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Cohomology of Classifying Spaces

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Introduction

The concept of the classifying space and characteristic classes are great tools in both geometry and topology.

Originally, the classifying space BG appeared as Grassmannian manifolds in discussing the equivalence of the fibre bundles of a fixed structure group G operating effectively on the fibre. And, the equivalence classes of such bundles on a CW-complex X are in one-to-one correspondence naturally with the homotopy classes of maps $f: X \to BG$ [39].

The classifying space BG of a topological group G is characterized as the base space of a universal G-bundle $G \to EG \to BG$ of ∞-connected total space EG. So, up to homotopy type, we may consider that the loop space ΩBG of BG is G, and BG is the de-looping of G.

For every associative H-space G, a classifying space BG is also constructed geometrically, and the construction is applied to give Eilenberg-Moore spectral sequences.

Lately, classifying space appeared in the theory of generalized cohomology. For each Brown functor F, i.e. a functor F satisfying wedge axiom and Mayer-Vietoris axiom, on the category of pointed finite CW-complexes. Under suitable condition, there exists a classifying space Y of F such that the functor F is naturally equivalent to the functor $[-, Y]_0$ of pointed homotopy classes. Then the original classifying space BG is that for the functor taking principal G-bundles over given base space [42].

The characteristic classes of fibre bundles are considered as a natural functor of fibre bundles to a cohomology class of the base spaces. For classical groups there are specially named characteristic classes, the Chern classes $c_n \in H^{2n}$ for unitary groups and complex general linear groups, the Stiefel-Whitney classes $w_n \in H^n(\ ; Z/2)$ for orthogonal groups and real general linear groups, the Pontrjagin classes $p_n \in H^{4n}$ for special orthogonal groups and real special linear groups, and others. By general theory of universal bundles, each characteristic class corresponds to an element of the cohomology $H^*(BG; -)$. The structure of the cohomology ring

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$H^*(BG;\,-)$ of suitable but essential coefficients are determined by A. Borel [7], [8], and the ring $H^*(BG;\,-)$ is a polynomial algebra on the above named characteristic classes.

For not classical Lie group G, the cohomology ring $H^*(BG; Z/p)$ is usually not a polynomial algebra if it has p-torsion, and the ring structure is very complicated. For such a case, one can apply Eilenberg-Moore spectral sequence

$$E_2 = \mathrm{Cotor}^{H^*(G;Z/p)}(Z/p, Z/p) \Longrightarrow H^*(BG; Z/p)$$

by the knowledge of the Hopf algebra structure of $H^*(G; Z/p)$.

The collapsing of the above spectral sequence is proved for exceptional groups in [21], [27], [28] and for projective classical groups $PG(4m+2)$ (where $G=U, Sp, SO$), $p=2$ in [20], [22], but the ring structure of $H^*(BG; Z/p)$ is not yet well-determined.

Here, we recall the case that G is a quotient G'/Γ of a classical group G' by a central subgroup Γ. The Hopf algebra structure of $H^*(G; Z/p)$ for such G was determined by Baum-Browder [6]. In general, the computations of E_2-term of the above spectral sequence seem difficult and complicated.

So, we propose to use an alternative spectral sequence

$$E_2 = \mathrm{Cotor}^{H^*(B\Gamma;Z/p)}(Z/p, H^*(BG'; Z/p)) \Longrightarrow H^*(BG; Z/p)$$

in place of the usual Eilenberg-Moore spectral sequence. The former one will serve a good information of the ring structure through the connection with the homomorphism $Bp^*: H^*(BG; Z/p) \to H^*(BG'; Z/p)$, where the image of Bp^* is contained in the subalgebra $PH^*(BG'; Z/p)$ of the primitive elements with respect to the action of $H^*(B\Gamma; Z/p)$ on $H^*(BG'; Z/p)$. In particular, $\mathrm{Im}\, Bp^* = PH^*(BG'; Z/p)$ if the spectral sequence collapses.

The present paper is an expositive note on classifying spaces and characteristic classes. The first two chapters are historical notes and the last two chapters are discussions on the above type of actions and applications to recovering fine structure of $H^*(BPG(4n+2); Z/2)$ [20] [22].

1. Classifying Spaces

1.1. Classifying spaces for fibre bundles

Let G be a topological group and consider principal G-bundles. A principal G-bundle

$$G \xrightarrow{i} E_n \xrightarrow{p} B_n$$

is called n-universal if the total space E_n is $(n-1)$-connected. The name "n-universal" is derived from the following "classifying theorem".

Theorem 1.1 (Steenrod [39]). *Let K be a CW-complex of* $\dim K<n$. *The operation of assigning to each map f: $K \to B_n$ its induced bundle sets up a* 1–1 *correspondence between homotopy classes of maps of K into B_n and equivalence classes of principal G-bundles over K.*

An ∞-universal bundle is called simply "universal bundle":

$$G \xrightarrow{i} EG \xrightarrow{p} BG$$

and the base space BG is called a "classifying space" of G.

If we consider the case that every BG are CW-complexes, then the above theorem shows that the classifying space BG is unique up to homotopy equivalence.

The equivalence classes of fibre bundles with fibre F, on which the structure group operates effectively, are in 1–1 correspondence with the equivalent classes of associated principal G-bundles. So, the classifying space BG also "classifies" the fibre bundles with such fibre F.

In his book [39], Steenrod constructed an n-universal bundle for each Lie group G. Using a faithful representation of G, G is considered as a subgroup of $O(N) \subset O(N+n)$.

Theorem 1.2. *$G \to E_n = O(N+n)/O(n) \to B_n = O(N+n)/(G \times O(n))$ is an n-universal G-bundle.*

In [25], Milnor constructed an n-universal bundle for arbitrary topological group G. Let

$$G * \cdots * G = \{t_0 g_0 + \cdots + t_n g_n \mid (t_0, \cdots, t_n) \in \Delta^n, g_i \in G\}$$

be the join of $(n+1)$-copies of G and give an action of G by

$$(t_0 g_0 + \cdots + t_n g_n) g = t_0 g_0 g + \cdots + t_n g_n g.$$

Theorem 1.3 (Milnor). *$G \to E_n \to B_n = E_n/G$ is an n-universal G-bundle.*

In both cases, the classifying space BG is a suitable limit of n-universal spaces B_n.

1.2. Classifying spaces of associative H-spaces

The natural inclusion of topological group G into the loop space ΩBG of the classifying space BG is a weak homotopy equivalence. So, we may consider that the classifying space functor B is the inverse of the loop space functor Ω, in the category of topological spaces homotopy equivalent to CW-complexes. The loop space ΩX is a homotopy associative H-space. Moreover, if we consider Moore type loop space it is a strictly associative H-space homotopy equivalent to the usual loop space.

Conversely, Dold-Lashof [14] and Rothenberg-Steenrod [32] constructed a classifying space BG for each associative H-space G. Their construction is done by giving a sequence of G-spaces, called a "G-resolution":

$$E_0=G\subset E_1\subset\cdots\subset E_n\subset E_{n+1}\subset\cdots,\qquad E=\bigcup_{n=0}^{\infty}E_n$$

such that E_n is contractible to a point in E_{n+1} and the G-action gives a relative homeomorphism $(D_n\times G, E_{n-1}\times G)\to(E_n, E_{n-1})$ for a suitable subset D_n with $E_{n-1}\subset D_n\subset E_n$. The quotient map $p\colon E\to E/G$ is a quasi-fibration, E is ∞-connected, and $B=E/G$ becomes a classifying space of G. The G-resolution is parallel to the algebraic bar construction, and the spectral sequence associated to the filtration $\{E_n\}$ is so-called "Eilenberg-Moore spectral sequence":

$$E_2=\mathrm{Cotor}^{H^*(G;Z/p)}(Z/p, Z/p)\Longrightarrow H^*(BG; Z/p).$$

Rothenberg-Steenrod [32] [33] proved that this is a multiplicative spectral sequence and showed its usefulness. For example, Borel's transgression theorem [7] is an easy consequence of this, and the cohomology of Eilenberg-MacLane space can be computed without difficulties.

The Eilenberg-MacLane space $K(\Gamma, n)$ is an important example of classifying space. $K(\Gamma, n)$ is homotopy equivalent to $\Omega K(\Gamma, n+1)$, or, we can say $K(\Gamma, n+1)=BK(\Gamma, n)$. We refer the mod 2 cohomology of $K(\Gamma, n)$ for the convenience of latter use.

When $\Gamma=Z$, infinite cyclic group, we may identify $K(Z, 1)=S^1$ and $K(Z, 2)=CP^\infty$. So, by indicating x_n an element of $H^n(\ ; Z/2)$,

$$H^*(K(Z, 1); Z/2)=\Lambda(x_1)\quad\text{and}\quad H^*(K(Z, 2); Z/2)=Z/2[x_2].$$

Let x_{2^k+1} be a transgression image of $x_2^{2^k}$, then we have

$$H^*(K(Z, 3); Z/2)=Z/2[x_3, x_5, \cdots, x_{2^k+1}, \cdots]. \tag{1.1}$$

When $\Gamma=Z/r$ the cyclic group of order r, $K(Z/r, 1)$ has an infinite dimensional lens space as an example, and

$$H^*(K(Z/r, 1); Z/2) = Z/2[x_1] \qquad \text{if } r \equiv 2 \pmod 4,$$
$$H^*(K(Z/r, 1); Z/2) = \Lambda(x_1) \otimes Z/2[x] \qquad \text{if } r \equiv 0 \pmod 4.$$

Then, for transgression image x_2 of x_1 and x_{2^k+1} of $x_2^{2^k}$ ($x_2 = x_1^2$ if $r \equiv 2 \pmod 4$) and for even r, we have

$$(1.2) \qquad H^*(K(Z/r, 2); Z/2) = Z/2[x_2, x_3, x_5, \cdots, x_{2^k+1}, \cdots].$$

1.3. Classifying space for generalized cohomology theory

For a generalized cohomology theory $h^* = \{h^n\}$ on a category of pointed CW-complexes, h^n is a Brown functor, Each Brown functor has a classifying space E provided the countability [13] or group structures [1] of the values. Then h^* has a spectrum $\{E_n\}$ as a sequence of classifying spaces and structure maps $\Sigma E_n \to E_{n+1}$. G. Segal [34] gave a geometrical construction of classifying spaces on categories. Milnor's classifying space is an example of Segal's one. Moreover, categories with some composition laws (Γ-categories) give spectra [35].

2. Characteristic Classes

2.1. Characteristic classes for classical groups

By the classification theorem for the fibre bundles with a structure group G, each characteristic class corresponds to an element of the cohomology of the classifying space BG. So, we shall consider the cohomology ring

$$H^*(BG; R)$$

with suitable coefficient ring R.

Let K be a maximal compact subgroup of a connected Lie group G, then it is well known that G is homeomorphic to the product of K and an euclidean space, and we have isomorphisms

$$H^*(G; R) \cong H^*(K; R)$$

and

$$H^*(BG; R) \cong H^*(BK; R).$$

So we shall discuss only for the case that G is compact.

For the classical groups G, the ring structure of $H^*(BG; R)$ are determined by A. Borel [7] [8] whose results are stated with a connection to a maximal torus T^l of G, $l=\text{rank } G$.

The classifying space BT^l of the torus T^l is equivalent to the product of l copies of BT or $CP^\infty=K(Z; 2)$ for $T=S^1=K(Z, 1)$, and

$$(2.1) \qquad H^*(BT^l; R)=R[t_1, t_2, \cdots, t_n], \qquad t_i \in H^2.$$

The Weyl group $\Phi(G)=N_G(T^l)/T^l$ acts on BT^l and the invariant subalgebra $H^*(BT^l; R)^{\Phi(G)}$ contains the image of the homomorphism

$$Bi^*: H^*(BG; R) \longrightarrow H^*(BT^l; R)$$

indcued by the natural map $Bi: BT^l \rightarrow BG$. The followings are the results due to A. Borel [7].

Theorem 2.1. *In the following cases, Bi^* are isomorphisms of $H^*(BG; R)$ onto $H^*(BT^l; R)^{\Phi(G)}$.*

(i) *$G=U(n)$, $l=n$ and R is arbitrary:*

$$Bi^*: H^*(BU(n); R) \cong H^*(BT^n; R)^{\Phi(U(n))}=R[\sigma_1, \sigma_2, \cdots, \sigma_n]$$

for the j-th elementary symmetric function σ_j of the variables $t_1, \cdots, t_n$.

(ii) *$G=SU(n)$, $l=n-1$ and R is arbitrary:*

$$Bi^*: H^*(BSU(n); R) \cong H^*(BT^l; R)^{\Phi(SU(n))}=R[\sigma_2, \cdots, \sigma_n]$$

by identifying $H^*(BT^l; R)$ with $H^*(BT^n; R)/(\sigma_1)$.

(iii) *$G=Sp(n)$, $l=n$ and R is arbitrary:*

$$Bi^*: H^*(BSp(n); R) \cong H^*(BT^n; R)^{\Phi(Sp(n))}=R[\bar\sigma_1, \bar\sigma_2, \cdots, \bar\sigma_n]$$

for the j-th elementary symmetric function $\bar\sigma_j$ of the variables $t_1^2, \cdots, t_n^2$.

(iv) *$G=SO(2n+1)$, $l=n$ and R is a field of characteristic $\neq 2$:*

$$Bi^*: H^*(BSO(2n+1); R) \cong H^*(BT^n; R)^{\Phi(SO(2n+1))}=R[\bar\sigma_1, \bar\sigma_2, \cdots, \bar\sigma_n].$$

(v) *$G=SO(2n)$, $l=n$ and R is a field of characteristic $\neq 2$:*

$$Bi^*: H^*(BSO(2n); R) \cong H^*(BT^n; R)^{\Phi(SO(2n))}=R[\bar\sigma_1, \cdots, \bar\sigma_{n-1}, \sigma_n].$$

By virtue of the theorem, the k-th Chern class $c_k \in H^{2k}(BU(n); R)$, $H^*(BSU(n); R)$ and the k-th symplectic Pontrjagin class $q_k \in H^{4k}(BSp(n); R)$ are defined by the following equations.

$$(2.2) \qquad Bi^*\left(\sum_{k=0}^n c_k\right)=\prod_{i=1}^n (1+t_i), \qquad c_0=1,$$

$$Bi^*\left(\sum_{k=0}^{n} q_k\right)=\prod_{i=1}^{n}(1+t_i^2), \qquad q_0=1.$$

Then we have

$$\begin{aligned} &H^*(BU(n);R)=R[c_1, c_2, \cdots, c_n], \\ (2.3) \qquad &H^*(BSU(n);R)=R[c_2, \cdots c_n], \qquad c_1=0, \\ &H^*(BSp(n);R)=R[q_1, q_2, \cdots, q_n]. \end{aligned}$$

The k-th Pontrjagin class $p_k \in H^{4k}(BSO(m);R)$ is defined by

$$(2.4) \qquad p_k=(-1)^k Bj^*(c_{2k})$$

for the homomorphism $Bj^*: H^*(BU(m);R) \to H^*(BSO(m);R)$ induced by the complexification $Bj: BSO(m) \to BU(m)$. Then we have

(2.5) (i) $Bi^*\left(\sum_{k=0}^{n} p_k\right)=\prod_{i=1}^{n}(1+t_i^2), \quad p_0=1$

for $Bi: BT^n \to BSO(m)$ *and* $m=2n+1$ *or* $m=2n$.

(ii) *For a coefficient field* R *of characteristic* $\neq 2$,

$$\begin{aligned} &H^*(BSO(2n+1);R)=R[p_1, p_2, \cdots, p_n], \\ &H^*(BSO(2n);R)=R[p_1, \cdots, p_{n-1}, \chi], \end{aligned}$$

where $Bi^*(\chi)=\sigma_n$, χ *is the Euler class and* $\chi^2=p_n$.

The equation (2.5), (i) follows from the correspondence between maximal tori $T^n=SO(m)\cap T^m$ of $SO(m)$ and T^m of $U(m)$.

By similar methods, we have

(2.6) (i) $Bj^*(q_k)=\sum_{i+j=k}(-1)^{i+k}c_i c_j$ *for* $Bj: BU(n) \to BSp(n)$.

(ii) $Bj^*(c_{2k-1})=0$ and $Bj^*(c_{2k})=(-1)^k q_k$ *for*: $BSp(n) \to BSU(2n)$.

(iii) $Bj^*(p_k)=\sum_{i+j=k}(-1)^i c_i c_j$, $Bj^*(\chi)=c_n$ *for* $Bj: BU(n) \to BSO(m)$, $m=2n$ *or* $m=2n+1$ *and for a coefficient field of characteristic* $\neq 2$.

There is a mod 2 version of the above results. The diagonal matrices of $O(n)$ form a subgroup Γ^n isomorphic to $(Z/2)^n$, $B\Gamma^n$ is equivalent to the product of n-copies of $RP^\infty=K(Z/2,1)$ and

$$(2.1)' \qquad H^*(B\Gamma^n; Z/2)=Z/2[u_1, u_2, \cdots, u_n], \qquad u_i \in H^1.$$

The group $\Phi'(O(n))=N_{O(n)}(\Gamma^n)/\Gamma^n$ acts on $H^*(B\Gamma^n;Z/2)$ as the permutation of u_i's. Let σ'_j be the j-th elementary symmetric function of the variables $u_1, \cdots, u_n$. Then we have

Theorem 2.2. *The natural map Bi: $B\Gamma^n \to BO(n)$ induces an isomorphism*

$$Bi^*: H^*(BO(n); Z/2) \cong H^*(B\Gamma^n; Z/2)^{\Phi'(O(n))} = Z/2[\sigma'_1, \sigma'_2, \cdots, \sigma'_n].$$

Let $\Gamma^{n-1} = \Gamma^n \cap SO(n)$. By identifying $H^*(B\Gamma^{n-1}; Z/2)$ with $H^*(B\Gamma^n; Z/2)/(\sigma'_1)$, Bi: $B\Gamma^{n-1} \to BSO(n)$ induces an isomorphism

$$Bi^*: H^*(BSO(n); Z/2) \cong H^*(B\Gamma^{n-1}; Z/2)^{\Phi'} = Z/2[\sigma'_2, \cdots, \sigma'_n],$$

where $\Phi' = N_{SO(n)}(\Gamma^{n-1})/\Gamma^{n-1} \cong \Phi'(O(n))$.

The Stiefel-Whitney class $w_k \in H^k(BO(n); Z/2)$ (or $H^k(BSO(n); Z/2)$) is defined by

$$(2.2)' \qquad Bi^*\left(\sum_{k=0}^{n} w_k\right) = \prod_{i=1}^{n} (1+u_i), \qquad w_0 = 1.$$

Then

$$(2.3)' \qquad \begin{aligned} &H^*(BO(n); Z/2) = Z/2[w_1, w_2, \cdots, w_n], \\ &H^*(BSO(n); Z/2) = Z/2[w_2, \cdots, w_n], \qquad w_1 = 0. \end{aligned}$$

Relations between Chern and Stiefel-Whitney classes are the followings.

(2.6)′ (i) $Bj^*(c_k) = w_k^2$ *for* Bj: $BO(n) \to BU(n)$.
(ii) $Bj^*(w_{2k-1}) = 0$ *and* $Bj^*(w_{2k}) = c_k$ *for* Bj: $BU(n) \to BSO(2n)$.

The squaring operations on characteristic classes are treated by use of the isomorphisms Bi^* in the above theorems, and we have the following Wu's formulae:

$$(2.7) \qquad \begin{aligned} &\mathrm{Sq}^i w_k = \sum_{j=0}^{i} \binom{k-j-1}{i-j} w_{k+i-j} w_j \quad (0 \le i \le k) \quad \text{in } H^*(BO(n); Z/2), \\ &\mathrm{Sq}^{2i} c_k = \sum_{j=0}^{i} \binom{k-j-1}{i-j} c_{k+i-j} c_j \quad (0 \le i \le k) \quad \text{in } H^*(BU(n); Z/2), \\ &\mathrm{Sq}^{4i} q_k = \sum_{j=0}^{i} \binom{k-j-1}{i-j} q_{k+i-j} q_j \quad (0 \le i \le k) \quad \text{in } H^*(BSp(n); Z/2). \end{aligned}$$

For reduced power operations, see [10].

2.2. Characteristic classes for simple Lie groups

In general, if a compact connected Lie group G has no p-torsion, the cohomology ring $H^*(G; Z/p)$ is an exterior algebra generated by odd dimensional elements. Then the cohomology ring $H^*(BG; Z/p)$ of the classifying space BG is determined by the following theorem.

Theorem 2.3 (Borel's transgression theorem [7]). *If $H^*(G; Z/p) = \Lambda(x_1, \cdots, x_n)$ the exterior algebra generated by elements x_i's of odd dimensionals, then we can choose x_i's to be transgressive and for the transgression images $y_i = \tau(x_i)$ we have*

$$H^*(BG; Z/p) = Z/p[y_1, \cdots, y_n].$$

Now consider the case that G is simply connected and compact. Such G is isomorphic to the direct sum of simple Lie groups. The compact simply connected simple Lie groups are $SU(n)$, $Sp(n)$, Spin(n) and exceptional groups G_2, F_4, E_6, E_7, E_8. The first two are torsion free and the remaining ones have 2-torsions. F_4, E_6, E_7 and E_8 have 3-torsions and E_8 has 5-torsion.

The mod 2 cohomology $H^*(B\,\mathrm{Spin}(n); Z/2)$ of the classifying space of the spinor group Spin(n) is computed by use of the Serre spectral sequence

$$E_2 = Z/2[w_2, w_3, \cdots, w_n] \otimes Z/2[t] \Longrightarrow H^*(B\,\mathrm{Spin}(n); Z/2)$$

associated with the fibering $B(Z/2) \to BSO(n) \to B\,\mathrm{Spin}(n)$, where the transgression τ is given by $\tau(t) = w_2$, and by use of Wu formula,

$$\tau(t^2) = \mathrm{Sq}^1 w_2 = w_3, \quad \tau(t^4) = \mathrm{Sq}^2 w_3 \equiv w_5, \quad \tau(t^8) \equiv \mathrm{Sq}^4 w_5 \equiv w_9,$$
$$\tau(t^{16}) \equiv \mathrm{Sq}^8 w_9 \equiv w_{17} + w_{13} w_4 + w_{11} w_6 + w_{10} w_7, \cdots.$$

Then, for $n \leq 9$, $H^*(B\,\mathrm{Spin}(n); Z/2)$ is a polynomial algebra generated by w_4, w_6, w_7, w_8 of $\dim \leq n$. But if $n \geq 10$ and $n-1$ is not a power of 2, then $H^*(B\,\mathrm{Spin}(n); Z/2)$ is no more a polynomial algebra. For example $H^*(B\,\mathrm{Spin}(10); Z/2) \simeq Z/2[w_4, w_6, w_7, w_8, w_{10}]/(w_{10} w_7)$. The general results on $H^*(B\,\mathrm{Spin}(n); Z/2)$ are given in [31] applying the theory of quadratic forms.

Next consider the exceptional groups with p-torsions.

There are two examples of polynomial algebras [8]:

$$H^*(BG_2; Z/2) = Z/2[x_4, x_6, x_7],$$
$$H^*(BF_4; Z/2) = Z/2[x_4, x_6, x_7, x_{16}, x_{24}].$$

For the other cases of the exceptional groups G with p-torsions, $H^*(BG; Z/p)$ is not a polynomial algebra which is a consequence of that the Hopf algebra $H^*(G; Z/p)$ is not primitively generated.

$H^*(BF_4; Z/3)$ is generated by generators of dimensions 4, 8, 9, 20, 21, 23, 25, 26, 36, 48 with 15 relations [43].

For remaining cases, the ring structure of $H^*(BG; Z/p)$ is not yet strictly fixed. They are investigated by use of Eilenberg-Moore spectral sequence

$$E_2 = \mathrm{Cotor}^{H^*(G;Z/p)}(Z/p; Z/p) \Longrightarrow H^*(BG; Z/p)$$

from the knowledge of the Hopf algebra structure of $H^*(G; Z/p)$ [29]. As the results, the above spectral sequences are collapses for all cases [44] [21] [27] [28].

Finally, we consider non-simply connected cases.

The classical groups $U(n)$, $SU(n)$, $Sp(n)$, $SO(2m)$ have the centers isomorphic to Z, Z/n, $Z/2$ and $Z/2$, respectively. The quotion groups of the classical groups by the centers or central subgroups are called the projective groups and denoted by the forms $PG(n)$ or $P'G(n)$.

The Hopf algebra structure of $H^*(PG(n); Z/p)$ is determined by Baum-Browder [6], and by use of Eilenberg-Moore spectral sequence, $H^*(BPG(4m+2); Z/2)$ are computed as the collapsing of the spectral sequence [20] [22]. But, in the statement of the ring structure of the E_2-term, there are some ambiguities which will be clarified in the subsequent chapters.

Note that there are results on $H^*(BG; Z/p)$ for $G = \mathrm{Ad}\, E_7 = E_7/(Z/2)$, $p=2$ and $G = PU(3)$, $p=3$ [24].

3. Action of $B\Gamma$ on BG

3.1. Comparison of actions

Consider a compact Lie group, a central subgroup Γ and the quotient group $\bar{G} = G/\Gamma$. We always assume that Γ is isomorphic to S^1 or discrete cyclic. Then we have

$$B\Gamma = K(Z, 2), \quad BB\Gamma = K(Z, 3) \qquad \text{if } \Gamma \cong S^1 = K(Z, 1),$$

and

$$B\Gamma = K(\Gamma, 1), \quad BB\Gamma = K(\Gamma, 2) \qquad \text{if } \Gamma \text{ is cyclic.}$$

From the exact sequence $1 \to \Gamma \xrightarrow{i} G \xrightarrow{p} \bar{G} = G/\Gamma$, it induces a fibration $B\Gamma \xrightarrow{Bi} BG \xrightarrow{Bp} B\bar{G}$. Furthermore we have the following homotopy commutative diagram:

$$\begin{array}{ccccccc} B\Gamma & \xrightarrow{Bi} & BG & \xrightarrow{Bp} & B\bar{G} & & \\ \downarrow & & \downarrow & & \| & & \\ \Omega(BB\Gamma) & \longrightarrow & F & \longrightarrow & B\bar{G} & \xrightarrow{f} & BB\Gamma. \end{array} \tag{3.1}$$

Here, the lower sequence is a fibre sequence induced by a map f which represents the transgression image of the fundamental class of $B\Gamma$. The vertical maps are weak homotopy equivalences.

Then we consider the action of $B\Gamma$ on BG replacing by the action

$$\mu\colon \Omega(BB\Gamma)\times F \longrightarrow F$$

of the loop $\Omega(BB\Gamma)$ on the homotopy fiber F. So, we have

(3.2) *The action induces a ring homomorphism of cohomology rings*

$$\phi=\mu^*\colon H^*(BG)\longrightarrow H^*(B\Gamma)\otimes H^*(BG)\cong H^*(B\Gamma\times BG),$$

with suitable coefficients, satisfying

$$(\phi\otimes 1)\phi=(1\otimes\phi)\phi$$

and

$$\phi(x)=1\otimes x+\textit{higher term},$$

that is, $H^*(BG)$ is a comodule algebra over the Hopf algebra $H^*(B\Gamma)$.

We shall use the following lemma to determine the actions for classical groups G.

Lemma 3.1. *Let G and Γ be as above, and let G' be a closed subgroup of G and Γ' a subgroup of $G'\cap\Gamma$. Then the natural maps $Bi\colon BG'\to BG$ and $Bj\colon B\Gamma'\to B\Gamma$ are compatible with the actions. Thus the following diagram commutes.*

$$\begin{array}{ccc} H^*(BG) & \xrightarrow{\phi} & H^*(B\Gamma)\otimes H^*(BG) \\ \downarrow{\scriptstyle Bi^*} & & \downarrow{\scriptstyle Bj^*\otimes Bi^*} \\ H^*(BG') & \xrightarrow{\phi} & H^*(B\Gamma')\otimes H^*(BG') \end{array}$$

3.2. The action on $BU(n)$

We start from the case

$\bar{G}=PU(n)$, that is, $G=U(n)$, $\Gamma=$(the center of $U(n)$)$\cong S^1$.

As is seen in the previous section

$$H^*(BU(n);Z)=Z[c_1, c_2, \cdots, c_n] \quad \text{and} \quad H^*(B\Gamma;Z)=Z[t]$$

for the k-th Chern class $c_k \in H^{2k}$ and the Euler class $t \in H^2$.

Proposition 3.2. *The action*

$$\phi=\mu^*: H^*(BU(n);Z) \longrightarrow H^*(B\Gamma;Z) \otimes H^*(BU(n);Z)$$

is determined by the following formula:

$$\phi(c_k)=\sum_{i+j=k} \binom{n-j}{i} t^i \otimes c_j.$$

Proof. Let T^n be the maximal torus of $U(n)$ which consists of diagonal matrices. Since Γ is a subgroup of T^n, it follows from Lemma 3.1 the following commutative diagram:

$$\begin{array}{ccc}
H^*(BU(n);Z) & \xrightarrow{\phi} & H^*(B\Gamma;Z)\otimes H^*(BU(n);Z) \\
\downarrow{\scriptstyle Bi^*} & & \downarrow{\scriptstyle 1\otimes Bi^*} \\
H^*(BT^n;Z) & \xrightarrow{\phi} & H^*(B\Gamma;\ Z)\otimes H^*(BT^n;Z)
\end{array}$$

Here, $H^*(BT^n;Z)=Z[t_1, t_2, \cdots, t_n]$ for canonical generators $t_i \in H^2$, Bi^* is injective and

$$Bi^*\left(\sum_{i=0}^n c_i\right)=\prod_{i=1}^n (1+t_i).$$

Since Γ acts diagonal-wise on T^n, $\phi(t_i)=1\otimes t_i+t\otimes 1$. Then

$$\begin{aligned}
(1\times Bi^*)\phi\left(\sum_{i=0}^n c_i\right) &= \phi Bi^*\left(\sum_{i=0}^n c_i\right)=\phi\prod_{i=1}^n (1+t_i) \\
&= \prod_{i=1}^n (1\otimes 1+t\otimes 1+1\otimes t_i) \\
&= \sum_{j=0}^n (1+t)^{n-j}\otimes Bi^* c_j \\
&= (1\otimes Bi^*)\sum_{j=0}^n \sum_{i=0}^j \binom{n-j}{i} t^i \otimes c_j.
\end{aligned}$$

So, the proposition follows from the injectivity of Bi^*. □

The natural map Bi: $BSU(n) \to BU(n)$ induces a projection

$$Bi^*: H^*(BU(n);Z) \longrightarrow H^*(BSU(n), Z)=Z[c_2, \cdots, c_n]$$

by giving the relation $c_1=0$.

Next consider the case that Γ is a central subgroup of $G=SU(n)$ or $U(n)$ which is cyclic of order n'. Since the center of $SU(n)$ is cyclic of order n, n' divides n when $\Gamma\subset SU(n)$.

Let p be a prime which divides n', then we have

$$H^*(B\Gamma; Z/p)=\Lambda(u)\otimes Z/p[t], \qquad u\in H^1,\ t\in H^2$$

where $u^2=t$ if $p=2$ and $n'\equiv 2 \pmod 4$ and $u^2=0$ otherwise.

Applying Lemma 3.1 we have the following commutative diagram

$$\begin{array}{ccc} H^*(BU(n); Z/p) & \xrightarrow{\phi} & H^*(BS^1; Z/p)\otimes H^*(BU(n); Z/p) \\ \downarrow{\scriptstyle Bi^*} & & \downarrow{\scriptstyle Bj^*\otimes Bi^*} \\ H^*(BG; Z/p) & \xrightarrow{\phi} & H^*(B\Gamma; Z/p)\otimes H^*(BG; Z/p) \end{array}$$

for the natural maps Bi: $BG\to BU(n)$, Bj: $B\Gamma\to BS^1$. Here, $Bj^*(t)=t$ $(=u^2$ if $p=2$ and $n'\equiv 2 \pmod 4)$. Then we have

Corollary 3.3. *Let Γ be a central cyclic group of order n' in $G=SU(n)$ or $U(n)$ and p be a prime dividing n'. Then the action ϕ of $H(B\Gamma; Z/p)$ on $H^*(BG; Z/p)$ is determined by the following formula*

$$\phi(c_k)=\sum_{i+j=k}\binom{n-j}{i}t^i\otimes c_j, \qquad (c_1=0 \text{ if } G=SU(n))$$

where $t=u^2$ if $p=2$ and $n'\equiv 2 \pmod 4$.

3.3. The action on $BSp(n)$ and $BSO(2m)$

The symplectic group $Sp(n)$ is a subgroup of $SU(2n)$, and the center Γ of $Sp(n)$ is of order 2 and a central subgroup of $SU(2n)$. The cohomology ring of $BSp(n)$ is

$$H^*(BSp(n); Z)=Z[q_1, q_2, \cdots, q_n]$$

and the natural map Bi: $BSp(n)\to BSU(2n)$ carries

$$Bj^*(c_{2k})=q_k \quad\text{and}\quad Bj^*(c_{2k-1})=0.$$

Then it follows from Lemma 3.1

Proposition 3.4. *The action*

$$\phi=\mu^*: H^*(BSp(n); Z/2)\longrightarrow H^*(B\Gamma; Z/2)\otimes H^*(BSp(n); Z/2)$$

is determined by the formula

$$\phi(q_k)=\sum_{i+j=k}\binom{n-j}{i}u^{4i}\otimes q_j$$

for the generator $u\in H^1$ *of* $H^*(B\Gamma;Z/2)=Z/2[u]$.

The center Γ of $SO(2m)$, $m\geq 2$, is isomorphic to $Z/2$ and its action on the subgroup Γ^{2m} of $SO(2m)$ is diagonal-wise. So, $\phi(u_i)=1\otimes u_i+u\otimes 1$ for the generators $u_1,\cdots,u_{2m}$ of $H^*(B\Gamma;Z/2)$. Then the proof of the following proposition is similar to that of Proposition 3.2.

Proposition 3.5. *The action*

$$\phi=\mu^*: H^*(BSO(2m);Z/2)\longrightarrow H^*(B\Gamma;Z/2)\otimes H^*(BSO(2m);Z/2)$$

is determined by the formula

$$\phi(w_k)=\sum_{i+j=k}\binom{n-j}{i}u^i\otimes w_j,\qquad w_1=0.$$

3.4. Comodule action of the polynomial algebra

The essntial part of the previous actions are that of the polynomial part. So, we consider a comodule algebra A over the primitively generated algebra $k[x]$, where k is a field of the characteristic p. Let

$$\phi: A\longrightarrow k[x]\otimes A$$

be the comodule action. Define linear maps

$$d_i: A\longrightarrow A\qquad \text{for } i=0,1,2,\cdots$$

by putting

$$\phi(a)=\sum_{i\geq 0}x^i\otimes d_i(a).$$

Then the properties of ϕ described in (3.2) are rewritten in the words of d_i. From the multiplicativity of ϕ, we have

$$d_k(ab)=\sum_{i+j=k}d_i(a)d_j(b)\qquad \textit{for } a,b\in A. \tag{3.3}$$

Since $\phi(x^k)=\sum_{i+j=k}\binom{k}{i}x^i\otimes x^j$, it follows from the relation $(1\otimes\phi)\phi=(\phi\otimes 1)\phi$

$$d_id_j(a)=\binom{i+j}{i}d_{i+j}(a),\qquad a\in A. \tag{3.4}$$

We have also

$$d_0=1. \tag{3.5}$$

In the case $A=H^*(BU(n); Z/p)=Z/p[c_1, \cdots, c_n]$, we have seen

$$\phi(c_q)=\cdots+t^q\otimes 1, \quad \text{that is,} \ d_q(c_q)=1$$

if q is the p-primary factor of n. So, we consider the following case.

(3.6) A and $k[x]$ are positively graded and there exists a homogeneous element $a_\# \in A$ such that $d_q(a_\#)=1$ for some power $q=p^t$ of p.

Lemma 3.6. *Assume* (3.6) *and put*

$$B=\{a \in A \mid d_i(a)=0 \ \textit{for} \ i \geq q\},$$

then the product gives a bijection $k[a_\#]\otimes B \cong A$.
Thus $B \cong A/(a_\#)$.

Proof. Consider a relation $\sum_{i=0}^r a_\#^i b_i=0$ for homogeneous elements b_i of B. If there is a maximal number k such that $d_k(b_r)\neq 0$, then

$$0=d_{rq+k}\left(\sum_{i=0}^r a_\#^i b_i\right)=d_{rq}(a_\#^r)d_k(b_r)=d_k(b_r)$$

which is a contradiction. So, $k[a_\#]\otimes B$ injects to A.

For arbitrary homogeneous element $a\neq 0$ of A, let k be the maximal number such that $d_k(a)\neq 0$. Put $k=k'+qj$ for $0\leq k'<q$. By (3.4), (ii), $d_s d_{qj}(a)=\binom{qj+s}{s}d_{s+qj}(a)$. Thus $d_s d_{qj}(a)=0$ for $s>k'$, that is $d_{qj}(a) \in B$, and $d_{k'}d_{qj}(a)=d_k(a)$.

Put $b=d_{qj}(a) \in B$ and $a'=a-a_\#^j b$, then

$$d_i(a')=d_i(a)-\sum_{s=0}^{k'} d_{i-s}(a_\#^j)d_s(b).$$

For $i>k$, $i-s>qj$ and $d_{i-s}(a_\#^j)=0$. So, $d_i(a')=d_i(a)=0$ for $i>k$. For $i=k$, $d_k(a')=d_k(a)-d_{qj}(a_\#^j)d_{k'}(b)=d_k(a)-(d_q(a_\#))^j d_k(a)=0$. Thus we have $a'=a-a_\#^j b$ with $b \in B$ and $d_i(a')=0$ for $i>k-1$.

Repeating this we have $a=\sum_{i=0}^j a_\#^i b_i$ for some $b_i \in B$. $\square$

We shall apply this lemma to $A=H^*(BG; Z/p)$ for classical groups G, where p is a prime factor of n for $G=U(n)$, $SU(n)$ and $p=2$ for $Sp(n)$ and $SO(n)$ of even n.

We write down A in the form

$$A=Z/p[a_1, a_2, \cdots, a_n] \qquad \text{for } G=U(n) \text{ and } Sp(n)$$

$$A=Z/p[a_2, \cdots, a_n] \qquad \text{for } G=SU(n) \text{ and } SO(n) \quad (a_1=0)$$

using a_i in place of c_i, q_i or w_i. Also the generator x of $Z/p[x]$ stands for t $(G=U(n), SU(n))$, u^4 $(G=Sp(n))$ or u $(G=SO(n))$.

By Propositions 3.2, 3.4, 3.5 and Corollary 3.3, we have

$$\phi(a_k)=\sum_{i+j=k}\binom{n-j}{i}x^i\otimes a_j. \tag{3.7}$$

In particular, we choose the element $a_\#$ of (3.6) as follows

(3.7)′ *For the p-primary factor* $q=p^t$ *of* n, *put* $a_\#=a_q$, *then*

$$\phi(a_q)=\sum_{i=0}^{q}x^i\otimes a_{q-i}.$$

Note that q is the least integer such that $d_q(a_q)\neq 0$.

The bijection $B\cong A/(a_q)$ of Lemma 3.6 induces a ring structure in B by giving new multiplication $*$ defined by the condition

$$b*b'\in B \quad \textit{and} \quad b*b'\equiv bb' \pmod{a_q} \qquad \textit{for } b, b'\in B. \tag{3.8}$$

Applying Lemma 3.6 and (3.4) we have directly the following

Proposition 3.7. *There exists uniquely a system* $\{\bar{a}_k\}$ *of generators of* $A=H^*(BG; Z/p)=Z/p[(\bar{a}_1,)\ \bar{a}_2, \cdots, \bar{a}_n]$ *satisfying*

(i) $\bar{a}_k=a_k$ *for* $k\leq q$,

(ii) $\bar{a}_{jq}\equiv a_{jq} \pmod{a_q}$ *for* $j>1$ *and* $d_i(\bar{a}_{jq})=0$ *for* $i\geq q$,

(iii) $\bar{a}_{jq-i}=d_i(\bar{a}_{jq})$ *for* $0\leq i<q$.

Then $\bar{a}_k\in B$ *if* $k\neq q$ *and*

$$B=Z/p[\cdots, \bar{a}_{k-1}, \bar{a}_{k+1}, \cdots, \bar{a}_n]$$

as a ring with the multiplication $*$. *We have*

$$d_i(\bar{a}_k)=\binom{k-i}{i}\bar{a}_{k-i}.$$

Note that the Cartan type formula (3.3) does not hold with respect to the multiplication $*$.

3.5. Primitive elements

We call an element a of A is *primitive* if $\phi(a)=1\otimes a$, and we denote by PA the subalgebra of A which consists of the primitive elements. Obviously, $PA\subset B$ for the submodule B of Lemma 3.6.

From the homotopy commutativity of the diagram

$$\begin{array}{ccc} B\Gamma\times BG & \xrightarrow{\mu} & BG \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow{\scriptstyle Bp} \\ BG & \xrightarrow{Bp} & B\bar{G}, \end{array}$$

we have easily

Proposition 3.8. *Each image of the induced homomorphism Bp^*: $H^*(B\bar{G}; Z/p)\to H^*(BG; Z/p)$ is primitive, that is,*

$$\mathrm{Im}\ Bp^*\subset PH^*(BG; Z/p).$$

We shall discuss $PH^*(BG; Z/2)$ for some classical groups G.
The case $G=Sp(2m+1)$ is trivial since $PA=B$.

Proposition 3.9. *Let $\bar{q}_i=\bar{a}_i$ be the elements of Proposition* 3.7, *then*

$$PH^*(BSp(2m+1); Z/2)=Z/2[\bar{q}_2, \bar{q}_3, \cdots, \bar{q}_{2m+1}].$$

Explicitly $\bar{q}_i$'s are:

$$\bar{q}_2=q_2+mq_1^2,\quad \bar{q}_3=q_3+q_2q_1,\quad \bar{q}_4=q_4+(m-1)q_2q_1^2+\binom{m}{2}q_1^4,$$

$$\bar{q}_5=q_5+q_4q_1+(m-1)\bar{q}_3q_1^2,\quad \bar{q}_6=q_6+mq_4q_1^2+\binom{m}{2}q_2q_1^4+\binom{m}{3}q_1^6,\ \cdots.$$

Next consider the cases, $G=U(4m+2)$, $Sp(4m+2)$, $SO(4m+2)$ and $q=p=2$. a_k stands for c_k, q_k or w_k.
For the generators $\bar{a}_k$ of $H^*(BG; Z/2)=Z/2[\bar{a}_k]$ in Proposition 3.7,

$$(3.9)\qquad \begin{aligned} &\phi(a_2)=1\otimes a_2+x\otimes a_1+x^2\otimes 1 && (\bar{a}_1=a_1,\ \bar{a}_2=a_2), \\ &\phi(\bar{a}_{2k})=1\otimes\bar{a}_{2k}+x\otimes\bar{a}_{2k-1} && (1<k\leq 2m+1), \end{aligned}$$

and the elements $\bar{a}_{2k-1}=d_1(\bar{a}_{2k})$ $(1\leq k\leq 2m+1)$ are primitive.

Explicitly, $\bar{a}_i$'s are:

$$\bar{a}_3=a_3+ma_1^3,\quad \bar{a}_4=a_4+m(a_2+a_1^2)a_2,$$

$$\bar{a}_5=a_5+a_4a_1+a_3a_2+a_3a_1^2,\quad \bar{a}_6=a_6+(a_4+a_3a_1)a_2,$$

$$\bar{a}_8=a_8+(m-1)(a_4a_2+a_4a_1^2+a_3a_1^3)a_2+\binom{m}{2}(a_2^3+a_2a_1^4+a_1^6)a_2,$$

$$\bar{a}_{10}=a_{10}+a_8a_2+a_7a_2a_1+(m-1)(\bar{a}_6a_2+\bar{a}_6a_1^2+\bar{a}_5a_1^3)a_2,\ \cdots.$$

We use the following notations:

$$b*c=bc+d_1(b)d_1(c)a_2 \qquad \text{for } b, c \in A,$$

and

$$D(b)=b*b+bd_1(b)a_1 \qquad \text{for } b \in A.$$

This multiplication $*$ restricted on $B=\{b \in A \mid d_i(b)=0, i>1\}$ coincides with that of (3.8), and we have easily

(3.10) (i) $*$ *is a commutative and associative multiplication.*
(ii) $b \in B$ *implies* $d_1(b) \in PA$ *and* $D(b) \in PA$.
(iii) $b, c \in B$ *imply* $b*c \in B$, *and* $b*c=bc$ *if* b *or* $c \in PA$.
(iv) $d_1(b*c)=b*d_1(c)+d_1(b)*c+d_1(b)d_1(c)a_1$.
(v) $d_1(b*c)d_1(d)+d_1(c*d)d_1(b)+d_1(d*b)d_1(c)=d_1(b)d_1(c)d_1(d)a_1$.

For each set $I=\{i_1, \cdots, i_r\}$ of integers $1<i_1, \cdots, i_r \leq 2m+1$, put

$$d(I)=i_1+\cdots+i_r, \quad l(I)=r,$$
$$a_I=\bar{a}_{2i_1}*\cdots*\bar{a}_{2i_r} \in B, \quad a_\phi=1$$

and

$$b_k=D(\bar{a}_{2k}) \in PA, \quad 1<k\leqq 2m+1.$$

Lemma 3.10. (i) *As a module over* $Z/2[a_1, \bar{a}_3, \bar{a}_5, \cdots, \bar{a}_{4m+1}, b_2, b_3, \cdots, b_{2m+1}]$ ($\subset PA$), *B has the following bases*:

$$\{1, a_I=\bar{a}_{2i_1}*\cdots*\bar{a}_{2i_r} \text{ for } 1<i_1<\cdots<i_r\leq 2m+1, l(I)=r\geq 1\}.$$

(ii) $H(B, d_1)=\operatorname{Ker} d_1/\operatorname{Im} d_1=Z/2[a_1, b_2, b_3, \cdots, b_{2m+1}]$.

Proof. (i) follows from that $a_I \equiv \bar{a}_{2i_1}\cdots\bar{a}_{2i_r}$ (mod a_2) and $b_k \equiv \bar{a}_{2k}^2$ (mod a_1, a_2) and from Proposition 3.7, Lemma 3.6. Then B is isomorphic to $A/(a_2)=Z/2[a_1, \bar{a}_3, \bar{a}_4, \cdots, \bar{a}_{4m+2}]$ as modules. Let $E=E(A/(a_2))$ be the algebra associated to the filtration given by (a_1). Then d_1 is derivative in E and $H(E, d_1)=Z/2[a_1, \bar{a}_4^2, \bar{a}_6^2, \cdots, \bar{a}_{2m+1}^2]$ as usual. Since $a_1, b_k \in \operatorname{Ker} d_1 = PA$, (ii) follows. □

By, (3.10), (ii), we have a primitive element $d_1(a_I)$ denoted by

$$y_I=y(i_1, \cdots, i_r)=d_1(\bar{a}_{2i_1}*\cdots*\bar{a}_{2i_r}) \in PA.$$

Note that
$$y_{\{k\}}=y(k)=\bar{a}_{2k-1}.$$

If I contains a pair of the same integer, say $I=\{k, k, j_1, \cdots, j_s\}$, then it follows from $b_k=D(\bar{a}_{2k})=\bar{a}_{2k}*\bar{a}_{2k}+\bar{a}_{2k}y(k)a_1$

(3.11) $$y(k, k, j_1, \cdots, j_s)=y(j_1, \cdots, j_s)b_k+y(k, j_1, \cdots, j_s)y(k)a_1.$$

Also it follows from (3.10), (v)

$$y(i_1,\cdots,i_r)y(j_1,\cdots,j_s)=y(i_1,\cdots,i_{r-1},j_1,\cdots,j_s)y(i_r)$$
$$+y(i_1,\cdots,i_{r-1})(y(i_r,j_1,\cdots,j_s)+y(j_1,\cdots,j_s)y(a_r)a_1)$$

for $s\geq 1$, $r\geq 2$. By induction on r, we have

(3.12) $$y_I y_J=\sum_{\phi\neq K\subset I} y_{(I-K)\cup J}y_K^* a_1^{l(K)-1}$$

where $y_K^*=y(k_1)\cdots y(k_t)$ *for* $K=\{k_1,\cdots,k_t\}$.

As the special case $s=1$, we have the following relation

(3.13) $$y_I y(j)=\sum_{\phi\neq K\subset I} y_{(I-K)\cup\{j\}}y_K^* a_1^{l(K)-1}.$$

We see also

(3.11)′ $$y(i,j)^2=y(i)^2b_j+y(j)^2b_i+y(i,j)y(i)y(j)a_1,$$

(3.14) modulo (a_1), the following relations hold:

$$y(i_1,\cdots,i_r)y(i_{r+1},\cdots,j_s)\equiv\sum_{k=1}^{r}y(i_1,\cdots,\hat{i}_k,\cdots,i_s)y(i_k),$$

$$y(k,k,j_1,\cdots,j_s)\equiv y(j_1,\cdots,j_s)b_k,$$

in particular,

$$y(i_1,\cdots,i_r)^2\equiv\sum_{k=1}^{r}y(i_k)^2b_{i_1}\cdots\hat{b}_{i_k}\cdots b_{i_r}.$$

Now we are ready to determine PA.

Proposition 3.11. *The subalgebra PA of the primitive elements of $A=Z/2[a_1,a_2,\bar{a}_3,\bar{a}_4,\cdots,\bar{a}_{4m+2}]$ is multiplicatively generated by the elements*

$$a_1,\ b_k\quad(1<k\leq 2m+1),\quad y_I=y(i_1,\cdots,i_r)\quad(1<i_1<\cdots<i_r\leq 2m+1)$$

and the relations are given by (3.12) *and* (3.11).

As a moduld over $C=Z/2[a_1,\bar{a}_3,\bar{a}_5,\cdots,\bar{a}_{4m+1},b_2,b_3,\cdots,b_{2m+1}]$, PA is spanned by the y_I's of $l(I)\geq 2$ with the relation (3.13).

For the case $a_1=0$, we may use the relation (3.14).

Proof. By Lemma 3.10, $PA=\operatorname{Ker} d_1$ is spanned by $y_I=d_1(a_I)$ and $Z/2[a_1,b_k]$. (3.12) and (3.11) show that the product of y_i's can be written in terms of a_1, $y(k)=\bar{a}_{2k-1}$, b_k and y_J's of distinct integers J with $l(J)\geq 2$.

So, it remains to prove that the relations among the y_I's of $l(J)\geq 2$, as module over C, are given by (3.13).

Consider a relation $\sum f_I y_I=0$ for $f_I \in C$. Since $d_I(f_I)=0$ it follows $\sum f_I a_I \in \operatorname{Ker} d_1$. By Lemma 3.10, $\sum f_I a_I=d_1(\sum g_J a_J)+z$ for some $z \in Z/2[a_1, b_k]$. So, the relations are given by $d_1(a_J)$ rewritten in a form $\sum h_K a_K$. The results is the same as (3.13). $\square$

Recalling the definitions of $\bar{a}_I$ and y_I, we have

(3.15) $$\bar{a}_{2k}\equiv a_{2k},\quad a_I=\bar{a}_{2i_1}*\cdots*\bar{a}_{2i_r}\equiv a_{2i_1}\cdots a_{2i_r} \bmod (a_2)$$

and

$$\bar{a}_{2k-1}\equiv a_{2k-1},\quad b_k\equiv a_{2k}^2,\quad y_I\equiv\sum_{k=1}^{r} a_{2i_1}\cdots a_{2i_k-1}\cdots a_{2i_r} \bmod (a_1, a_2).$$

In the remaining part of this section, we shall consider the case $n=q=4$: $A=Z/2[a_1, a_2, a_3, a_4]$. The action ϕ is given by

$$\phi(a_1)=1\otimes a_1,\quad \phi(a_2)=1\otimes a_2+x\otimes a_1,$$
$$\phi(a_3)=1\otimes a_3+x^2\otimes a_1$$

and

$$\phi(a_4)=1\otimes a_4+x\otimes a_3+x^2\otimes a_2+x^3\otimes a_1+x^4\otimes 1.$$

We have the following primitive elements.

(3.16) a_1, $b_2=a_2^2+a_2a_1^2$ *and* $b_3=a_3^2+a_3a_2a_1+a_4a_1$ *are primitive.*

Then $B=\{a \in A \mid d_i(a)=0 \text{ for } i\geq 4\}$ has a basis $\{1, a_2, a_3, a_2a_3\}$ as a module over $Z/2[a_1, b_2, b_3]$.

Put $C=Z/2[\theta_1, \theta_2]$ and define $d\colon C\otimes B\to C\otimes B$ by

$$d(\theta\otimes b)=\theta\theta_1\otimes d_1(b)+\theta\theta_2\otimes d_2(b).$$

Then $dd=0$ since $d_1d_1=0$, $d_1d_2=d_3=d_2d_1$ and $d_2d_2=\binom{4}{2}d_4=0$ by (3.4). Direct computations show

Lemma 3.12. (i) $PA=Z/2[a_1, b_2, b_3]$.
(ii) $H(C\otimes B, d)=1\otimes PA+Z/2[\theta_1, \theta_2]\otimes Z/2[b_2, b_3]\otimes \Delta(\lambda)$

with the relation $a_1\theta_1=a_1\theta_2=0$, *where* λ *is the class of* $\theta_1\otimes a_3+\theta_2\otimes a_2$.

4. Cohomology Rings of Classifying Spaces for Quotients of Classical Groups

4.1. An Elienberg-Moore spectral sequence

Consider a fibering

$$(\Omega B \longrightarrow) F \xrightarrow{i} E \xrightarrow{p} B.$$

B is a classifying space of ΩB, and we may replace B by the base space $B=E'/G$ of a ΩB-resolution $E'=\bigcup_{n=0}^{\infty} E'_n$ of Dold-Lashof. Taking a field k as the coefficients, we have a bar resolution $\{H_*(E'_n, E'_{n-1}) \cong \otimes^{n-1} \tilde{H}_*(\Omega B) \otimes H_*(\Omega B)\}$ of k over $H_*(\Omega B)$, and the Eilenberg-Moore spectral sequence $E_2=$ $\mathrm{Cotor}^{H^*(\Omega B)}(k, k) \Rightarrow H^*(B)$ is obtained from

$$H_*(B_n, B_{n-1}) \cong H_*(E'_n, E'_{n-1}) \underset{H^*(\Omega B)}{\otimes} k.$$

Give a filtration $\{E_n\}$ of E by $E_n=p^{-1}(E'_n)$, then we have

$$H_*(E_n, E_{n-1}) \cong H_*(E'_n, E'_{n-1}) \underset{H_*(\Omega B)}{\otimes} H_*(F).$$

Passing to the dual, we get a spectral sequence (*cf.* [38]).

$$E_2=\mathrm{Cotor}^{H^*(\Omega B; k)}(k, H^*(F; k)) \Longrightarrow H^*(E; k).$$

We shall apply this to the lower sequence of (3.1):

$$(4.1) \quad E_2=\mathrm{Cotor}^{H^*(B\Gamma; Z/p)}(Z/p, H^*(BG; Z/p)) \Longrightarrow H^*(B(G/\Gamma); Z/p).$$

In order to compute the E_2-term of (4.1), we use an economical injective resolution $H^*(BB\Gamma; Z/p) \otimes H^*(B\Gamma; Z/p)$ of Z/p over $H^*(B\Gamma; Z/p)$.

For example, let $p=2$ and $\Gamma \cong S^1$, then

$$H^*(BB\Gamma; Z/2)=Z/2[z_3, z_5, \cdots, z_{2^k+1}, \cdots], \quad H^*(B\Gamma; Z/2)=Z/2[x_2]$$

and the differential is given by $d=(\mu \otimes 1)(1 \otimes \theta \otimes 1)(1 \otimes \phi)$, where μ is the multiplication in $H^*(BB\Gamma; Z/2)$, ϕ is the comultiplication in $H^*(B\Gamma; Z/2)$ and θ is defined by $\theta(x_2^{2^k})=z_{2^{k+1}+1}$ and $\theta(x^i)=0$ for i not a power of 2.

In the case $p=2$ and $\Gamma \cong Z/2m$, d is defined similarly by adding $\theta(u_1)=z_2$ for $H^*(B\Gamma; Z/2)=\Lambda(u_1) \otimes Z/2[x_2]$ and $H^*(BB\Gamma; Z/2)=Z/2[z_2, z_3, z_5, \cdots, z_{2^k+1}, \cdots]$.

Then the E_2-terms of these cases are computed by

$$(4.2) \qquad \begin{aligned} &E_2 = H(E_1, d) \quad \textit{for } E_1 = H^*(BB\Gamma; Z/2) \otimes H^*(BG; Z/2) \\ &\qquad\qquad \textit{and} \quad d = (\mu \otimes 1)(1 \otimes \theta \otimes 1)(1 \otimes \phi) \end{aligned}$$

where ϕ is the action $H^*(BG; Z/2) \to H^*(B\Gamma; Z/2) \otimes H^*(BG; Z/2)$.

Note that the following holds in the spectral sequence (4.1).

$$(4.3) \qquad E_2^{*,0} = PH^*(BG; Z/\mathrm{p}) \supset E_\infty^{*,0} = \mathrm{Im}\, Bp^*$$

for the induced homomorphism Bp^*: $H^*(B(G/\Gamma); Z/p) \to H^*(BG; Z/p)$ and $E_\infty^{0,*}$ corresponds to the image of $H^*(BB\Gamma; Z/p) \to H^*(B(G/\Gamma); Z/p)$ modulo decomposable terms.

According to the section 3.3, we write

$$A = H^*(BG; Z/2) = Z/2[a_1, a_2, a_3, \cdots, a_n] = B \otimes Z/2[a_q]$$

for

$$B = \{a \in A \mid d_i(a) = 0 \text{ for } i \geq q\},$$

where $a_i = c_i$, q_i resp. w_i for $G = U(n)$ (or $SU(n)$), $Sp(n)$ resp. $SO(n)$, $a_1 = 0$ if $G = SU(n)$ or $SO(n)$, and $q = 2^t$ is the 2-primary factor of n.

Since $\phi(a_q) = 1 \otimes a_q + \cdots + x^q \otimes 1$, $\phi(a_q^{2^r}) = \cdots + x^{2^r q} \otimes 1$ and

$$d(1 \otimes a_q^{2^r}) = \theta(x^{2^r q}) \otimes 1 + \text{lower terms in } E_1.$$

Here, $\theta(x^{2^r q}) = z_{2^{r+1}q+1}$, $z_{2^{r+2}q+1}$ resp. $z_{2^r q+1}$ for $G = U(n)$ (or $SU(n)$), $Sp(n)$ resp. $SO(n)$. This shows that in computing the E_2-term $H(E_1, d)$ of (4.2), $Z/2[a_q]$ is cancelled with the part $Z/2[\theta(a_q), \theta(a_{2q}), \cdots]$ of $H^*(BB\Gamma; Z/2)$, provided that the remaining part together with B is closed under the differential d. Then we have

Theorem 4.1. *The E_2-term of the spectral sequence* (4.2) *is isomorphic to $H(C \otimes B, d)$ for the subalgebra C of $H^*(BB\Gamma; Z/2)$ given as follows:*

$$\begin{aligned} &C = Z/2[z_3, z_5, \cdots, z_{q+1}] && \textit{for } G = U(n),\ \Gamma \cong S^1, \\ &C = Z/2[z_2, z_3, z_5, \cdots, z_{q+1}] && \textit{for } G = U(n),\ SU(n),\ \Gamma \cong Z/2m, \\ &C = Z/2[z_2, z_3, z_5, \cdots, z_{2q+1}] && \textit{for } G = Sp(n), \\ &C = Z/2[z_2, z_3, z_5, \cdots, z_{q/2+1}] && \textit{for } G = SO(n). \end{aligned}$$

4.2. Tensor products

We shall apply tensor products to show the collapsing of (4.1).

The tensor products define homomorphisms

$$(4.4) \qquad t: G(m) \times H(n) \longrightarrow G(mn), \quad t(A, B) = A \otimes B,$$

for classical groups, where G stands for U, SU, Sp when $H=U$, and for SO, U, SU, Sp when $H=O$. The correspondence of the characteristic classes in the induced homomorphism

$$Bt^*: H^*(BG(mn); Z/2) \longrightarrow H^*(BG(m); Z/2) \otimes H^*(BH(n); Z/2)$$

is given in terms of maximal tori or maximal 2-groups as in the section 2. For example, when $G=H=U$, by identifying $\sum_{i=0}^{n} c_i$ with $\prod_{j=1}^{n}(1+t_i)$,

$$Bt^*\left(\prod_{k=1}^{mn}(1+t_k)\right)=\prod_{i=1}^{m}\prod_{j=1}^{n}(1\otimes 1+t_i\otimes 1+1\otimes t_j).$$

In particular,

(4.5) *if m is a power of 2, we have for the restriction t_2: $H(n)\rightarrow G(mn)$ of t on the second factor,*

$$Bt_2^*(c_{mk})=c_k^m\ (G=U),\quad Bt_2^*(q_{mk})=c_k^{2m}\ (G=Sp)\quad \textit{for}\ \ H=U;$$
$$Bt_2^*(w_{mk})=w_k^m\ (G=SO),\quad Bt_2^*(c_{mk})=w_k^{2m}\ (G=U),$$
$$Bt_2^*(q_{mk})=w_k^{4m}\ (G=Sp)\quad \textit{for}\ \ H=0,$$

and Bt_2^* is trivial on c_i, q_i, w_i if $i\not\equiv 0$ (mod m).

Let Γ be a central subgroup of scalar matrices of $G(m)$, then $t(\Gamma\times 1)$ is a central subgroup of $G(mn)$ isomorphic to Γ, denoted by the same symbol. Then t induces a homomorphism

(4.4)′ $$t: G(m)/\Gamma\times H(n)\longrightarrow G(mn)/\Gamma.$$

Next consider the restrictions to the first factors:

$$t_1: G(m)\longrightarrow G(mn)\quad \text{and}\quad G(m)/\Gamma\longrightarrow G(mn)/\Gamma.$$

The induced homomorphism t_{1*}: $\pi_i(G(m))\rightarrow\pi_i(G(mn))$ is the n-times of the natural injection homomorphisms. So, if $n\not\equiv 0$ (mod p), t_{1*} is a mod p isomorphism up to the connectedness of $G(mn)/G(m)$. Passing to $G(m)/\Gamma$, $B(G(m)/\Gamma)$ and then $H^*(\ ; Z/p)$, we have

(4.6) *If $n\not\equiv 0$ (mod p), Bt_1^*: $H^*(B(G(mn)/\Gamma); Z/p)\cong H^*(B(G(m)/\Gamma); Z/p)$ for $*<d(m+1)$, where $d=1, 2, 4$ for $G=SO$, U, Sp, respectively.*

The map f of (3.1) represents the transgression image

$$x_2=\tau(u_1)\in H^2(B(G/\Gamma); \Gamma)\qquad (G=G(mn), G(m); \Gamma\neq S^1)$$

of the fundamental class u_1. Clearly, $Bt_1^*(x_2)=x_2$. On the other hand

$Bt_2^*(x_2)=0$ since Bt_2^* is factored through $H^*(B(G(mn)/\Gamma);\Gamma)\to H^*(BG(mn);\Gamma)$. Thus we have

$$Bt^*(x_2)=x_2\otimes 1. \tag{4.6)'$$

4.3. Cohomology ring of $BPU(4m+2)$

Consider the case $G=U(4m+2)$, $G/\Gamma=PU(4m+2)$, $\Gamma\cong S^1$.
It follows from Theorem 4.1, Lemma 3.10 and Proposition 3.11 that

$$\begin{aligned} E_2&=\mathrm{Cotor}^{H^*(B\Gamma;Z/2)}(Z/2,\ H^*(BU(4m+2);Z/2))\\ &=1\otimes PH^*(BU(4m+2);Z/2)+Z/2[z_3]\otimes Z/2[c_1,b_2,b_3,\cdots,b_{2m+1}],\end{aligned}\tag{4.7}$$

where $b_k=D(\bar{c}_{2k})=\bar{c}_{2k}^2+\bar{c}_{2k}\bar{c}_{2k-1}c_1+\bar{c}_{2k-1}^2c_2\in PH^{8k}(BU(4m+2);Z/2)$, *and* $PH^*(BU(4m+2);Z/2)$ *is multiplicatively generated by* c_1, b_k $(1<k\le 2m+1)$ *and* $y_I=d_1(c_I)$ *for* $c_I=\bar{c}_{2i_1}*\cdots*\bar{c}_{2i_r}$ $(1<i_1<\cdots<i_r\le 2m+1)$.

Since $d(1\otimes c_I)=z_3\otimes d_1(c_I)=z_3\otimes y_I$ in E_1, we have

$$z_3y_I=0 \qquad in\ (4.7). \tag{4.7)'$$

The result on $H^*(BPU(4m+2);Z/2)$ can be stated by use of the following commutative diagram:

$$\begin{array}{ccc} H^*(BPU(4m+2);Z/2) & \xrightarrow{Bt^*} & H^*(BPU(2);Z/2)\otimes H^*(BU(2m+1);Z/2)\\ \downarrow{\scriptstyle Bp^*} & & \downarrow{\scriptstyle \text{proj.}}\\ H^*(BU(4m+2);Z/2) & \xrightarrow{Bt_2^*} & H(BU(2m+1);Z/2).\end{array}\tag{4.8}$$

Since $PU(2)\cong SU(2)/(Z/2)\cong SO(3)$, we may write

$$H^*(BPU(2);Z/2)=Z/2[w_2,w_3].$$

Proposition 4.2. *$H^*(BPU(4m+2);Z/2)$ is multiplicatively generated by*

$$x_2\in H^2,\quad x_3\in H^3,\quad b_k\in H^{8k}\qquad for\ 1<k\le 2m+1$$

and

$$y_I=y(i_1,\cdots,i_r)\in H^{4d(I)-2}\qquad for\ 1<i_1<\cdots<i_r\le 2m+1.$$

The above generators are chosen such that

$$\begin{aligned}&Bp^*(x_2)=c_1,\quad Bp^*(b_k)=b_k,\quad Bp^*(y_I)=y_I,\quad x_3y_I=0,\\ &Bt^*(x_2)=w_2\otimes 1,\quad Bt^*(x_3)=w_3\otimes 1\quad and\quad Bt^*(y_I)=0.\end{aligned}$$

Then the relations are generated by (3.11), (3.12) *and* $x_3 y_I = 0$.

Proof. (4.6)′ and (4.7) imply the existence of x_2 and $x_3 = \mathrm{Sq}^1 x_2$.

By (4.5), $Bt_2^*(c_{2k-1}) = 0$ and $Bt_2^*(c_{2k}) = c_k^2$. Then it follows from (3.15) that $Bt_2^*(b_k) \equiv c_k^4 \bmod (c_1^2)$. So, by the commutativity of (4.8), Bt^* is injective on $B_k = Z/2[x_2, x_3, b_2, b_3, \cdots, b_k]$, provided the existence of $b_2, \cdots, b_k \in H^*(PBU(4m+2); Z/2)$. Since the filtration degree in the spectral sequence (4.1) is given by x_3, a possible non-trivial differential image is in B_k. These show, by induction on the degree, that (4.1) collapses. It follows from (4.7)

$$\mathrm{Im}\, Bp^* = PH^*(BU(4m+2); Z/2)$$

and

$$\mathrm{Ker}\, Bp^* = x_3\, Z/2[x_2, x_3, b_2, b_3, \cdots, b_{2m+1}].$$

We choose b_k by the condition $Bp^*(b_k) = b_k$.

Choose an element y_I' with $Bp^*(y_I') = y_I$. In E_2, $x_3 y_I'$ represents $z_3 y_I$ which is trivial by (3.7)′. So, $x_3 y_I' = x_3^2 f$ for some f. Then by putting $y_I = y_I' - x_3 f$, we have

$$Bp^*(y_I) = y_I \quad \text{and} \quad x_3 y_I = 0.$$

Applying Bt^* to the relation $x_3 y_I = 0$, we have $w_3 Bt^*(y_I) = 0$ which implies $Bt^*(y_I) = 0$ since $H^*(BPU(2); Z/2) \otimes H^*(BU(2m+1); Z/2)$ is a polynomial algebra.

The relations corresponding to (3.12) and (3.11) hold modulo $\mathrm{Ker}\, Bp^*$. Since each term of (3.12) and (3.11) contains some y_I, they are annihilated by x_3. But $\mathrm{Ker}\, Bp^*$ is not annihilated by x_3. So the relations hold without modulo, completing the proof of the proposition. □

Corollary 4.3. $\mathrm{Im}\, Bp^* = PH^*(BU(4m+2); Z/2)$ *and we have an extension*

$$0 \longrightarrow Z/2[x_2, x_3, b_2, \cdots, b_{2m+1}] \xrightarrow{\times x_3} H^*(BPU(4m+2); Z/2) \xrightarrow{Bp^*} \mathrm{Im}\, Bp^* \longrightarrow 0.$$

Remark 4.4. (i) We can use the natural map $Bj\colon BPU(4m+2) \to BPSp(2m+1)$ in place of Bt, where $H^*(BPSp(2m+1); Z/2) = Z/2[x_2, x_3, \bar{q}_2, \cdots, \bar{q}_{2m+1}]$ by [19].

(ii) In [20], the result on $H^*(BPU(4m+2); Z/2)$ is stated by a module isomorphism with the ring $\mathrm{Cotor}^{H^*(PU(4m+2); Z/2)}(Z/2, Z/2)$ giving by generators and relations. But the notations are slightly different with us. The generators $y(I)$ for $I = (i_1, \cdots, i_r)$ in [20] corresponds to our y_J

for $J=(i_1+1, \cdots, i_r+1)$, x'_{8j+8} to b_{j+1}, a_2 to x_2 and a_3 to x_3. The statement of the relation $y(I)y(J)=\sum f_i y(I_i)$ in [20] has some ambiguity. The relation holds only for I, J of the length ≥ 2, and also the relation of the type (3.13) have to be added.

Similar remarks valid for the results on $H^*(BPSO(4m+2); Z/2)$ in [20] and on $H^*(BPSp(4m+2); Z/2)$ in [22].

4.4. Cohomology rings of $BPSO(4m+2)$ and $BPSp(4m+2)$

The discussions of the previous section 4.2 can be applied to the case $G=SO(4m+2)$, $G/\Gamma=PSO(4m+2)$, $\Gamma\cong Z/2$.

In this case,

$$\begin{aligned} E_2 &= \mathrm{Cotor}^{H^*(B\Gamma;Z/2)}(Z/2, H^*(BSO(4m+2); Z/2)) \\ &= 1\otimes PH^*(BSO(4m+2); Z/2)+Z/2[z_2]\otimes Z/2[b_2, b_3, \cdots, b_{2m+1}] \end{aligned}$$

for $b_k=D(\overline{w}_{2k})=\overline{w}_{2k}^2+\overline{w}_{2k-1}^2 w_2 \in PH^{4k}(BSO(4m+2); Z/2)$.

For the generator $y_I=d_1(w_I)=d_1(\overline{w}_{2i_1}*\cdots*\overline{w}_{2i_r})$ of $PH^*(BSO(4m+2); Z/2)$, the relation $z_2 y_I=0$ holds in E_2.

In the commutative diagram

$$\begin{array}{ccc} H^*(BPSO(4m+2); Z/2) & \xrightarrow{Bt^*} & H^*(BP'SO(2); Z/2)\otimes H^*(BO(2m+1); Z/2) \\ \downarrow{\scriptstyle Bp^*} & & \downarrow{\scriptstyle \text{proj.}} \\ H^*(BSO(4m+2); Z/2) & \xrightarrow{Bt_2^*} & H^*(BO(2m+1); Z/2), \end{array}$$

$H^*(BP'SO(2); Z/2)=Z/2[t]$, $t \in H^2$, where $P'SO(2)=SO(2)/(Z/2)\cong S^1$.

Then the discussions parallel to the previous section imply

Proposition 4.5. *$H^*(BPSO(4m+2); Z/2)$ is multiplicatively generated by*

$$x_2 \in H^2, \quad b_k \in H^{4k} \qquad \textit{for } 1<k\leq 2m+1$$

and

$$y_I=y(i_1, \cdots, i_r) \in H^{2d(I)-1} \qquad \textit{for } 1<i_1<\cdots<i_r\leq 2m+1.$$

The above generators are chosen such that

$$\begin{aligned} &Bp^*(x_2)=0, \quad Bp^*(b_k)=b_k, \quad Bp^*(y_I)=y_I, \quad x_2 y_I=0, \\ &Bt^*(x_2)=t\otimes 1 \quad \textit{and} \quad Bt^*(y_I)=0. \end{aligned}$$

Then the relations are generated by

$$x_2 y_I=0, \quad y(k, k, j_1, \cdots, j_s)=y(j_1, \cdots, j_s)b_k \qquad \textit{for } s\geq 1$$

and

$$y(i_1, \cdots, i_r)y(i_{r+1}, \cdots, i_s)=\sum_{k=1}^{r} y(i_i, \cdots, \hat{y}_k, \cdots, y_s)y(i_k) \quad \textit{for } s>r\geq 2.$$

Corollary 4.6. Im $Bp^*=PH^*(BSO(4m+2); Z/2)$ *and we have an extension*

$$0\to Z/2[x_2, b_2, \cdots, b_{2m+1}]\xrightarrow{\times x_2} H^*(BPSO(4m+2); Z/2)\xrightarrow{Bp^*}\text{Im } Bp^*\to 0.$$

We have

$$y(i_1, \cdots, i_r)^2=\sum_{k=1}^{r} y(i_k)^2 b_{i_1}\cdots b_{i_k}\cdots b_{i_r}.$$

Next consider the case $G=Sp(4m+2)$, $G/\Gamma=PSp(4m+2)$, $\Gamma\cong Z/2$. In this case,

$$\begin{aligned} E_2 &= \text{Cotor}^{H^*(B\Gamma;Z/2)}(Z/2, H^*(BSp(4m+2); Z/2)) \\ &= Z/2[z_2, z_3]\otimes PH^*(BSp(4m+2); Z/2) \\ &\qquad + Z/2[z_2, z_3, z_5, b_2, b_3, \cdots, b_{2m+1}] \end{aligned}$$

for $b_k=D(\bar{q}_{2k})=\bar{q}_{2k}^2+\bar{q}_{2k}\bar{q}_{2k-1}q_1+\bar{q}_{2k-1}^2 q_2 \in PH^{16k}(BSp(2m+1); Z/2)$.
As before consider the following two homomorphisms

$$Bp^*\colon H^*(BPSp(4m+2); Z/2)\longrightarrow H^*(BSp(4m+2); Z/2)$$

and

$$\begin{aligned} Bt^*\colon & H^*(BPSp(4m+2); Z/2) \\ &\longrightarrow H^*(BPSp(2); Z/2)\otimes H^*(BU(2m+1); Z/2). \end{aligned}$$

Since Bp^* is not enough to pick up the first factor of E_2, we apply

$$\begin{aligned} Bt'^*\colon & H^*(BPSp(4m+2); Z/2) \\ &\longrightarrow H^*(BSp(1); Z/2)\otimes H^*(BU(4m+2); Z/2) \end{aligned}$$

the projection of which to the second factor is $Bt_2'^*$. $Bt_2'^*$ is equivalent to Bp^* since $Bt_2'^*=Bj^*\circ Bp^*$ for the injection Bj^* of $H^*(BSp(4m+2); Z/2)$ into $H^*(BU(4m+2); Z/2)$.

Since $PSp(1)\cong SO(3)$ and $PSp(2)\cong SO(5)$, we may write

$$\begin{aligned} H^*(BPSp(1); Z/2) &= Z/2[w_2, w_3], \\ H^*(BPSp(2); Z/2) &= Z/2[w_2, w_3, w_4, w_5], \end{aligned}$$

where $w_3=\text{Sq}^1 w_2$ and $w_5=\text{Sq}^2 w_3+w_3 w_2$ by (2.7). It follows from (4.6)′

$$Bt^*(x_i)=w_i\otimes 1 \ (i=2, 3, 5) \quad \text{and} \quad Bt'^*(x_i)=w_i\otimes 1 \ (i=2, 3).$$

Other generators are chosen such that

$$Bp^*(b_k)=b_k\ (1<k\leq 2m+1) \quad\text{and}\quad Bp^*(y_I)=y_I.$$

Provided the existence of such generators, it follows from (4.5) that $Z/2[b_2, \cdots, b_{2m+1}]$ is mapped injectively by

$$Bt_2^*: H^*(BPSp(4m+2); Z/2) \longrightarrow H^*(BU(2m+1); Z/2),$$

and then $Z/2[x_2, x_3, x_5, b_2, \cdots, b_{2m+1}]$ injectively by Bt^*.

Similarly, $Z/2[w_2, w_3]\otimes PH^*(BSp(4m+2); Z/2)$ is mapped injectively by Bt'^*.

By use of these facts, the collapsing of (4.1) is proved.

The relations are fixed mudulo (x_2, x_3).

Proposition 4.7. *$H^*(BPSp(4m+2); Z/2)$ is multiplicatively generated by*

$$x_i \in H^i \ \textit{for}\ i=2, 3, 5, \qquad b_k \in H^{16k}\ \textit{for}\ 1<k\leq 2m+1$$

and

$$y_I = y(i_1, \cdots, i_r) \in H^{8d(I)-4} \qquad \textit{for}\ 1<i_1<\cdots<i_r\leq 2m+1.$$

These generators are chosen such that

$$Bt^*(x_i)=w_i\ (i=2, 3, 5), \qquad Bt'^*(x_i)=w_i\ (i=2, 3).$$

$$Bp^*(b_k)=b_k, \qquad Bp^*(y_I)=y_I \quad\textit{and}\quad x_5 y_I\equiv 0 \bmod (x_2, x_3).$$

Then the relations (3.12) *and* (3.11) *hold modulo* (x_2, x_3).

Corollary 4.8. Im $Bp^* = PH^*(BSp(4m+2); Z/2)$ *and we have an extension*

$$0 \longrightarrow Z/2[x_2, x_3, x_5, b_2, \cdots, b_{2m+1}] \xrightarrow{\times x_5} H^*(BPSp(4m+2); Z/2)$$
$$\longrightarrow Z/2[x_2, x_3]\otimes PH^*(BSp(4m+2); Z/2) \longrightarrow 0.$$

4.5. Cohomology rings of $BPU(4)$ and $BPSp(4)$

We shall consider the classifying spaces for $PG(4)$, $G=SO$, U, Sp.

Since Spin $(4)\cong Sp(1)\times Sp(1)$, we have $PSO(4)\cong SO(3)\times SO(3)$, and

$$H^*(BPSO(4); Z/2)=Z/2[w_2, w_3]\otimes Z/2[w_2, w_3]. \tag{4.9}$$

In fact, $Bp^*H^*(BPSO(4); Z/2)=PH^*(BSO(4); Z/2)=Z/2[w_2, w_3]$.

It is known that $SU(4)$ is a double cover of $SO(6)$. So, $PU(4)\cong PSO(6)$, and it follows from Proposition 4.3.

(4.10) $H^*(BPU(4); Z/2) \cong H^*(BPSO(6); Z/2)$
$$= Z/2[x_2, b_2, b_3, y(2), y(3), y(2, 3)]/R$$

where $x_2 \in H^2$, $b_2 \in H^8$, $b_3 \in H^{12}$, $y(2) \in H^3$, $y(3) \in H^5$, $y(2, 3) \in H^9$ *and* R *is the ideal generated by*

$$x_2 y(2), \quad x_2 y(3), \quad x_2 y(2, 3) \quad \textit{and} \quad y(2, 3)^2 + y(2)^2 b_3 + y(3)^2 b_2.$$

Review this by means of the spectral sequence (4.1):

$$E_2 = \mathrm{Cotor}^{H^*(B\Gamma; Z/2)}(Z/2, H^*(BU(4); Z/2)) \Longrightarrow H^*(BPU(4); Z/2).$$

The E_2-term is computed by Theorem 4.1 and Lemma 3.12:

(4.11) $E_2 = Z/2[c_1, b_2, b_3] + Z/2[z_3, z_5] \otimes \Delta(z_9) \otimes Z/2[b_2, b_3]$

where $b_2 = c_2^2 + c_3 c_1$, $b_3 = c_3^2 + c_3 c_2 c_1 + c_4 c_1^2$ *and the relations*

$$c_1 z_3 = 0, \quad c_1 z_5 = 0, \quad c_1 z_9 = 0 \quad \textit{and} \quad z_9^2 = z_3^2 b_3 + z_5^2 b_2$$

hold.

Here, the elements z_3, z_5 and z_9 correspond to θ_1, θ_2 and $\lambda = [z_3 \otimes c_3 + z_5 \otimes c_2]$ of Lemma 3.12 respectively, and the relations $z_9 = \lambda$, $c_1 z_9 = 0$ and $z_9^2 = z_3^2 b_3 + z_5^2 b_2$ are given as the boundary of the following elements in $E_1 = Z/2[z_3, z_5, z_9, \cdots] \otimes Z/2[c_1, c_2, c_3, c_4]$, respectively:

$$1 \otimes c_4, \quad 1 \otimes (c_3 c_2 + c_4 c_1) \quad \text{and} \quad z_9 \otimes (c_4 + c_3 c_1) + z_5 \otimes c_4 c_2 + z_3 \otimes f,$$

where $f = c_4(c_3 + c_2 c_1 + c_1^3) + c_3(c_3 c_1 + c_2^2 + c_2 c_1^2)$.

Obviously the above spectral sequence collapses and

$$Bp^* H^*(BPU(4); Z/2) = PH^*(BU(4); Z/2) = Z/2[c_1, b_2, b_3].$$

Next consider the group $PSp(4) = Sp(4)/\Gamma$, $\Gamma \cong Z/2$, and the spectral sequence

$$E_2 = \mathrm{Cotor}^{H^*(B\Gamma; Z/2)}(Z/2, H^*(BSp(4); Z/2)) \Longrightarrow H^*(BPSp(4); Z/2).$$

The computation of the E_2-term is similar to that for $BPU(4)$:

(4.12) $E_2 = Z/2[z_2, z_3] \otimes Z/2[q_1, b_2, b_3] + Z/2[z_2, z_3, z_5, z_9] \otimes \Delta(z_{17}) \otimes Z/2[b_2, b_3]$

where $b_2 = q_2^2 + q_3 q_1$, $b_3 = q_3^2 + q_3 q_2 q_1 + q_4 q_1^2$ *and the relations*

$$q_1 z_5 = 0, \quad q_1 z_9 = 0, \quad q_1 z_{17} = 0 \quad \textit{and} \quad z_{17}^2 = z_5^2 b_3 + z_9^2 b_2$$

hold.

Since the elements z_i ($i=2, 3, 5, 9, 17$) correspond to the images of the generators of $H^*(BB\Gamma; Z/2)=Z/2[z, \mathrm{Sq}^1z, \mathrm{Sq}^2\mathrm{Sq}^1z, \cdots]$, they are permanent cycles, and we can choose representatives $x_i \in H^i(BPSp(4); Z/2)$ of z_i such that

$$x_3=\mathrm{Sq}^1x_2, \quad x_5=\mathrm{Sq}^2x_3+x_3x_2, \quad x_9=\mathrm{Sq}^4x_5$$

and

$$x_{17}=\mathrm{Sq}^8x_9+x_9x_5x_3+x_5^3x_2.$$

Through the inclusion i: $SU(4)\to Sp(4)$, the center Γ of $Sp(4)$ is a central subgroup of order 2 in $SU(4)$. By identifying $SU(4)/\Gamma=SO(6)$, we have an inclusion j: $SO(6)\to PSp(4)$ which is 2-connected. Then the induced homomorphism

$$Bj^*: H^*(BPSp(4); Z/2)\longrightarrow H^*(BSO(6); Z/2)=Z/2[w_2, w_3, w_4, w_5, w_6]$$

is an isomorphism of H^2. By a routine computation using (2.7) we see

(4.13) (i) $Bj^*(x_2)=w_2, \quad Bj^*(x_3)=w_3, \quad Bj^*(x_5)=w_5,$

$Bj^*(x_9)=w_6w_3+w_5w_4,$

and $Bj^*(x_{17})=w_6^2(w_5+w_3w_2)+(w_6w_3+w_5w_4)w_4^2.$

(ii) Bj^* maps $Z/2[x_2, x_3, x_5, x_9]\otimes\Delta(x_{17})$ *injectively.*

As a consequence of this, we see that q_1 is a permanent cycle. Let x_4 be a representative of q_1. Comparing Bj^* with $Bi^*: H^*(BSp(4); Z/2)\to H^*(BSU(4); Z/2)$ through Bp^*, we see that $Bj^*(x_4)\equiv 0 \bmod (w_2^2)$ since $Bi^*(q_1)=c_1^2=0$ and $Bp^*(w_4)=c_2$. Replacing x_4 by $x_4+x_2^2$ if it is necessary, we have

(4.14) *There exists an element* $x_4 \in H^4(BPSp(4); Z/2)$ *such that*

$$Bp^*(x_4)=q_1 \quad \textit{and} \quad Bj^*(x_4)=0.$$

On the other hand, consider the tensor product t: $PSp(1)\times O(4)\to \to PSp(4)$ and the induced homomorphism

$$Bt^*: H^*(BPSp(4); Z/2)\longrightarrow H^*(BPSp(1); Z/2)\otimes H^*(BO(4); Z/2)$$
$$=Z/2[w_2, w_3]\otimes Z/2[w_1, w_2, w_3, w_4].$$

By (4.6)′, $Bt^*(x_2)=w_2\otimes 1$, and hence $Bt^*(x_3)=Bt^*(\mathrm{Sq}^1x_2)=\mathrm{Sq}^1(w_2\otimes 1) =w_3\otimes 1$. Also we have $Bt_2^*(x_4)=w_1^4$ by (3.5). Thus

(4.15) (i) $Bt^*(x_i)=w_i\otimes 1$ *for* $i=2, 3,$

$Bt^*(x_4)=1\otimes w_1^4+(\textit{other terms}).$

(ii) *Bt^* maps $Z/2[x_2, x_3, x_4]$ injectively.*

(4.13), (ii) and (4.15), (ii) show that the elements of degree 17 in (4.12) are not bounded in the spectral sequence. So, b_2 is a permanent cycle and it is an image of Bp^*.

In $H^*(BSp(4); Z/2)$, we have by use of (2.7) and Cartan formula,

$$\mathrm{Sq}^8(b_2) = \mathrm{Sq}^8(q_2^2 + q_3 q_1) = b_3 + b_2 q_1^2.$$

Then

(4.16) *there exist $b_2 \in H^{16}(BPSp(4); Z/2)$ and $b_3 = \mathrm{Sq}^8(b_2) + b_2 q_1^2 \in H^{24}(BPSp(4); Z/2)$ such that $Bp^*(b_i) = b_i$ for $i = 2, 3$.*

We have proved that all the generators of (4.12) are permanent cycles and the spectral sequence collapses. Then the following theorem is established except the relations (4.17).

Theorem 4.9. *$H^*(BPSp(4); Z/2)$ is multiplicatively generated by*

$$x_i \in H^i \quad \text{for } i = 2, 3, 4, 5, 9, 17, \quad b_2 \in H^{16} \text{ and } b_3 \in H^{24}$$

satisfying (4.13) (ii), (4.14) *and* (4.16). *The relations*

$$x_4 x_5 = 0, \quad x_4 x_9 = 0, \quad x_4 x_{17} = 0 \tag{4.17}$$

and $x_{17}^2 \equiv x_9^2 b_2 + x_5^2 b_3$ modulo higher decomposables hold.

Corollary 4.10.

$$Bp^* H^*(BPSp(4); Z/2) = PH^*(BSp(4); Z/2) = Z/2[q_1, b_2, b_3]$$

and

$$H^*(BPSp(4); Z/2) = A_1 + A_2 \quad \text{for } A_1 = Z/2[z_2, z_3, z_4, b_2, b_3],$$

$$A_2 = Z/2[x_2, x_3, x_5, x_9, b_2, b_3] \otimes \Delta(x_{17}) \quad \text{and} \quad A_1 \cap A_2 = Z/2[z_2, z_3, b_2, b_3].$$

Proof of 4.17. Since $Bj^*(x_4) = 0$ by (4.14), $x_4 x_i$, $\mathrm{Sq}^k x_4 \in \mathrm{Ker}\, Bj^*$. By (4.13), (ii) and (4.12),

$$\mathrm{Ker}\, Bj^* = \{x_4, x_4 x_2, x_4 x_3, x_4^2, x_4 x_3 x_2, x_4 x_3^2, x_4 x_2^3, \cdots\}.$$

Then, for some $a \in Z/2$,

$$\mathrm{Sq}^1 x_4 = 0, \quad \mathrm{Sq}^2 x_4 = a \cdot x_4 x_2 \quad \text{and} \quad \mathrm{Sq}^3 x_4 = \mathrm{Sq}^1 \mathrm{Sq}^2 x_4 = a \cdot x_4 x_3.$$

Now, assume that $x_4 x_5 \neq 0$, then $x_4 x_5 = x_4 x_3 x_2$ and

$$x_4(\mathrm{Sq}^2 \mathrm{Sq}^1 x_2) = x_4(\mathrm{Sq}^2 x_3) = x_4(x_5 + x_3 x_2) = 0.$$

By (4.15), (ii),

$$Bt^*(x_2)=w_2\otimes 1+1\otimes f \qquad \text{for some } f\in H^2(BO(4);\boldsymbol{Z}/2)$$

and

$$Bt^*(x_4)=1\otimes w_1^4+(\textit{other terms}).$$

In $H^*(BPSp(1);\boldsymbol{Z}/2)=H^*(BSO(3);\boldsymbol{Z}/2)=\boldsymbol{Z}/2[w_2, w_3]$, $w_5=0$ and

$$\mathrm{Sq}^2\mathrm{Sq}^1 w_2=\mathrm{Sq}^2 w_3=w_3w_2.$$

So,

$$\begin{aligned}0&=Bt^*(x_4(\mathrm{Sq}^2\mathrm{Sq}^1x_2))=Bt^*(x_4)\mathrm{Sq}^2\mathrm{Sq}^1Bt^*(x_2)\\&=(1\otimes w_1^4+\cdots)(w_3w_2\otimes 1+1\otimes\mathrm{Sq}^2\mathrm{Sq}^1 f).\end{aligned}$$

It follows that $w_1^4(\mathrm{Sq}^2\mathrm{Sq}^1 f)=0$, $\mathrm{Sq}^2\mathrm{Sq}^1 f=0$ and then $w_3w_2\otimes w_1^4=0$ which is a contradiction. We have proved the first relation

$$x_4x_5=0.$$

By use of Adem relations and Cartan formula,

$$\begin{aligned}&\mathrm{Sq}^1x_5=\mathrm{Sq}^1\mathrm{Sq}^2x_3+\mathrm{Sq}^1(x_3x_2)=\mathrm{Sq}^3x_3+x_3(\mathrm{Sq}^1x_2)=x_3^2+x_3^2=0,\\&\mathrm{Sq}^2x_5=\mathrm{Sq}^2\mathrm{Sq}^2x_3+\mathrm{Sq}^2(x_3x_2)=\mathrm{Sq}^3\mathrm{Sq}^1x_3+(\mathrm{Sq}^2x_3)x_2+x_3x_2^2=x_5x_2,\\&\mathrm{Sq}^3x_5=\mathrm{Sq}^1\mathrm{Sq}^2x_5=\mathrm{Sq}^1(x_5x_2)=x_5x_3.\end{aligned}$$

Then

$$\begin{aligned}0&=\mathrm{Sq}^4(x_4x_5)=x_4^2x_5+(\mathrm{Sq}^2x_4)(\mathrm{Sq}^2x_5)+x_4(\mathrm{Sq}^4x_5)\\&=(x_4+a\cdot x_2^2)x_4x_5+x_4x_9=x_4x_9.\end{aligned}$$

Next,

$$\begin{aligned}&\mathrm{Sq}^4x_9=\mathrm{Sq}^4\mathrm{Sq}^4x_5=\mathrm{Sq}^7\mathrm{Sq}^1x_5+\mathrm{Sq}^6\mathrm{Sq}^2x_5=\mathrm{Sq}^6(x_5x_2)=x_5^2x_3+x_9x_2^2,\\&\mathrm{Sq}^5x_9=\mathrm{Sq}^5\mathrm{Sq}^4x_5=\mathrm{Sq}^7\mathrm{Sq}^2x_5=\mathrm{Sq}^7(x_5x_2)=x_5^2x_2^2,\\&\mathrm{Sq}^6x_9=\mathrm{Sq}^6\mathrm{Sq}^4x_5=\mathrm{Sq}^7\mathrm{Sq}^3x_5=\mathrm{Sq}^7(x_5x_3)=x_5^2(x_5+x_3x_2)+x_9x_3^2.\end{aligned}$$

So, these elements are annihilated by x_4, $\mathrm{Sq}^ix_4=ax_4x_i$ $(i=2, 3)$, and

$$\begin{aligned}0&=\mathrm{Sq}^8(x_4x_9)=x_4^2(\mathrm{Sq}^4x_9)+(\mathrm{Sq}^3x_4)(\mathrm{Sq}^5x_9)+(\mathrm{Sq}^2x_4)(\mathrm{Sq}^6x_9)+x_4(\mathrm{Sq}^8x_9)\\&=x_4(x_{17}+x_9x_5x_3+x_5^3x_2)=x_4x_{17}.\end{aligned}$$

□

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