. Since it is known by [Mi-2, Theorem 6.6]

$$c_2(\mathcal{T}_N) \cdot (\mu^* K_M)^{n-2} \geq 0,$$

it is sufficient to show

$$c_1(\mathcal{T}_N)^2 \cdot (\mu^* K_M)^{n-2} = 0.$$

By definition,

$$\begin{aligned} &c_1(\mathcal{T}_N)^2 \cdot (\mu^* K_M)^{n-2} \\ &= (\mu^* K_M)^n + 2(\sum a_i E_i) \cdot (\mu^* K_M)^{n-1} + (\sum a_i E_i)^2 \cdot (\mu^* K_M)^{n-2}, \end{aligned}$$

and the first two terms vanish from the assumption and by the projection formula. Since M is smooth in codimension 2, $(\sum_i a_i E_i)^2 \cdot (\mu^* H)^{n-2} = 0$ for any ample $\mathbf{Q}$-divisor H, and in tern, $(\sum_i a_i E_i)^2 \cdot (\mu^* K_M)^{n-2} = 0$. Therefore we have $c_1(\mathcal{T}_N)^2 \cdot (\mu^* K_M)^{n-2} = 0$. Q.E.D.

Corollary 6.5. *Let M be an n dimensional compact minimal variety and $\mu : N \longrightarrow M$ a resolution. Assume that $\mu^*(K_M)$ is cohomologous to zero. Then we have*

$$\{(n-1)c_1(\mathcal{T}_N)^2 - 2nc_2(\mathcal{T}_N)\} \cdot (\mu^* H)^{n-2} \leq 0,$$

for any ample divisor H on M.

In the rest of this section, we shall prove (6.1).

Lemma 6.6. *The torsion free sheaf $\mu_* E$ is H-semistable.*

Proof. Let $\mathcal{S} \subset \mu_* E$ be a coherent subsheaf of positive rank. We consider the diagram

$$\begin{array}{ccc} \mu^* \mathcal{S} & \longrightarrow & \mu^* \mu_* E \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow \\ E & \xrightarrow{\text{identity}} & E. \end{array}$$

We set $\mathcal{T} := \rho(\mu^* \mathcal{S})$. Since $\mathcal{T}$ is a coherent subsheaf of E with positive rank, by the assumption, we obtain

$$\frac{1}{\mathrm{rk}(\mathcal{S})}\{c_1(\mathcal{T}) \cdot (\mu^* H)^{n-1}\} \leq \frac{1}{r}\{c_1(E) \cdot (\mu^* H)^{n-1}\}.$$

Since μ is a proper birational morphism between normal varieties, there exists a analytic subset M_1 of M with $\operatorname{codim} M_1 \geq 2$ such that μ is an isomorphism on $N \setminus \mu^{-1}(M_1)$. Now, for sufficiently large integers $m_1, \cdots, m_{n-1}$, choose a general complete intersection curve C of $| \; m_i H \; |$'s so that $C \subset M \setminus M_1$. Then

$$c_1(\mathcal{S}) \cdot C = c_1(\mathcal{T}) \cdot (\mu^{-1}C)$$

and

$$c_1(\mu_* E) \cdot C = c_1(E) \cdot (\mu^{-1}C).$$

Therefore, from the inequality above, we have finished the proof.

Q.E.D.

Combining (6.6) and [Mi-2, Theorem 2.5 and Corollary 3.6 (see also Remark 2.6)], we obtain the following proposition.

Proposition 6.7 (see also [Mi-2,Lemma 4.1]). *Let C be a compact smooth curve as in the proof of* (6.6) *and set $\tilde{C} := \mu^{-1}C(\simeq C)$. Then, for any divisor with* $\deg D > 0$, *we have*

$$\begin{aligned} &H^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E) - D)) \\ &= H^0(\tilde{C}, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) - D)) = 0 \end{aligned}$$

for every positive integer t, where Sym *denotes symmetric tensorial power.*

For sufficiently large integers $m_2, \cdots, m_{n-1}$, choose a general complete intersection surface X of $| \; m_i H \; |$'s so that $X \cap M_1$ is a set of finite points and that $S := \mu^{-1}X$ is a compact smooth surface.

Lemma 6.8. *For any non-zero effective divisor D,*

$$\begin{aligned} &H^0(S, \operatorname{Sym}^{rt} E(-tc_1(E) - D)) \\ &= H^0(S, \operatorname{Sym}^{rt} E * (-tc_1(E*) - D)) = 0 \end{aligned}$$

for every positive integer t.

Proof. Choose an integer m_1 large enough, and we take a general complete intersection curve C of $| \; m_i H \; |$'s as in (6.7). Then, from (6.7),

$$\begin{aligned} &H^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E) - D)) \\ &= H^0(\tilde{C}, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) - D)) = 0. \end{aligned}$$

Since C is general, and since

$$\operatorname{Sym}^{rt} E(-tc_1(E) - D)$$

and

$$\operatorname{Sym}^{rt} E^*(-tc_1(E^*) - D)$$

is a vector bundle, we complete the proof. Q.E.D.

The following lemma can be proved completely same way as [Mi-2, Corollary 4.2].

Lemma 6.9. *Let things be as in* (6.7) *and L a fixed Cartier divisor. Then the dimensions*

$$h^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E) + L))$$

and

$$h^0(\tilde{C}, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + L))$$

are bounded by a polynomial of degree $r-1$ in t.

Proposition 6.10. *The dimensions*

$$h^0(S, \operatorname{Sym}^{rt} E(-tc_1(E))$$

and

$$h^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S))$$

are bounded by a polynomial of degree $r-1$.

Proof. Consider the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(S, \operatorname{Sym}^{rt} E(-tc_1(E) - \tilde{C})) &\longrightarrow H^0(S, \operatorname{Sym}^{rt} E(-tc_1(E))) \\ &\longrightarrow H^0(\tilde{C}, \operatorname{Sym}^{rt} E(-tc_1(E))). \end{aligned}$$

Then, by (6.8) and (6.9), the first desired statement is completed.

On the other hand, since $K_S = \mu^* K_X + \sum_i a_i E_i$ where E_i is the exceptional divisor of $\mu\mid_S \colon S \longrightarrow X$, and since $\tilde{C} \cap (\cup_i E_i) = \emptyset$ for a sufficiently general member C of $\mid m_i H \mid$'s,

$$\begin{aligned} &\operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S - \tilde{C}) \mid_{\tilde{C}} \\ &= \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + \mu^*(K_X - m_1 H)) \mid_{\tilde{C}} . \end{aligned}$$

Let choose a positive integer m_1 so that $m_1 H - K_X$ is ample. Then, by (6.7), and from the argument of (6.8),

$$H^0(S, \operatorname{Sym}^{rt} E^*(-tc_1(E^*) + K_S - \tilde{C})) = 0.$$

Now consider the exact sequence

$$\begin{aligned}0 &\to H^0(S, \mathrm{Sym}^{rt} E^*(-tc_1(E^*) + K_S - \tilde{C})) \\ &\to H^0(S, \mathrm{Sym}^{rt} E^*(-tc_1(E^*) + K_S)) \\ &\to H^0(\tilde{C}, \mathrm{Sym}^{rt} E^*(-tc_1(E^*) + K_S)).\end{aligned}$$

Then, by (6.9), the second desired statement is also completed. Q.E.D.

Now (6.1) easily follows from the argument of [Mi-2, Theorem 4.3] which uses Riemann-Roch theorem.

§7. Local decomposition theorem

Let M be an n dimensional compact projective algebraic variety with only canonical singularities whose canonical divisor K_M is ample, and $\mu : N \longrightarrow M$ a Hironaka's resolution of singularities. Then, by (6.2), the tangent sheaf $\mathcal{T}_N$ of N is $\mu^* K_M$-semistable and it is easy to see that $\mathcal{T}_N$ admits the unique filtration of coherent sheaves

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t = \mathcal{T}_N$$

such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ is a torsion free sheaf of positive rank, $\mu^* K_M$-stable and

$$\frac{1}{\mathrm{rk}(\mathcal{S}_i)}\{c_1(\mathcal{S}_i) \cdot (\mu^* K_M)^{n-1}\} = \frac{1}{n}\{c_1(\mathcal{T}_N) \cdot (\mu^* K_M)^{n-1}\}$$

for any i. We call such a filtration as $\mu^* K_M$-stable filtration of $\mathcal{T}_N$. In this section, we shall show the following theorem. Note that we sometimes consider the regular part M_{reg} of M as an open subset of N via μ.

Theorem 7.1. *M_{reg} admits a C^∞ Einstein-Kähler metric $\tilde{\gamma}$ and there exists a holomorphic vector bundle $\mathbf{S}_i$ on M_{reg} such that*

$$(\mathcal{S}_i/\mathcal{S}_{i-1}) \mid_{M_{\mathrm{reg}}} = \mathcal{O}(\mathbf{S}_i)$$

and that $\mathcal{T}_N$ orthogonally decomposes

$$\mathcal{T}_N = \mathbf{S}_1 \oplus \cdots \oplus \mathbf{S}_t$$

on M_{reg} with respect to $\tilde{\gamma}$. Moreover, for any point x of M_{reg}, there exists an open neighborhood U of x such that $(U, \tilde{\gamma})$ is isometric to the direct product of Einstein-Kähler manifolds

$$(U, \tilde{\gamma}) = (U_1, \tilde{\gamma}_1) \times \cdots \times (U_k, \tilde{\gamma}_k).$$

Here U_i is a complex submanifold of U characterized by $\mathcal{T}_{U_i} = \mathbf{S}_i \mid_{U_i}$ and $\tilde{\gamma}_i = \tilde{\gamma} \mid_{U_i}$.

We shall prove (7.1) using a degenerate Monge-Ampère equation. By the definition of M, K_N is written as $K_N = \mu^* K_M + \sum_i a_i E_i$ for non-negative rational numbers $\{a_i\}$. Here $\{E_i\}$ run all exceptional divisor of μ. We denote $\sum_i a_i E_i = aE$ for certain non-negative rational number a and a non-zero effective divisor E. Let $\gamma \in 2\pi c_1(\mu^* K_M)$ be a C^∞ d-closed real (1.1)-form on N which is positive semi-definite everywhere and positive definite outside the support of E. Let $\{\delta_i\}$ be sufficiently small positive rational numbers so that $\mu^* K_M - \sum_i \delta_i E_i$ is $\mathbf{Q}$-ample. We choose a hermitian fibre metric $\mathbf{h}_i$ of the holomorphic line bundle $[E_i]$ such that a C^∞ d-closed real (1.1)-form

$$\gamma - \sum_i \delta_i \sqrt{-1}\partial\bar{\partial} \log \mathbf{h}_i$$

is positive. We write $\sum_i \delta_i E_i = \delta_0 F$ for a certain positive rational number δ_0 and a non-zero effective divisor F. Let $\mathbf{h}$ and ρ be hermitian fibre metrics of $[E]$ and $[F]$ induced by $\{\mathbf{h}_i\}$ respectively, and s (resp. σ) a section of $[E]$ (resp. $[F]$) whose the zero set is just E (resp. F). Now, for positive numbers ϵ and t , we consider a equation

$$\begin{aligned} &(\gamma - t\delta_0 \sqrt{-1}\partial\bar{\partial} \log \rho + \sqrt{-1}\partial\bar{\partial} u)^n \\ &= (\| s \|_{\mathbf{h}}^2 + \epsilon)^a (\| \sigma \|_\rho^2 + \epsilon)^{t\delta_0} e^u \Phi^n \end{aligned} \tag{7.2}$$

where Φ is a Kähler form on N such that

$$\sqrt{-1}\partial\bar{\partial} \log \Phi^n = \gamma + a\sqrt{-1}\partial\bar{\partial} \log \mathbf{h}. \tag{7.3}$$

By [Y-2], the equation (7.2) always has the unique solution $u_{t,\epsilon}$ for any positive ϵ and t. The following fact is due to Enoki.

Fact 7.4 ([E,Lemma 3.1]). *We set*

$$\begin{aligned} \delta(\epsilon) : &= a\sqrt{-1}\partial\bar{\partial} \log \mathbf{h} + t\delta_0 \sqrt{-1}\partial\bar{\partial} \log \rho \\ &+ \sqrt{-1}\partial\bar{\partial} \log[(\| s \|_{\mathbf{h}}^2 + \epsilon)^a (\| \sigma \|_\rho^2 + \epsilon)^{t\delta_0}] \end{aligned}$$

for $\epsilon > 0$. Then there exists a positive C^∞ function χ_ϵ on N which is uniformly bounded and $\chi_\epsilon \longrightarrow 0$ in L^1 sense as ϵ goes to zero. Moreover $-\delta(\epsilon) \leq \chi_\epsilon \Phi$ for any positive ϵ.

Now we finish the proof of (7.1). Since the proof is almost same as [E, Proposition 3.2], we shall only mention changes to be made.

Proof of (7.1). We set

$$\tilde{\gamma}_{t,\epsilon} := \gamma - t\delta_0\sqrt{-1}\partial\bar{\partial}\log\rho + \sqrt{-1}\partial\bar{\partial}u_{t,\epsilon}.$$

By simple computation, we have

$$\mathrm{Ric}\,\tilde{\gamma}_{t,\epsilon} = -\tilde{\gamma}_{t,\epsilon} - \delta(\epsilon). \tag{7.5}$$

Let $\mathcal{S}$ be a proper coherent subsheaf of $\mathcal{T}_N$ with positive rank r and $W(\mathcal{S})$ the minimal analytic subset of N such that

$$\mathcal{S}\mid_{(N\backslash W(\mathcal{S}))} = \mathcal{O}(\mathbf{S})$$

for certain holomorphic vector bundle $\mathbf{S}$. Now, by Gauss-Codazzi's equation, and by (7.4) and (7.5), we obtain

$$\begin{aligned}\int_{N\backslash W(\mathcal{S})} \| A_{t,\epsilon} \|^2\, \tilde{\gamma}_{t,\epsilon}^n \leq{}& -2\pi n\int_N c_1(\mathcal{S})\wedge\tilde{\gamma}_{t,\epsilon}^{n-1} \\ &+ 2\pi r\int_N c_1(\mathcal{T}_N)\wedge\tilde{\gamma}_{t,\epsilon}^{n-1} + r\int_N \delta(\epsilon)\wedge\tilde{\gamma}_{t,\epsilon}^{n-1} \\ &+ n\int_N \chi_\epsilon\Phi\wedge\tilde{\gamma}_{t,\epsilon}^{n-1},\end{aligned}$$

where $A_{t,\epsilon}$ is the second fundamental form of $\tilde{\gamma}_{t,\epsilon}\mid_{\mathbf{S}}$ and $\| A_{t,\epsilon} \|$ is its norm with respect to $\tilde{\gamma}_{t,\epsilon}$. Let $\epsilon \longrightarrow 0$. Then, by (3.7) and (7.4), and also by Fatou's lemma, we have

$$\begin{aligned}&\int_{N\backslash W(\mathcal{S})} \| A_t \|^2\, \tilde{\gamma}_t^n \\ &\leq -[2\pi n c_1(\mathcal{S})\cdot\{2\pi(\mu^*K_M) - t\delta_0\sqrt{-1}\partial\bar{\partial}\log\rho\}^{n-1}][N] \\ &+ [2\pi r c_1(\mathcal{T}_N)\cdot\{2\pi(\mu^*K_M) - t\delta_0\sqrt{-1}\partial\bar{\partial}\log\rho\}^{n-1}][N] \\ &+ r\int_N (a\sqrt{-1}\partial\bar{\partial}\log\mathbf{h} + t\delta_0\sqrt{-1}\partial\bar{\partial}\log\rho) \\ &\wedge(\gamma - t\delta_0\sqrt{-1}\partial\bar{\partial}\log\rho)^{n-1}.\end{aligned}$$

Next let $t \longrightarrow 0$. Then, by (5.6) and Fatou's lemma again, we have

$$\begin{aligned}\frac{1}{(2\pi)^n rn}\int_{N\backslash W(\mathcal{S})} \| A_0 \|^2\, \tilde{\gamma}^n \leq{}& -\frac{1}{r}\{c_1(\mathcal{S})\cdot(\mu^*K_M)^{n-1}\} \\ &+ \frac{1}{n}\{c_1(\mathcal{T}_N)\cdot(\mu^*K_M)^{n-1}\} \\ &+ \frac{1}{(2\pi)^n n}\int_N a\sqrt{-1}\partial\bar{\partial}\log\mathbf{h}\wedge\gamma^{n-1}.\end{aligned}$$

Since

$$\sqrt{-1}\partial\bar{\partial}\log \mathbf{h} = -\sqrt{-1}\partial\bar{\partial}\log \parallel s \parallel_{\mathbf{h}}^2$$

on M_{reg}, and since E is an exceptional divisor for μ, by Poincaré-Lelong's formula, the last term vanishes. Now the required statement follows from the last part of the proof of [Ko, Theorem 8.3] and de Rham's decomposition theorem. Q.E.D.

§8. Global decomposition theorem

Let (B, ω) be an n dimensional complete simply connected Kähler manifold with $\mathrm{Ric}\,\omega = -\omega$ and we assume that

$$\mathrm{Isom}(B, \omega) := \{\text{all biholomorphic maps which preserve} \quad \omega\}$$

acts B transitively. Let Γ be a discrete subgroup of $\mathrm{Isom}(B, \omega)$ and we assume that Γ satisfies the following condition.

Condition 8.1. *The quotient variety $M := B/\Gamma$ is compact. Let*

$$\Gamma_x := \{\gamma \in \Gamma \mid \gamma(x) = x\}.$$

We assume the cardinality of Γ_x is finite for any $x \in B$.

The following lemma is easy to see.

Lemma 8.2. *Let*

$$F_\Gamma := \{x \in B \mid \gamma(x) = x\}$$

for $\gamma \neq$ identity $\in \Gamma$. Then, for any element x of F_Γ, there exists an open neighborhood U of x and a nowhere vanishing holomorphic n form η_U on U which is Γ_x-invariant.

From (8.2), it follows $\mathrm{codim} F_\Gamma \geq 2$, and M has only canonical singularities. Moreover M is a V-manifold and K_M is ample. Let y be a point of the singularities of M and $x \in \pi^{-1}(y)$ where $\pi : B \longrightarrow M$ be the natural projection. We choose an open neighborhood W of x so small that there exists a Γ_x-invariant holomorphic n form η_W on W. We set $V := \pi(W)$, $\eta_V := \pi_* \eta_W$. Let

$$\mu : N \longrightarrow M$$

be a Hironaka's resolution of singularities. Since $\mu^* \eta_V$ is square integrable, $\mu^* \eta_V$ can be extended to a holomorphic n form on $\mu^{-1}(V)$ and we denote it $\mu^* \eta_V$ again (for instance, see [L]). Since

$$K_N = \mu^* K_M + \sum_i a_i E_i$$

for certain non-negative integers a_i, $\mu^*\eta_V$ vanishies along the exceptional divisor E_i with order a_i. Therefore, in particular,

$$\omega^n = f \parallel s \parallel_{\mathbf{h}}^{2a} \Phi^n$$

on M_{reg}. Here we denote $\pi_*\omega$ by ω again and f is a C^∞ function on M_{reg} satisfying $C^{-1} \leq f \leq C$ for a positive number C, and $\mathbf{h}$ is a hermitian fibre metric as in Section 7. Let $\tilde{\gamma}$ be a metric in (7.1)

Proposition 8.3. *We have $\tilde{\gamma} = \omega$ on M_{reg}.*

Proof. Let $\{\tilde{\gamma}_t\}$ be the metrics in the proof of (7.1). For any positive number t and ϵ, by (3.8), we have

$$\tilde{\gamma}_t^n \leq C_1 \parallel s \parallel_{\mathbf{h}}^{2a} \parallel \sigma \parallel_\rho^{2t\delta_0} \Phi^n,$$

where C_1 is a positive constant depending only on t, and where ρ is a hermitian fibre metric as in Section 7. Therefore we get

$$0 < \tilde{\gamma}_t^n / \omega^n \leq C_2 \parallel \sigma \parallel_\rho^{2t\delta_0}$$

for certain positive number C_2. We set $v_t := \tilde{\gamma}_t^n/\omega^n$. Since $\mathrm{Ric}\tilde{\gamma}_t = -\tilde{\gamma}_t$ and since $\mathrm{Ric}\omega = -\omega$, by the same computation as [Y-1], we obtain

$$\frac{1}{2n}\Delta_\omega v_t \geq v_t^{1+\frac{1}{n}} - v_t,$$

where Δ_ω is the Laplacian with respect to ω. Choose $x_0 \in M_{\mathrm{reg}}$ so that $v_t(x_0) = \sup_{M_{\mathrm{reg}}} v_t$. By the maximal principle, we have $v_t(x_0) \leq 1$. Now let $t \longrightarrow 0$, we have $\tilde{\gamma}^n \leq \omega^n$. Because

$$\begin{aligned}\int_{M_{\mathrm{reg}}} \tilde{\gamma}^n &= (2\pi\mu^*K_M)^n[N] \\ &= (2\pi K_M)^n[M] = \int_{M_{\mathrm{reg}}} \omega^n,\end{aligned}$$

we get $\tilde{\gamma}^n = \omega^n$. Since both $\tilde{\gamma}$ and ω are Einstein-Kähler metrics, we finally obtain $\tilde{\gamma} = \omega$. Q.E.D.

Combining (7.1) and (8.3), we obtain the following result.

Corollary 8.4. *If $\mathcal{T}_N$ is not μ^*K_M-stable, we have*

$$(B, \omega) = (B_1, \omega_1) \times \cdots \times (B_t, \omega_t)$$

isometrically ($t \geq 2$). *Therefore, in particular, if* (B, ω) *is irreducible,* $\mathcal{T}_N$ *is* $\mu^* K_M$*-stable.*

Remark 8.5. In the course of proof of (8.3), we have obtained more general statement as follows. Let (M, ω) be an n dimensional compact Kähler V-manifold with Ric $\omega = -\omega$. Then $\tilde{\gamma} = \omega$.

References

[A] T. Aubin, Equation du type Monge-Ampère sur les variétés kählerienne compactes, C. R. Acad. Paris, **283** (1976), 459–464.

[B-K] S. Bando and R. Kobayashi, in preprint.

[Bo] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective varieties, Math. USSR Izv., **13** (1978), 499–555.

[Ch-O] B. Y. Chen and K. Ogiue, Some characterization of complex space forms in terms of Chern classes, Quart. J. Math. Oxford, **26** (1975), 459–464.

[D] S. K. Donaldoson, Infinite determinants, stable bundles and curvature, Duke Math. J., **54** (1987), 231–247.

[E] I. Enoki, Stability and negativity for tangent sheaves of minimal Kähler spaces, Springer Lecture Notes, **1339** (1987), 118–126.

[G] D. Gieseker, On a theorem of Bogomolov on Chern classes of stable bundles, Amer. J. Math., **101** (1979), 77–85.

[G-T] D. Gilberg and N. S. Trudinger, Elliptic partial differntial equation of second order, Springer-Verlag, second edition (1983).

[K-M-M] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, Adv. Stud. in Pure Math., **10** Alg. Geom. Sendai (1987), 283–360.

[Ko-R] R. Kobayashi, Einstein-Kähler V-metrics on open Satake V-sur faces with isolated quotiet singularities, Math. Ann., **272** (1985), 395–398.

[Ko-S] S. Kobayashi, Differential geometry of complex vector bundles, Publ. Math. Soc. Japan, **15** (1987).

[L] H. Laufer, On rational singularities, Amer. J. Math., **94** (1972), 597–608.

[Lü] M. Lübke, Chern klassen von Hermite-Einstein-Vektorbünden, Math. Ann., **260** (1982), 133–141.

[Mi-1] Y. Miyaoka, On the Chern numbers of surfaces of general type, Invent. Math., **42** (1977), 213–224.

[Mi-2] ———, Algebraic surface with positive indices, Progress in Math., **39** Birkhäuser (1984), 281–301.

[Mi-3] ———, The Chern classes and Kodaira dimension of minimal varieties, Adv. Stud. in Pure Math., **10** Alg. Geom. Sendai (1985), 449–476.

[Mi-4] ______, The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Ann., **268** (1984), 159–171.

[M-R] V. B. Mehta and A. Ramanathan, Semi-stable sheaves on projective varieties and their restriction to curves, Math. Ann., **258** (1982), 213–224.

[Sg] K. Sugiyama, On tangent sheaves of minimal varieties, in this volume.

[Ts-1] H. Tsuji, Existence and degenerationb of Kähler- Einstein metrics on minimal algebraic varieties of genertal type, Math. Ann., **281** (1986), 257–293.

[Ts-2] ______, in preprint.

[Ts-3] ______, in preprint.

[U-Y] K. Uhlenbeck and S.-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, Comm. Pure Appl. Math., **39** (1986), 257–293.

[V] Van de Ven, On the Chern numbers of surfaces of general type, Invent. Math., **36** (1976), 285–293.

[Y-1] S.-T. Yau, A general Schwarz lemma for Kähler manifolds, Amer. J. Math, **100** (1978), 197–203.

[Y-2] ______, On the Ricci curvature of compact Kähler manifolds and the complex Monge-Ampère equation 1, Comm. Pure Appl. Math., **31** (1978), 339–411.

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