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Eta Invariants and Automorphisms of Compact Complex Manifolds

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Dedicated to Professor Akio Hattori on his sixtieth birthday

§1. Introduction

Let M be an m-dimensional compact complex manifold, G the group of all automorphisms of M and $\mathfrak{g}$ the complex Lie algebra of all holomorphic vector fields on M. In [F1, FMo] we defined a complex Lie algebra character $\mathcal{F} : \mathfrak{g} \to \mathbb{C}$ with properties that $\mathcal{F}$ depends only on the complex structure of M, and that the vanishing of $\mathcal{F}$ is a necessary condition for M to admit an Einstein-Kähler metric. $\mathcal{F}$ can be lifted to a group character $\widehat{\mathcal{F}} : G \to \mathbb{C}/\mathbb{Z}$. For these we refer the reader to a survey [FMaS], Chapters 1 and 3 in this volume; but brief reviews of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ will be given respectively in this section and at the beginning of Section 3.

In this paper we apply the theory of eta invariants of [APS] and [D] to obtain an interpretation of $\widehat{\mathcal{F}}$ in terms of eta invariants (Theorem 3.7) and a localization formula for $\widehat{\mathcal{F}}(a)$ in terms of the fixed point set of an automorphism $a \in G$ (Theorem 3.10). We also compute a few examples.

An unsolved question, which motivated this study, is whether M admits an Einstein-Kähler metric if $c_1(M) > 0$ and $\mathfrak{g} = \{0\}$. Note that if $\mathfrak{g} = \{0\}$ then $\mathcal{F} = 0$ trivially. Our study began with an attempt to know whether $\widehat{\mathcal{F}}$ can play any role even if $\mathfrak{g} = \{0\}$. If $c_1(M) > 0$ and $\mathfrak{g} = \{0\}$ then G is a finite group and the imaginary part $\operatorname{Im}\widehat{\mathcal{F}} : G \to \mathbb{R}$ vanishes identically, but the real part $\operatorname{Re}\widehat{\mathcal{F}} : G \to \mathbb{R}/\mathbb{Z}$ may not do. Our aim is therefore to find an example of a compact complex manifold with $\mathfrak{g} = \{0\}$ and with $\widehat{\mathcal{F}} \neq 0$. We mention however that it is not known whether $\widehat{\mathcal{F}} \neq 0$ implies the nonexistence of an Einstein-Kähler metric,

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while it is obvious from the definition of $\mathcal{F}$ that $\mathcal{F} \neq 0$ implies the nonexistence of an Einstein-Kähler metric. Let us recall the definition of $\mathcal{F}$.

Let h be a Hermitian metric of M, and $\omega = (i/2\pi)h_{\alpha\bar{\beta}}dz^{\alpha} \wedge d\bar{z}^{\beta}$ be its fundamental 2-form. Let $\mathcal{Y}$ be a holomorphic vector field on M. The character $\mathcal{F}$ can be expressed by

$$\mathcal{F}(\mathcal{Y}) = (m+1)\frac{i}{2\pi}\int_M \operatorname{div}(\mathcal{Y})\, c_1(h)^m,$$

where $\operatorname{div}(\mathcal{Y})$ is the complex divergence defined by

$$d(i(\mathcal{Y})\,\omega^m) = \operatorname{div}(\mathcal{Y})\,\omega^m$$

and $c_1(h)$ is the first Chern form with respect to h:

$$c_1(h) = -\,\frac{i}{2\pi}\,\partial\bar{\partial}\,\log\det\,(h_{\alpha\bar{\beta}}).$$

If h is an Einstein-Kähler metric, i.e. $c_1(h) = k\omega$ for some real constant k, then, from the Stokes theorem

$$\mathcal{F}(\mathcal{Y}) = (m+1)\frac{ik^m}{2\pi}\int_M d(i(\mathcal{Y})\,\omega^m) = 0.$$

Thus $\mathcal{F} \neq 0$ implies the nonexistence of an Einstein-Kähler metric. This statement is strengthened as follows: $\mathcal{F} \neq 0$ implies the nonexistence of a Hermitian metric such that for some real constant k'

$$c_1(h)^m = k'\omega^m,$$

for we can rewrite the above formula of $\mathcal{F}$ as

$$\mathcal{F}(\mathcal{Y}) = -(m+1)\int_M \mathcal{Y}(\frac{c_1(h)^m}{\omega^m})\ \omega^m.$$

Note that $c_1(h)^m/\omega^m$ is a smooth function globally well defined on M.

In Section 4 we shall exhibit an example of a rational surface with $\mathfrak{g} = \{0\}$ and with $\widehat{\mathcal{F}} \neq 0$. The first Chern class of this surface, however, is not positive. We could not find an example with an additional assumption $c_1(M) > 0$.

Finally we mention the recent existence results of Einstein-Kähler metrics of positive Ricci curvature. First of all, in [Sa, KS1, KS2] Sakane and Koiso considered a certain class of $\mathbb{P}^1$-bundles with $\mathbb{C}^*$-action which restrict to actions on the fibers $\mathbb{P}^1$. Let $\mathcal{Y}$ be the holomorphic vector field induced from this action. They showed that $\mathcal{F}(\mathcal{Y}) = 0$ becomes a sufficient condition for the existence of an Einstein-Kähler metric of positive Ricci curvature. In the proof of this result it is shown that if $\mathcal{F}(\mathcal{Y}) \neq 0$ then the regularity of the solution to the Einstein equation is spoiled along the zero set zero($\mathcal{Y}$) of $\mathcal{Y}$. Note that zero($\mathcal{Y}$) is the fixed point set of the $\mathbb{C}^*$-action. Secondly Siu [Si], Tian [T] and Tian and Yau [TY] proved that every differentiable type of $\mathbb{P}^2 \sharp k\overline{\mathbb{P}^2}$ with $3 \leq k \leq 8$ admits an Einstein-Kähler metric of positive Ricci curvature. The methods of their proofs show that if M has good symmetries one can prove the existence of an Einstein-Kähler metric.

§2. Signature operators and eta invariants

In this section we review the definitions and known results on the eta invariants. We refer the reader to [APS] and [D] for the detail. Let X be a real $2l$-dimensional compact oriented smooth manifold with boundary Y, and ξ a complex vector bundle over X. Assume that the metrics and metric connections of X and ξ are product near the boundary. Let τ be an involution of $\wedge^* T^* X \otimes \xi$ defined by

$$\tau(\alpha) = i^{q(q-1)+l} * \alpha \qquad \text{for} \quad \alpha \in \wedge^q T^* X \otimes \xi$$

where $* : \wedge^q T^* X \otimes \xi \rightarrow \wedge^{2l-q} T^* X \otimes \xi$ is the Hodge star operator defined by the metric and the orientation of X. Let $\wedge^+$ be the subbundle of $\wedge^{\text{even}} T^* X \otimes \xi$ which consists of $+1$-eigenvectors of τ, and $\wedge^-$ the subbundle of $\wedge^{\text{odd}} T^* X \otimes \xi$ which consists of -1-eigenvectors of τ. Let $\Gamma(\wedge^+ : P)$ denote the the subspace of $\Gamma(\wedge^+)$ satisfying the boundary condition $P(f|\partial X) = 0$ where P denotes the spectral projection corresponding to the eigenvalues $\lambda \geq 0$ of A_ξ defined below. Then ξ-valued signature operator $D_\xi : \Gamma(\wedge^+ : P) \rightarrow \Gamma(\wedge^-)$ is defined by

$$D_\xi \phi = (d_\xi + d_\xi^*)\phi = (d_\xi - * d_\xi *)\phi$$

for $\phi \in \Gamma(\wedge^+)$, where d_ξ denotes the covariant exterior differential operator induced from the connection of ξ.

Definition 2.1. sign(X, ξ) is defined to be the index of D_ξ, namely,

$$\operatorname{sign}(X, \xi) = \dim \ker D_\xi - \dim \operatorname{coker} D_\xi.$$

By an automorphism of a vector bundle we mean a diffeomorphism of the total space such that it descends to a diffeomorphism of the base space and that it maps a fiber to a fiber isomorphically. When an automorphism a acts on $\wedge^* T^* X \otimes \xi$ commuting with D_ξ, $\mathrm{sign}(a, X, \xi)$ is defined to be the a-index of D_ξ, namely,

$$\mathrm{sign}(a, X, \xi) = \mathrm{tr}(a|\ker D_\xi) - \mathrm{tr}(a|\operatorname{coker} D_\xi).$$

Note that $\mathrm{sign}(X, \xi) = \mathrm{sign}(1, X, \xi)$.

Let $C = Y \times I$ be a collar neighborhood of the boundary $Y = \partial X$ and $\gamma : C \to Y$ the projection. Let $\tilde{\xi}$ be the restriction of ξ to Y. Then $\tau_+ = 1 + \tau$ induces an isomorphism

$$\tau_+ : \gamma^*(\wedge^{\mathrm{even}} T^* Y \otimes \tilde{\xi}) \xrightarrow{\sim} \wedge^+|_C.$$

We define a first order self-adjoint elliptic differential operator

$$A_{\tilde{\xi}} : \Gamma(\wedge^{\mathrm{even}} T^* Y \otimes \tilde{\xi}) \to \Gamma(\wedge^{\mathrm{even}} T^* Y \otimes \tilde{\xi})$$

by

$$A_{\tilde{\xi}}\phi = i^l(-1)^{q+1}(*d_{\tilde{\xi}} - d_{\tilde{\xi}}*)\phi \tag{2.2}$$

for $\phi \in \Gamma(\wedge^{2q} T^* Y \otimes \tilde{\xi})$ where $* : \wedge^q T^* Y \otimes \tilde{\xi} \to \wedge^{2l-1-q} T^* Y \otimes \tilde{\xi}$ is the Hodge star operator for Y and $d_{\tilde{\xi}} : \Gamma(\wedge^q T^* Y \otimes \tilde{\xi}) \to \Gamma(\wedge^{q+1} T^* Y \otimes \tilde{\xi})$ is the covariant exterior differential operator as before.

The following proposition can be shown by the same computations as in [APS], p.63.

Proposition 2.3. *On the collar C, the signature operator D_ξ is of the form*

$$D_\xi = \sigma \circ \tau_+ \circ \left(\frac{\partial}{\partial u} + A_\xi\right) \circ \tau_+^{-1},$$

where $u \in I$ is the normal coordinate and $\sigma = \sigma(D_\xi)(du) : \wedge^+|_C \xrightarrow{\sim} \wedge^-|_C$ is the isomorphism defined by the principal symbol of D_ξ.

The eta invariants of $A_{\tilde{\xi}}$ are defined as follows.

Definition 2.4. Let a be an automorphism of $\wedge^{\mathrm{even}} T^* Y \otimes \tilde{\xi}$, and suppose that a commutes with $A_{\tilde{\xi}}$. Then the equivariant eta function $\eta_{\tilde{\xi}}(a, s)$ of $A_{\tilde{\xi}}$ is defined by

$$\eta_{\tilde{\xi}}(a, s) = \sum_{\lambda \neq 0} \mathrm{sign}(\lambda)\, \mathrm{tr}(a|_{F_\lambda})|\lambda|^{-s},$$

where λ are non-zero eigenvalues of $A_{\widetilde{\xi}}$ and F_λ is the λ-eigenspace. $\eta_{\widetilde{\xi}}(a, s)$ is meromorphically continued to the whole s-plane and the equivariant eta invariant $\eta_{\widetilde{\xi}}(a)$ is defined by

$$\eta_{\widetilde{\xi}}(a) = \eta_{\widetilde{\xi}}(a, 0),$$

see [D], Theorem 1.2. Note that $\eta_{\widetilde{\xi}}(1)$ is the ordinary non-equivariant eta invariant of $A_{\widetilde{\xi}}$. Note moreover that

$$\eta_{\widetilde{\xi}_1 \oplus \widetilde{\xi}_2}(a) = \eta_{\widetilde{\xi}_1}(a) + \eta_{\widetilde{\xi}_2}(a),$$

since $A_{\widetilde{\xi}_1 \oplus \widetilde{\xi}_2} = A_{\widetilde{\xi}_1} \oplus A_{\widetilde{\xi}_2}$ with respect to the direct sum connection.

Suppose now that a compact Lie group K acts on X and ξ, preserving the orientation, the metrics and the metric connections of X and ξ, and that an element $a \in K$ acts freely on Y. Let $\Omega \subset X$ be the fixed point set of a ($\Omega \cap Y = \emptyset$) which is the disjoint union of connected closed submanifolds N. The normal bundle $TN^\perp$ of N is decomposed into the Whitney sum of subbundles

$$TN^\perp = TN^\perp(-1) \oplus TN^\perp(\theta_1) \oplus \cdots \oplus TN^\perp(\theta_s)$$

where a acts on $TN^\perp(-1)$ via multiplication by -1 and on complex vector bundle $TN^\perp(\theta_j)$ via multiplication by $e^{i\theta_j}$, $\theta_j \neq \pi$. Further $\xi|_N$ is decomposed into the Whitney sum of subbundles

$$\xi|_N = \xi(\psi_1) \oplus \ \ldots \ \oplus \xi(\psi_r),$$

where a acts on $\xi(\psi_j)$ via multiplication by $e^{i\psi_j}$, see [D], p.901. On the other hand a induces automorphisms of $\wedge^* T^* X \otimes \xi$ and $\wedge^* T^* Y \otimes \widetilde{\xi}$, which commute with $A_{\widetilde{\xi}}$.

We shall see that $\operatorname{sign}(X, \xi)$ and $\operatorname{sign}(a, X, \xi)$ are computed in terms of certain characteristic forms (or classes) and eta invariants. We first define characteristic forms and classes which we need.

Definition 2.5. (a) $\operatorname{ch}(\xi)$ is the Chern character form of a complex vector bundle ξ with respect to the connection of ξ. Note that

$$\operatorname{ch}(\xi) = \operatorname{rank}_{\mathbb{C}} \xi \ + \ c_1(\xi) \ + \ \text{higher terms},$$

where $c_1(\xi)$ is the first Chern form of ξ, and that

$$\operatorname{ch}(\xi_1 \oplus \xi_2) = \operatorname{ch}(\xi_1) + \operatorname{ch}(\xi_2),$$
$$\operatorname{ch}(\xi_1 \otimes \xi_2) = \operatorname{ch}(\xi_1) \wedge \operatorname{ch}(\xi_2)$$

with respect to the direct sum and tensor product connections.

(b) The $\mathcal{L}$-polynomial of the Pontrjagin classes $p_1, p_2, \dots$ is defined by

$$\begin{aligned}\mathcal{L}(p) &= \prod_j \frac{x_j/2}{\tanh(x_j/2)} \\ &= 1 + \frac{1}{12}p_1 + \dots,\end{aligned}$$

where p_i is the i-th symmetric function of the x_j^2. $\mathcal{L}(X) = \mathcal{L}(p(TX))$ is the $\mathcal{L}$-form of tangent bundle TX with respect to the metric of X. Note that the evaluation of $\mathcal{L}(N)$ at the fundamental cycle $[N]$ of a closed submanifold N is independent of the metric and that $\mathcal{L}(N) = 1$ if N is a point.

(c) $\mathrm{ch}(\xi|_N, a)$ is an element of $\prod_{q=0}^{\infty} H^{2q}(N; \mathbb{C})$ defined by

$$\mathrm{ch}(\xi|_N, a) = \sum_{j=1}^{r} e^{i\psi_j}\, \mathrm{ch}(\xi(\psi_j)),$$

where $\mathrm{ch}(\xi(\psi_j))$ is the ordinary Chern character of $\xi(\psi_j)$. Note that

$$\begin{aligned}\mathrm{ch}(\xi_1 \oplus \xi_2|_N, a) &= \mathrm{ch}(\xi_1|_N, a) + \mathrm{ch}(\xi_2|_N, a), \\ \mathrm{ch}(\xi_1 \otimes \xi_2|_N, a) &= \mathrm{ch}(\xi_1|_N, a)\ \mathrm{ch}(\xi_2|_N, a).\end{aligned}$$

(d) Finally, the $\mathcal{M}^\theta$-polynomial of the Chern classes $c_i(TN^\perp(\theta))$ is defined by

$$\mathcal{M}^\theta = \prod_j \frac{\tanh(i\theta/2)}{\tanh((x_j + i\theta)/2)},$$

where $c_i(TN^\perp(\theta))$ is the i-th elementary symmetric function of the x_j. Note that $\mathcal{M}^\theta(TN^\perp(\theta)) = 1$ if $TN^\perp(\theta) = \{0\}$.

Theorem (cf. Theorem 3.10 in [APS] and Theorem 2.5 in [D]). *Under the notations* $n = \frac{1}{2}\dim_{\mathbb{R}} N$, $d = \frac{1}{2}\operatorname{rank}_{\mathbb{R}} TN^\perp(-1)$ *and* $c(\theta_j) = \operatorname{rank}_{\mathbb{C}} TN^\perp(\theta_j)$, *we have*

$$\operatorname{sign}(X, \xi) = 2^l \int_X \mathrm{ch}(\xi)\mathcal{L}(X) - \eta_{\tilde{\xi}}(1), \tag{2.6}$$

$$\operatorname{sign}(a, X, \xi) = \sum_{N \subset \Omega} 2^{n-d}\ \mathrm{ch} \cdot \mathcal{L}(a)[N] - \eta_{\tilde{\xi}}(a), \tag{2.7}$$

where $\mathrm{ch}\cdot\mathcal{L}(a) \in \prod_{q=0}^{\infty} H^{2q}(N;\mathbb{C})$ *is defined by*

$$\mathrm{ch}\cdot\mathcal{L}(a) = \mathrm{ch}(\xi|_N, a)\prod_j(-i\cot(\theta_j/2))^{c(\theta_j)}\mathcal{L}(N)$$
$$\mathcal{L}(TN^{\perp}(-1))^{-1}e(TN^{\perp}(-1))\prod_j \mathcal{M}^{\theta_j}(TN^{\perp}(\theta_j)),$$

and $e(TN^{\perp}(-1))$ *is the Euler class of* $TN^{\perp}(-1)$.

Now assume that the compact Lie group K is a finite group and that K acts freely on Y. Since the K-action preserves the orientation, metrics and connections of Y and $\widetilde{\xi}$, there exist an orientation and a metric on Y/K, a complex vector bundle $\widehat{\xi} = \widetilde{\xi}/K$ over Y/K and a connection in $\widehat{\xi}$ such that the projection $\pi : Y \to Y/K$ is a Riemannian covering and that $\pi^*\widehat{\xi}$ is isomorphic to $\widetilde{\xi}$ as a vector bundle with a connection.

Definition 2.8. A first order self-adjoint elliptic differential operator $A_{\widehat{\xi}} : \Gamma(\wedge^{\mathrm{even}}T^*(Y/K)\otimes\widehat{\xi}) \to \Gamma(\wedge^{\mathrm{even}}T^*(Y/K)\otimes\widehat{\xi})$ is defined just as in 2.2 and the eta invariant $\eta_{\widehat{\xi}}(1)$ is also defined as in 2.4. Note that $A_{\widehat{\xi}}$ is locally the same as $A_{\widetilde{\xi}}$.

The eigenvalues of $A_{\widehat{\xi}}$ is closely related to those of $A_{\widetilde{\xi}}$ and the next proposition can be proved similarly to (3.6) in [D].

Proposition 2.9. $\eta_{\widehat{\xi}}(1) = \frac{1}{|K|}\sum_{a\in K}\eta_{\widetilde{\xi}}(a)$.

In later sections, we consider the signatures and the eta invariants for virtual vector bundles ξ.

Definition 2.10. Let ξ_1 and ξ_2 be complex vector bundles over X which satisfy the conditions in Definitions 2.1, 2.4 and 2.5. For the virtual bundle $\xi = \xi_1 - \xi_2$, we define $\mathrm{sign}(a, X, \xi)$, $\eta_\xi(a)$, $\mathrm{ch}(\xi)$ and $\mathrm{ch}(\xi|_N, a)$ by

$$\begin{gathered}\mathrm{sign}(a,X,\xi) = \mathrm{sign}(a,X,\xi_1) - \mathrm{sign}(a,X,\xi_2),\\ \eta_\xi(a) = \eta_{\xi_1}(a) - \eta_{\xi_2}(a),\\ \mathrm{ch}(\xi) = \mathrm{ch}(\xi_1) - \mathrm{ch}(\xi_2) \qquad \text{and}\\ \mathrm{ch}(\xi|_N, a) = \mathrm{ch}(\xi_1|_N, a) - \mathrm{ch}(\xi_2|_N, a).\end{gathered}$$

Then, the next proposition is obvious.

Proposition 2.11. *The formulae 2.6, 2.7 and Proposition 2.9 hold for the virtual bundles ξ, $\widetilde{\xi}$ and $\widehat{\xi}$.*

§3. $\widehat{\mathcal{F}}$ and eta invariants

Let M be an m-dimensional compact complex manifold, G the group of all biholomorphic automorphisms of M and $\mathfrak{g}$ the complex Lie algebra of all holomorphic vector fields on M. In [FMo] it is shown that the Lie algebra character $\mathcal{F}$ can be expressed in terms of Simons character of a certain foliation.

Theorem ([FMo]). *For $\mathcal{Y} \in \mathfrak{g}$, let $\mathrm{Fol}_{\mathcal{Y}}$ be a complex foliation of codimension m on $M \times S^1$ defined by the vector field $\frac{\partial}{\partial t} + 2\,\mathrm{Re}\,\mathcal{Y}$ where t is the coordinate of S^1 and $\mathrm{Re}\,\mathcal{Y}$ denotes the real part of $\mathcal{Y}$. Then*

$$\mathcal{F}(\mathcal{Y}) = -S_{c_1^{m+1}}(\nu(\mathrm{Fol}_{\mathcal{Y}}))[M \times S^1] \mod \mathbb{Z}$$

where $S_{c_1^{m+1}}(\nu(\mathrm{Fol}_{\mathcal{Y}})) \in H^{2m+1}(M \times S^1; \mathbb{C}/\mathbb{Z})$ is the Simons character of c_1^{m+1} for the normal bundle $\nu(\mathrm{Fol}_{\mathcal{Y}})$ of $\mathrm{Fol}_{\mathcal{Y}}$ with any basic connection.

Note that our notation of $\mathcal{F}$ differs from f in [FMo] by $i/2\pi$. For $g \in G$, let M_g be the mapping torus $M_g = M \times [0,1]/\sim$ where $(p,0) \sim (g(p),1)$. Let Fol_g be the complex foliation defined by the $[0,1]$-directed vector field. Then, by definition,

$$\widehat{\mathcal{F}}(g) = S_{c_1^{m+1}}(\nu(\mathrm{Fol}_g))[M_g]$$

where $S_{c_1^{m+1}}(\nu(\mathrm{Fol}_g))$ is the Simons character for the normal bundle $\nu(\mathrm{Fol}_g)$ with any Bott connection. As is seen in [F2] (see also [FMaS]), $\widehat{\mathcal{F}} : G \to \mathbb{C}/\mathbb{Z}$ is a Lie group homomorphism. The above theorem further shows that its infinitesimal Lie algebra homomorphism is equal to $\mathcal{F}$ up to a constant multiple.

Remark 3.1. Let $g = \exp(2\,\mathrm{Re}\,\mathcal{Y})$. Since (M_g, Fol_g) is isomorphic to $(M \times S^1, \mathrm{Fol}_{-\mathcal{Y}})$, it follows from the above theorem that

$$\widehat{\mathcal{F}}(\exp(2\,\mathrm{Re}\,\mathcal{Y}) = -\mathcal{F}(\mathcal{Y}) \mod \mathbb{Z}.$$

Remark 3.2. If $g^p = 1$, then $p\widehat{\mathcal{F}}(g) = \widehat{\mathcal{F}}(g^p) = \widehat{\mathcal{F}}(1) = 0$. Hence $\widehat{\mathcal{F}}(g)$ is of the form q/p for some integer q.

We assume for the rest of this paper that K is a cyclic subgroup of G generated by an element g of order p. Let $X = M \times D^2$ and

$Y = \partial X = M \times S^1$, and let $q_M : X \to M$, $q_{D^2} : X \to D^2$ and $q_Y : Y \to M$ be the projections. We consider the following action of K on X:

$$g(z, re^{i\theta}) = (g(z), re^{i(\theta + 2\pi/p)})$$

for $(z, re^{i\theta}) \in X = M \times D^2$; $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. The next lemma is obvious.

Lemma 3.3. *The fixed point set $\Omega_X \subset X$ of the $a = g^k$-action on X for $a \in K$ coincides with the fixed point set $\Omega_M \subset M = M \times \{0\} \subset X$ of the a-action on M.*

Since M and D^2 carry the complex structures, we may regard TM, TD^2 and $TX = q_M^* TM \oplus q_{D^2}^* TD^2$ as holomorphic tangent bundles, i.e. the tangent bundles of type (1,0). We first give a rotationally symmetric Hermitian metric on D^2 such that it is a product metric of $S^1 \times [0, \varepsilon)$ near the boundary $\partial D^2 = S^1$. We then give a Hermitian metric on X which is the product of a given K-invariant Hermitian metric on M and the Hermitian metric on D^2. This metric on X is obviously product near the boundary Y and invariant under the K-action on X defined above. Let ∇ be the Hermitian connection of TX, which is uniquely determined under the conditions that the connection form of ∇ is of type (1,0) and that ∇ preserves the Hermitian metric of TX. Note that ∇ is not necessarily torsion free. It is obvious that ∇ is the direct sum connection of the Hermitian connections of TM and TD^2 and that ∇ is K-invariant. Since the K-action is free near the boundary of X, X/K carries a complex structure and a Hermitian metric near the boundary. The connection ∇, which is K-invariant, descends to the Hermitian connection of $T(X/K)$. By restriction we obtain a connection, which we also denote by ∇, on $T(X/K)|_{Y/K}$.

Note that Y/K is diffeomorphic to the mapping torus M_g and that $T(X/K)|_{M_g}$ is orthogonally decomposed into $T(X/K)|_{M_g} = \nu(\mathrm{Fol}_g) \oplus \mathcal{E}$ where $\mathcal{E}$ denotes the trivial complex line bundle consisting of all Fol_g-directed vectors. Under this decomposition ∇ splits as

$$\nabla = \nabla|_{\nu(\mathrm{Fol}_g)} \oplus \nabla^0$$

where ∇^0 is the globally flat connection of $\mathcal{E}$. Note that $\nabla|_{\nu(\mathrm{Fol}_g)}$ is a basic connection for the foliation Fol_g. From this decomposition and 4.18 in [S] we obtain:

Proposition 3.4. $\quad S_{c_1^{m+1}}(\nu(\mathrm{Fol}_g)) = S_{c_1^{m+1}}(T(X/K)|_{M_g}, \nabla).$

Definition 3.5. Let ξ be a virtual K-vector bundle over X defined by

$$\xi = \bigotimes^{m+1}(q_M^* TM - \mathcal{E}^m),$$

where $\mathcal{E}^m$ denotes a trivial K-vector bundle over X. Namely,

$$\xi = \bigoplus_{k:\mathrm{even}} \binom{m+1}{k} mk\, q_M^*(\bigotimes^{m+1-k} TM) - \bigoplus_{k:\mathrm{odd}} \binom{m+1}{k} mk\, q_M^*(\bigotimes^{m+1-k} TM).$$

Let $\widetilde{\xi}$ be a virtual K-vector bundle over Y defined by $\widetilde{\xi} = \xi|_Y = \bigotimes^{m+1}(q_Y^* TM - \mathcal{E}^m)$, and $\widehat{\xi}$ a virtual complex vector bundle over $M_g = Y/K$ defined by

$$\widehat{\xi} = \widetilde{\xi}/K = \bigotimes^{m+1}(\nu(\mathrm{Fol}_g) - \mathcal{E}^m).$$

We endow connections of ξ, $\widetilde{\xi}$ and $\widehat{\xi}$ which are induced from the Hermitian connection of TM and $\nabla|_{\nu(\mathrm{Fol}_g)}$.

Definition 3.6. Let $\eta_{\widehat{\xi}}(1)$ be the eta invariant defined as in Definitions 2.8 and 2.10 for the virtual bundle $\widehat{\xi}$ over M_g.

Our first main result is:

Theorem 3.7. $2^{m+1}\widehat{\mathcal{F}}(g) = \eta_{\widehat{\xi}}(1) \mod \mathbb{Z}.$

Proof. Since M_g is a stably almost complex manifold there exists a compact $(2m+2)$-dimensional almost complex manifold W such that $\partial W = M_g$, see [Mo]. Let ξ^W be a virtual bundle over W defined by

$$\xi^W = \bigotimes^{m+1}(TW - \mathcal{E}^{m+1}).$$

We may assume that W is isomorphic to X/K near the boundary M_g as an almost complex manifold together with a Hermitian metric; hence the metric is product near M_g. Let ∇^W be the Hermitian connection of TW which coincides with ∇ of $T(X/K)$ near M_g. Let $c(TW)$, $\mathrm{ch}(TW)$ and $\mathrm{ch}(\xi^W)$ be the Chern and the Chern character forms with respect to ∇^W. Since $\mathrm{sign}(W, \xi^W)$ is an integer, we then obtain from 2.6 and 2.11

$$\eta_{\overline{\xi}}(1) = 2^{m+1} \int_W \mathrm{ch}(\xi^W)\mathcal{L}(W) \quad \mathrm{mod}\ \mathbb{Z},$$

where $\overline{\xi} = \xi^W|_{M_g}$. From the properties of the Chern character forms we have

$$\begin{aligned}\mathrm{ch}(\xi^W) &= \{\mathrm{ch}(TW) - \mathrm{ch}(\mathcal{E}^{m+1})\}^{m+1} \\ &= \{c_1(TW)\}^{m+1} + \text{higher terms.}\end{aligned}$$

Since the leading term of $\mathcal{L}$ is equal to 1,

$$\int_W \mathrm{ch}(\xi^W)\mathcal{L}(W) = \int_W \{c_1(TW)\}^{m+1}.$$

Hence it follows from this, the definition of $\widehat{\mathcal{F}}$, Propositions 3.4 and 5.15 in [S] that

$$\begin{aligned}\widehat{\mathcal{F}}(g) &= S_{c_1^{m+1}}(\nu(\mathrm{Fol}_g))[M_g] \\ &= S_{c_1^{m+1}}(T(X/K)|_{M_g}, \nabla)[M_g] \\ &= S_{c_1^{m+1}}(TW, \nabla^W)[\partial W] \\ &= \int_W \mathrm{ch}(\xi^W)\mathcal{L}(W) \qquad \mathrm{mod}\,\mathbb{Z}.\end{aligned}$$

On the other hand, since $TW|_{M_g} = \nu(\mathrm{Fol}_g) \oplus \mathcal{E}^1$ we have

$$\begin{aligned}\overline{\xi} &= \xi^W|_{M_g} \\ &= \bigotimes^{m+1}(\nu(\mathrm{Fol}_g) \oplus \mathcal{E}^1 - \mathcal{E}^{m+1}) = \overline{\xi_1} - \overline{\xi_2},\end{aligned}$$

where

$$\overline{\xi_1} = \bigoplus_{k:\text{even}} \binom{m+1}{k}(m+1)k \bigotimes^{m+1-k} (\nu(\mathrm{Fol}_g) \oplus \mathcal{E}^1),$$

$$\overline{\xi_2} = \bigoplus_{k:\text{odd}} \binom{m+1}{k}(m+1)k \bigotimes^{m+1-k} (\nu(\mathrm{Fol}_g) \oplus \mathcal{E}^1).$$

Since there exists a complex vector bundle ξ_3 over M_g such that $\overline{\xi_1} = \widehat{\xi}_1 \oplus \xi_3$ and $\overline{\xi}_2 = \widehat{\xi}_2 \oplus \xi_3$ where

$$\widehat{\xi}_1 - \widehat{\xi}_2 = \widehat{\xi} = \bigotimes^{m+1}(\nu(\mathrm{Fol}_g) - \mathcal{E}^m),$$

it follows from the property of eta invariants (see 2.4) that

$$\begin{aligned}\eta_{\overline{\xi}}(1) &= \eta_{\overline{\xi}_1}(1) - \eta_{\overline{\xi}_2}(1)\\ &= \eta_{\widehat{\xi}_1}(1) + \eta_{\xi_3}(1) - \eta_{\widehat{\xi}_2}(1) - \eta_{\xi_3}(1)\\ &= \eta_{\widehat{\xi}_1}(1) - \eta_{\widehat{\xi}_2}(1) = \eta_{\widehat{\xi}}(1).\end{aligned}$$

This completes the proof.

Corollary 3.8. *If p is odd, then $\widehat{\mathcal{F}}(g)$ vanishes if and only if $\eta_{\widehat{\xi}}(1)$ is an integer.*

Proof. In view of Remark 3.2 it is clear that $\widehat{\mathcal{F}}(g) = 0$ if and only if $2^{m+1}\widehat{\mathcal{F}}(g) = 0$. This completes the proof.

Now, for $1 \leq k \leq p-1$, we apply 2.7 to the case where $a = g^k$, $X = M \times D^2$ and $\xi = \bigotimes^{m+1}(q_M^* TM - \mathcal{E}^m)$. Let $\Omega(k)$ be the fixed point set of the g^k-action on X. Note that $\Omega(k)$ coincides with the fixed point set of the g^k-action on M (cf. Lemma 3.3) and is the disjoint union of connected closed complex submanifolds N of M. Let $TN^\perp$ be the normal bundle of N in M, and

$$TN^\perp = TN^\perp(-1) \oplus TN^\perp(\theta_1) \oplus \cdots \oplus TN^\perp(\theta_s)$$

the decomposition into the Whitney sum of complex subbundles where g^k acts on $TN^\perp(-1)$ via multiplication by -1 and on $TN^\perp(\theta_j)$ via multiplication by $e^{i\theta_j}$, $\theta_j \neq \pi$. The normal bundle of N in X is the Whitney sum of $TN^\perp$ and a trivial complex line bundle $\mathcal{E}$ where g^k acts on $\mathcal{E}$ via multiplication by $e^{2\pi k i/p}$. Hence it follows from 2.7 that

$$\eta_{\widetilde{\xi}}(g^k) = 2^{n-d} \sum_{N \subset \Omega(k)} \mathrm{ch} \cdot \mathcal{L}(g^k)[N] - \mathrm{sign}(g^k, X, \xi), \tag{3.9}$$

where $\mathrm{ch} \cdot \mathcal{L}(g^k) \in \prod_{q=0}^{\infty} H^{2q}(N; \mathbb{C})$ is defined as in 2.7 for the normal bundle $TN^{\perp} \oplus \mathcal{E}$ of N in X, namely,

$$\operatorname{ch}\cdot\mathcal{L}(g^k) = \operatorname{ch}(\xi|_N, g^k)(-i\cot\frac{\pi k}{p})\prod_j(-i\cot\frac{\theta_j}{2})^{c(\theta_j)}$$
$$\mathcal{L}(N)\mathcal{L}(TN^{\perp}(-1))^{-1}e(TN^{\perp}(-1))\prod_j\mathcal{M}^{\theta_j}(TN^{\perp}(\theta_j)),$$

where $\xi|_N = \bigotimes^{m+1}(TM|_N - \mathcal{E}^m)$.

Our second theorem is

Theorem 3.10. *With the notations as above we have*

$$2^{m+1}\widehat{\mathcal{F}}(g) = \frac{1}{p}\sum_{k=1}^{p-1}2^{n-d}\sum_{N\subset\Omega(k)}\operatorname{ch}\cdot\mathcal{L}(g^k)[N] \mod \mathbb{Z}.$$

Proof. By 2.6, 3.9, Proposition 2.9 and Theorem 3.7 we have

$$\begin{aligned}2^{m+1}\widehat{\mathcal{F}}(g) &= \frac{1}{p}\sum_{k=1}^{p}\eta_{\tilde{\xi}}(g^k) \mod \mathbb{Z}\\ &= \frac{1}{p}[\sum_{k=1}^{p-1}2^{n-d}\sum_{N\subset\Omega(k)}\operatorname{ch}\cdot\mathcal{L}(g^k)[N] + \int_X \operatorname{ch}(\xi)\mathcal{L}(X)\\ &\quad - \sum_{k=1}^{p}\operatorname{sign}(g^k, X, \xi)] \mod \mathbb{Z}.\end{aligned}$$

So the theorem follows from Lemma 3.11 and Lemma 3.12 below.

Lemma 3.11. $\int_X \operatorname{ch}(\xi)\mathcal{L}(X) = 0.$

Proof. Since the connection in ξ is induced from the connection in TM, it follows from the property of Chern character form that

$$\begin{aligned}\operatorname{ch}(\xi) &= \operatorname{ch}(\bigotimes^{m+1}(q_M^* TM - \mathcal{E}^m))\\ &= (q_M^*\operatorname{ch}(TM) - m)^{m+1}\\ &= (q_M^* c_1(TM) + \text{higher terms})^{m+1}\\ &= q_M^* c_1(TM)^{m+1}.\end{aligned}$$

Since $\dim M = m$, $c_1(TM)^{m+1}$ vanishes identically. This completes the proof.

Lemma 3.12. $\sum_{k=1}^{p} \mathrm{sign}(g^k, X, \xi) = 0 \mod p$.

Proof. From the definition of $\mathrm{sign}(g^k, X, \xi)$ (cf. 2.1 and 2.10), it suffices to show the following simple lemma.

Lemma 3.13. *For any finite dimensional $K(= \mathbb{Z}_p)$-module V,*

$$\sum_{k=1}^{p} \mathrm{tr}(g^k|_V) = 0 \mod p.$$

Proof. Apply the next (3.14) to the eigenvalues $\lambda_1, \dots, \lambda_{\dim V}$ of $g|_V$.

$$\text{If} \quad \lambda^p = 1, \text{then} \quad \sum_{k=1}^{p} \lambda^k = 0 \mod p. \tag{3.14}$$

This completes the proof.

When p is odd and $\Omega(k)$ is independent of k, we have a slightly simpler formula. Note that this situation occurs if p is an odd prime integer. Suppose that the fixed point set $\Omega := \Omega(k) \subset M$ is a disjoint sum of connected closed complex submanifolds N. Let $\bigoplus_j TN_j^{\perp}$ be the decomposition of the normal bundle of N in M where g acts on the complex vector bundle $TN_j^{\perp}$ via multiplication by $e^{i\tau_j}$. We define σ_k by

$$\sigma_k = \sum_{N \subset \Omega} 2^n(-i \cot \frac{\pi k}{p}) \prod_j (-i \cot \frac{k\tau_j}{2})^{c_j}$$
$$(\mathrm{ch}(TN) + \sum_j e^{k\tau_j} \mathrm{ch}(TN_j^{\perp}) - m)^{m+1} \mathcal{L}(N) \prod_j \mathcal{M}^{k\tau_j}(TN_j^{\perp})[N],$$

where $n = \dim_{\mathbb{C}} N$ and $c_j = \mathrm{rank}_{\mathbb{C}} TN_j^{\perp}$.

Corollary 3.15. *Assume that p is odd and $\Omega = \Omega(k)$ is independent of k. Then,*

$$2^{m+1}\widehat{\mathcal{F}}(g) = \frac{1}{p} \sum_{k=1}^{p-1} \sigma_k \mod \mathbb{Z}.$$

In particular, $\widehat{\mathcal{F}}(g)$ vanishes if and only if $\sum_{k=1}^{p-1} \sigma_k$ is a multiple of p.

Proof. The corollary follows from Theorem 3.10, Corollary 3.8 and the following facts:

(i) since p is odd, $TN^{\perp}(-1) = \{0\}$;

(ii) $\xi|_N = \bigotimes^{m+1}(TM|_N - \mathcal{E}^m) = \bigotimes^{m+1}(TN \oplus (\bigoplus_j TN_j^{\perp}) - \mathcal{E}^m)$ and hence, by the property of $\mathrm{ch}(\xi|_N, g^k)$ we have

$$\mathrm{ch}(\xi|_N, g^k) = (\mathrm{ch}(TN) + \sum_j e^{ik\tau_j}\,\mathrm{ch}(TN_j^{\perp}) - m)^{m+1}.$$

This completes the proof.

§4. Examples

Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates on the complex projective plane $\mathbb{P}^2$, and M the surface obtained by blowing up $\mathbb{P}^2$ at one point, say $[1 : 0 : 0]$. Note that $c_1(M) > 0$. The Lie algebra of all holomorphic vector fields on M is not reductive, and by Matsushima's theorem M does not admit an Einstein-Kähler metric. The last statement also follows from $\mathcal{F} \neq 0$. To see this we consider the $\mathbb{C}^*$-action $[z_0 : z_1 : z_2] \to [z_0 : c\, z_1 : c\, z_2]$, $c \neq 0$, on $\mathbb{P}^2$. This action lifts to a $\mathbb{C}^*$-action on M. Let $w_1 = z_1/z_0$, $w_2 = z_2/z_0$ be the inhomogeneous coordinates on $\mathbb{P}^2$, and $\mathcal{X} = 2\pi i(w_1\partial/\partial w_1 + w_2\partial/\partial w_2)$ be the holomorphic vector field which generates the $\mathbb{C}^*$-action. Then $\mathcal{X}$ also lifts a holomorphic vector field on M, which we denote by $\mathcal{Y}$. The zero set of $\mathcal{Y}$ consists of the line $C = p^{-1}(\{[0 : z_1 : z_2]\})$ and the exceptional curve $E = p^{-1}([1 : 0 : 0])$ where $p : M \to \mathbb{P}^2$ denotes the projection. One can then apply the localization formula for $\mathcal{F}$, cf. Theorem 2.6 in [FMaS], to obtain

$$\mathcal{F}(\mathcal{Y}) = \frac{(1+3a)^3}{(1+a)}[C] + \frac{(-1+b)^3}{(-1-b)}[E] = 4,$$

where a and b denote the positive generators of $H^2(C;\mathbb{Z})$ and $H^2(E;\mathbb{Z})$ respectively. Consider now a $\mathbb{Z}_p$-action generated by an element g defined by $g([z_0 : z_1 : z_2]) = [z_0 : e^{2\pi i/p}z_1 : e^{2\pi i/p}z_2]$ with p odd prime. One sees that $g = \exp(-\mathcal{Y}/p)$, and hence by Remark 3.1

$$\widehat{\mathcal{F}}(g) = \frac{4}{p} \mod \mathbb{Z}.$$

We can alternatively derive this using Corollary 3.15. Modulo terms of degree higher than 2 we have

$$\begin{gathered}
\mathrm{ch}(TC) = 1 + 2a, \qquad \mathrm{ch}(TE) = 1 + 2b, \\
\mathrm{ch}(TC^{\perp}) = 1 + a, \qquad \mathrm{ch}(TE^{\perp}) = 1 - b, \\
\mathcal{M}^{-2\pi k/p}(TC^{\perp}) = 1 + i\,\mathrm{cosec}(\frac{-2\pi k}{p})a, \\
\mathcal{M}^{2\pi k/p}(TE^{\perp}) = 1 - i\,\mathrm{cosec}(\frac{2\pi k}{p})b.
\end{gathered}$$

Using these we can deduce by straightforward computations

$$\begin{aligned}
\sigma_k = 22e^{2\pi ki/p} + 4e^{4\pi ki/p} - 6e^{6\pi ki/p} \\
- 26e^{-2\pi ki/p} - 20e^{-4\pi ki/p} - 6e^{-6\pi ki/p}.
\end{aligned}$$

Since $\sum_{k=1}^{p-1} e^{2\pi lki/p} \equiv -1 \mod p$ for any integer l, it follows that

$$8\widehat{\mathcal{F}}(g) = \frac{32}{p} \mod \mathbb{Z}.$$

Further, since $(p, 8) = 1$, we obtain again

$$\widehat{\mathcal{F}}(g) = \frac{4}{p} \mod \mathbb{Z}.$$

We now take p to be 3, and consider the $\mathbb{Z}_3$-action given by

$$g([z_0 : z_1 : z_2]) = [z_0 : \omega z_1 : \omega z_2],$$

where $\omega = e^{2\pi i/3}$. Let $q_1 = [1:1:0]$, $q_2 = [1:0:1]$ and $q_3 = [1:1:1]$. We further blow up M at q_1, $g(q_1)$, $g^2(q_1)$, q_2, $g(q_2)$, $g^2(q_2)$, q_3, $g(q_3)$ and $g^2(q_3)$. Since we blew up $\mathbb{P}^2$ at 10 points, the resulting manifold, which we denote by $\widehat{M}$, does not have positive first Chern class. Obviously the action of $\mathbb{Z}_3$ on M lifts to the one on $\widehat{M}$.

Proposition 4.1. *There is no non-zero holomorphic vector field on $\widehat{M}$, and*

$$\widehat{\mathcal{F}}(g) = 4/3 \mod \mathbb{Z}.$$

Proof. Let a be an element in the identity component of the group of all automorphisms of $\widehat{M}$. Since the self-intersection of each exceptional curve is -1, a leaves each exceptional curve invariant and descends to an automorphism of $\mathbb{P}^2$ which leaves the 10 points fixed. But such an automorphism on $\mathbb{P}^2$ must be an identity, and thus $a = 1$. This proves the first assertion. To prove the second assertion, note that the fixed point set of the g-action on $\widehat{M}$ is equal to the one on M. So the computation of $\widehat{\mathcal{F}}$ for $\widehat{M}$ using Corollary 3.15 reduces to quite the same computation as in the case of M. This completes the proof.

We now examine a few cases where $c_1(M) > 0$. The following lemma is useful.

Lemma 4.2. *In the situation of Definition* 3.6, *let g be an automorphism of M of order* 2. *Then $\eta_{\widehat{\xi}}(1) = 0$. In particular, if H is a subgroup of the group of automorphisms generated by elements of order* 2, *then $2^{m+1}\widehat{\mathcal{F}}(h) = 0$ for all $h \in H$.*

Proof. Consider a symmetry $\phi : M \times I \to M \times I$ defined by $\phi(z,t) = (z, 1-t)$. Then ϕ descends to M_g since

$$\phi(z,0) = (z,1) \qquad \text{and}$$
$$\phi(g(z),1) = (g(z),0) \sim (g^2(z),1) = (z,1).$$

Since ϕ is an orientation reversing isometry and hence anti-commutes with $A_{\widehat{\xi}}$, the eigenvalues of $A_{\widehat{\xi}}$ are symmetric with respect to 0 and therefore the eta function $\eta_{\widehat{\xi}}$ is identically zero by the definition (see Definition 2.4). This completes the proof.

This lemma may not imply that $\widehat{\mathcal{F}}(g) = 0$ for an element of order 2 because of the factor 2^{m+1} in Theorem 3.7, but at least in the surface case this is true, see [FMa].

A compact complex surface with $c_1(M) > 0$ and with no non-zero holomorphic vector field is obtained by blowing up $\mathbb{P}^2$ at k-points, $4 \leq k \leq 8$, in general position. When these points are in sufficiently general position it is known that the group of all automorphisms is generated by elements of order 2 ([K]).

Let us consider the Fermat hypersurface M of degree m in $\mathbb{P}^m$. It is known by Siu [Si] and Tian [T] that there exists a Kähler-Einstein metric on M. By Lemma 4.2 the eta invariants vanishes modulo $\mathbb{Z}$ on the subgroup S_{m+1} consisting of the elements corresponding to the permutations of the $m+1$ coordinates. Consider now a $(\mathbb{Z}_m)^{m+1}$-action generated by

$$g_j([z_0 : z_1 : \cdots : z_m]) = [z_0 : z_1 : \cdots : \tau z_j : \cdots : z_m], \quad j = 0, \ldots, m,$$

where $\tau = e^{2\pi i/m}$. Since $g_j^m = 1$, $\widehat{\mathcal{F}}(g_j)$ must be of the form $\widehat{\mathcal{F}}(g_j) = q/m$ with an integer q, and by symmetry we also have $\widehat{\mathcal{F}}(g_i) = \widehat{\mathcal{F}}(g_j)$. Hence

$$\widehat{\mathcal{F}}(g_0) = \widehat{\mathcal{F}}(g_1^{-1} \dots g_m^{-1}) = -m\widehat{\mathcal{F}}(g_0) = -q \equiv 0 \mod \mathbb{Z}.$$

Using Corollary 3.15, one can check the vanishing of $\widehat{\mathcal{F}}(g)$ for the following examples which appear in the tables of classification of Fano threefolds in [MM]:

1) the blow-up of $\mathbb{P}^3$ with center the intersection of two cubics,

$$z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0,$$
$$a_0 z_0^3 + a_1 z_1^3 + a_2 z_2^3 + a_3 z_3^3 = 0,$$

where $a_i \neq a_j$ for $i \neq j$, with the action

$$g([z_0 : z_1 : z_2 : z_3]) = [\omega z_0 : z_1 : z_2 : z_3];$$

2) the blow-up of $\mathbb{P}^3$ with center a twisted cubic,

$$\{[u^3 : u^2 v : uv^2 : v^3] \mid [u : v] \in \mathbb{P}^1\},$$

with the action

$$g([z_0 : z_1 : z_2 : z_3]) = [z_0 : \omega z_1 : \omega^2 z_2 : z_3];$$

3) the blowing up of $\mathbb{P}^3$ with center the disjoint union of a line, $\{z_1 = z_2 = 0\}$, and the twisted cubic with the action defined as in 2);

4) V_7 being the blow-up of $\mathbb{P}^3$ at a point, $[1 : 0 : 0 : 0]$, the blow-up of V_7 with center the strict transform of a twisted cubic passing through the center of the blowing up $V_7 \to \mathbb{P}^3$, with the action defined as in 2);

5) the blow-up of a cubic threefold

$$z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$$

in $\mathbb{P}^4$ with center a cubic curve,

$$z_0 = 0, \qquad z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0,$$

with the action

$$g([z_0 : z_1 : z_2 : z_3 : z_4]) = [z_0 : \omega z_1 : z_2 : z_3 : z_4];$$

6) V_2, i.e. the smooth hypersurface,

$$z_0^6 + \cdots + z_3^6 + z_4^2 = 0,$$

of degree 6 in the weighted projective space $\mathbb{P}(z_0, \ldots, z_4)$, where $\deg z_i = 1$ for $0 \leq i \leq 3$, and $\deg z_4 = 3$, with the action

$$g([z_0 : \cdots : z_4]) = [\omega z_0 : z_1 : \cdots : z_4].$$

These are not at all exhaustive in the tables of [MM], but we selected them among those which do not have non-zero holomorphic vector field and have an automorphism of odd order such that no subgroup generated by elements of order 2 contains it.

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