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Fundamental Groups of Semisimple Symmetric Spaces

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Dedicated to Professor R. Takahashi on his 60th birthday

Abstract
The aim of this report is to determine the fundamental group of an arbitrary irreducible semisimple symmetric space G/H when G is a connected semisimple Lie group with trivial center. The fundamental group $\pi_1(G/H)$ is well-known if G/H is Riemannian. Therefore, we restrict our attention to the case where G/H is non-Riemannian so both G and H are not compact. The result is summarized in Table 4.

§ 1. Preliminaries

Let $\mathfrak{g}$ be a semisimple Lie algebra and let σ be its involution. Then we obtain a direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ for σ. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a (semisimple) symmetric pair. Let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. Since $\theta\sigma$ is also an involution of $\mathfrak{g}$, we obtain a direct sum decomposition $\mathfrak{g}=\mathfrak{h}^a+\mathfrak{q}^a$ for $\theta\sigma$. The pair $(\mathfrak{g}, \mathfrak{h}^a)$ is the associated symmetric pair of $(\mathfrak{g}, \mathfrak{h})$ (cf. [B, p. 102]). Let G be the adjoint group $\mathrm{Int}\,\mathfrak{g}$. Then σ is lifted to G. We denote its lifting by the same letter. Let K be the maximal compact subgroup of G corresponding to $\mathfrak{k}$. Put $G^\sigma=\{g \in G; \sigma(g)=g\}$ and $G^{\theta\sigma}=\{g \in G; \theta\sigma(g)=g\}$. Then G/G^σ and $G/G^{\theta\sigma}$ are (semisimple) symmetric spaces. By definition, $\mathfrak{h}$ and $\mathfrak{h}^a$ are the Lie algebra of G^σ and that of $G^{\theta\sigma}$, respectively.

The aim of this report is to answer the following problem.

Problem. Determine the fundamental group of G/G^σ.

A symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible if the representation of $\mathfrak{h}$ on $\mathfrak{q}$ via the adjoint representation is irreducible. Moreover, a symmetric space G/H is irreducible if the corresponding symmetric pair is irreducible. Then it is sufficient to treat irreducible symmetric spaces to answer Problem. At this stage, we recall the following lemma (cf. [B, Prop. 53.2]).

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Lemma 1. *The symmetric space G/G^{σ} is a vector bundle over K/K^{σ} with fibres isomorphic to $\mathfrak{p}\cap\mathfrak{q}$.*

Corollary. $\pi_1(G/G^{\sigma})\simeq\pi_1(G/G^{\theta\sigma})\simeq\pi_1(K/K^{\sigma})$.

Proof. By Lemma 1, we have $\pi_1(G/G^{\sigma})\simeq\pi_1(K/K^{\sigma})$. On the other hand, $K^{\sigma}=K^{\sigma\theta}$. This implies that $\pi_1(G/G^{\theta\sigma})\simeq\pi_1(K/K^{\sigma})$.

We note some remarks on this subject.

(i) If G/G^{σ} is an irreducible compact symmetric space, $\pi_1(G/G^{\sigma})$ is determined by E. Cartan. (For the sake of completeness, we contain this result in Tables 1, 2).

(ii) If G/G^{σ} is a Riemannian symmetric space of non-compact type, then $\pi_1(G/G^{\sigma})=1$. This follows from the Cartan decomposition $G=K\exp(\mathfrak{p})$.

(iii) Consider the case where $\mathfrak{g}$ is a complex simple Lie algebra and $\mathfrak{h}$ is its real form. Then $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$. So we know $\pi_1(G/G^{\sigma})\simeq\pi_1(K/K^{\sigma})$ from Corollary and (i).

(iv) Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair considered in (iii). In this case, $\mathfrak{h}^{a}$ is a complexification of $\mathfrak{k}\cap\mathfrak{h}=\mathfrak{k}\cap\mathfrak{h}^{a}$. So $\pi_1(G/G^{\theta\sigma})\simeq\pi_1(G/G^{\sigma})$ is also determined. Note that there is a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{k}\cap\mathfrak{h}$ is its maximal compact subalgebra. So $G/G^{\theta\sigma}$ is regarded as a "complexification of a Riemannian symmetric space".

(v) Consider the case where G/G^{σ} is a group space. In this case, there is a simple Lie algebra $\mathfrak{g}_1$ such that $\mathfrak{g}=\mathfrak{g}_1\times\mathfrak{g}_1$ and $\sigma(x, y)=(y, x)$ for any $x, y\in\mathfrak{g}_1$. Let G_1 be the adjoint group of $\mathfrak{g}_1$. Then $G=G_1\times G_1$ and the map of G to G_1 defined by $(g, h)\mapsto gh^{-1}$ induces an isomorphism of G/G^{σ} to G_1. Then $\pi_1(G/G^{\sigma})=\pi_1(G_1)$ is determined by E. Cartan. (For the sake of completeness, we also summarize the fundamental groups of connected non-compact real simple Lie groups with trivial center in Table 3.)

According to (i)–(v), it is sufficient to restrict our attention to the case where $\mathfrak{g}$ is a non-compact real form of a complex simple Lie algebra and $\mathfrak{h}$ is not a maximal compact subalgebra of $\mathfrak{g}$. In the sequel, we always assume this condition.

In general, K is not the adjoint group of $\mathfrak{k}$. But, if the Cartan involution θ of $\mathfrak{g}$ is an outer automorphism, then $\mathfrak{k}$ is semisimple and K is its adjoint group. So the determination of $\pi_1(K/K^{\sigma})$ is reduced to the compact case (i). Next consider the case where θ is an inner automorphism. In this case, since K is not necessarily the adjoint group of $\mathfrak{k}$, in order to determine $\pi_1(K/K^{\sigma})$, we need its concrete form (cf. Table 3). Let $\mathfrak{k}_s$ be the semisimple part of $\mathfrak{k}$. If $\mathfrak{k}=\mathfrak{k}_s$, that is, $\mathfrak{k}$ is semisimple but not abelian, then $\pi_1(K/K^{\sigma})$ is a finite group. On the other hand, if $\mathfrak{k}\neq\mathfrak{k}_s$, that is, $\mathfrak{k}$ is reductive but not semisimple, then $\pi_1(K/K^\sigma)$ is not necessarily a finite group. In fact, the center of K is a one dimensional torus. In this case, we need some computation to determine the torsion part of $\pi_1(K/K^\sigma)$. For the reasons stated above, it is better to decompose into the following cases:

Case (I) The Cartan involution θ is an outer automorphism of $\mathfrak{g}$.

Case (IIa) The Cartan involution θ is an inner automorphism of $\mathfrak{g}$ and K is simple but not abelian.

Case (IIb) The Cartan involution θ is an inner automorphism of $\mathfrak{g}$ and K is semisimple but not simple.

Case (IIIa) $\mathfrak{k}_s \neq \mathfrak{k}$ and $\mathfrak{k}_s$ is simple.

Case (IIIb) $\mathfrak{k}_s \neq \mathfrak{k}$ and $\mathfrak{k}_s$ is semisimple but not simple.

We are going to explain how $\pi_1(G/G^\sigma)$ is computed shortly. As explained before, the determination of $\pi_1(G/G^\sigma)$ for Case (I) is easy. For the other cases, we compute $\pi_1(G/G^\sigma)$ by case by case discussion using the concrete form of K. In almost all cases, it is sufficient to investigate the compact symmetric space K/K^σ instead of G/G^σ and it is not difficult to compute $\pi_1(K/K^\sigma)$. But in the case where $\mathfrak{g}$ is one of the exceptional Lie algebras $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(8)}$, we cannot determine $\pi_1(G/G^\sigma)$ if we only consider K/K^σ. The reason is as follows. Consider the semispinor group $Ss(4n)$ $(n>2)$. Then there are two involutions τ, τ' with the following property: Put $X=Ss(4n)/Ss(4n)^\tau$, $X'=Ss(4n)/Ss(4n)^{\tau'}$. Then X is isomorphic to $SO(4n)/U(2n)$ and therefore is simply connected and $X\to X'$ is a double covering. On the other hand, if $\mathfrak{g}$ is one of $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(8)}$, the maximal compact subgroup K is related with semispinor groups. In fact, $K=(Ss(12)\times SU(2))/Z_2$ if $\mathfrak{g}=\mathfrak{e}_{7(-5)}$, and $K=Ss(16)$ if $\mathfrak{g}=\mathfrak{e}_{8(8)}$ (cf. Table 3). These two cases are discussed in [S].

A classification of simple Lie groups are accomplished by Goto-Kobayashi [GK]. Their classification is based on the detailed study on the fundamental groups of adjoint groups. For a similar reason, it is possible to classify the global irreducible semisimple symmetric spaces by using the results in Table 4.

§ 2. The case of universal linear groups

If G is a real form of a simply connected complex simple Lie group, the fundamental group of G/G^σ is computed in a simple way for any involution σ of G. In this section, we shall discuss this subject.

Retain the notation of § 1. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g}_c$ be its complexification. Let G_c be a simply connected Lie group with the Lie algebra $\mathfrak{g}_c$. Then the real analytic subgroup of G_c corresponding to $\mathfrak{g}$ is called a universal linear group corresponding to $\mathfrak{g}$ and is denoted by G_{ul}. By definition, for a given Lie algebra, its universal linear group is unique up to isomorphism. Let K_{ul} be a maximal compact subgroup of G_{ul}. Since K_{ul} is semisimple or reductive, put $L=[K_{ul}, K_{ul}]$ and $T=$the center of K_{ul}. By definition,$K_{ul}=LT$.

Proposition 2. *Assume that* $\mathfrak{g}_c$ *is simple.*

(i) *If K_{ul} is semisimple, then $\tilde{G}=G_{ul}$ or $\tilde{G}$ is a double covering of G_{ul}, where $\tilde{G}$ is the universal cover of G.*

(ii) *If K is not semisimple, then L is simply connected.*

This result is well-known but the author does not find its proof in a literature. (One of its proofs is to check all the cases by using Table 3.)

Let σ be an involution of $\mathfrak{g}$ and let $(\mathfrak{g}, \mathfrak{h})$ be the corresponding symmetric pair. Constant use of the notation of § 1. Then $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ such that $\sigma(\mathfrak{k})=\mathfrak{k}$. By definition, σ can be lifted to G_{ul} and $\tilde{G}$. So we denote the liftings by the same letter. We may take K_{ul} such that $\mathfrak{k}$ is its Lie algebra.

Proposition 3. *Assume that* $\mathfrak{g}_c$ *is simple. Let σ be an involution of* $\mathfrak{g}$ *such that* $\sigma(\mathfrak{k})=\mathfrak{k}$.

(i) *If $\mathfrak{k}$ is semisimple, then $G_{ul}/(G_{ul})_0^\sigma$ is simply connected and $\#((G_{ul})^\sigma/(G_{ul})_0^\sigma)\leq 2$. Here $(G_{ul})_0^\sigma$ is the identity component of $(G_{ul})_0^\sigma$.*

(ii) *If $\mathfrak{k}$ is not semisimple and $\sigma(t)=t$ for all $t \in T$, then $G_{ul}/(G_{ul})^\sigma$ is simply connected.*

(iii) *If $\mathfrak{k}$ is not semisimple and $\sigma(t)=t^{-1}$ for all $t \in T$, then $\pi_1(G/(G_{ul})_0^\sigma)$ $=\boldsymbol{Z}$.*

Proof. First note that $\tilde{G}/\tilde{G}^\sigma$ is simply connected (cf. [L, Chap. IV, Th. 3.5]). In particular $\tilde{G}^\sigma$ is connected.

(i) If $\tilde{G}$ is linear, we have nothing to prove. So assume that $\tilde{G}$ is not linear. Then according to Proposition 2, (i), there is a central element $z \in \tilde{G}$ such that $\tilde{G}/\{1, z\}=G_{ul}$. Since σ induces involutions on both $\tilde{G}$ and G_{ul}, we find that $\sigma(z)=z$. Put $H=\{g \in \tilde{G}: g^{-1}\sigma(g) \in \{1, z\}\}$. By definition, $\tilde{G}/H\simeq G_{ul}/(G_{ul})^\sigma$. Now suppose that there is an element $g_0 \in \tilde{G}$ such that $\sigma(g_0)=zg_0$. Then $H=\tilde{G}^\sigma \cup g_0\tilde{G}^\sigma$. So we conclude that $(G_{ul})^\sigma$ has at most two connected components. Moreover, since $G_{ul}/(G_{ul})_0^\sigma$ $\simeq\tilde{G}/\tilde{G}^\sigma$, we find that $G_{ul}/(G_{ul})^\sigma$ is simply connected. Next consider the case where $\sigma(g)\neq zg$ for all $g \in \tilde{G}$. Then $H=\tilde{G}^\sigma$ and therefore $G_{ul}/(G_{ul})_0^\sigma$ is simply connected.

(ii) From the assumption, we find that $(TL)^\sigma=TL^\sigma$. Then $K_{ul}/(K_{ul})^\sigma\simeq L/L^\sigma$. It follows from Proposition 2, (ii) and a theorem of

E. Cartan on compact symmetric spaces that L/L^σ is simply connected. Hence $K_{ul}/(K_{ul})^\sigma$ and therefore $G_{ul}/(G_{ul})^\sigma$ is simply connected.

(iii) By definition, L^σ is a maximal compact subgroup of $(G_{ul})_0^\sigma$. Hence $\pi_1(G_{ul}/(G_{ul})_0^\sigma) \simeq \pi_1(TL/L^\sigma)$. By the assumption, T is a one dimensional torus. Therefore we identify T with $\{z \in C; |z|=1\}$. Define a map ϕ of $T \times L/L^\sigma$ to TL/L^σ by $\phi(t, mL)=tmL$. This is a finite covering. Take an element $x_0=L^\sigma$ of L/L^σ. Then there is an integer $n>0$ such that $\phi^{-1}(x_0)=\{y_k=(t_0^k, x_kL^\sigma); 0 \leq k \leq n\}$, where $t_0=\exp(2\pi i/n)$. Now take a path $c(\theta)=(c_1(\theta), c_2(\theta))$ $(0 \leq \theta < 1)$ on $T \times L/L^\sigma$ such that $c(j)=y_j$ $(j=0, 1)$. We may take $c_1(\theta)=\exp(2\pi i\theta/n)$. Then $\phi \circ c$ defines a homotopy class $[\phi \circ c]$ of $\pi_1(TL/L^\sigma, x_0)$. In virtue that $\pi_1(L/L^\sigma)=1$, $\pi_1(T)=\boldsymbol{Z}$, we find that $[\phi \circ c]$ is a generator of $\pi_1(TL/L^\sigma, x_0)$ and furthermore, $\boldsymbol{Z}[\phi \circ c] = \boldsymbol{Z}$. q.e.d.

Remark. The statement of Proposition 3, (i) is useful in the definition of principal series for semisimple symmetric space (cf. [O]).

§ 3. Tables

We use the notation of Helgason's book [H] without any comment.

(0) As for the results of Tables 1–3, the readers consult [C], [GK], [SS], [TM] and their references.

(1) *Table 1.* In this table, $\mathfrak{g}$ means a compact simple Lie algebra and $G=\mathrm{Int}\,\mathfrak{g}$.

(2) *Table 2.* The meaning of $\mathfrak{g}$ and G is the same as in the case (1). Take an involutive automorphism σ of G and put $K=\{g \in G; \sigma(g)=g\}$.

(3) *Table 3.* In this table, $\mathfrak{g}$ means a real simple Lie algebra, $G=\mathrm{Int}\,\mathfrak{g}$ and K means a maximal compact subgroup of G. By the Cartan decomposition, $\pi_1(G)=\pi_1(K)$. We refer to [TM] for the determination of K in the case where $\mathfrak{g}$ is one of $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(8)}$.

(4) *Table 4.* In this table, $(\mathfrak{g}, \mathfrak{h})$ means an irreducible symmetric pair. (A classification of irreducible symmetric pairs was accomplished by M. Berger [B].)

Remark. In Tables 1 and 3, the notation E_6, E_7, E_8, F_4, G_2 mean simply connected compact Lie groups with Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$, respectively.

In Table 3, the notation $(K_1 \times K_2)/\boldsymbol{Z}_2$ is used. For example, $(SO(2p) \times SO(2q))/\boldsymbol{Z}_2$, $(SU(6) \times SU(2))/\boldsymbol{Z}_2$, etc. Now explain its meaning. Take central elements $z_i \in K_i$ $(i=1, 2)$ of order 2. Put $Z=\{(1, 1), (z_1, z_2)\}$. Then $(K_1 \times K_2)/Z$ is written as $(K_1 \times K_2)/\boldsymbol{Z}_2$. The meaning of $(E_6 \times SO(2))/\boldsymbol{Z}_3$ is similar.

Full proofs will be published elsewhere.

Table 1. The fundamental group of a compact simple group

$\mathfrak{g}$	G	$\pi_1(G)$
$\mathfrak{su}(n)$	$SU(n)/Z_n$	Z_n
$\mathfrak{so}(2n+1)$	$SO(2n+1)$	Z_2
$\mathfrak{sp}(n)$	$Sp(n)/Z_2$	Z_2
$\mathfrak{so}(2n)$ $(n>2)$	$SO(2n)/Z_2$	Z_4 $(n$: odd) $Z_2\times Z_2$ $(n$: even)
$\mathfrak{e}_6$	E_6/Z_3	Z_3
$\mathfrak{e}_7$	E_7/Z_2	Z_2
$\mathfrak{e}_8$	E_8	1
$\mathfrak{f}_4$	F_4	1
$\mathfrak{g}_2$	G_2	1

Table 2. Fundamental groups of irreducible compact symmetric spaces

$(\mathfrak{g}, \mathfrak{k})$	$\pi_1(G/K)$
$(\mathfrak{su}(n), \mathfrak{so}(n))$	Z_n
$(\mathfrak{su}(2n), \mathfrak{sp}(n))$	Z_n
$(\mathfrak{su}(p+q), \mathfrak{su}(p)+\mathfrak{su}(q)+\mathfrak{so}(2))$	Z_d $(d=(p, q))$
$(\mathfrak{so}(p+q), \mathfrak{so}(p)+\mathfrak{so}(q))$	Z_2 $(p\neq q)$ Z_4 $(p=q$: odd) $Z_2\times Z_2$ $(p=q$: even)
$(\mathfrak{sp}(n), \mathfrak{u}(n))$	Z_2
$(\mathfrak{sp}(p+q), \mathfrak{sp}(p)+\mathfrak{sp}(q))$	1 $(p\neq q)$ Z_2 $(p=q)$
$(\mathfrak{so}(2n), \mathfrak{u}(n))$	1 $(n$: odd) Z_2 $(n$: even)
$(\mathfrak{e}_6, \mathfrak{sp}(4))$	Z_3
$(\mathfrak{e}_6, \mathfrak{su}(6)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_6, \mathfrak{so}(10)+\mathfrak{so}(2))$	1
$(\mathfrak{e}_6, \mathfrak{f}_4)$	Z_3
$(\mathfrak{e}_7, \mathfrak{su}(8))$	Z_2
$(\mathfrak{e}_7, \mathfrak{so}(12)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_7, \mathfrak{e}_6+\mathfrak{so}(2))$	Z_2
$(\mathfrak{e}_8, \mathfrak{so}(16))$	1
$(\mathfrak{e}_8, \mathfrak{e}_7+\mathfrak{su}(2))$	1
$(\mathfrak{f}_4, \mathfrak{sp}(3)+\mathfrak{su}(2))$	1
$(\mathfrak{f}_4, \mathfrak{so}(9))$	1
$(\mathfrak{g}_2, \mathfrak{so}(4))$	1

Table 3. Concrete forms of maximal compact subgroups and fundamental groups of non-compact real simple Lie groups

$\mathfrak{g}$	K	$\pi_1(G)$
$\mathfrak{sl}(2n, \boldsymbol{R})$ $(n>1)$	$SO(2n)/\boldsymbol{Z}_2$	$\boldsymbol{Z}_4$ $(n$: odd$)$ $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ $(n$: even$)$
$\mathfrak{sl}(2n+1, \boldsymbol{R})$	$SO(2n+1)$	$\boldsymbol{Z}_2$
$\mathfrak{su}^*(2n)$ $(n>2)$	$Sp(n)/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
$\mathfrak{su}(p, 1)$	$U(p)/\boldsymbol{Z}_{p+1}$	$\boldsymbol{Z}$
$\mathfrak{su}(p, q)$ $(p, q>1)$	$S(U(p)\times U(q))/\boldsymbol{Z}_{p+q}$	$\boldsymbol{Z}\times\boldsymbol{Z}_d$ $(d=(p, q))$
$\mathfrak{so}(2p, 1)$ $(p>1)$	$SO(2p)$	$\boldsymbol{Z}_2$
$\mathfrak{so}(2, 2q-1)$ $(q>1)$	$SO(2)\times SO(2q-1)$	$\boldsymbol{Z}\times\boldsymbol{Z}_2$
$\mathfrak{so}(2p, 2q-1)$ $(p, q>1)$	$SO(2p)\times SO(2q-1)$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
$\mathfrak{sp}(n, \boldsymbol{R})$ $(n>2)$	$U(n)/\boldsymbol{Z}_2$	$\boldsymbol{Z}$ $(n$: odd$)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(n$: even$)$
$\mathfrak{sp}(p, q)$ $(p, q>0)$	$(Sp(p)\times Sp(q))/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
$\mathfrak{so}(2p-1, 1)$ $(p>2)$	$SO(2p-1)$	$\boldsymbol{Z}_2$
$\mathfrak{so}(2p-1, 2q-1)$ $(p, q>1)$	$SO(2p-1)\times SO(2q-1)$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
$\mathfrak{so}(2p, 2)$ $(p>1)$	$(SO(2p)\times SO(2))/\boldsymbol{Z}_2$	$\boldsymbol{Z}\times\boldsymbol{Z}_2$
$\mathfrak{so}(2p, 2q)$ $(p, q>1)$	$(SO(2p)\times SO(2q)/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_4$ $(p$ or q: odd$)$ $\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ $(p, q$: even$)$
$\mathfrak{so}^*(2n)$ $(n>3)$	$U(n)/\boldsymbol{Z}_2$	$\boldsymbol{Z}$ $(n$: odd$)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(n$: even$)$
$\mathfrak{e}_{6(6)}$	$Sp(4)/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
$\mathfrak{e}_{6(2)}$	$(SU(6)/\boldsymbol{Z}_3\times SU(2))/\boldsymbol{Z}_2$	$\boldsymbol{Z}_6$
$\mathfrak{e}_{6(-14)}$	$(Spin(10)\times SO(2))/\boldsymbol{Z}_4$	$\boldsymbol{Z}$
$\mathfrak{e}_{6(-26)}$	F_4	1
$\mathfrak{e}_{7(7)}$	$SU(8)/\boldsymbol{Z}_4$	$\boldsymbol{Z}_4$
$\mathfrak{e}_{7(-5)}$	$(Ss(12)\times SU(2))/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
$\mathfrak{e}_{7(-25)}$	$(E_6\times SO(2))/\boldsymbol{Z}_3$	$\boldsymbol{Z}$
$\mathfrak{e}_{8(8)}$	$Ss(16)$	$\boldsymbol{Z}_2$
$\mathfrak{e}_{8(-24)}$	$(E_7\times SU(2))/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
$\mathfrak{f}_{4(4)}$	$(Sp(3)\times SU(2))/\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
$\mathfrak{f}_{4-(20)}$	$Spin(9)$	1
$\mathfrak{g}_{2(2)}$	$SO(4)$	$\boldsymbol{Z}_2$

Table 4. Fundamental groups of semisimple symmetric spaces

Case (I)

$(\mathfrak{sl}(n, \boldsymbol{R}), \mathfrak{sl}(i, \boldsymbol{R})+\mathfrak{sl}(n-i, \boldsymbol{R})+\boldsymbol{R})$ $(\mathfrak{sl}(n, \boldsymbol{R}), \mathfrak{so}(i, n-i))$ $(0<i\leq n/2, 2<n)$	$\boldsymbol{Z}_2$ $(2i<n)$ $\boldsymbol{Z}_4$ $(2n=n, i$: odd$)$ $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ $(2i=n, i$: even$)$
$(\mathfrak{sl}(2n, \boldsymbol{R}), \mathfrak{sp}(n, \boldsymbol{R}))$ $(\mathfrak{sl}(2n, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{C})+\mathfrak{so}(2))$ $(n>1)$	1 $(n$: odd$)$ $\boldsymbol{Z}_2$ $(n$: even$)$
$(\mathfrak{su}^*(2n), \mathfrak{su}^*(2i)+\mathfrak{su}^*(2n-2i)+\boldsymbol{R})$ $(\mathfrak{su}^*(2n), \mathfrak{sp}(i, n-i))$ $(0<i\leq n/2, 2<n)$	1 $(2i<n)$ $\boldsymbol{Z}_2$ $(2i=n)$
$(\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$ $(\mathfrak{su}^*(2n), \mathfrak{sl}(n, \boldsymbol{C})+\mathfrak{so}(2))$ $(2<n)$	$\boldsymbol{Z}_2$
$(\mathfrak{so}(2p-1, 2q-1), \mathfrak{so}(k)+\mathfrak{so}(2p-k-1, 2q-1))$ $(0<k<2p-1, 0<q)$	$\boldsymbol{Z}_2$
$(\mathfrak{so}(2p-1, 2q-1), \mathfrak{so}(k, h)+\mathfrak{so}(2p-k-1, 2q-h-1)$ $(0<k<2p-1, 0<h<2q-1)$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$
$(\mathfrak{so}(2n+1, 2n+1), \mathfrak{sl}(2n+1, \boldsymbol{R})+\boldsymbol{R})$ $(\mathfrak{so}(2n+1, 2n+1), \mathfrak{so}(2n+1, \boldsymbol{C}))$ $(n>0)$	$\boldsymbol{Z}_2$
$(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$ $(\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5)+\boldsymbol{R})$ $(\mathfrak{e}_{6(6)}, \mathfrak{sp}(2, 2))$	$\boldsymbol{Z}_2$
$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4, \boldsymbol{R}))$ $(\mathfrak{e}_{6(6)}, \mathfrak{sl}(6, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$	$\boldsymbol{Z}_2$
$(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$ $(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3, 1))$	1
$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9, 1)+\boldsymbol{R})$ $(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$	1

Case (IIa)

$(\mathfrak{so}(1, 2n), \mathfrak{so}(1, h)+\mathfrak{so}(2n-h))$ $(2<n, 0<h<2n)$	$\boldsymbol{Z}_2$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(12)+\mathfrak{su}(2))$ $(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)}+\mathfrak{so}(2))$	1
$(\mathfrak{e}_{7(7)}, \mathfrak{so}(6, 6)+\mathfrak{sl}(2, \boldsymbol{R}))$ $(\mathfrak{e}_{7(7)}, \mathfrak{su}(4, 4))$	$\boldsymbol{Z}_2$
$(\mathfrak{e}_{7(7)}, \mathfrak{sl}(8, \boldsymbol{R}))$	$\boldsymbol{Z}_4$
$(\mathfrak{e}_{7(7)}, \mathfrak{su}^*(8))$ $(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)}+\boldsymbol{R})$	$\boldsymbol{Z}_4$
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)}+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{8(8)}, \mathfrak{so}(8, 8))$	$\boldsymbol{Z}_2$
$(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(16))$ $(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(7)}+\mathfrak{sl}(2, \boldsymbol{R}))$	$\boldsymbol{Z}_2$
$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(1,8))$	1
$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2,1)+\mathfrak{su}(2))$	1

Case (IIb)

$(\mathfrak{so}(2p, 2q-1), \mathfrak{so}(k)+\mathfrak{so}(2p-k, 2q-1))$ $(1<p, q, 0<k<2p)$	Z_2
$(\mathfrak{so}(2p, 2q-1), \mathfrak{so}(k, h)+\mathfrak{so}(2p-k, 2q-h-1))$ $(1<p, q, 0\leq k\leq 2p, 0<h<2q-1)$	Z_2 $(k=0$ or $2p)$ $Z_2\times Z_2$ $(0<k<2p)$
$(\mathfrak{sp}(p, q), \mathfrak{sp}(k, h)+\mathfrak{sp}(p-k, q-h))$ $(0<p, q, 0\leq k\leq p, 0<h<q)$	1 $(2k\neq p$ or $2h\neq q)$ Z_2 $(2k=p$ and $2h=q)$
$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n)+\boldsymbol{R})$ $(\mathfrak{sp}(n, n), \mathfrak{sp}(n, \boldsymbol{C}))$	Z_2
$(\mathfrak{sp}(p, q), \mathfrak{su}(p, q)+\mathfrak{so}(2))$ $(0<p, q)$	Z_2
$(\mathfrak{so}(2p, 2q), \mathfrak{so}(k, h)+\mathfrak{so}(2p-k, 2q-h))$ $(1<p, q)$	Z_2 $(k=0, 2p$ or $h=0, 2q)$ $Z_2\times Z_2$ $\begin{pmatrix}0<k<2p, 0<h<2q\\ k\neq p \text{ or } h\neq q\end{pmatrix}$ $Z_2\times Z_4$ $\begin{pmatrix}k=p, h=q\\ p \text{ or } q \text{ odd}\end{pmatrix}$ $Z_2\times Z_2\times Z_2$ $\begin{pmatrix}k=p, h=q\\ p \text{ and } q \text{ even}\end{pmatrix}$
$(\mathfrak{so}(2p, 2q), \mathfrak{su}(p, q)+\mathfrak{so}(2))$ $(1<p, q)$	1 $(p$: odd or q: odd) Z_2 $(p, q$ even)
$(\mathfrak{so}(2n, 2n), \mathfrak{sl}(2n, \boldsymbol{R})+\boldsymbol{R})$ $(n>1)$ $(\mathfrak{so}(2n, 2n), \mathfrak{so}(2n, \boldsymbol{C})$	Z_4 $(n$: odd) $Z_2\times Z_2$ $(n$: even)
$(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10)+\mathfrak{so}(2))$	1
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(4, 6)+\mathfrak{so}(2))$ $(\mathfrak{e}_{6(2)}, \mathfrak{su}(2, 4)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3, 3)+\mathfrak{sl}(2, \boldsymbol{R}))$	Z_2
$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(3, 1))$ $(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$	Z_3
$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(4, \boldsymbol{R}))$	Z_6
$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)}+\mathfrak{so}(2))$	Z_2
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(4, 8)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4, 4))$	$Z_2\times Z_2$
$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(2, 6))$ $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)}+\mathfrak{so}(2))$	Z_2
$(\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(12)+\mathfrak{sl}(2, \boldsymbol{R}))$	Z_2
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(16))$	Z_2
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(4, 12)$ $(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-5)}+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-25)}+\mathfrak{sl}(2, \boldsymbol{R}))$	Z_2
$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$	Z_2
$(\mathfrak{f}_{4(4)}, \mathfrak{so}(4, 5))$ $(\mathfrak{f}_{4(4)}, \mathfrak{sp}(1, 2)+\mathfrak{su}(2))$	1
$(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$	Z_2

Case (IIIa)

$(\mathfrak{sl}(2, \boldsymbol{R}), \mathfrak{so}(1, 1))$	$\boldsymbol{Z}$
$(\mathfrak{su}(1, n), \mathfrak{su}(1, h)+\mathfrak{su}(n-h)+\mathfrak{so}(2))$ $(0<h<n)$	1
$(\mathfrak{so}(2, 2n-1), \mathfrak{so}(k, h)+\mathfrak{so}(2-k, 2n-h-1))$ $(1<n, 0\leq h\leq 2n-1)$	$\boldsymbol{Z}_2$ $(k=0, 2)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(k=1)$
$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{sp}(i, \boldsymbol{R})+\mathfrak{sp}(n-i, \boldsymbol{R}))$ $(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{su}(i, n-i)+\mathfrak{so}(2))$ $(0<i\leq n/2, 2<n)$	1 $(2i<n)$ $\boldsymbol{Z}_2$ $(2i=n)$
$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{R})+\boldsymbol{R})$ $(n>2)$	$\boldsymbol{Z}$ $(n$: odd$)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(n$: even$)$
$(\mathfrak{sp}(2n, \boldsymbol{R}), \mathfrak{sp}(n, \boldsymbol{C}))$ $(n>1)$	$\boldsymbol{Z}$ $(n$: odd$)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(n$: even$)$
$(\mathfrak{so}(2, 2n), \mathfrak{so}(k, h)+\mathfrak{so}(2-k, 2n-h))$ $(1<n, 0\leq h\leq 2n)$	$\boldsymbol{Z}_2$ $(k=0, 2)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(k=1)$
$(\mathfrak{so}(2, 2n), \mathfrak{su}(1, n)+\mathfrak{so}(2))$ $(2<n)$	1
$(\mathfrak{so}^*(2n), \mathfrak{so}^*(2i)+\mathfrak{so}^*(2n-2i))$ $(\mathfrak{so}^*(2n), \mathfrak{su}(i, n-i)+\mathfrak{so}(2))$ $(3<n)$	1 $(2i<n)$ $\boldsymbol{Z}_2$ $(2i=n)$
$(\mathfrak{so}^*(2n), \mathfrak{so}(n, \boldsymbol{C}))$ $(3<n)$	$\boldsymbol{Z}$ $(n$: odd$)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(n$: even$)$
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n)+\boldsymbol{R})$ $(2<n)$	$\boldsymbol{Z}$ $(n$: odd$)$ $\boldsymbol{Z}\times\boldsymbol{Z}_2$ $(n$: even$)$
$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$	$\boldsymbol{Z}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(2, 8)+\mathfrak{so}(2))$	1
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2, 4)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{6(-14)}, \mathfrak{sp}(2, 2))$	$\boldsymbol{Z}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(1, 5)+\mathfrak{sl}(2, \boldsymbol{R}))$ $(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10)+\mathfrak{so}(2))$	1
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}^*(8))$	$\boldsymbol{Z}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}(2, 10)+\mathfrak{sl}(2, \boldsymbol{R}))$ $(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)}+\mathfrak{so}(2))$	1
$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(2, 6))$ $(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(12)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)}+\boldsymbol{R})$	$\boldsymbol{Z}$

Case (IIIb)

$(\mathfrak{su}(p, q), \mathfrak{su}(k, h)+\mathfrak{su}(p-k, q-h)+\mathfrak{so}(2))$ $(p, q>1)$	1 $(2k\neq p$ or $2h\neq q)$ $\boldsymbol{Z}_2$ $(2k=p$ and $2h=q)$
$(\mathfrak{su}(p, q), \mathfrak{so}(p, q))$	$\boldsymbol{Z}\times\boldsymbol{Z}_d$ $(d=(p, q))$
$(\mathfrak{su}(2p, 2q), \mathfrak{sp}(p, q))$	$\boldsymbol{Z}\times\boldsymbol{Z}_d$ $(d=(p, q))$
$(\mathfrak{su}(n, n), \mathfrak{so}^*(2n))$ $(\mathfrak{su}(n, n), \mathfrak{sp}(n, \boldsymbol{R}))$ $(1<n)$	$\boldsymbol{Z}_n$
$(\mathfrak{su}(n, n), \mathfrak{sl}(n, \boldsymbol{C})+\boldsymbol{R})$	$\boldsymbol{Z}\times\boldsymbol{Z}_n$

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