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Complex Analytic Singularities
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On the Equations $x^p + y^q + z^r - xyz = 0$

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To the memory of the late Professor Takehiko Miyata

Introduction

We know two strange dualities—the duality of fourteen exceptional unimodular singularities [A] and the duality of fourteen hyperbolic unimodular singularities [N1]. The first purpose of this article is to recall and compare them. The second is to give explanations for the second duality from various viewpoints. The third is to study deformations of $T_{p,q,r}$, the singularity defined by the equation in the title, or more generally cusp singularities by means of hyperbolic Inoue surfaces (Inoue-Hirzebruch surfaces).

This article is organized as follows. In Section 1 we recall a basic notion of modality of singularities and a classification of zero and one modal hypersurface singularities. In Section 2 we recall the duality of exceptional unimodular singularities. In Section 3 we recall the duality of hyperbolic unimodular singularities. A remarkable fact is that both of the dualities take place for the same pairs of triples—the fourteen Dolgachev (or Gabrielov) triples. A typical pair of the second duality is

$$T_{3,4,4}\colon\quad x^3 + y^4 + z^4 - xyz = 0,$$
$$T_{2,5,6}\colon\quad x^2 + y^5 + z^6 - xyz = 0.$$

Sections 4–7 are devoted to studying the second duality. In Section 4 we recall hyperbolic Inoue surfaces and the duality of cycles of rational curves on them. The exceptional sets of $T_{3,4,4}$ and $T_{2,5,6}$ are cycles of rational curves and both cycles appear on one and the same hyperbolic Inoue surface. In Section 5 we shall give a number-theoretic explanation for the duality. We will see that the duality is essentially the relationship between a complete module and its dual in a real quadratic field. In Section 6 we shall provide a geometric explanation for the duality by means of general theory of surfaces of class VII_0. In Section 7 we shall

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give a lattice-theoretic explanation for the duality. This is more or less well known to specialists. Two lattices $L(\tau_{3,4,4})$ and $L(\tau_{2,5,6})$ are orthogonal complements of each other in the $K3$ lattice $(-E_8)\oplus(-E_8)\oplus H\oplus H$. In Section 8 we study deformations of hyperbolic Inoue surfaces. We see that for $p, q\geqq 3$, $r\geqq 4$ there is a bijective correspondence between any two of the following three objects;

1) proper subdiagrams of $\tau_{p,q,r}$ containing $\tau_{3,3,3}$,

2) (isomorphism classes of) elliptic deformations of $T_{p,q,r}$ with Degree 3,

3) (deformation classes of) "blown-up" hyperbolic or parabolic Inoue surfaces with a "blown-up" dual cycle $(p-1, q-1, r-1)$.

In this connection we conjecture that the parameters s_i, t_j, u_k in the family

$$x^3\prod_{i=1}^{p-3}(x+s_i)+y^3\prod_{j=1}^{q-3}(y+t_j)+z^3\prod_{k=1}^{r-3}(z+u_k)-xyz=0$$

are affine coordinates of the points on the dual cycle where hyperbolic or parabolic Inoue surfaces are blown up.

Contents

§ 1. Modality of singularities

(1.1) Let us consider the following problem. Let f be a germ of a holomorphic function at the origin (0, 0, 0) of $\boldsymbol{C}^3$ with isolated critical zero at the origin, and find a normal form of f up to biholomorphic coordinate transformations of $\boldsymbol{C}^3$ at the origin. Following Arnold we consider the problem in the following manner. Consider $X_0=\{(x, y, z)\in \boldsymbol{C}^3; f(x, y, z)=0\}=f^{-1}(0)$ and arbitrary deformation of X_0. In other words, consider

$$f_t(x, y, x)=f(x, y, z)+t_1h_1+\cdots+t_kh_k$$

for h_j fixed holomorphic functions, t_j complex parameters with $|t_j|$ sufficiently small. Let $X_t=f_t^{-1}(0)$, $t=(t_1, \cdots, t_k)$. We ask, for instance, what X_0 is if any X_t is smooth or isomorphic to X_0 itself. The answer for it is quite simple, indeed, up to equivalence (coordinate transformations at the origin),

$$f=x^2+y^2+z^2, \qquad X_0=f^{-1}(0)\cong A_1.$$

The proof of it goes as follows. We take a Weierstrass normal form of f

$$f=x^m+\sum_{j=0}^{m-1} a_j(y, z)x^j$$

with $a_j(0, 0)=0$. Define

$$f_t=f+tx^2$$

and $X_t=f_t^{-1}(0)$. Then for $t\neq 0$,

$$\begin{aligned} f_t&=tx^2+a_1x+a_0+x^3(\cdots)\\ &=u^2+a_0^*+x^3(\cdots) \qquad (u=\sqrt{t}(x+a_1/2t), \exists a_0^*)\\ &=u^2(1+b_1u+\cdots)+a_0^{**}(y, z) \qquad (\exists b_1, \cdots, a_0^{**})\\ &=v^2+a_0^{**}(y, z) \qquad v=u(1+\cdots)^{1/2}\end{aligned}$$

where $a_0^{**}(0, 0)=a_{0y}^{**}(0, 0)=a_{0z}^{**}(0, 0)=0$. So X_t is singular at the origin. Hence by assumption $X_t=X_0$ which implies that $m=2$. We infer

$$f(x, y, z)=x^2+y^2+z^2$$

up to coordinate tranformation. The next problem is what are the singularities of X_0 if any small deformations are smooth or A_1 (that is, $x^2+y^2+z^2=0$). The answer for it is $X_0\cong A_2$.

Thus we are led to the following

(1.2) **Theorem** [A]. *Suppose that* $\sharp$ (*isomorphism classes of deformations of* X_0) *is finite for a given isolated hypersurface singularity* $X_0=f^{-1}(0)$. *Then* f *is one of the following*

$$\begin{aligned} A_k&: \ x^{k+1}+y^2+z^2\\ D_k&: \ x^2y+y^{k-1}+z^2\\ E_6&: \ x^3+y^4+z^2\\ E_7&: \ x^3+xy^3+z^2\\ E_8&: \ x^3+y^5+z^2\end{aligned}$$

(1.3) Let us consider the following finite $\boldsymbol{C}$-module

$$M_f := \boldsymbol{C}[[x, y, z]]/(f_x, f_y, f_z, f)$$

for a holomorphic function f with an isolated critical zero at o, i.e., the set $\{f=f_x=f_y=f_z=0\}$ is $\{o=(0, 0, 0)\}$. Consider

$$F(x, y, z)=f+t_1h_1+\cdots+t_kh_k$$

for a basis $h_1, \cdots, h_k$ of M_f and define

$$\mathscr{X}=\{(x, y, z; t_1, \cdots, t_k); F(x, y, z)=0, |t_j|<\varepsilon\}.$$

Let π be the natural projection of $\mathscr{X}$ to D_ε^k, $\mathscr{X}_t=\pi^{-1}(t)$, $t=(t_1, \cdots, t_k)$, where $D_\varepsilon=\{t \in \boldsymbol{C}; |t|<\varepsilon\}$. Then it is known that any (small) deformation is equivalent (or isomorphic) to one of the fibers $\mathscr{X}_t$.

(1.4) **Definition** [A]. *The modality* of an isolated singularity $X=f^{-1}(0)$ is the minimal dimension of an analytic subset S of D_ε^k such that any isomorphism class of deformations of X is one of the fibers $\mathscr{X}_t$, $t \in S$.

(1.5) **Theorem** [A]. *Any* 0-*modal hypersurface isolated singularity is one of* A_k, D_k, E_6, E_7 *and* E_8. *Any* 1-*modal* (*unimodular*) *singularity is one of the following*

1) *simply elliptic singularities* $T_{2,3,6}$, $T_{2,4,4}$, $T_{3,3,3}$

2) 14 *exceptional singularities* $S_{p,q,r}$ *with* (p, q, r) *one of the Gabrielov triples* (*see* (3.3))

3) *cusp singularities* $T_{p,q,r}$ *with* $(1/p)+(1/q)+(1/r)<1$ *where* $T_{p,q,r}$: $x^p+y^q+z^r-txyz=0$ $(t\neq 0)$, *t can be chosen to be* 1 *in the case* 3).

§ 2. Strange duality of exceptional singularities

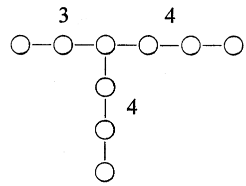
We consider the following germs S and S' of isolated singularities at the origins;

$$S:\ x^2z+yz^2+y^4=0, \qquad S':\ x^4+xy^4+z^2=0.$$

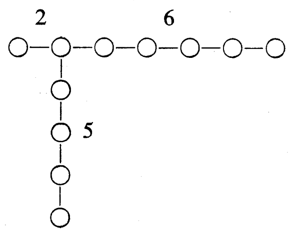
The singularities S and S' are among the 14 exceptional singularities. Let $f=x^2z+yz^2+y^4$, $g=x^4+xy^4+z^2$. Let $S_t=f^{-1}(t)$, $S_t'=g^{-1}(t)$ $(t\neq 0)$. Then $b_2(S_t)=11$, $b_2(S_t')=13$ and there are bases $e_1, \cdots, e_{11}$ and $f_1, \cdots, f_{13}$ of $H_2(S_t, \boldsymbol{Z})$ and $H_2(S_t', \boldsymbol{Z})$ such that their intersection diagrams are $\tau_{3,4,4}\oplus H$ and $\tau_{2,5,6}\oplus H$ where

$$H=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

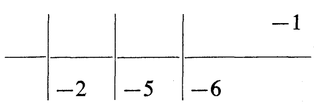
$\tau_{3,4,4}$:

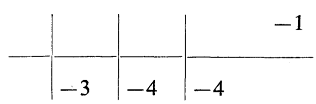


$\tau_{2,5,6}$:



We call (3, 4, 4) and (2, 5, 6) the Gabrielov numbers of S and S' and write $\mathrm{Gab}(S)=(3, 4, 4)$ and $\mathrm{Gab}(S')=(2, 5, 6)$ respectively. On the other hand we have resolutions of S and S' with exceptional sets consisting of 4 nonsingular rational curves as below;





where each line denotes a nonsingular rational curve, a negative integer beside it denotes the selfintersection number of the curve. We call (2, 5, 6) and (3, 4, 4) the Dolgachev numbers of S and S' and we write $\mathrm{Dolg}(S)=(2, 5, 6)$ etc.. So we have

$$\mathrm{Gab}(S)=\mathrm{Dolg}(S'), \qquad \mathrm{Dolg}(S)=\mathrm{Gab}(S').$$

For a Dolgachev triple (p, q, r) of an exceptional singularity U we define $\Delta(U)=pqr-pq-qr-rp$. Then we have

$$\Delta(S)=\Delta(S').$$

This is part of the strange duality of Arnold-Gabrielov of 14 exceptional singularities. See [A].

Here is another observation. The polynomials f and g are quasihomogeneous. Namely by defining degrees of variables x, y and z

$$(\deg x, \deg y, \deg z)=(6, 5, 4)$$
$$(\deg x, \deg y, \deg z)=(4, 3, 8)$$

for f and g respectively, the polynomials f and g are homogeneous of degree 16 (equal!). Moreover the sums of degrees of variables are both 15 (=the degree of f minus one). The duality was recently generalized to Kodaira singularities by Ebeling-Wahl [EW].

§ 3. Duality of hyperbolic singularities

(3.1) Let $T_{p,q,r}$ be a germ of an isolated singularity at the origin defined by

$$T_{p,q,r}: x^p+y^q+z^r-xyz=0$$

where $(1/p)+(1/q)+(1/r)<1$. We define $\deg(T_{p,q,r})=p+q+r$, $\operatorname{index}(T_{p,q,r})=(p-1, q-1, r-1)$, $\Delta(T_{p,q,r})=pqr-pq-qr-rp$.

Let $T=T_{3,4,4}$, $T^*=T_{2,5,6}$. We shall show that there is a duality between T and T^*. First we resolve the singularities. Their exceptional sets in their minimal resolutions are cycles of nonsingular rational curves,

$$C=C_1+C_2, \qquad D=D_1+D_2+D_3$$

with selfintersection numbers $C_1^2=-3$, $C_2^2=-4$, $D_1^2=-2$, $D_2^2=-3$, $D_3^2=-3$. By blowing up the first once at one of the intersections of C_1 and C_2, we obtain a cycle of three nonsingular rational curves C_1', C_2', C_3' with $C_1'^2=-1$, $C_2'^2=-4$, $C_3'^2=-5$. Now we define

$$\operatorname{cycle}(T)=(1, 4, 5), \qquad \operatorname{cycle}(T^*)=(2, 3, 3).$$

Then the first duality between T and T^* is

$$\operatorname{index}(T)=\operatorname{cycle}(T^*), \qquad \operatorname{cycle}(T)=\operatorname{index}(T^*).$$

The second is

$$\Delta(T)=\Delta(T^*).$$

Moreover the intersection matrices of C and D are

$$(C_iC_j)=\begin{pmatrix}-3 & 2\\ 2 & -4\end{pmatrix}, \qquad (D_iD_j)=\begin{pmatrix}-2 & 1 & 1\\ 1 & -3 & 1\\ 1 & 1 & -3\end{pmatrix}$$

whose determinants are equal to $\Delta(T)$ or $\Delta(T^*)$ up to sign.

Next we consider modified continued fractions arising from the sequences of selfintersection numbers of C and D. Let $\omega=[[\overline{3, 4}]]$. By definition

$$\omega=3-\cfrac{1}{4-\cfrac{1}{3-\cfrac{1}{4-\cfrac{1}{\ddots}}}} \qquad =3-\cfrac{1}{4-\cfrac{1}{\omega}}=(3+\sqrt{6})/2.$$

Then the modified continued fraction expansion of $(1/\omega)$ is given by

$$1/\omega=[[1,2,\overline{3,2,3}]]$$

where 3, 2, 3 is the periodic part of the expansion and the first 1 and 2 have no particular meaning, indeed, the 1 comes first simply because $(1/\omega)<1$. Since (3, 2, 3) is a cyclic permutation of (2, 3, 3), we may identify (2, 3, 3) and (3, 2, 3). Conversely if we start with $\omega^*=[[\overline{3,2,3}]]$, then we obtain $1/\omega^*=[[1,2,\overline{4,3}]]$. This is the third duality.

Next we reconsider the cycles C and D. The cycles C and D are so-called fundamental divisors of the singularities T and T^*. An important fact is for instance that the embedding dimension of any cusp singularity S is equal to $\max(3,-Z^2)$ for the fundamental divisor Z of S. So we define $\operatorname{Deg}(T)=-C^2$, $\operatorname{Deg}(T^*)=-D^2$. Then

$$\operatorname{Deg}(T)=-(C_1+C_2)^2=3+4-4=3,$$
$$\operatorname{Deg}(T^*)=-(D_1+D_2+D_3)^2=2+3+3-2-2-2=2.$$

Now the fourth duality is

$$\operatorname{Deg}(T)=\sharp\,(\text{irreducible components of } D),$$
$$\operatorname{Deg}(T^*)=\sharp\,(\text{irreducible components of } C).$$

Here we define *length* $(T)=\sharp$ (*irreducible components of* C) etc..

There is still a duality between T and T^*. To state it, we need to take another pair $T_{2,3,9}$ and $T_{3,3,4}$. The exceptional sets of $T_{2,3,9}$ and $T_{3,3,4}$ are cycles of three nonsingular rational curves with selfintersection numbers -2, -2, -3 and a rational curve with a node with selfintersection number -3. By blowing up the second at the node of the rational curve, we obtain a cycle of two rational curves with selfintersection numbers -1, -7. By blowing up again at one of the intersection points of two curves, we have a cycle of three rational curves with selfintersection numbers -1, -2, -8. So we define

$$\operatorname{cycle}(T_{2,3,9})=(2,2,3), \qquad \operatorname{cycle}(T_{3,3,4})=(1,2,8).$$

Thus we have the same duality as before,

$$\mathrm{cycle}\,(T_{2,3,9})=\mathrm{index}\,(T_{3,3,4}), \qquad \mathrm{index}\,(T_{2,3,9})=\mathrm{cycle}\,(T_{3,3,4}).$$

Now the fifth duality is

$$\deg(T)+\deg(T^*)=24,$$
$$\deg(T_{2,3,9})+\deg(T_{3,3,4})=24.$$

(3.2) **Theorem.** *Let $\mathfrak{T}$ be the set of all $T_{p,q,r}$ with length less than 4. Then there is a bijection* i *of $\mathfrak{T}$ onto itself such that for any T of $\mathfrak{T}$*

0) $\mathrm{i}(\mathrm{i}(T))=T$,
1) $\mathrm{index}\,(T)=\mathrm{cycle}\,(\mathrm{i}(T))$,
2) $\deg(T)+\deg(\mathrm{i}(T))=24$,
3) $\Delta(T)=\Delta(\mathrm{i}(T))$,
4) *a duality about continued fraction expansions holds,*
5) $\mathrm{Deg}\,(T)=\mathrm{length}\,(\mathrm{i}(T))$.

We notice that $\sharp(\mathfrak{T})=14$ and $T_{p,q,r}$ belongs to $\mathfrak{T}$ iff $S_{p,q,r}$ is one of the 14 exceptional unimodular singularities and that $T_{p,q,r}$ and $T_{s,t,u}$ are dual iff $S_{p,q,r}$ and $S_{s,t,u}$ are dual.

(3.3) **Table of 14 triples**

dual	self-dual
(2, 3, 8) $\overset{}{\underset{\text{dual}}{\longleftrightarrow}}$ (2, 4, 5)	(2, 3, 7), (3, 3, 6)
(2. 3, 9) $\longleftrightarrow$ (3, 3, 4)	(2, 4, 6), (3, 4, 5)
(2, 4, 7) $\longleftrightarrow$ (3, 3, 5)	(2, 5, 5), (4, 4, 4)
(2, 5, 6) $\longleftrightarrow$ (3, 4, 4)	

where we mean by self-dual that (p, q, r) is dual to (p, q, r).

§ 4. Hyperbolic Inoue surfaces

(4.1) Let K be a real quadratic field with conjugation $(\ )'$, M a free $\boldsymbol{Z}$ module of rank two (called a complete module) in K. Let $U^+(M) =\{\alpha \in K;\ \alpha M=M \text{ and } \alpha>0,\ \alpha'>0\}$, V a subgroup of $U^+(M)$ of finite index. It is known that $U^+(M)$, a fortiori, V is infinite cyclic. Let $\boldsymbol{H}$ be the upper half plane $\{z \in \boldsymbol{C};\ \mathrm{Im}\,(z)>0\}$. We define actions of M and V on $\boldsymbol{H}\times\boldsymbol{H}$ and $\boldsymbol{H}\times\boldsymbol{C}$ by

$$m:\ (z_1, z_2)\longrightarrow(z_1+m,\ z_2+m')$$
$$\alpha:\ (z_1, z_2)\longrightarrow(\alpha z_1,\ \alpha' z_2),$$

and $G(M, V)$ to be the group generated by those actions of M and V. Then the actions of $G(M, V)$ on $\boldsymbol{H}\times\boldsymbol{H}$ and $\boldsymbol{H}\times\boldsymbol{C}$ are free and properly discontinuous so that we have as quotient spaces nonsingular surfaces

$$X'(M, V)=\boldsymbol{H}\times\boldsymbol{H}/G(M, V)$$
$$S'(M, V)=\boldsymbol{H}\times\boldsymbol{C}/G(M, V)$$
$$X'(M)=\boldsymbol{H}\times\boldsymbol{H}/\text{the group of actions of } M.$$

The surface $S'(M, V)$ is compactified by adding two points ∞ and ∞_- and we obtain a singular normal surface $S_{\mathrm{sing}}(M, V)$. By the natural inclusion of $\boldsymbol{H}\times\boldsymbol{H}$ into $\boldsymbol{H}\times\boldsymbol{C}$ we may consider $X'(M, V)$ as a subset of $S_{\mathrm{sing}}(M, V)$. We may assume that $X(M, V)$, the interior of the closure of $X'(M, V)$ in $S_{\mathrm{sing}}(M, V)$, contains ∞. We have

$$X(M, V)=X'(M, V)\cup\{\infty\}.$$

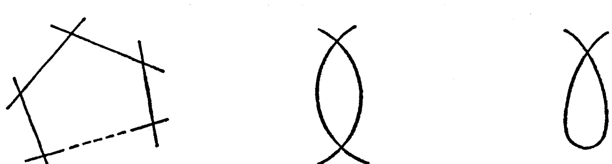
I shall give the one-dimensional analogue of $X(M, V)$. We take $K=\boldsymbol{Q}$, $M=\boldsymbol{Z}$, $V=\{1\}$ and define an action of M on $\boldsymbol{H}$ by

$$m(\in M)\colon z\longrightarrow z+m.$$

Then the quotient X' is a punctured disc $D'=D-\{0\}$ by $\exp(2\pi\sqrt{-1}\,z)$. $S'(M, V)=\boldsymbol{C}/M\ (\cong\boldsymbol{C}^*)$, $S_{\mathrm{sing}}=\boldsymbol{P}^1$. The interior of the closure of X' in S_{sing} is the unit disc D.

(4.2) **Definition.** The germ $(X(M, V), \infty)$ at ∞ is called a cusp singularity of type (M, V).

The surface $S_{\mathrm{sing}}(M, V)$ has two cusp singularities at ∞ and ∞_- which can resolved by replacing ∞ and ∞_- by C and D cycles of rational curves [H]. Here we mean by a cycle of rational curves a connected curve $C=C_1+C_2+\cdots+C_n$ such that $n\geqq 3$ $C_jC_k=0$ ($j\neq k$, $k\pm 1 \bmod n$), $C_jC_{j+1}=1$ (for any $j \bmod n$), $n=2$ C_1 and C_2 meet at two distinct points transversally, each C_j is a nonsingular rational curve ($n\geqq 2$), $n=1$ C_1 is a rational curve with a node.



($n\geqq 3$) ($n=2$) ($n=1$)

By resolving the singularities ∞ and ∞_- of $S_{\mathrm{sing}}(M, V)$, we obtain a nonsingular surface $S(M, V)$ which has an infinite cyclic fundamental group and no exceptional curves of the first kind. *This is the second example of a surface of class* VII_0 *with* $b_2>0$, which was constructed by Masahisa Inoue in 1974. We call this surface *a hyperbolic Inoue surface* from various reasons. As we saw, any hyperbolic Inoue surface $S(M, V)$ has two cycles C and D of rational curves. It is not difficult to check except 3) the following

(4.3) **Proposition.** 1) *The intersection matrices* (C_jC_k) *and* (D_jD_k) *are negative definite.*

2) $C^2=-\#$ *(irreducible components of* D*),*
$D^2=-\#$ *(irreducible components of* C*),*
$b_2(S)=\#$ *(irreducible components of* $C+D$*).*

3) $H_2(C, Z)$ *and* $H_2(D, Z)$ *are primitive sublattices of* $H_2(S(M, V), Z)$*, and* $H_2(D, Z)=H_2(C, Z)^{\perp}$ *(the orthogonal complement).*

4) $|\det(C_jC_k)|=|\det(D_jD_k)|$.

See (7.9) for the definitions of lattices, primitive sublattices. We also notice that the sequences of selfintersection numbers of irreducible components of C and D are related by modified continued fraction expansions of a real quadratic irrationality ω and $1/\omega$. To be precise, we define

(4.4) **Definition.** For a cycle C of rational curves

$$\mathrm{Zykel}(C)=\begin{cases}(-C_1^2, -C_2^2, \cdots, -C_n^2) & (n\geqq 2),\\ (-C_1^2+2) & (n=1).\end{cases}$$

(4.5) **Lemma.** *Let* ω *be a real quadratic irrationality with* $\omega>2$, $1>\omega'>0$. *Then there exist* p_j, q_j $(\geqq 3)$ *and* n $(\geqq 1)$ *such that*

$$\omega=[[\overline{p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \underbrace{2, \cdots, 2}_{(q_2-3)}, \cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)}}]]$$

$$1/\omega=[[1, 2, \overline{\underbrace{2, \cdots, 2}_{(p_1-3)}, q_1, \underbrace{2, \cdots, 2}_{(p_2-3)}, q_2, \cdots, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_n-3)}, q_n}]].$$

With these preparations we can state the relation between C and D as follows;

(4.6) **Proposition.** *For two cycles* C *and* D *on a hyperbolic Inoue surface* $S(M, V)$ *there exist* p_j, q_j $(\geqq 3)$ *and* n $(\geqq 1)$ *such that*

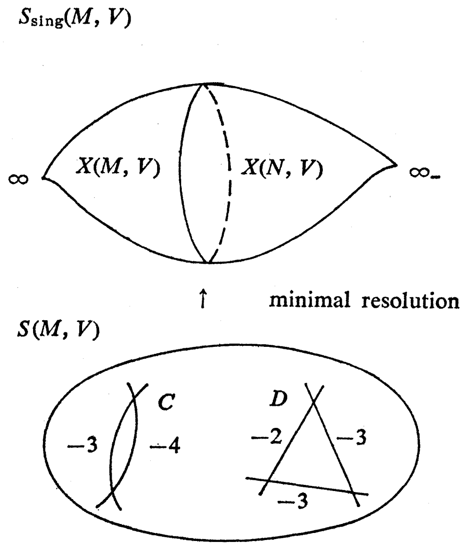
$$\text{Zykel}\,(C)=(p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \underbrace{2, \cdots, 2}_{(q_2-3)}, \cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)})$$

$$\text{Zykel}\,(D)=(\underbrace{2, \cdots, 2}_{(p_1-3)}, q_1, \underbrace{2, \cdots, 2}_{(p_2-3)}, q_2, \cdots, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_n-3)}, q_n)$$

and $M=(\boldsymbol{Z}+\boldsymbol{Z}\omega)\beta$

$$\omega=[[\overline{p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)}}]]$$

for some β $(\in K)$ *with* $\beta>0$, $\beta'>0$.



$(M=\boldsymbol{Z}+\boldsymbol{Z}(3+\sqrt{6})/2)$

(4.7) **Example.** Let $M=\boldsymbol{Z}+\boldsymbol{Z}\omega$, $\omega=(3+\sqrt{6})/2$, and $V=U^+(M)$. Then V is an infinite cyclic group generated by α, $\alpha=5-2\sqrt{6}$. The surface $S(M, V)$ has two cycles C and D

$$C=C_1+C_2, \qquad D=D_1+D_2+D_3$$

with $C_1^2=-3$, $C_2^2=-4$, $D_1^2=-2$, $D_2^2=-3$, $D_3^2=-3$. We have

$$\omega=[[\overline{3, 4}]], \qquad 1/\omega=[[1, 2, \overline{3, 2, 3}]].$$

This is the case we treated in Section 3.

Next we consider a double covering $S(M, V^2)$ of $S(M, V)$ where

$V^2=\{\beta^2;\ \beta\in V\}$. Then we have two cycles C' and D' of rational curves on $S(M, V^2)$, each being a double unramified covering of C or D respectively as well as $S(M, V^2)$. Then we have

$$\mathrm{Zykel}\,(C')=(3, 4, 3, 4), \qquad \mathrm{Zykel}\,(D')=(3, 2, 3, 3, 2, 3)$$

which is also a special case of (4.6).

(4.8) We say that two complete modules M and N are *strictly equivalent* if there is γ in K such that $\gamma>0$, $\gamma'>0$ and $M=\gamma N$. Up to strict equivalence, we may assume that $M=\boldsymbol{Z}+\boldsymbol{Z}\omega$ with $\omega>2$, $1>\omega'>0$. Then we define with the help of (4.6)

$$\omega^*=[[\overline{\underbrace{2,\cdots,2}_{(p_1-3)}, q_1, \underbrace{2,\cdots,2}_{(p_2-3)}, q_2,\cdots, q_{n-1}, \underbrace{2,\cdots,2}_{(p_n-3)}, q_n}]].$$

It is easy to see that $\omega^*=(\omega-1)/(\omega-2)$.

(4.9) **Lemma.** *Let $M=\boldsymbol{Z}+\boldsymbol{Z}\omega$, $N=\boldsymbol{Z}+\boldsymbol{Z}\omega^*$ with the notations in* (4.8). *Then $(S_{\mathrm{sing}}(M, V), \infty_-)\cong(X(N, V), \infty)$ for any subgroup V of $U^+(M)$ of finite index.*

(4.10) **Definition.** Two cusp singularities $(X(M, V), \infty)$ and $(X(N, U), \infty)$ are *dual* if $V=U$ and there exist a real quadratic irrationality ω with $\omega>2$, $1>\omega'>0$ such that M and N are respectively strictly equivalent to $\boldsymbol{Z}+\boldsymbol{Z}\omega$ and $\boldsymbol{Z}+\boldsymbol{Z}\omega^*$ where $\omega^*=(\omega-1)/(\omega-2)$.

This definition is equivalent to saying that two cusp singularities are dual iff *they are obtained from one and the same hyperbolic Inoue surface* by contracting two cycles of rational curves on it.

(4.11) **Proposition.** *Let M be a complete module, M^* the dual of M, that is, $M^*=\{x\in K;\ \mathrm{tr}\,(xy)\in\boldsymbol{Z}$ for any y in $M\}$. Then two cusp singularities $(X(M, V), \infty)$ and $(X(M^*, V), \infty)$ are dual for any subgroup V of $U^+(M)$ of finite index.*

This proposition is essentially due to K. Ueno. (See [N2] (2.21).) This fact was pointed out to us also by van der Geer.

§ 5. A number-theoretic explanation for the second duality

The purpose of this section is to give an explanation for why the duality (3.2) 1) holds true.

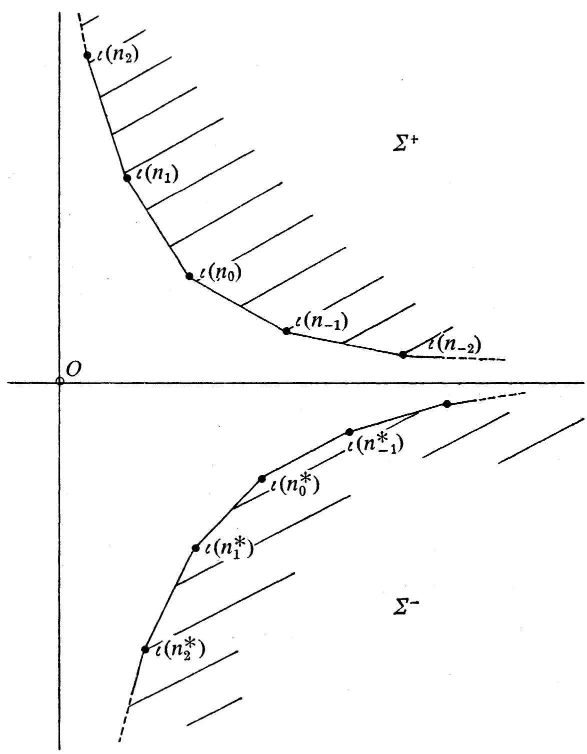
(5.1) Let $M=\boldsymbol{Z}+\boldsymbol{Z}\omega$ be a complete module with $\omega>2$, $1>\omega'>0$ and V a subgroup of $U^+(M)$ of finite index. We embed M into $\boldsymbol{R}^2$ by a mapping

$$\iota: M \longrightarrow \boldsymbol{R}^2$$
$$m \longrightarrow (m, m').$$

Consider the convex hulls of the images of M in the first and fourth quadrants,

$$\Sigma^+ = \text{convex hull of } \{\iota(m);\ m \in M,\ m>0,\ m'>0\},$$
$$\Sigma^- = \text{convex hull of } \{\iota(m);\ m \in M,\ m>0,\ m'<0\}.$$

Let $\partial\Sigma^+$ ($\partial\Sigma^-$) be the boundary of Σ^+ (Σ^-). Then $\partial\Sigma^\pm$ is a one dimensional polygon as the picture shows below.



The polygons $\partial\Sigma^\pm$ consist of infinitely many edges, each connecting two points of $\iota(M)$. Let us number them consecutively. Let

$$\mathrm{Sk}^+(M) = \iota^{-1}(\partial\Sigma^+ \cap \iota(M)) = \{n_j;\ j \in \boldsymbol{Z}\}$$
$$\mathrm{Sk}^-(M) = \iota^{-1}(\partial\Sigma^- \cap \iota(M)) = \{n^*_j;\ j \in \boldsymbol{Z}\}$$

where $n_0 = 1$, $n^*_0 = (\omega-1)/\omega^*$, $n_j < n_k$, $n^*_j < n^*_k$ $(j>k)$.

These polygons are called the *Cohn's support polygons*. Of particular importance is that they describe the minimal resolutions of the cusp singularities $(X(M, V), \infty)$ and $(X(N, V), \infty)$ where $N=\boldsymbol{Z}+\boldsymbol{Z}\omega^*$, $\omega^*=(\omega-1)/(\omega-2)$. See [O].

We recall that for $m^* \in (M^*)^+$, (hence in particular for $m^* \in \mathrm{Sk}^+(M^*)$) one can define a holomorphic function on $X(M, V)$ by

$$F_{m^*}(z_1, z_2)=\sum_{\beta \in Vm^*} \exp(2\pi\sqrt{-1}(\beta z_1+\beta' z_2)).$$

In this respect the following lemma is very important for the investigation in this section.

(5.2) **Lemma.** *Let* $M=\boldsymbol{Z}+\boldsymbol{Z}\omega$, $\omega>2$, $1>\omega'>0$. *Then*

$$\mathrm{Sk}^+(M^*)=\{f(n_j^*);\ n_j^* \in \mathrm{Sk}^-(M)\}$$

where $f(n^*)=(n^*/(\omega-\omega'))'$.

(5.3) We define two cone decompositions of $M\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ as follows;

$$\mathrm{Dec}^+(M)=\{\boldsymbol{R}_+n_j+\boldsymbol{R}_+n_{j+1},\ \boldsymbol{R}_+n_j\ (j \in \boldsymbol{Z}),\ \{0\}\}$$
$$\mathrm{Dec}^-(M)=\{\boldsymbol{R}_+n_j^*+\boldsymbol{R}_+n_{j+1}^*,\ \boldsymbol{R}_+n_j^*\ (j \in \boldsymbol{Z}),\ \{0\}\}.$$

By the general theory of torus embeddings [O], we have two complex spaces locally of finite type associated to $\mathrm{Dec}^{\pm}(M)$

$$T_M\,\mathrm{emb}\,(\mathrm{Dec}^+(M)) \quad \text{and} \quad T_M\,\mathrm{emb}\,(\mathrm{Dec}^-(M)).$$

For simplicity we denote them by $T(\mathrm{Dec}^+(M))$ and $T(\mathrm{Dec}^-(M))$ in what follows. $X'(M)$ and $X'(N)$ (See (4.1).) are naturally embedded into $T(\mathrm{Dec}^+(M))$ and $T(\mathrm{Dec}^-(M))$ as open subsets. Let $\mathscr{D}^+(M)$ or $\mathscr{D}^-(M)$ be the interior of the closure of the image of $X'(M)$ or $X'(N)$ in $T(\mathrm{Dec}^+(M))$ or $T(\mathrm{Dec}^-(M))$ respectively.

(5.4) We have a dictionary of correspondence between objects in $\mathrm{Dec}^+(M)$ and $T(\mathrm{Dec}^+(M))$ as follows;

$\mathrm{Dec}^+(M)$	$T(\mathrm{Dec}^+(M))$
n_j (or $\boldsymbol{R}_+n_j$, a cone of dim 1)	C_j, a nonsingular rational curve in $\mathscr{D}^+(M)$
$\boldsymbol{R}_+n_j+\boldsymbol{R}_+n_{j+1}$ (a cone of dim 2)	$p_j=C_j\cdot C_{j+1}$, the transversal intersection of C_j and C_{j+1}
no cone of dim 2 containing n_j and n_k $(k\neq j, j\pm 1)$	C_j and C_k don't meet $(k\neq j, j\pm 1)$

$n_{j-1}+n_{j+1}=a_j n_j$ ($^\exists a_j \in \boldsymbol{N}$)	$C_j^2=-a_j$
α: a generator of V	g: an automorphism of $\mathscr{D}^+(M)$ inducing on $X'(M)$
	$g\|_{X'(M)}$: $(z_1, z_2)\to(\alpha z_1, \alpha' z_2)$
$\alpha n_j=n_{j+r}$ for any j	$g(C_j)=C_{j+r}$ for any j

(5.5) **Lemma.** *The group $\{g^n; n \in \boldsymbol{Z}\}$ operates on $\mathscr{D}^+(M)$ freely and properly discontinuously. We have a natural holomorphic mapping h^+ of $\mathscr{D}^+(M)/V:=\mathscr{D}^+(M)/\{g^n; n \in \boldsymbol{Z}\}$ onto $X(M, V)$. The mapping h^+ is a minimal resolution of ∞.*

(5.6) We assume that $\sharp\{n_j^*; j \in \boldsymbol{Z}\} \bmod V=3$. Then we have a dictionary for $\mathrm{Dec}^-(M)$ and $T(\mathrm{Dec}^-(M))$, $\mathscr{D}^-(M)$ as follows;

$\mathrm{Dec}^-(M)$	$T(\mathrm{Dec}^-(M))$
n_j^*	D_j, a rational curve in $\mathscr{D}^-(M)$
$(p-1)n_{3j}^*=n_{3j-1}^*+n_{3j+1}^*$	$D_{3j}^2=-(p-1)$
$(q-1)n_{3j+1}^*=n_{3j}^*+n_{3j+2}^*$	$D_{3j+1}^2=-(q-1)$
$(r-1)n_{3j+2}^*=n_{3j+1}^*+n_{3j+3}^*$	$D_{3j+2}^2=-(r-1)$
α: a generator of V	g: an automorphism of $\mathscr{D}^-(M)$
$\alpha n_j^*=n_{j+3}^*$	$g(D_j)=D_{j+3}$

where $p, q \geqq 3$, $r \geqq 4$.

We know n_j^* explicitly. For instance see [O, p. 161].

In correspondence with D_j or n_j^*, we define holomorphic functions f_j on $X(M, V)$ by

$$\begin{aligned} f_j &= F_{f(n_j^*)}(z_1, z_2) \\ &= \sum_{\beta \in V\cdot f(n_j^*)} \exp\left(2\pi\sqrt{-1}(\beta z_1+\beta' z_2)\right) \end{aligned}$$

where $f(n_j^*)=(n_j^*/(\omega-\omega'))'$, $V\cdot f(n_j^*)=\{vf(n_j^*); v \in V\}$.

(5.7) **Theorem.**

1) *f_j is holomorphic on $X(M, V)$ and $f_j(\infty)=0$. We have $f_j=f_k$ iff $j \equiv k$* mod 3.

2) *The mapping F: $(X(M, V), \infty)\to(\boldsymbol{C}^3, 0)$*

$$(z_1, z_2)\longmapsto(f_0, f_1, f_2)$$

is a holomorphic embedding.

3) *We have* $f_0^p+f_1^q+f_2^r-f_0f_1f_2=0$ (mod *higher order*).

4) *There exist holomorphic functions* $\hat{f}_j$ *on* $(X(M, V), \infty)$ *such that*

$$\hat{f}_j \equiv f_j \bmod \mathfrak{m}^2 \quad (\mathfrak{m}: \textit{maximal ideal of } \infty)$$

$$\hat{f}_0^p+\hat{f}_1^q+\hat{f}_2^r-\hat{f}_0\hat{f}_1\hat{f}_2=0.$$

(5.8) **Theorem.** *Under the assumption in* (5.6), $(X(M, V), \infty)$ *is isomorphic to* $T_{p,q,r}$ *where* $p, q \geqq 3$, $r \geqq 4$.

(5.9) Next we consider the case where $\# \{n_j^*; j \in \boldsymbol{Z}\} \bmod V=2$. We define $n_{2j-(1/2)}^*$, $\mathrm{Dec}^{-\prime}(M)$ as follows;

$$n_{2j-(1/2)}^* = n_{2j}^* + n_{2j-1}^*$$

$$\mathrm{Dec}^{-\prime}(M) = \left\{ \begin{matrix} \boldsymbol{R}_+ n_{2j-1}^* + \boldsymbol{R}_+ n_{2j-(1/2)}^*, \ \boldsymbol{R}_+ n_{2j-(1/2)}^* + \boldsymbol{R}_+ n_{2j}^* \\ \boldsymbol{R}_+ n_{2j}^* + \boldsymbol{R}_+ n_{2j+1}^* \text{ and their faces} \end{matrix} \right\}.$$

Then we define $\mathscr{D}^{-\prime}(M)$ to be the interior of the closure of $X'(M)$ in $T(\mathrm{Dec}^{-\prime}(M))$. ($X'(M)$ is embedded into $T(\mathrm{Dec}^{-\prime}(M))$ too.) We can also lift the action of g on $\mathscr{D}^-(M)$ to $\mathscr{D}^{-\prime}(M)$ which we denote by the same g.

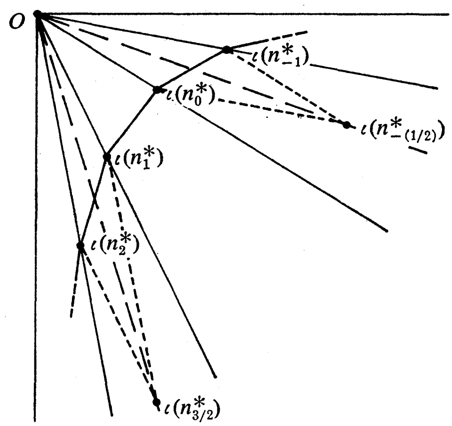
Now we have two dictionaries.

$\mathrm{Dec}^-(M)$	$T(\mathrm{Dec}^-(M))$
n_j^*	$D_j(\subset \mathscr{D}^-(M))$
$(q-2)n_{2j}^* = n_{2j-1}^* + n_{2j+1}^*$	$D_{2j}^2 = -(q-2)$
$(r-2)n_{2j+1}^* = n_{2j}^* + n_{2j+2}^*$	$D_{2j+1}^2 = -(r-2)$

$\mathrm{Dec}^{-\prime}(M)$	$T(\mathrm{Dec}^{-\prime}(M))$
n_j^*	$D'_j(\subset \mathscr{D}^{-\prime}(M))$
$n_{2j-(1/2)}^*$	$D'_{2j-(1/2)}$
$(2-1)n_{2j-(1/2)}^* = n_{2j-1}^* + n_{2j}^*$	$(D'_{2j-(1/2)})^2 = -(2-1)$
$(q-1)n_{2j}^* = n_{2j-(1/2)}^* + n_{2j+1}^*$	$(D'_{2j})^2 = -(q-1)$
$(r-1)n_{2j+1}^* = n_{2j}^* + n_{2j+(3/2)}^*$	$(D'_{2j+1})^2 = -(r-1)$

where $q \geqq 4$, $r \geqq 5$.

$\iota(\mathrm{Dec}^-(M))$ and $\iota(\mathrm{Dec}^{-\prime}(M))$



(5.10) **Lemma.** *The quotient surface* $\mathscr{D}^{-\prime}(M)/V:=\mathscr{D}^{-\prime}(M)/\{g^n;\ n\in \boldsymbol{Z}\}$ *is a blowing-up of* $\mathscr{D}^-(M)/V:=\mathscr{D}^-(M)/\{g^n;\ n\in\boldsymbol{Z}\}$ *with center the image of* p_{-1}.

(5.11) **Theorem.** *Define holomorphic functions* f_j $(j=-1/2, 0, 1)$ *on* $X(M, V)$ *and a holomorphic mapping* F *of* $X(M, V)$ *into* $\boldsymbol{C}^3$ *by*

$$\begin{aligned} f_j &= F_{f(n_j^*)}(z_1, z_2)\\ &= \sum_{\beta\in V\cdot f(n_j^*)} \exp\left(2\pi\sqrt{-1}\,(\beta z_1+\beta' z_2)\right)\\ F(z_1, z_2) &= (f_{-(1/2)}, f_0, f_1). \end{aligned}$$

Then

1) *F is a holomorphic embedding of* $(X(M, V), \infty)$ *into* $(\boldsymbol{C}^3, 0)$.
2) *We have* $f_{-1/2}^2+f_0^q+f_1^r-f_{-1/2}f_0f_1=0$ (mod *higher order*).
3) *There exist holomorphic functions* $\hat{f}_j$ *on* $(X(M, V), \infty)$ *such that*

$$\hat{f}_j=f_j \bmod \mathfrak{m}^2, \qquad \hat{f}_{-1/2}^2+\hat{f}_0^q+\hat{f}_1^r-\hat{f}_{-1/2}\hat{f}_0\hat{f}_1=0.$$

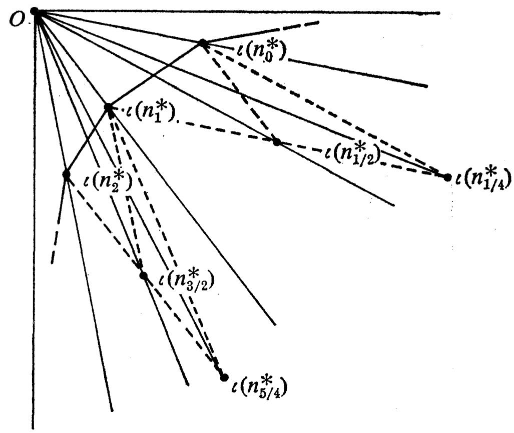
(5.12) **Theorem.** *Under the assumption in* (5.9), $(X(M, V), \infty)$ *is isomorphic to* $T_{2,q,r}$ *where* $q\geqq 4$, $r\geqq 5$.

(5.13) Finally we consider the case $\#\{n_j^*\} \bmod V=1$. We define $n_{j+(1/2)}^*$, $n_{j+(1/4)}^*$ and $\mathrm{Dec}^{-\prime\prime}(M)$ as follows,

$$n_{j+(1/2)}^*=n_j^*+n_{j+1}^*, \qquad n_{j+(1/4)}^*=n_j^*+n_{j+(1/2)}^*,$$

$$\mathrm{Dec}^{-\prime\prime}(M)=\left\{\begin{matrix} \boldsymbol{R}_+n_{j-(1/2)}+\boldsymbol{R}_+n_j^*,\ \boldsymbol{R}_+n_j^*+\boldsymbol{R}_+n_{j+(1/4)}^*, \\ \boldsymbol{R}_+n_{j+(1/4)}^*+\boldsymbol{R}_+n_{j+(1/2)}^*, \text{ and their faces}\end{matrix}\right\}.$$

$\iota(\mathrm{Dec}^{-\prime\prime}(M))$



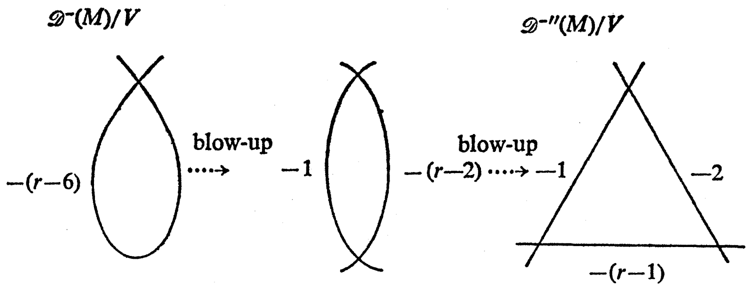
(5.14) **Lemma.** *Let $\mathscr{D}^{-\prime\prime}(M)$ be the interior of the closure of $X'(M)$ embedded in $T_M\,\mathrm{emb}\,(\mathrm{Dec}^{-\prime\prime}(M))$, g the lifting of the automorphism g of $\mathscr{D}^-(M)$ corresponding to a generator of V in (5.3). Then the surface $\mathscr{D}^{-\prime\prime}(M)/V := \mathscr{D}^{-\prime\prime}(M)/\{g^n; n \in Z\}$ is a succession of two blowing-ups of $\mathscr{D}^-(M)/V$ with centers a node of the unique rational curve in $\mathscr{D}^-(M)/V$ and a singular point of the total transform of the curve.*

(5.15) We have two dictionaries;

$\mathrm{Dec}^-(M)$	$T(\mathrm{Dec}^-(M))$
n_j^*	$D_j(\subset \mathscr{D}^-(M))$
$(r-4)n_j^* = n_{j-1}^* + n_{j+1}^*$	$D_j^2 = -(r-4)$

$\mathrm{Dec}^{-\prime\prime}(M)$	$T(\mathrm{Dec}^{-\prime\prime}(M))$
n_j^*	$D_j''(\subset \mathscr{D}^{-\prime\prime}(M))$
$n_{j+(1/2)}^*$	$D_{j+(1/2)}''$
$n_{j+(1/4)}^*$	$D_{j+(1/4)}''$
$(2-1)n_{j+(1/4)}^* = n_j^* + n_{j+(1/2)}^*$	$(D_{j+(1/4)}'')^2 = -(2-1)$
$(3-1)n_{j+(1/2)}^* = n_{j+(1/4)}^* + n_{j+1}^*$	$(D_{j+(1/2)}'')^2 = -(3-1)$
$(r-1)n_{j+1}^* = n_{j+(1/2)}^* + n_{j+(5/4)}^*$	$(D_{j+1}'')^2 = -(r-1)$

where $r \geqq 7$.



We remark that $D_0^2=-(r-4)$ in $\mathscr{D}^-(M)$ but $\bar{D}_0^2=-(r-6)$ where $\bar{D}_0$ is the image of D_0 in $\mathscr{D}^-(M)/V$.

(5.16) **Theorem.** *We define holomorphic functions* f_j ($j=1/4$, $1/2$, 1) *on* $X(M, V)$ *and a holomorphic mapping* F *of* $X(M, V)$ *into* $\boldsymbol{C}^3$ *by*

$$f_j=F_{f(n_j^*)}(z_1, z_2), \qquad F(z_1, z_2)=(f_{1/4}, f_{1/2}, f_1).$$

Then

1) F *is a holomorphic embedding of* $(X(M, V), \infty)$ *into* $(\boldsymbol{C}^3, 0)$.
2) *We have* $f_{1/4}^2+f_{1/2}^3+f_1^r-f_{1/4}f_{1/2}f_1=0$ (mod *higher order*).
3) *There exist holomorphic functions* $\hat{f}_j$ *on* $(X(M, V), \infty)$ *such that*

$$\hat{f}_j=f_j \bmod \mathfrak{m}^2, \qquad \hat{f}_{1/4}^2+\hat{f}_{1/2}^3+\hat{f}_1^r-\hat{f}_{1/4}\hat{f}_{1/2}\hat{f}_1=0.$$

(5.17) **Theorem.** *Under the assumption in* (5.13), $(X(M, V), \infty)$ *is isomorphic to* $T_{2,3,r}$ *where* $r\geqq 7$.

(5.18) Now we are in a position to give an explanation for the duality (3.2) 1). Let $M=\boldsymbol{Z}+\boldsymbol{Z}\omega$, $N=\boldsymbol{Z}+\boldsymbol{Z}\omega^*$, $\omega=(3+\sqrt{6})/2$, $\omega^*=(\omega-1)/(\omega-2)=(7+2\sqrt{6})/5$. Then $V:=U^+(M)=U^+(N)=\{\alpha^n;\, n\in\boldsymbol{Z}\}$ where $\alpha=5-2\sqrt{6}$, $1>\alpha>0$. Let $T=(X(M, V), \infty)$, $T^*=(X(N, V), \infty)$. T^* is the dual of T by definition. Then by [H] or (5.5), we have minimal resolutions of T and T^*

$$\mathscr{D}^+(M)/V\ (\cong\mathscr{D}^-(N)/V), \qquad \mathscr{D}^-(M)/V\ (\cong\mathscr{D}^+(N)/V).$$

Their exceptional sets are cycles of rational curves, respectively

$$C_0,\ C_1 \quad\text{and}\quad D_0,\ D_1,\ D_2$$

with $C_0^2=-3$, $C_1^2=-4$ and $D_0^2=-3$, $D_1^2=-2$, $D_2^2=-3$. In view of (5.6)–(5.8), we have holomorphic functions $\hat{f}_j$ ($j=0, 1, 2$) on T such that

$$\hat{f}_j = F_{f(n_j^*)} \bmod \mathfrak{m}^2 \quad (j=0, 1, 2)$$
$$\hat{f}_0^4 + \hat{f}_1^3 + \hat{f}_2^4 - \hat{f}_0\hat{f}_1\hat{f}_2 = 0$$

together with correspondence

$$\begin{array}{ll} n_0^* \leftarrow\cdots\rightarrow D_0, & D_0^2 = -(4-1), \\ n_1^* \leftarrow\cdots\rightarrow D_1, & D_1^2 = -(3-1), \\ n_2^* \leftarrow\cdots\rightarrow D_2, & D_2^2 = -(4-1). \end{array}$$

The singularity $T = (X(M, V), \infty)$ is thus isomorphic to $T_{4,3,4}$ by (5.8).

On the other hand we have in view of (5.9)–(5.12) holomorphic functions $\hat{g}_j$ $(j = -(1/2), 0, 1)$ on T^* such that

$$\hat{g}_j = F_{h(n_j)} \bmod \mathfrak{m}^2 \quad (j = -(1/2), 0, 1)$$
$$\hat{g}_{-(1/2)}^2 + \hat{g}_0^5 + \hat{g}_1^6 - \hat{g}_{-(1/2)}\hat{g}_0\hat{g}_1 = 0$$

together with correspondence

$$\begin{array}{ll} n_{-(1/2)} \leftarrow\cdots\rightarrow C'_{-(1/2)}, & (C'_{-(1/2)})^2 = -(2-1), \\ n_0 \quad \leftarrow\cdots\rightarrow C'_0, & (C'_0)^2 = -(5-1), \\ n_1 \quad \leftarrow\cdots\rightarrow C'_1, & (C'_1)^2 = -(6-1) \end{array}$$

where $h(n) = ((\omega^* - 1)n/(\omega^* - \omega^{*\prime}))'$ and C'_j is a rational curve in $\mathscr{D}^{-\prime}(N)$. Then the singularity $T^* = (X(N, V), \infty)$ is isomorphic to $T_{2,5,6}$ in view of (5.11). The definition of $n_{-(1/2)}$ corresponds to a blowing-up $\mathscr{D}^{-\prime}(N)/V$ of $\mathscr{D}^{-}(N)/V$, which fits the definition of cycle(T) in (3.1). This explains (3.2) 1).

(5.19) **Remark.** As was noted in (5.2), we have

$$\mathrm{Sk}^+(M^*) = \{f(n_j^*); j \in \boldsymbol{Z}\} = \{(n_j^*/(\omega - \omega'))'; j \in \boldsymbol{Z}\}.$$

Similarly one checks

$$\mathrm{Sk}^+(N^*) = \{h(n_j); j \in \boldsymbol{Z}\} = \{((\omega^* - 1)n_j/(\omega^* - \omega^{*\prime}))'; j \in \boldsymbol{Z}\}.$$

A more natural explanation than in (5.18) will be possible by taking $T^* := X(M^*, V)$ instead of $X(N, V)$. See [N2]. We chose the above explanation since we insisted on Cohn's support polygons and $\mathrm{Sk}^{\pm}(M)$.

§ 6. A geometric explanation for the second duality

(6.1) **Theorem** [N5]. *Let S be a* VII_0 *surface (i.e. a compact complex surface with* $b_1 = 1$ *having no exceptional curves of the first kind). Suppose* *that S has two cycles A and B of rational curves. Then S is a hyperbolic Inoue surface.*

In view of (4.3) and (4.6), (6.1) shows that there is a duality between two cycles A and B of rational curves on a VII_0 surface. However the argument for the proof proceeds in the reverse order in reality. We make an essential use of the duality in order to prove (6.1). So it is worthy of mentioning

(6.2) **Theorem.** *Let S be a* VII_0 *surface with A, B two cycles of rational curves. Then*

1) *the intersection matrices (A_jA_k) and (B_jB_k) are negative definite,*
2) $A^2=-\#$ *(irreducible components of B),*
   $B^2=-\#$ *(irreducible components of A),*
   $b_2=\#$ *(irreducible components of $A+B$),*
3) *there exist positive integers p_j, q_j $(\geqq 3)$ and n $(j=1,\cdots,n)$ such that*

$$\mathrm{Zykel}\,(A)=(p_1, \underbrace{2,\cdots,2}_{(q_1-3)}, p_2, 2,\cdots, p_n, \underbrace{2,\cdots,2}_{(q_n-3)})$$

$$\mathrm{Zykel}\,(B)=(\underbrace{2,\cdots,2}_{(p_1-3)}, q_1, 2,\cdots, q_{n-1}, \underbrace{2,\cdots,2}_{(p_n-3)}, q_n),$$

4) *$H_2(A, \boldsymbol{Z})$ and $H_2(B, \boldsymbol{Z})$ are primitive sublattices of a unimodular lattice $H_2(S, \boldsymbol{Z})$, each being the orthogonal complement of the other in $H_2(S, \boldsymbol{Z})$,*
5) $|\det(A_jA_k)|=|\det(B_jB_k)|$.

The proof of this is essentially based on the following fact.

(6.3) **Lemma** [N5]. *Let S, A and B be the same as in* (6.2). *Then there exists a proper smooth family $\pi\colon \mathcal{S}\to D$ over the unit disc D with two π-flat divisors $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ such that*

1) $(\mathcal{S}_0, \mathcal{A}_0, \mathcal{B}_0)\cong(S, A, B)$,
2) *$\mathcal{A}_t$ and $\mathcal{B}_t$ are nonsingular elliptic curves for $t\neq 0$,*
3) *$\mathcal{S}_t$ is a blown-up primary Hopf surface with two elliptic curves whose proper transforms are $\mathcal{A}_t$ and $\mathcal{B}_t$, and the centers of blowing-ups are on the two elliptic curves or their proper transforms.*

See [N4, N5] for the details. See [N4, § 8 Adderndum] for a proof of (3.2) 2).

(6.4) **Corollary.** *There exists a $\boldsymbol{Z}$ basis L_j, M_k $(1\leqq j\leqq s, 1\leqq k\leqq r)$ of $H^2(S, \boldsymbol{Z})$ such that*

$$L_j^2 = M_k^2 = -1, \quad L_i L_j = M_k M_l = L_i M_k = 0 \quad (i \neq j, k \neq l)$$
$$K_S = L_1 + \cdots + L_s + M_1 + \cdots + M_r,$$
$$A_\lambda = M_{\lambda-1} - M_\lambda - \sum_{j \in I_\lambda} L_j, \quad B_\nu = L_\nu - L_{\nu+1} - \sum_{k \in J_\nu} M_k$$

where

$$I_\lambda = [R_j + 1, R_{j+1}] \ (\lambda = S_j + 1 \textit{ for some } j) \textit{ or } \phi \ (\textit{otherwise})$$
$$J_\nu = [S_{j-1} + 1, S_j] \ (\nu = R_j \textit{ for some } j) \textit{ or } \phi \ (\textit{otherwise})$$
$$R_j = \sum_{i=1}^{j} (p_i - 2), \quad S_j = \sum_{i=1}^{j} (q_i - 2), \quad [a, b] = \{k \in \mathbf{Z};\ a \leqq k \leqq b\},$$
$$S_0 = 0, \quad M_0 = M_r, \quad L_{1+s} = L_1, \quad r = S_n, \quad s = R_n.$$

(6.5) **Example.** In the example in (4.7), let

$$A_\lambda = C_\lambda, \qquad B_\nu = D_{\nu+2}.$$

Then

$$A_1 = M_2 - M_1 - L_1, \quad A_2 = M_1 - M_2 - L_2 - L_3,$$
$$B_1 = L_1 - L_2 - M_1, \quad B_2 = L_2 - L_3, \quad B_3 = L_3 - L_1 - M_2.$$

Any intersection relation between A_λ and B_ν follows from $L_i L_j = -\delta_{ij}$, $M_k M_l = -\delta_{kl}$, $L_i M_k = 0$. These expressions of A_λ and B_ν yield all of the duality (6.2).

§ 7 Lattices $L(\tau_{p,q,r})$

(7.1) By [L2] the deformation theory for a singular hyperbolic Inoue surface with one cusp T and with the dual cycle D of T preserved is equivalent to the deformation theory for the cusp singularity T. Hence we study deformations of a hyperbolic Inoue surface instead of deformations of T.

(7.2) **Lemma** [L2]. *Let Y be a singular hyperbolic Inoue surface with one cusp T, the dual cycle D. Suppose that D consists of three rational curves with selfintersection numbers $-(p-1)$, $-(q-1)$, $-(r-1)$ $(p, q \geqq 3$, $r \geqq 4)$. Then there exists a proper flat family*

$$\pi: \mathcal{Y} \longrightarrow \Delta$$

over a disc Δ with a π-flat Cartier divisor $\mathcal{D}$ of $\mathcal{Y}$ such that $(\mathcal{Y}_0, \mathcal{D}_0) = (Y, D)$, $\mathcal{D} \cong D \times \Delta$, $\mathcal{Y}_t$ $(t \neq 0)$ is a nonsingular surface. Moreover the pair $(\mathcal{Y}_t, \mathcal{D}_t)$ is a blown-up projective plane $\mathbf{P}^2$ and the proper transform of a cycle consisting of three lines.

(7.3) Let L_0, L_1 and L_2 be three lines forming a cycle on the projective plane $\boldsymbol{P}^2$, P_j nonsingular points of the cycle such that $P_j \in L_0$ $(1 \leqq j \leqq p)$, $P_j \in L_1$ $(p+1 \leqq j \leqq p+q)$, $P_j \in L_2$ $(p+q+1 \leqq j \leqq p+q+r)$. Blow up $\boldsymbol{P}^2$ at these points to obtain a surface Y' and exceptional curves E_j $(1 \leqq j \leqq p+q+r)$, D_j the proper transform of L_j $(1 \leqq j \leqq 3)$. Then $E_jE_k = -\delta_{jk}$, $D_0^2 = -(p-1)$, $D_1^2 = -(q-1)$, $D_2^2 = -(r-1)$. A fiber $(\mathscr{Y}_t, \mathscr{D}_t)$ is isomorphic to one of such pairs (Y', D) by (7.2).

(7.4) **Lemma** [L2]. *We have an exact sequence,*

$$0 \to H^1(\mathscr{D}_t, \boldsymbol{Z}) \to H_2(\mathscr{Y}_t \backslash \mathscr{D}_t, \boldsymbol{Z}) \to H_2(\mathscr{Y}_t, \boldsymbol{Z}) \to H^2(\mathscr{D}_t, \boldsymbol{Z}) \to 0.$$

(7.5) It follows that

$$L(T) := H_2(\mathscr{Y}_t \backslash \mathscr{D}_t, \boldsymbol{Z})/\boldsymbol{Z} \cong H_2(\mathscr{D}_t, \boldsymbol{Z})^{\perp} \qquad \text{in } H_2(\mathscr{Y}_t, \boldsymbol{Z})$$
$$(\cong \text{ the orthogonal complement of } H_2(D, \boldsymbol{Z}) \text{ in } H_2(Y', \boldsymbol{Z})).$$

Let us study $L(T)$. Then by (7.2) and (7.3)

$$\begin{aligned}
&H_2(Y', \boldsymbol{Z}) = \boldsymbol{Z}h \oplus \bigoplus_{i=1}^{p+q+r} \boldsymbol{Z}E_i,\\
&D_0 = L_0 - E_1 - \cdots - E_p \sim h - E_1 - \cdots - E_p,\\
&D_1 = L_1 - E_{p+1} - \cdots - E_{p+q} \sim h - E_{p+1} - \cdots - E_{p+q},\\
&D_2 = L_2 - E_{p+q+1} - \cdots - E_{p+q+r} \sim h - E_{p+q+1} - \cdots E_{p+q+r},
\end{aligned}$$

where h denotes the pull back of the class of a line on $\boldsymbol{P}^2$. So we define

$$\begin{aligned}
&e = h - E_1 - E_{p+1} - E_{p+q+1}, \quad e_i = E_i - E_{i+1} \quad (1 \leqq i \leqq p-1),\\
&f_{j-p} = E_j - E_{j+1} \quad (p+1 \leqq j \leqq p+q-1),\\
&g_{k-p-q} = E_k - E_{k+1} \quad (p+q+1 \leqq k \leqq p+q+r-1).
\end{aligned}$$

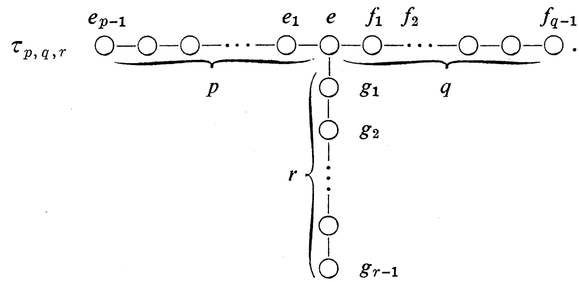
Then e, e_i, f_j and g_k have length -2, i.e., $e^2 = e_i^2 = f_j^2 = g_k^2 = -2$, and

$$\begin{aligned}
&(e, e_i) = \delta_{i1}, \quad (e, f_j) = \delta_{j1}, \quad (e, g_k) = \delta_{k1},\\
&(e_i, e_{i+1}) = (f_j, f_{j+1}) = (g_k, g_{k+1}) = 1,\\
&(e_i, e_{i'}) = (f_i, f_{j'}) = (g_k, g_{k'}) = 0 \quad \text{(otherwise)},\\
&(e_i, f_j) = (e_i, g_k) = (f_j, g_k) = 0,
\end{aligned}$$

where $e^2 = (e, e)$ etc., (,) is the intersection form on $H_2(Y', \boldsymbol{Z})$. They form a $\boldsymbol{Z}$-basis of $L(T)$.

In what follows we denote by $L(\tau_{p,q,r})$ the free $\boldsymbol{Z}$ module generated by e, e_i, f_j and g_k $(1 \leqq i \leqq p-1,\ 1 \leqq j \leqq q-1,\ 1 \leqq k \leqq r-1)$ with bilinear form defined as above for any triple (p, q, r) with $(1/p)+(1/q)+(1/r)<1$. To indicate the bilinear form on $L(\tau_{p,q,r})$ we define a graph in the following manner. Each vertex of the graph denotes one of e, e_i, f_j and g_k. Two vertices e' and e'' are connected by a single edge iff two vectors v' and v'' corresponding to e' and e'' in $L(\tau_{p,q,r})$ have $(v', v'')=1$.

The graph thus defined is



This fact was observed in [L2].

(7.6) **Proposition** (Gabrielov [A]). *Let* $f=x^p+y^q+z^r-xyz$, $(1/p)+(1/q)+(1/r)<1$. *Let* $\mathscr{X}_t=f^{-1}(t)$. *Then*

$$(H_2(\mathscr{X}_t, \boldsymbol{Z}), \textit{intersection form})\cong(\boldsymbol{Z}, 0)\oplus L(\tau_{p,q,r})$$

where $(\boldsymbol{Z}, 0)$ *denotes* $\boldsymbol{Z}$ *with intersection form equal to zero.*

In view of (5.8), (5.12) and (5.17), we may assume that $\mathscr{X}_t$ is embedded into $\mathscr{Y}_t$ for any t by a suitable choice of $\pi: \mathscr{Y}\to\Delta$. The isomorphism in (7.6) would be induced from the inclusion homomorphism of $H_2(\mathscr{X}_t, \boldsymbol{Z})$ to $H_2(\mathscr{Y}_t\backslash\mathscr{D}_t, \boldsymbol{Z})\cong H^1(\mathscr{D}_t, \boldsymbol{Z})\oplus H_2(\mathscr{D}_t, \boldsymbol{Z})^{\perp}$.

(7.7) The bilinear form on $L(\tau_{p,q,r})$ is nondegenerate of rank $p+q+r-2$. It has a positive eigenvalue and $(p+q+r-3)$ negative eigenvalues. Therefore $L(\tau_{p,q,r})$ is canonically embedded into the dual $L(\tau_{p,q,r})^*$ $(:=\mathrm{Hom}\,(L(\tau_{p,q,r}), \boldsymbol{Z}))$ with finite index. Rather surprising is that the finite group $L(\tau_{p,q,r})^*/L(\tau_{p,q,r})$ is related to the automorphism group of $T_{p,q,r}$.

(7.8) **Proposition** (Pinkham-Wahl [P]). *Let G be the group of monomial automorphisms of* $T_{p,q,r}$,

$$G=\{g: (x, y, z)\to(\alpha x, \beta y, \gamma z);\ \alpha^p=\beta^q=\gamma^r=\alpha\beta\gamma,\ \alpha, \beta, \gamma\in\boldsymbol{C}^*\}.$$

Then G is isomorphic to $L(\tau_{p,q,r})^*/L(\tau_{p,q,r})$. *Moreover the quotient of* $T_{p,q,r}$ *by G is a cusp singularity dual to* $T_{p,q,r}$.

(7.9) *A lattice* is by definition a free Z module of finite rank with a nondegenerate integer-valued bilinear form. A free Z-submodule M of a lattice L is called *a sublattice* of L if the bilinear form on M is the restriction of that on L. *A primitive sublattice* M of L is by definition a sublattice of L with L/M free. A lattice M is said to be (*primitively*) *embedded* into a lattice L if there exists a monomorphism $j: M \to L$ of Z modules such that the bilinear form on M is the pull back of that on L (and $L/j(M)$ is free).

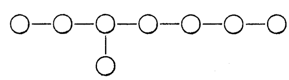
(7.10) **Proposition.** *Suppose* $(1/p)+(1/q)+(1/r)<1$.

1) *$L(\tau_{p,q,r})$ is primitively embedded into $L:=(-E_8)\oplus(-E_8)\oplus H\oplus H$ if $p+q+r\leqq 19$.*

2) *If $p+q+r\leqq 17$, then the primitive embedding of $L(\tau_{p,q,r})$ into L is unique, that is, for two arbitrary primitive embeddings f and g of $L(\tau_{p,q,r})$ into L there exists an automorphism h of L such that $f=hg$, h keeps the bilinear form on L invariant.*

3) *Assume $p+q+r\leqq 15$. Then $L(\tau_{p,q,r})$ is isomorphic to $L(\tau_{p',q',r'})$ iff $(p, q, r)=(p', q', r')$.*

In the above we mean by E_8 and H the E_8 lattice and a lattice of rank 2 with bilinear form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

E_8 

By Brieskorn, $L(\tau_{2,7,7})\cong L(\tau_{3,3,10})$, $L(\tau_{3,9,10})\cong L(\tau_{4,5,13})$. So (7.10) 3) fails for $p+q+r=16$.

(7.11) **Theorem** (Lattice-theoretic duality). *Two singularities $T_{p,q,r}$ and $T_{s,t,u}$ are dual if and only if $L(\tau_{p,q,r})$ is primitively embedded into $(-E_8)\oplus(-E_8)\oplus H\oplus H$ so that $L(\tau_{p,q,r})^{\perp}$ (:= the orthogonal complement of $L(\tau_{p,q,r})$) may coincide with $L(\tau_{s,t,u})$ embedded primitively.*

(7.10) and (7.11) follows from [Ni]. See also [P, Theorem 1]. The relation (3.2) 2) is rank $(E_8\oplus E_8\oplus H\oplus H)=20$.

§ 8 Deformations of hyperbolic Inoue surfaces

(8.1) **Theorem.** *Let Y be a singular hyperbolic Inoue surface with a cusp T and its dual cycle D. Let $\pi: \mathcal{Y}\to\Delta$ be a proper flat morphism such that $\mathcal{Y}_0=\pi^{-1}(0)\cong Y$, and such that there is a π-flat divisor $\mathcal{D}$ of $\mathcal{Y}$ with $\mathcal{D}_0=D$, $\mathcal{D}\cong D\times\Delta$. Suppose that $\mathcal{Y}_t$ $(t\neq 0)$ has a nonrational singularity.*

Then the minimal resolution of $\mathscr{Y}_t$ is a blown-up parabolic or a blown-up hyperbolic Inoue surface with blown-up dual cycle equal to the given D (together with the intersection numbers). The centers of blowing-ups are on the union of an elliptic curve and a cycle of rational curves or the union of two cycles of rational curves.

Proof. Let $f_t: \mathscr{X}_t \to \mathscr{Y}_t$ be the minimal resolution of $\mathscr{Y}_t$, $\bar{p}_g(\mathscr{Y}_t):=\dim R^1 f_{t*}\mathcal{O}_{\mathscr{X}_t}$. By the assumption, $\bar{p}_g(\mathscr{Y}_t)\neq 0$. Since $\bar{p}_g(\mathscr{Y}_t)$ is upper semi-continuous with respect to t, we have $\bar{p}_g(\mathscr{Y}_t)=1$. By the spectral sequence

$$E_2^{p,q}=H^p(Y, R^q f_{0*}\mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X)$$

we see

$$H^p(Y, \mathcal{O}_Y)=0 \ (p=1, 2), \qquad H^1(X, \mathcal{O}_X)\cong H^0(Y, R^1 f_{0*}\mathcal{O}_X)\cong \boldsymbol{C},$$

where $X=\mathscr{X}_0$. By the upper semi-continuity, we have $H^p(\mathscr{Y}_t, \mathcal{O}_{\mathscr{Y}_t})=0$. Therefore

$$H^1(\mathscr{X}_t, \mathcal{O}_{\mathscr{X}_t})\cong R^1 f_{t*}\mathcal{O}_{\mathscr{X}_t}\cong \boldsymbol{C}, \qquad H^2(\mathscr{X}_t, \mathcal{O}_{\mathscr{X}_t})=0.$$

The dualising sheaf ω_Y of Y is equal to $\mathcal{O}_Y(-D)$. Since the singularity of Y is Gorenstein, so are the singularities of $\mathscr{Y}_t$. The dualising sheaf $\omega_{\mathscr{Y}_t}$ of $\mathscr{Y}_t$ is $\mathcal{O}_{\mathscr{Y}_t}(-\mathscr{D}_t)$ because $H^1(Y, \mathcal{O}_Y)=0$ and therefore liftings of line bundles are unique. Consequently $\omega_{\mathscr{X}_t}=\mathcal{O}_{\mathscr{X}_t}(-\mathscr{C}_t-\mathscr{D}_t)$ for an effective divisor $\mathscr{C}_t$ on $\mathscr{X}_t$. This implies that $P_m(\mathscr{X}_t)=0$ for $m>0$. By the classification of surfaces [Ko, Theorem 55], we see that $\mathscr{X}_t$ is either a (blown-up) ruled surface of genus one or a surface with $b_1=1$. Suppose that $\mathscr{X}_t$ is a blown-up ruled surface of genus one. Therefore we have a surjective morphism $h: \mathscr{X}_t \to E$ for an elliptic curve E, with general fiber a connected nonsingular rational curve. Therefore there is no cycle of rational curves in any fiber of h. The surface has however a cycle $\mathscr{D}_t$ of rational curves. The cycle is therefore mapped onto E, which contradicts that the genus of E is greater than that of any irreducible component of $\mathscr{D}_t$. Thus $\mathscr{X}_t$ is a surface with $b_1=1$. In view of [N5] $\mathscr{C}_t$ is an elliptic curve or a cycle of rational curves. By [N5, (7.1) and (8.1)], $\mathscr{X}_t$ is either a blown-up parabolic Inoue surface or a blown-up hyperbolic Inoue surface. Consequently the centers of blowing-ups are on the union of (proper transforms of) $\mathscr{C}_t$ and $\mathscr{D}_t$. This completes the proof of (8.1). Q.E.D.

(8.2) **Corollary.** *The singularities of $\mathscr{Y}_t$ are either a simply elliptic singularity and some singularities of type A_k or a cusp singularities and some singularities of type A_k.*

Proof. By $\bar{p}_g(\mathscr{Y}_t)=1$, $\mathscr{Y}_t$ has a unique elliptic singularity, which is either a simply elliptic or a cusp singularity by (8.1). If $\mathscr{Y}_t$ has a singularity besides it, $\mathscr{X}_t$ has an exceptional set A of the singularity. Since $\mathscr{X}_t$ is a parabolic or hyperbolic Inoue surface blown up with centers on the union of $\mathscr{C}_t$ and $\mathscr{D}_t$, any connected component of A is a chain of (-2) curves. Therefore it can be blown down to a singularity of type A_k. Q.E.D.

(8.3) **Conjecture.** *The minimal resolution $\mathscr{X}_t$ of $\mathscr{Y}_t$ is minimal along $\mathscr{C}_t$. In other words, no exceptional curve of the first kind on $\mathscr{X}_t$ meets $\mathscr{C}_t$.*

(8.4) **Proposition.** *With the same notations as in* (8.1), *let ∞_t be the unique elliptic singularity of $\mathscr{Y}_t$. If* $\mathrm{Deg}(\mathscr{Y}_t, \infty_t)$ *is constant, then $\mathscr{X}_t$ is minimal along $\mathscr{C}_t$.*

Proof. Since Degree-preserving deformations of the singularities can be simultaneously resolved (possibly by a finite base change), we have a smooth proper family $\tilde{\pi}\colon \mathscr{S}\to\tilde{\Delta}$ such that $\mathscr{S}_s$ is a minimal resolution of $\mathscr{Y}_{h(s)}$ where $h\colon \tilde{\Delta}\to\Delta$ is the base change. We have only to prove that $\mathscr{S}_s$ is minimal along $\tilde{\mathscr{C}}_s :=$ exceptional set of $\infty_{h(s)}$. The divisor $\tilde{\mathscr{C}}$ of $\mathscr{S}$ is a proper flat family of deformations of $C=\tilde{\mathscr{C}}_0$. In view of (6.4), there is a basis L_j, M_k of $H^2(S, \boldsymbol{Z})$ such that

$$C_\lambda=M_{\lambda-1}-M_\lambda-\sum_{j\in I_\lambda} L_j, \qquad D_\nu=L_\nu-L_{\nu+1}-\sum_{k\in J_\nu} M_k$$

where $S=\mathscr{S}_0$. Let C'_i $(1\leq i\leq m)$ be irreducible components of $\tilde{\mathscr{C}}_s$ $(s\neq 0)$. Notice that $\tilde{\mathscr{C}}_s$ has the same number of irreducible components for $s\neq 0$, $|s|$ small enough. Since $\tilde{\mathscr{C}}_s$ is obtained by connected sums of irreducible components of C in the underlying differentiable manifold of S, we have a decomposition of $[1, n]$ into mutually disjoint subsets $\Lambda_1, \Lambda_2, \cdots, \Lambda_m$ such that

$$C'_i=\sum_{\lambda\in\Lambda_i} C_\lambda=M_{\lambda_i}-M_{\lambda_{i+1}}-\sum_{\lambda\in\Lambda_i,}\sum_{j\in I_\lambda} L_j,$$
$$\Lambda_i=[\lambda_i+1, \lambda_{i+1}] \quad (\subset \boldsymbol{Z}/n\boldsymbol{Z}).$$

In view of (6.1) or (6.2.3), $\mathscr{S}_s$ is not minimal. Let E be an exceptional curve of the first kind on $\mathscr{S}_s$. By the diffeomorphism of S with $\mathscr{S}_s$, we may view L_j, M_k as basis of $H^2(\mathscr{S}_s, \boldsymbol{Z})$. We can express E as a $\boldsymbol{Z}$-linear combination of L_j and M_k. Since $E^2=K_{\mathscr{S}_s}E=-1$, we have $E=L_j$ or M_k for some j or k. Since E is a curve, we have $EC'_i\geqq 0$, $ED_\nu\geqq 0$ where we view D $(=\mathscr{D}_s)$ as a cycle on $\mathscr{S}_s$. If $E=L_\nu$, then $ED_\nu=-1$ which is absurd. So $E=M_\lambda$ for some λ. If $\lambda=\lambda_i$, then $EC'_i=-1$ which is absurd.

Hence $\lambda \neq \lambda_i$ for any i. This implies that $EC'_i=0$, and that $\mathscr{S}_s$ is minimal along $\mathscr{C}_s$. Q.E.D.

(8.5) For an n-vector $a=(a_1, \cdots, a_n)$ we define $|a|=a_1+\cdots+a_n+n$. For two s-vectors $a=(a_1, \cdots, a_s)$ and $b=(b_1, \cdots, b_s)$ we define $a \leqq b$ (*a is smaller than b*) if $a_j \leqq b_j$ for any j (possibly by a cyclic permutation). For an n-vector $a=(a_1, \cdots, a_n)$, an $(n+1)$-vector b is a *blowing-up* of a if

$$b=(a_1, \cdots, a_{j-1}, a_j+1, 1, a_{j+1}+1, a_{j+2}, \cdots, a_n) \quad (n \geqq 2) \quad \text{or}$$
$$(1, a_1+1, a_2, \cdots, a_{n-1}, a_n+1) \quad (n \geqq 2) \text{ or } (1, a_1+2)\ (n=1)$$

(possibly by a cyclic permutation). For an s-vector a, a set $\boldsymbol{k}$ of t vectors $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_t$ and a t-vector b we define $a \oplus \boldsymbol{k} \leqq b$ if $s \leqq t$, and if $a'':=(a'_1+|\boldsymbol{k}_1|, a'_2+|\boldsymbol{k}_2|, \cdots, a'_t+|\boldsymbol{k}_t|)$ is smaller than b for a suitable $(t-s)$-fold blowing-up $a'=(a'_1, \cdots, a'_t)$ of a. The vectors $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_t$ are of arbitrary size. For an integral n-vector $a=(a_1, \cdots, a_n)$ with $a_j \geqq 0$, we define $A(a)$ to be the disjoint union of $A_{a_1}, \cdots, A_{a_n}$, where $A_0=\phi$ and define $A(\boldsymbol{k})=$ the disjoint union of $A(\boldsymbol{k}_1), \cdots, A(\boldsymbol{k}_t)$ for a set $\boldsymbol{k}=\{\boldsymbol{k}_1, \cdots, \boldsymbol{k}_t\}$.

(8.6) **Conjecture.** *Let T be a cusp singularity, T^* the dual of T. A cusp singularity U plus $A(\boldsymbol{k})$ a disjoint union of rational double singularities is a small deformation of T iff* Zykel $(U^*) \oplus \boldsymbol{k} \leqq$ Zykel (T^*) *where U^* is the dual of U.*

A simply elliptic singularity $E(n)$ plus $A(\boldsymbol{k})$ is included in (8.6) by viewing Zykel $(E(n)^*)=(2, \cdots, 2)$ (n-times). Only if part of (8.6) is true if (8.3) is true. Therefore (8.6) is proved for Degree $T \leqq 2$ by means of (8.4). [Ka] asserts that (8.6) is true for Degree $T \leqq 4$. It is also true for Degree $T \leqq 5$ if a gap in [L2, III (3.2)] is overcome.

(8.7) **Theorem.** *Let T be a cusp singularity, T^* the dual of T. Suppose that a cusp singularity U has the same Degree as T. Then U plus $A(\boldsymbol{k})$ a disjoint union of A_k singularities is a small deformation of T iff* Zykel $(U^*) \oplus \boldsymbol{k} \leqq$ Zykel (T^*) *where U^* is the dual of U.*

Proof. This is a corollary to [W, Theorem 3.9 and 5.4]. In fact, with the same notation as in [W], let Y be a partial resolution of T with $A(\boldsymbol{k})$ singularities obtained by blowing down some chains of (-2) curves in the minimal resolution X of T. Then by [W, Theorem 3.9], $BL_Y \rightarrow L_{A(\boldsymbol{k})}$ is smooth. Since $L_{A(\boldsymbol{k})} \cong D_{A(\boldsymbol{k})}$ is smooth, BL_Y is smooth. If U is a (Degree-preserving) small deformation of T, then $U \oplus A(\boldsymbol{k})$ is therefore a small deformation of T. Then it is easy to check that U is obtained by [W, Theorem 5.4] iff Zykel $(U^*) \leqq$ Zykel (T^*). Q.E.D.

(8.8) **Theorem**. *Let T be a cusp singularity. Then a cusp singularity U (or a simply elliptic singularity) is a small deformation of T if* Zykel $(U^*)\leqq$ Zykel (T^*).

Proof. This is a corollary to [W, Theorems 5.4 and 5.6]. We shall prove

Assertion. *Given a cusp singularity U,* Zykel $(U^*)\leqq$ Zykel (T^*) *iff U is obtained from T by the following two operations W_1 and W_2*

$$W_1:\ C(d_1, d_2, d_3, \cdots, d_r)\to C(d_1+d_2-2, d_3, \cdots, d_r)$$
$$W_2:\ C(d_1, c_1, \cdots, c_t, d_2, \cdots, d_r)\to C(d_1+d_2-1, d_3, \cdots, d_r)$$

where $c_1, \cdots, c_t$ are integers $\geqq 2$ such that

$$n^2/(nq-1)=c_1-1/c_2-1/c_3-1/\cdots-1/c_t$$

for relatively prime n and q, $0<q<n$. (See [ibid.].)

Here we denote a cusp singularity T' with Zykel $(T')=(d_1, \cdots, d_t)$ by $C(d_1, \cdots, d_t)$ following [W].

One sees immediately from [W, (2.11.2)] that

$$(c_1, \cdots, c_t)$$
$$=(a_s-1, 2^{b_{s-1}-3}, a_{s-1}, 2^{b_{s-2}-3}, \cdots, a_2, 2^{b_1-3}, a_{1+2}, 2^{a_1-3}, \cdots, b_{s-1}, 2^{a_s-3})$$

for some $a_j, b_j\geqq 3$ where 2^n stands for $(2, \cdots, 2)$ (n-times).

These operations induce operations on the dual cycles via (6.2) 3). For simplicity, we assume $d_j\geqq 3$, $a_s\geqq 4$. Then W_k induces an operation W_k^* on the dual cycles as follows;

$$W_1^*:\ C(2^{d_1-3}, 3, 2^{d_2-3}, 3, \cdots, 2^{d_r-3}, 3)\to C(2^{d_1+d_2-5}, 3, 2^{d_3-3}, \cdots, 2^{d_r-3}, 3)$$
$$W_2^*:\ C(2^{d_1-3}, 3, 2^{a_s-4}, b_{s-1}, 2^{a_{s-1}-3}, \cdots, b_1, 2^{a_1-1}, a_1, 2^{b_1-3}, \cdots,$$
$$a_s, 2^{d_2-3}, 3, \cdots, 2^{d_r-3}, 3) \quad (=:C)$$
$$\to C(2^{d_1+d_2-4}, 3, 2^{d_3-3}, 3, \cdots, 2^{d_r-3}, 3) \quad (=:C').$$

It is easy to see that C is a blowing-up of C', hence in particular $C\geqq C'$.

The blowing-down operations of the dual cycles are

$$B_1:\ C(e_1, e_2, \cdots, e_l)\to C(e_1-1, e_2, \cdots, e_l) \quad (e_1\geqq 3),$$
$$B_2:\ C(e_1, e_2, e_3, \cdots, e_l)\to C(e_1, 1, e_3, \cdots, e_l)$$
$$\to C(e_1-1, e_3-1, e_4, \cdots, e_l) \quad (e_1, e_3\geqq 3)$$

or more generally

$$B_3\colon C(e_1, b_s, 2^{a_s-m}, b_{s-1}, 2^{a_{s-1}-3}, \cdots, b_1, 2^{a_1-1}, a_1, 2^{b_1-3}, \cdots, a_s, e_2, \cdots, e_l) \to C(e_1, b_s-1, m-2, e_2, \cdots, e_l)$$

where $a_j, b_j \geqq 3$, $m \geqq 4$. One sees $W_1^* = B_1$ and $W_2^* = B_3$. Thus the proof of the assertion is complete. Q.E.D.

(8.9) Let $T = T_{p,q,r}$ $(p, q \geqq 3, r \geqq 4)$. In view of (8.7), $U = T_{p',q',r'}$ plus $A(k)$ where $k = \{k_1, k_2, k_3\}$ is a small deformation of T iff $|k_1| \leqq p-p'$, $|k_2| \leqq q-q'$, $|k_3| \leqq r-r'$. This is equivalent to that the Dynkin diagram of $U \oplus A(k)$ is a proper subdiagram of $\tau_{p,q,r}$ containing $\tau_{3,3,3}$. This establishes the bijective correspondence between 1) and 2) in the introduction. We compare this with the following. In view of [L2], any elliptic deformation U of T with same Degree is realized by blowing down a lifting of C on a deformation $\mathscr{X}_t$ of a nonsingular hyperbolic Inoue surface with C lifted, D invariant. The surface $\mathscr{X}_t$ is a blown-up parabolic or a hyperbolic Inoue surface with dual cycle D with Zykel$(D) = (p-1, q-1, r-1)$. In view of (8.4), the centers of blowing-ups thereby are on D. We may assume that the minimal model $\mathscr{X}_t^{\min}$ of $\mathscr{X}_t$ is a parabolic or a hyperbolic Inoue surface with dual cycle D' with Zykel$(D') = (p'-1, q'-1, r'-1)$ where $3 \leqq p' \leqq p$, $3 \leqq q' \leqq q$, $3 \leqq r' \leqq r$. If $p' = q' = r' = 3$, then $\mathscr{X}_t^{\min}$ is a parabolic Inoue surface and conversely in view of [N5]. Let $\mathscr{C}_t$ be a lifting of C to $\mathscr{X}_t$, U the singularity obtained by blowing down $\mathscr{C}_t$. We see that U is isomorphic to $T_{p',q',r'}$ by (5.11). If $p' = q' = r' = 3$, then U is a simply elliptic singularity $T_{3,3,3}$. We choose and fix affine coordinates on three irreducible components of D'. Then we may set centers of blowing-ups of $\mathscr{X}_t^{\min}$ as

$$\begin{aligned} &P_i \in D_1', \quad Q_j \in D_2', \quad R_k \in D_3' \quad \text{and} \\ &P_i\colon s = s_i \quad (1 \leqq i \leqq p-p'), \\ &Q_j\colon t = t_j \quad (1 \leqq j \leqq q-q'), \\ &R_k\colon u = u_k \quad (1 \leqq k \leqq r-r'). \end{aligned}$$

Suppose that Zykel$(D') \oplus k \leqq$ Zykel(D), that is, $|k_1| \leqq p-p'$, $|k_2| \leqq q-q'$, $|k_3| \leqq r-r'$. Choose centers of blowing-ups by

$$(\sharp) \quad \begin{aligned} &s_{i_\lambda+1} = s_{i_\lambda+2} = \cdots = s_{i_{\lambda+1}} \quad (0 \leqq \lambda \leqq l-1) \\ &t_{j_\lambda+1} = t_{j_\lambda+2} = \cdots = t_{j_{\lambda+1}} \quad (0 \leqq \lambda \leqq m-1) \\ &u_{k_\lambda+1} = u_{k_\lambda+2} = \cdots = u_{k_{\lambda+1}} \quad (0 \leqq \lambda \leqq n-1) \end{aligned}$$

where
$$\begin{aligned}
\boldsymbol{k}_1&=(i_1-1,\, i_2-i_1-1,\, i_3-i_2-1,\, \cdots,\, i_l-i_{l-1}-1),\\
\boldsymbol{k}_2&=(j_1-1,\, j_2-j_1-1,\, j_3-j_2-1,\, \cdots,\, j_m-j_{m-1}-1),\\
\boldsymbol{k}_3&=(k_1-1,\, k_2-k_1-1,\, k_3-k_2-1,\, \cdots,\, k_n-k_{n-1}-1),
\end{aligned}$$

$i_0=j_0=k_0=0$ and there are no further coincidence of centers of blowing-ups. Then the exceptional set of the singularity $A(\boldsymbol{k})$ disjoint from D (the proper transform of D') appears on the blowing-up of a hyperbolic or a parabolic Inoue surface. Thus we have established the bijective correspondence between 2) and 3) in the introduction.

(8.10) **Corollary.** *There is a bijective correspondence between any two of the following three objects;*

1) *proper subdiagrams of* $\tau_{p,q,r}$ *containing* $\tau_{3,3,3}$,

2) *(isomorphism classes of) not necessarily connected elliptic deformation of* $T_{p,q,r}$ *with Degree three,*

3) *(deformation classes of) blown-up hyperbolic or parabolic Inoue surfaces whose dual cycle* D' *satisfies* Zykel$(D')\leqq(p-1,\, q-1,\, r-1)$ *and is blown up into* $(p-1,\, q-1,\, r-1)$.

(8.11) We consider the following family

$$\text{(b)}\qquad x^3\prod_{i=1}^{p-3}(x+s_i)+y^3\prod_{j=1}^{q-3}(y+t_j)+z^3\prod_{k=1}^{r-3}(z+u_k)-xyz=0$$

over $S:=\{(s_i, u_j, t_k)\in \boldsymbol{C}^{p+q+r-9};\, |s_i|, |u_j|, |t_k|<\varepsilon\}$. If we choose s_i, t_j, u_k as in (8.9) (♯), then we obtain $A(\boldsymbol{k})$ singularities on the hypersurface. In fact, for instance, it has a singularity A_{i_1-1} at $x=-s_1(=-s_2=\cdots=-s_{i_1})$, $y=z=0$. Moreover if $s_{p'-2}=\cdots=s_{p-3}=t_{q'-2}=\cdots=t_{q-3}=u_{r'-2}=\cdots=u_{r-3}=0$, then it has $T_{p',q',r'}$ at $x=y=z=0$. Thus for any $\boldsymbol{k}=\{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\}$ with $|\boldsymbol{k}_1|\leqq p-p'$, $|\boldsymbol{k}_2|\leqq q-q'$, $|\boldsymbol{k}_3|\leqq r-r'$, the hypersurface has the singularity $T_{p',q',r'}\oplus A(\boldsymbol{k})$. As we have seen above, the bijective correspondence between 2) and 3) is sharpened by the parameters s_i, u_j, t_k. This fact suggests the following

(8.12) **Conjecture.** *The parameters* s_i, u_j, t_k *in the family* (b) *are affine coordinates of the centers of blowing-ups on three irreducible components of D of hyperbolic or parabolic Inoue surfaces.*

(8.13) Let $T=T_{p,q,r}$ $(p, q, \geqq 3, r\geqq 4)$. We recall from [N6] a description of $\mathrm{Ext}^1(\Omega_T^1, \mathcal{O}_T)$, the space of infinitesimal deformations of T. Compare also [B]. By [S], we have an exact sequence

$$0\to \mathrm{Ext}^1(\Omega_T^1, \mathcal{O}_T)\to H^1(W, \Theta_W)\xrightarrow{dF} H^1(W, \mathcal{O}_W^3)$$

where $F=(f_0, f_1, f_2)$ is the holomorphic mapping in (5.7), $W=T\backslash\{\infty\}$, Θ_W is the sheaf of germs of holomorphic vector fields over W. We may assume $W=T\backslash\{\infty\}\cong \boldsymbol{H}^2/G(M, V)=(\boldsymbol{H}^2/M)/V$. Then by means of group cohomology groups, we can describe

$$H^1(W, \Theta_W)\cong H^1(V, H^0(\boldsymbol{H}^2/M, \Theta_{\boldsymbol{H}^2/M})),$$
$$H^1(W, \mathcal{O}_W)\cong H^1(V, H^0(\boldsymbol{H}^2/M, \mathcal{O}_{\boldsymbol{H}^2/M})).$$

For the description of dF, see [B, p. 419] or [N6, (4.2)].

(8.14) **Theorem** [N6, (5.3)]. *Let $T=T_{p,q,r}$ $(p, q\geqq 3, r\geqq 4, (p, q, r)\neq (3, 3, 4))$. Then as a subspace of $H^1(V, H^0(\boldsymbol{H}^2/M, \Theta_{\boldsymbol{H}^2/M}))$, $\mathrm{Ext}^1(\Omega_T^1, \mathcal{O}_T)$ is spanned by*

$$\begin{aligned}
&\theta(-in_0^*)\delta_0 \ (=:\theta_{i,0}) \quad (1\leqq i\leqq p-2),\\
&\theta(-jn_1^*)\delta_1 \ (=:\theta_{j,1}) \quad (1\leqq j\leqq q-2),\\
&\theta(-kn_2^*)\delta_2 \ (=:\theta_{k,2}) \quad (1\leqq k\leqq r-2),\\
&\theta(-pn_0^*)\delta_0+\theta(-qn_1^*)\delta_1+\theta(-rn_2^*)\delta_2\\
&\theta(-(p-1)n_0^*)\delta_0+\theta(-n_1^*-n_2^*)(\delta_1+\delta_2) \quad (=:\theta_{p-1,0})\\
&\theta(-(q-1)n_1^*)\delta_1+\theta(-n_2^*-n_3^*)(\delta_2+\delta_3) \quad (=:\theta_{q-1,1})\\
&\theta(-(r-1)n_2^*)\delta_2+\theta(-n_3^*-n_4^*)(\delta_3+\delta_4) \quad (=:\theta_{r-1,2})
\end{aligned}$$

where $\theta(-n^*)=\exp(2\pi i(-n^*z_1-(n^*)'z_2))$, $\delta_j=(n_j^*)'(\partial/\partial z_1)-n_j^*(\partial/\partial z_2)$.

(8.15) On the other hand, as is well known,

letting $H=x^p+y^q+z^r-xyz$,

$$\begin{aligned}
\mathrm{Ext}^1(\Omega_T^1, \mathcal{O}_T)&\cong \boldsymbol{C}[x, y, z]/(H_x, H_y, H_z, H)\\
&=\boldsymbol{C}[x, y, z]/(x^p, y^q, z^r, xyz, H_x, H_y, H_z)
\end{aligned}$$

is spanned by $1, x, \cdots, x^{p-1}, y, \cdots, y^{q-1}, z, \cdots, z^{r-1}$. Comparing the actions of the monomial automorphism group of $T_{p,q,r}$ (see (7.8)), both the expressions of $\mathrm{Ext}^1(\Omega_T^1, \mathcal{O}_T)$ are probably related in the following manner;

$$x^i\leftrightarrow\theta_{p-i,0}, \quad y^j\leftrightarrow\theta_{q-j,1}, \quad z^k\leftrightarrow\theta_{r-k,2},$$
$$1\leftrightarrow\theta(-pn_0^*)\delta_0+\theta(-qn_1^*)\delta_1+\theta(-rn_2^*)\delta_2.$$

(8.16) **Problem.** *Give the exact relation between two expressions of* $\mathrm{Ext}^1(\Omega_T^1, \mathcal{O}_T)$. *What is the geometry behind this isomorphism? In other words, study the deformation* (8.11) (b) *by means of transcendental expression of* $T=\boldsymbol{H}^2/G(M, V)\cup\{\infty\}$.

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