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Complete Symmetric Varieties II

Intersection theory

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Introduction

This paper is a continuation of our "Complete symmetric varieties" [5]. We explain here a method suitable to solve general enumerative problems on a symmetric homogeneous space. In the classical enumerative theory of conics the following class of problems was studied. One gives a "condition" on a conic, i.e. the condition of passing through a point, being tangent to a line, being osculating to a curve in a given family etc.

The set of conics satisfying this type of conditions will be an algebraic subvariety, its codimension is called the dimension of the condition.

If one imposes a number of independent conditions, such that the sum of their dimensions is equal to 5 (the dimension of the space of conics), then the set of conics satisfying the given conditions is finite and the main question of enumerative geometry of conics to compute its cardinality. Of course here we are talking about conics as a way of example but at least for this general approach, a general homogeneous variety fits in the discussion. The method of Chasles, Schubert used classically is to construct an algebra with the set of conditions; compute any condition as the linear combination of basic ones and reduce any enumerative problem to the ones involving the basic conditions. For instance Chasles computes the number of conics tangent to 5 general conics as follows. He proves that the condition of being tangent to a conic is $2(\alpha+\beta)$ where α is "passing through a point", β is "being tangent to a line" the number requested is then $2^5(\alpha+\beta)^5$ and each monomial $\alpha^i\beta^{5-i}$ can be computed by direct geometric arguments. The idea of equivalence of conditions is justified by the principle of conservation of number, we transform an enumerative problem into another one which has the same number of solutions through this principle. We refer to Kleiman's treatment for further comments and informations on this theory [12].

It is not hard to formalize this set of ideas although, as we shall see,

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there are great difficulties to carry out the program in general (for a symmetric variety this is instead possible).

A way of formalizing is the following: Let G/H be a homogeneous variety of dimension n, G a connected algebraic group. Let $Y_1, Y_2 \subseteq G/H$ be two subvarieties of codimension r and $n-r$. If we take a generic element $g \in G$ we have that gY_1 intersects Y_2 in a finite number of points with transversal intersection. This number is, in fact, constant for g in an open set of G, since $gY_1 \cap Y_2$ can be interpreted as the fiber at g of the projection morphism $\pi: W \to G$, where W is the set of triples (x_1, x_2, g), $gx_1 = x_2$, $x_1 \in Y_1$, $x_2 \in Y_2$ (see [11]). One can take the computation of the number of elements of $gY_1 \cap Y_2$ as a basic enumerative problem. We set (Y_1, Y_2) equal to the previous number and extend this to a pairing $\mathscr{Z}^r(G/H) \times \mathscr{Z}^{n-r}(G/H) \to \mathbf{Z}$ between cycles of complementary codimension. Of course two cycles $a, b \in \mathscr{Z}^r(G/H)$ should be considered equivalent, from the enumerative point of view, if $(a, u) = (b, u)$ for every $u \in \mathscr{Z}^{n-r}(G/H)$, i.e. we set $\mathscr{B}^r(G/H) = \{a \in \mathscr{Z}^r(G/H) / (a, u) = 0$ for every $u \in \mathscr{Z}^{n-r}(G/H)\}$ and $C^r(G/H) = \mathscr{Z}^r(G/H)/\mathscr{B}^r(G/H)$. The basic pairing factors through $C^r(G/H)$ giving rise to a non degenerate pairing

$$C^r(G/H) \times C^{n-r}(G/H) \longrightarrow \mathbf{Z}.$$

We consider the set $C^r(G/H)$ as the "space of conditions of dimension r". The main problem, from a theoretical point of view, with this definition is to be able to introduce an intersection product which may finally justify the algebra of Chasles, Schubert. A naive approach is to consider two cycles $\Sigma n_i A_i$, $\Sigma m_j B_j$ and try to set as intersection $\Sigma n_i A_i \cap \Sigma m_j g B_j$ for a generic $g \in G$, the theorem of Kleiman [11] shows that at least for generic g, the intersection is proper; what is not true, in general, is that this cycle, for g in a non empty open set, varies in an equivalence class of cycles of $C^*(G/H)$ and finally that in this way we can introduce a ring structure in the space of conditions $C^*(G/H)$. Consider in fact the following example: Let $G = V$ a 3-dimensional vector space acting on itself by translation. It is clear that two lines (resp. planes) are equivalent if and only if they are parallel. Consider now the quadratic $xy - z = 0$, we want to intersect it with a generic translate of the plane $x = 0$. Such a translate is $x = \lambda$, λ a parameter. Then we get $x\lambda - z = 0$, $x = \lambda$ a family of inequivalent lines. On the other hand the previous approach works perfectly in the case in which G/H is a complete variety, in this case the theory of Bruhat cells, generalizing Schubert's cycles theory shows that $C^*(G/H)$ can be identified with the Chow ring $A^*(G/H)$ or with the cohomology ring $H^*(G/H)$ [3]. It is a remarkable fact that the previous method works also in the special case of a symmetric variety, as for instance the variety of non degenerate quadrics in $\mathbf{P}^n$.

The corresponding theory is the object of this paper and we shall now explain it.

We assume now that G/H is symmetric, G an adjoint group. In [5] we have constructed a canonical G-equivariant compactification X of G/H, we shall consider all possible compactifications X' which lie over X; i.e. we consider G varieties X' with a dense open orbit isomorphic to G/H and a G-equivariant morphism $\pi: X' \to X$ extending the identity on G/H. In Section 5 we classify all such varieties and in particular the ones which are proper and smooth, let us indicate by $\mathscr{C}$ this class. Our main result is:

i) The class $\mathscr{C}$ is a directed family and a cofinal subset of $\mathscr{C}$ is formed by the varieties obtained from X by a sequence of blow ups of closures of codimension 2 orbits.

ii) If $X' \in \mathscr{C}$, X' is paved by affine cells, the Chow ring $A^*(X')$ is isomorphic to the cohomology $H^*(X')$.

iii) The space of conditions $C^*(G/H)$ is a ring under the intersection product previously discussed

iv) $C^*(G/H) = \varinjlim_{X' \in \mathscr{C}} A^*(X') = \varinjlim_{X' \in \mathscr{C}} H^*(X')$.

To discuss further the features of this theory let T^1 be a maximal anisotropic torus in G, W^1 be the Weyl group of the symmetric variety, $\bar{T}^1 = T^1/H \cap T^1 \subseteq G/H$. W^1 acts on $\bar{T}^1$ and if we consider a G equivariant embedding $G/H \subseteq X'$ the closure of $\bar{T}^1$ in X' is a torus embedding on which W^1 acts extending its action on $\bar{T}^1$.

We have then

v) The torus embedding Z_0, closure of $\bar{T}^1$ in X is associated to the r.p.p.d. of Weyl chambers of the root system of the symmetric variety.

vi) There is a 1–1 correspondence between equivariant embeddings of G/H over X and W^1 stable torus embedding of $\bar{T}^1$ lying over Z_0, given by associating to a G/H embedding X' the closure of $\bar{T}^1$.

vii) The previous correspondence preserves the properties of being normal, complete, smooth (more generally one has the same type of singularities).

viii) For any G-equivariant embedding X' of G/H and the corresponding torus embedding $Z' \subseteq X'$ of $\bar{T}^1$ one has that each G orbit of X' intersects Z' exactly in a (non empty) W^1 equivalence class of $\bar{T}^1$ orbits.

If we consider in particular normal complete embeddings, the corresponding $\bar{T}^1$ embedding is described by a r.p.p.d. stable under the action of W^1 and refining the r.p.p.d. of Weyl chambers; thus, equivalently, the torus embedding is described by a r.p.p.d. refining the fundamental Weyl chamber.

As we have already mentioned in our previous paper this set of ideas has been influenced by the work of Luna and Vust [13], [19]. Here we do not use their general method but rather develop the theory in a more direct way with an explicit link with torus embeddings. The advantage for us is a presentation that reduces everything to the basic model X.

This presentation works in a characteristic free way at least assuming that X exists, which seems to be a general fact although we have not tried to discuss this in detail for all cases (cf. [5]). On the other hand this approach does not seem suitable for getting all the results of Vust [19] in the sense that we only describe the embeddings which lie over X.

Let us discuss now some of the technical aspects of the theory. We need first of all to prove a theorem (in the style of Hironaka's resolution of singularities) for torus embeddings:

ix) Given two embeddings of a torus T, Y_1, Y_2 with Y_2 complete and smooth we can construct a torus embedding Y', obtained from Y_2 by a sequence of blow ups of codimension 2 orbit closures, a T stable open set U of Y' and a proper T-equivariant morphism $U \to Y_1$.

We will need this to prove a basic transversality result which is the clue for the study of the "ring of conditions $C^*(G/H)$". The result is the following:

x) Given a cycle $Y \subseteq G/H$ we can find a smooth G-equivariant compactification X' of G/H (lying over X) such that the closure $\bar{Y}$ of Y in X' has proper intersection with the cycle Z' of X' sum of all closures of the G orbits in X'.

It will be clear that this is the main result needed to treat the so called "Halphen conditions". Let us briefly discuss this point. In the beginning of the theory the first difficulty to be overcome was the following: in order to apply the principle of conservation of number one has to make sure not to change the number of solutions of a given problem in a deformation. This is of course immediate by general principles if the ambient variety is compact, not so in the non compact case. Of course the variety of non degenerate conics has a natural compactification in the variety of all conics, a $\mathbf{P}^5$, but if we consider in this $\mathbf{P}^5$ the hypersurface of conics tangent to a given one we see that it contains the Veronese surface of double lines (which is the unique closed orbit under the action of the projective group). This is the main reason for which one cannot apply the theorem of Bezout for the computation of the problem of tangency to 5 conics and instead one has to pass to the model of complete conics X (which is obtained by blowing up the Veronese surface).

In X the hypersurface given by the same condition does not contain the unique closed orbit in X (called the set of Halphen conics) nevertheless it is clear that one can find some other condition which is satisfied by all elements in this closed orbit.

For instance, given a net of conics, one may consider the variety of all conics hyperosculating one of the elements of the net.

This condition defines a hypersurface in X containing the Halphen conics [16].

Thus one is led to find a new model where the previous condition is not satisfied by all the elements in closed orbits. This is the method to be performed for conditions of dimension 1.

More generally one can easily see (cf. Section 5) that for a general condition the correct model is one in which the cycle given by the condition has proper intersection with every orbit. The existence of such a model is in fact the content of our main technical result. The informations so obtained are then collected in a formal theorem that identifies the ring $C^*(G/H)$ as a limit of Chow rings of equivariant compactifications. The actual computation of $C^*(G/H)$ can, at least in principle, be performed since at each stage we are blowing up a codimension 2 orbit closure which is the transversal intersection of two smooth orbit closures, thus the normal bundle is known and one can at least theoretically work by induction.

It would be in fact interesting to give a more explicit presentation of $C^*(G/H)$ in the spirit at least of the enumerative algorithm described at the end of [5].

Finally we wish to thank D. Luna, T. Vust for discussing with us parts of their general theory, E. Sernesi for useful information on the Chow ring and H. Kraft for organizing a meeting in Basel on embeddings in which we had the opportunity to sketch some of these results.

§ 1. Generalities on torus embeddings

Standard references for this section are [4], [10], [14].

1.1. Let T be a fixed n-dimensional torus, defined over an algebraically closed field K; $X(T)$ its character group, $\check{X}(T)=\mathrm{Hom}(X(T), Z)$ and $V\simeq\mathbf{R}^n$ the n-dimensional vector space of linear forms on $X(T)\otimes_Z \mathbf{R}$.

Definition. A torus embedding consists of a variety Y with a T action (a T variety), having a dense orbit isomorphic to T (as a T variety).

If Y is affine, the coordinate ring $\mathcal{O}(Y)$ of Y is a T stable subalgebra of $\mathcal{O}(T)$.

Since $\mathcal{O}(T)=\oplus_{\chi\in X(T)} K\chi$ one has $\mathcal{O}(Y)=\oplus_{\chi\in S} K\chi$, where S is a subsemigroup of $X(T)$ which spans $X(T)\otimes_Z \mathbf{R}$ over $\mathbf{R}$.

If Y is normal we associate to Y the cone

$$C_Y=\{\varphi \in V \mid (\varphi, \chi)\geqslant 0,\ \chi \in S\}.$$

Proposition. i) *C_Y does not contain any line (a pointed cone).*
(ii) *There exist primitive vectors $v_1, v_2, \cdots, v_m \in \check{X}(T)$ such that*

$$C_Y=\left\{\sum_{i=1}^{m} \alpha_i v_i \mid \alpha_i \geqslant 0\right\}.$$

We will always assume that the v_i's are irredundant, under this assumption they are well determined and we write $C_Y=C(v_1, \cdots, v_m)$; we call a cone of this type a "rational pointed cone".

If $\sigma=C(v_1, \cdots, v_m)$ we set $\mathring{\sigma}=\sum_{i=1}^{m} \alpha_i v_i$, $\alpha_i>0$ and notice that $\mathring{\sigma}$ is the interior of σ, considered as a subset of the linear space spanned by $v_1, \cdots, v_m$.

Theorem. *The given construction establishes a 1–1 correspondence between normal affine torus embeddings and rational pointed cones.*

1.2. Given a rational pointed cone $\sigma=C(v_1, \cdots, v_m)$, for any subset $v_{i_1}, \cdots, v_{i_K}$ of $v_1, \cdots, v_m$ we call the rational pointed cone $C(v_{i_1}, \cdots, v_{i_K})$ a "face" of σ.

If Y is a normal affine torus embedding, for every T stable affine open set U of Y we have a corresponding cone C_U.

Proposition. i) *The correspondence $U \rightarrow C_U$ is a bijection between the open T stable affine sets of Y and the faces of C_Y, preserving the inclusion relations.*

ii) *Every open T stable affine set U of Y contains a unique T-orbit Z_U closed in U.*

iii) *Every T orbit Z in Y is of the form Z_U for a unique T stable affine set U.*

iv) $\operatorname{Codim}_Y Z_U=\dim C_U$.

Remark. Explicitly $\mathcal{O}(Y)=\bigoplus_{\chi \in S} K\chi$, $\mathcal{O}(U)=\bigoplus_{\chi \in S'} K\chi$.

$$C_Y=C(v_1, \cdots, v_m), \qquad C(U)=C(v_{i_1}, \cdots, v_{i_k})$$
$$S=\{\chi \mid (v_i, \chi)\geqslant 0,\ i=1, \cdots, m\}$$
$$S'=\{\chi \mid (v_{i_t}, \chi)\geqslant 0,\ t=1, \cdots, k\}$$
$$\mathcal{O}(Z_U)=\bigoplus_{\substack{(v_{i_t}, \chi)=0 \\ i=1,\cdots,k}} K\chi.$$

1.3. Let Y be a normal affine torus embedding, $C_Y=C(v_1, \cdots, v_m)$ its associated cone.

Propositon. *Y is smooth if and only if $v_1, \cdots, v_m$ can be completed to an integral basis of $\check{X}(T)$.*

In this case we will refer to $v_1, \cdots, v_m$ as a "partial basis", C_Y will be then called a non singular cone. We notice the following:

Remark. Y smooth has a T fix point if and only if $n=m$ and $v_1, \cdots, v_n$ is a basis of $\check{X}(T)$. In this case Y is isomorphic to A^n and the T-action is given by the characters $\varphi_1 \cdots \varphi_n$, the dual basis to $v_1 \cdots v_n$. In these cases we will talk of a torus embedding A^n.

1.4.

Definition. A rational polyhedral decomposition (r.p.p.d.) is a finite collection $\Delta=\{\sigma_\alpha\}$ of rational pointed cones σ_α such that for each α, β we have $\sigma_\alpha \cap \sigma_\beta$ is a face of σ_α and σ_β and belongs to the collection Δ. If $\sigma \in \Delta$ and τ is a face of σ we have $\tau \in \Delta$.

If Y is a general normal torus embedding, for each T stable affine open set U_α of Y we consider its associated rational pointed cone σ_α. The main theorem on torus embeddings is:

Theorem. i) *The collection $\{\sigma_\alpha\}$ associated to Y is an r.p.p.d.*

ii) *In this way we establish a 1–1 correspondence between normal torus embeddings and r.p.p.d.'s.*

iii) *Y is complete if and only if $\bigcup_\alpha \sigma_\alpha = V$.*

Remark. i) The main step consists in showing that Y is covered by T stable affine open sets.

ii) The theorem, and 1.3, imply that Y is smooth if and only if each σ_α is spanned by a partial basis.

iii) If we fix a basis $v_1, \cdots, v_n$ of $\check{X}(T)$ and we consider the r.p.p.d. Δ whose elements are the cones $\sigma=\{\pm v_{i_1}, \cdots, \pm v_{i_k}\}$ for any subset $(i_1, \cdots, i_k)$ of $(1, \cdots, n)$ and for any choice of signs, then the associated torus embedding is isomorphic to $\mathbf{P}^n$.

In this case we will talk of a torus embedding $\mathbf{P}^n$.

1.5. In this section we want to discuss the T equivariant morphisms between torus embeddings. Given two torus embeddings Y_1, Y_2 a morphism $\pi\colon Y_1 \to Y_2$ will be implicitly assumed to be T equivariant.

Remark. There exists at most one morphism $\pi\colon Y_1 \to Y_2$ up to translation by T.

Theorem. i) *If Y_1, Y_2 correspond to two r.p.p.d's Δ_1, Δ_2 there exists* *a morphism* $\pi\colon Y_1 \to Y_2$ *if and only if each* $\sigma \in \Delta_1$ *is contained in a cone* $\tau \in \Delta_2$.

ii) $\pi\colon Y_1 \to Y_2$ *is an open immersion if and only if* $\Delta_1 \subseteq \Delta_2$.

iii) $\pi\colon Y_1 \to Y_2$ *is proper if and only if* $\bigcup_{\sigma \in \Delta_1} \sigma = \bigcup_{\tau \in \Delta_2} \tau$

1.6. If Y is a torus embedding, Δ its r.p.p.d. and $Z \subseteq Y$ a T orbit one can associate to Z an element σ_Z of Δ as follows: One shows easily that there is a unique T stable affine open set U containing Z as a closed subvariety. The cone C_U is the element of Δ associated to Z. As in 1.2 we have $\operatorname{codim}_Y Z = \dim \sigma_Z$.

Suppose now that Y is smooth. One can show that also $\bar{Z}$ is smooth, therefore performing the blow up of Y along $\bar{Z}$ one obtains a new smooth torus embedding Y_Z and a proper morphism $\pi\colon Y_Z \to Y$. The r.p.p.d. Δ' associated to Y_Z is obtained as follows:

Let $\sigma_Z = C(v_1, v_2, \cdots, v_h)$ ($v_1, \cdots, v_h$ a partial basis), set $u = v_1 + v_2 + \cdots + v_h$.

If $\tau \in \Delta$, $\tau \not\supseteq \sigma$ we have $\tau \in \Delta'$; if $\tau \in \Delta$, $\tau = C(v_1, \cdots, v_h, w_1, \cdots, w_t)$ we replace τ and its faces with $\tau_1, \tau_2, \cdots, \tau_h$ where $\tau_i = C(v_1, \cdots, v_{i-1}, u, v_{i+1}, \cdots, v_h, w_1, \cdots, w_t)$ and their faces. We will say that Δ' is the blow up of Δ along σ_Z.

§ 2. Equivariant resolutions

2.1. We keep the notation of Section 1. The elements of $X(T)$ can be thought as linear functions on V integral on $\check{X}(T)$.

Fixing such an element $\varphi \in X(T)$ we set

$$\pi_\varphi = \{x \in V \mid \varphi(x) = 0\}.$$

Definition. Let $\sigma = C(v_1, \cdots, v_m)$ be a rational pointed cone.

i) σ crosses π_φ if there exist two indices i, j such that $\varphi(v_i) < 0$, $\varphi(v_j) > 0$.

ii) Set $M = M_\sigma = \max |\varphi(v_i)|$. We say that σ is indefinite if there exist two indices i, j such that $M = \varphi(v_i) = -\varphi(v_j)$. σ is definite otherwise.

iii) We associate to σ the quadruple (M, p, m, h) where $M = M_\sigma$, $p = \min(p_-, p_+)$, $m = \max(p_-, p_+)$, $p_+ = \{\#i \mid \varphi(v_i) = M\}$, $p_- = \{\#i \mid \varphi(v_i) = -M\}$. σ is definite if and only if $p = 0$. $h = \infty$ if σ is indefinite. If σ is definite we associate to it a sign $\varepsilon = \pm 1$. $\varepsilon = +1$ if $p_+ > 0$, $\varepsilon = -1$ if $p_- > 0$, and we set $h = \{\#i \mid \operatorname{sign} \varphi(v_i) = -\varepsilon\}$.

Remark. If σ is definite, σ does not cross π_φ if and only if $h = 0$.

2.2. We shall order the quadruples P_σ lexicographically. Remark that if τ is a face of σ, $P_\tau < P_\sigma$.

We will assume now that σ is a non singular rational pointed cone.

Lemma. i) *Assume σ is indefinite $M=\varphi(v_{i_1})=-\varphi(v_{i_2})$. The blow up of σ along the face $C(v_{i_1}, v_{i_2})$ replaces σ by σ_1, σ_2 with $P_{\sigma_1}, P_{\sigma_2}<P_\sigma$.*

ii) *Assume σ is definite and crosses π_φ. Let $|\varphi(v_{i_1})|=M$ and sign $\varphi(v_{i_2})=-\varepsilon$. The blow up of σ along the face $C(v_{i_1}, v_{i_2})$ replaces σ by σ_1, σ_2 with $P_{\sigma_1}, P_{\sigma_2}<P_\sigma$.*

Proof. i) Assume $i_1=1$, $i_2=2$. We have $\sigma_1=C(v_1, v_1+v_2, v_3, \cdots, v_m)$, $\sigma_2=C(v_2, v_1+v_2, v_3 \cdots, v_m)$. Assume $p=p_+$, $m=p_-$ then if $m=p$ we have P_{σ_1}, P_{σ_2} are of type $(M, p-1, m, h')$. If $m>p$, $P_{\sigma_1}=(M, p, m-1, \infty)$, $P_{\sigma_2}=(M, p-1, m, h')$.

ii) We may assume without loss of generality that $\varphi(v_1)=M$ and $-M<\varphi(v_2)<0$. Blowing up $C(v_1, v_2)$ we obtain $\sigma_1=C(v_1, v_1+v_2, v_3, \cdots, v_m)$ and $\sigma_2=C(v_2, v_1+v_2, v_3, \cdots, v_m)$. σ_1 is definite and $P_{\sigma_1}=(M, 0, m, h-1)$. If $m>1$ we have σ_2 definite and $P_{\sigma_2}=(M, 0, m-1, h)$, if $m=1$ we have $M_{\sigma_2}<M_\sigma$.

2.3. Let us give a r.p.p.d. $\Delta=\{\sigma_\alpha\}$ in which each σ_α is non singular.

Consider the set Δ^c of the σ_α's which cross π_φ. If $\Delta^c\neq\phi$ set $P_\Delta=\max_{\sigma\in\Delta^c} P_\sigma$ Set $q_\Delta=\{\#\sigma\in\Delta^c\,|\,P_\sigma=P_\Delta\}$.

Lemma. *We can perform on Δ a sequence of blow ups along 2 dimensional faces so that, if $\Delta'=\{\sigma_\beta'\}$ is the resulting r.p.p.d. each σ_β' does not cross π_φ.*

Proof. If $\Delta^c=\phi$ there is nothing to be proved. Otherwise we perform induction on (P_Δ, q_Δ) ordered lexicographically.

Assume first that there is an indefinite σ with $P_\sigma=P_\Delta$, we have a face $\sigma'=C(v_1, v_2)$ of σ with $\varphi(v_1)=-\varphi(v_2)=M_\sigma$. Any other $\tau\in\Delta$ having σ' as a face is still indefinite (and in Δ^c) and $P_\tau\leqslant P_\Delta$. We blow up along σ' and by 1.6 we have a new r.p.p.d. Δ'. If all the $\tau\in\Delta$ for which $P_\tau=P_\Delta$ contain σ' we have, from Lemma 2.2, that $P_{\Delta'}<P_\Delta$ otherwise $P_{\Delta'}=P_\Delta$, but $q_{\Delta'}<q_\Delta$.

If there is no indefinite σ with $P_\sigma=P_\Delta$, we choose σ with $P_\sigma=P_\Delta$ and a face $\sigma'=C(v_1, v_2)$ in σ with (without loss of generality) $\varphi(v_1)=M, -M<\varphi(v_2)<0$. We blow up along σ' and obtain a new r.p.p.d. Δ' by 1.6. Remark first of all that any $\tau\in\Delta$ having σ' as a face is definite. If all the $\tau\in\Delta$ for which $P_\tau=P_\Delta$ contain σ' we have by Lemma 2.2 that $P_{\Delta'}<P_\Delta$, otherwise $P_{\Delta'}=P_\Delta$, but $q_{\Delta'}<q_\Delta$.

We finish by induction.

2.4.

Theorem. *Given a r.p.p.d. $\Delta=\{\sigma_\alpha\}$ in which each σ_α is a non singular* *cone and another arbitrary r.p.p.d.* $\Gamma=\{\tau_\beta\}$ *we can perform on* Δ *a sequence of blow ups along 2 dimensional faces so that the resulting r.p.p.d.* $\Delta'=\{\sigma'_\alpha\}$ *has the property that for each* Υ, $\mathring{\sigma}_\gamma$ *is either contained in one of the* $\tau'_\beta s$ *or is in the complement of* $\bigcup_\beta \tau_\beta$.

Proof. Since Γ is a finite set it is sufficient to show that given a τ_β we can find $\Delta'=\{\sigma'_\alpha\}$ with $\mathring{\sigma}'_\gamma\subset\tau_\beta$ or $\mathring{\sigma}'_\gamma\subset\mathscr{C}(\tau_\beta)$ for each Υ.

Now τ_β is defined to be the set of points $x\in V$ satisfying $\varphi_i(x)\geqslant 0$ $i=1,\cdots,k$ for suitable linear functions $\varphi_i\in X(T)$. We can find, by Lemma 2.3, a $\Delta'=\{\sigma'_\gamma\}$ such that each σ'_γ does not cross any of the hyperplanes π_{φ_i}, $i=1,\cdots,k$. This implies the claim.

By a completely analogous argument we get:

Proposition. *Let, for each* $\sigma_\alpha\in\Delta$, *be given an r.p.p.d.* $\{\tau_\beta^{(\alpha)}\}=\Delta^{(\alpha)}$ *such that* $\bigcup\tau_\beta^{(\alpha)}=\sigma_\alpha$. *Then there exists a sequence of blow ups of* Δ *along 2 dimensional faces such that the resulting r.p.p.d.* $\Delta=\{\sigma'_\gamma\}$ *has the property that each* σ'_γ *is contained in one of the* $\tau_\beta^{(\alpha)}$*'s.*

2.5.

Corollary. *Let* Y_1, Y_2 *be two torus embeddings. Assume* Y_1 *is smooth. Put* $Y=Y_1\times_T Y_2$. *Then there exists a torus embedding* Z *obtained from* Y_1 *by a sequence of blow ups along closures of codimension 2 orbits and a open T-stable subset* $U\subset Z$ *with a proper T-equivariant morphism* $\pi\colon U\to Y$.

Proof. First normalize Y_2, call the normalization $\bar{Y}_2$. Associate to Y_1, $\bar{Y}_2$ their respective r.p.p.d. $\Delta_1=\{\sigma_\alpha\}$, $\Delta_2=\{\tau_\gamma\}$ and notice that since Y_1 is smooth each σ_α is a non singular rational pointed cone.

Since by 1.6 the formal blow ups along 2 dimensional faces correspond to actual blow ups along closures of codimension 2 orbits, by Theorem 2.4 we find a torus embedding Z obtained from Y_1 by a sequence of blow ups along closures of codimension 2 orbits such that, if $\Delta'=\{\sigma'_\beta\}$ is the r.p.p.d. associated to Z, each σ'_β either lies in a cone of Δ_2 or in the complement of $\bigcup_{\tau_\gamma\in\Delta_2}\tau_\gamma$. Let $U\subset Z$ be the open T-stable subset corresponding to the r.p.p.d. Δ'' of σ'_β's which are contained in some τ_γ. Clearly we have T-equivariant morphisms $\pi_1\colon U\to Y_1$ $\pi_2\colon U\to Y_2$ by 1.5, hence a T-equivariant morphism $\pi\colon U\to Y$. Since

$$\bigcup_{\sigma'_\beta\in\Delta''}\sigma'_\beta=\Big(\bigcup_{\sigma_\alpha\in\Delta_1}\sigma_\alpha\Big)\cap\Big(\bigcup_{\tau_\gamma\in\Delta_2}\tau_\gamma\Big)$$

again by 1.5 we get the properness of Y.

2.6. We fix a torus embedding $\mathbf{P}^n$

Corollary. *For any torus embedding Y there exists a suitable Z obtained from $\mathbf{P}^n$ by a sequence of blow ups along closures of codimension 2 orbits and a T-stable open subset $U \subset Z$ together with a proper morphism π: $U \to Y$. If X is complete $U = Z$.*

Proof. Clear from the above once we notice that, if we let $\Delta = \{\sigma_\alpha\}$ be the r.p.p.d. corresponding to $\mathbf{P}^n$, $\bigcup_{\sigma_\alpha \in \Delta} \sigma_\alpha = V$.

We now fix a torus embedding A^n.

Lemma. *Let H be any hypersurface of A^n. There exists a torus embedding Y, obtained from A^n by a sequence of blow ups along closures of codimension 2 orbits, such that: if we let: $Y \to A^n$ be the canonical projection, the variety $H' = \overline{\pi^{-1}(H \cap T)}$ (T is the dense orbit in A^n) does not contain any T fix point.*

Proof. By Corollary 2.4 it is sufficient to exhibit a torus embedding Y and a proper T-equivariant morphism π: $Y \to A^n$ such that the variety $H' = \overline{\pi^{-1}(H \cap T)}$ does not contain any T-fixpoint.

Let f be an equation of H. Write $f = \sum_{i=0}^N a_{I_i} X^{I_i}$ where $X^I = X_1^{j_1} \cdots X_n^{j_n} (I = (j_1, \cdots, j_n))$, $a_I \neq 0$.

Consider the rational map $A^n \xrightarrow{\varphi} \mathbf{P}^n$ of coordinates $(X^{I_0}, \cdots, X^{I_N})$. Set Y equal to the closure of the graph of φ in $A^n \times \mathbf{P}^N$. We can clearly give a torus action on $\mathbf{P}^N$ such that φ is T-equivariant and the T-fixpoints of $\mathbf{P}^N$ are the $N+1$ points $r_0, \cdots, r_n$ of coordinates $(0, 0, \cdots, 0, 1, 0, \cdots, 0)$.

Clearly Y is a torus embedding proper over A^n and its fixpoints lie over the points r_i.

We may work locally on the standard affine open charts of $\mathbf{P}^N$. Consider for instance the chart where the first coordinate is non zero.

In $A^n \times A^N$ we are considering the closure of the graph of the function on T,

$$\left(X_1, \cdots, X_n, \frac{X^{I_1}}{X^{I_0}}, \cdots, \frac{X^{I_N}}{X^{I_0}}\right).$$

Using coordinates $(X_1, \cdots, X_n, Y_1, \cdots, Y_N)$ we have the equations $Y_i X^{I_0} = X^{I_i}$ $i = 1, \cdots, N$.

In this chart the only T fixpoint is the origin, but the equation of H' is given by

$$a_{I_0} + \sum_{j=1} a_{I_j} Y_j \quad \text{(with } a_{I_0} \neq 0\text{) so } H' \text{ does not contain}$$

the origin.

Proposition. *Given a smooth torus embedding Y and an hypersurface $H \subset Y$; there exists a torus embedding Z, obtained from Y by a sequence of blow ups along closures of codimension 2 orbits, such that if π; $Z \to Y$ denotes the canonical T-equivariant projection $H' = \overline{\pi^{-1}(H \cap U)}$ does not contain any T-fixpoint.*

Proof. Let $\Delta = \{\sigma_\alpha\}$ be the r.p.p.d. associated to Y_1 so each σ_α is a non singular cone.

If $\{r_1, \cdots, r_t\}$ is the set of T-fixpoints in Y, Δ contains exactly t n-dimensional cones, $\sigma_1, \cdots, \sigma_t$ and if $U_1, \cdots, U_t$ are the open affine T-stable subsets associated to $\sigma_1, \cdots, \sigma_t$, $r_i \in U_i$, $1 \leqslant i \leqslant T$ and each U_i is an A^n (1.3). By the lemma we can find, for each i $1, \cdots, t$, a smooth torus embedding $\bar{Z}_i$ and a proper morphism ψ_i: $\bar{Z}_i \to U_i$ such that $H_i = \overline{\psi_i^{-1}(H \cap T)}$ does not contain any T fixpoint in $\bar{Z}_i$.

Let $\Delta_i = \{\sigma_\beta^{(i)}\}$ be the r.p.p.d. associated to $\bar{Z}_i$. By Proposition 2.4 we can find an r.p.p.d. $\Delta' = \{\tau_\gamma\}$ obtained from Δ by a sequence of blow ups along two dimensional faces such that: for each τ_γ, either τ_γ lies in some σ_α which is not a face of one of the σ_i's, $i = 1, \cdots, t$, or $\tau_\gamma \subset \sigma_\beta^{(i)}$ for some $\sigma_\beta^{(i)}$.

Let Z be the torus embedding associated to Δ'. Z is obtained from Y by a sequence of blow ups along closures of codimension 2 orbits.

Let π: $Z \to Y$ be the canonical projection. Setting $Z_i = \pi^{-1}(U_i)$, $i = 1, \cdots, t$ we clearly have a proper T-equivariant morphism φ_i: $Z_i \to \bar{Z}_i$. Setting $H' = \overline{\pi^{-1}(H \cap T)}$, $H' \cap Z_i$ does not contain any T-fixpoint. Since each T-fixpoint in Z lies over a fix point in Y we get the claim.

§ 3. Regular configurations

3.1.

Definition. i) Given a smooth variety X, a finite family $\mathscr{S}$ of hypersurfaces $\mathscr{S} = \{S_i\}_{i \in I}$, $S_i \subset X$ will be called a *regular configuration* if the following two properties are satisfied:

a) Each S_i is smooth

b) If $x \in S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ is a point of intersection of k distinct S_i's, their intersection is transversal in x.

ii) Given a regular configuration in X and a subset J of I we set

$$S_J = \bigcap_{j \in J} S_j.$$

We call such a variety S_J a *coordinate variety*.

Remark. If $J \subseteq I$, the coordinate variety S_J is smooth. If S_J is non empty it has pure codimension $|J|$ in X.

Examples. i) If X is a smooth torus embedding the family of the closures of the codimension one orbits is a regular configuration.

ii) Given a principal T bundle P on a smooth variety Y and a smooth torus embedding X the variety $P\times_T X$ is equipped with the regular configuration $\{\Sigma_i\}$, $\Sigma_i=P\times_T S_i$, S_i the closure of a codimension one orbit.

iii) If X has a regular configuration $\{S_i\}$ and $Y\subset X$ is a smooth subvariety, the family $\{S_i\}$, $S_i'=S_i\cap Y$ is a regular configuration in Y if

a) $S_i'\neq\phi$ for each i

b) The intersection of Y with any coordinate variety S_J is transversal in each point.

iv) If $\{S_i\}_{i\in I}$ is a regular configuration so is any subfamily.

We want to show now that every regular configuration can be obtained through the previous examples.

Let $\{S_i\}_{i\in I}$ be a regular configuration in X, let $n=|I|$. For each i set $\mathscr{L}_i=\mathcal{O}(S_i)$ and $s_i\in H^0(X,\mathscr{L}_i)$ a section with divisor S_i.

Let $V=\oplus_{i\in I}\mathscr{L}_i$. We have a natural action of $T=G_m$ on V, and an associated principal bundle P, by acting as scalars independently on each summand. Furthermore we have a section

$$s\in H^0(X, V),\qquad s=s_1\oplus s_2\oplus\cdots\oplus s_n.$$

Let $v_1,\cdots,v_n$ be the canonical basis of $X(G_m^n)$ given by the n coordinate 1-parameter subgroups.

We define the r.p.p.d. Δ formed by all the cones $C(v_{i_1},\cdots,v_{i_k})$ for which $S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_k}\neq\phi$. The corresponding torus embedding Y is a T stable open set of the canonical embedding A^n associated to $C(v_1,\cdots,v_n)$.

Given a point $\beta=(\alpha_1,\cdots,\alpha_n)\in A^n$, setting $J=\{i\,|\,\alpha_i=0\}$ we have $p\in Y$ if and only if $S_J\neq\phi$.

The fiber bundle $A=P\times_T Y$ is the open set of V described fiberwise in the same way.

The regular configuration in $Y\subseteq A^n$ is given by the n hypersurfaces of Y defined by intersecting Y with the coordinate hyperplanes of A^n. The induced configuration in A is formed therefore by the hypersurfaces $\Sigma_i=A\cap(\oplus_{j\neq i}\mathscr{L}_j)$.

Proposition. i) *$\Sigma_J\neq\phi$ if and only if $S_J\neq\phi$.*

ii) *$S(X)\subseteq A$.*

iii) *$S(X)$ meets Σ_J transversally in S_J.*

Proof. One easily sees that the statements are of local nature and thus one can assume that all the $\mathscr{L}_i$'s are trivial bundles. In this case the verification is clear.

3.2. Let X, $\{S_i\}$ be a regular configuration. We have seen how to associate to X a torus embedding Y, a principal T bundle P on X and an embedding $j\colon X \hookrightarrow P\times_T Y$. Suppose now Z is a smooth torus embedding with a morphism $\pi\colon Z\to Y$. As with Y the variety $P\times_T Z$ is smooth and inherits a regular configuration from the one in Z, furthermore we have a morphism $\bar{\pi}\colon P\times_T Z\to P\times_T Y$. Consider the variety $X_Z=\bar{\pi}^{-1}(j(X))$. We claim:

Proposition. i) *X_Z is smooth*

ii) *X_Z is transversal to the regular configuration of $P\times_T Z$*

iii) *The projection $\bar{\pi}\colon X_Z\to X=j(X)$ is birational.*

iv) *If π is proper so is $\bar{\pi}$.*

Proof. The statements are essentially of local nature. Let us take an open set U of X where the line bundles $\mathcal{L}_i$ are trivial: $\mathcal{L}_i=X\times A^1$, the sections s_i are thus associated to functions f_i and the map $\varphi\colon U\to A^n$ given by the coordinates f_i is smooth. Of course we do not assume that all the hypersurfaces S_i meet U necessarily, we know in any case that φ maps U inside the open set Y.

Set $U_Z=\bar{\pi}^{-1}(j(U))$. We can identify U_Z with the fiber product:

$$\begin{tikzcd} U_Z \arrow[r, dashed] \arrow[d, dashed] & Z \arrow[d] \\ U \arrow[r, "\varphi"] & Y \end{tikzcd}$$

Thus U_Z is smooth, the projection to U is birational and proper if π is so. The map $U_Z\to Z$ is also smooth. This last statement is equivalent to the transversality of X_Z to the regular configuration of $P\times_T Z$.

3.3. We should notice a special case of the construction of the previous paragraph. Suppose Z is obtained from Y by blowing up a T stable subvariety W of codimension k; then $P\times_T Z$ is obtained from $P\times_T Y$ by blowing up $P\times_T W$ (still of codimension k).

Since $X=j(X)\subseteq P\times_T Y$ is transversal to the regular configuration an elementary property of blow ups (cf. [9]) shows that $X_Z=\bar{\pi}^{-1}(j(X))$ is also obtained by blowing up the smooth subvariety of codimension k in X, $X\cap P\times_T W$. We should summarize:

Theorem. *Given a regular configuration $\{S_i\}_{i=1,\dots,n}$ in X we construct a torus embedding Y of $T=G_m^n$, a principal bundle P on X and an embedding $j\colon X\to P\times_T Y$ such that:*

i) *For any smooth torus embedding Z over Y the fiber product X_Z of the diagram*

$$\begin{array}{ccc} & & P\times_T Z \\ & & \downarrow \\ X & \longrightarrow & P\times_T Y \end{array}$$

is smooth, inherits a regular configuration from $P\times_T Z$, is birational over $\mathbf{X}$ and proper if Z is proper over Y.

ii) *If Z is obtained from Y by a sequence of blow ups of orbit closures, $\mathbf{X}_Z$ is obtained from $\mathbf{X}$ by a sequence of blow ups of coordinate varieties.*

iii) *If Z_1, Z_2 are two torus embeddings over Y and $\psi\colon Z_1\to Z_2$ is a T equivariant morphism one has an induced morphism $\bar{\psi}\colon \mathbf{X}_{Z_1}\to \mathbf{X}_{Z_2}$. We shall say in this case that $\mathbf{X}_{Z_1}$ dominates $\mathbf{X}_{Z_2}$.*

§ 4. Transversality

4.1. Let $(X, \mathscr{S})$ be a regular configuration and $Y\subset X$ a subscheme.

Definition. i) Y is transversal to $\mathscr{S}=\{S_i\}$ in a point $p\in Y$ if the following property is satisfied. Let $S_1, S_2, \cdots, S_n$ be all the hypersurfaces of $\mathscr{S}$ passing through p and $x_1, x_2, \cdots, x_n$ be their local equations in a neighborhood of p, then $x_1, x_2, \cdots, x_n$ is a regular sequence in the local ring $\mathcal{O}_{p,Y}$ of Y in p.

ii) We say that Y is transversal to $\mathscr{S}$ if it is transversal to $\mathscr{S}$ in each point $p\in Y$.

The geometric interpretation of this notion is the following:

Proposition. *If Y is of pure codimension k, Y is transversal to $\mathscr{S}$ implies that given a coordinate subvariety S_J the intersection $Y\cap S_J$ is proper,* (briefly y intersects $\mathscr{S}$ properly).

Proof. Let $S_J=S_1\cap S_2\cap\cdots\cap S_r$. Assume $p\in Y\cap S_J$ and Y is transversal to $\mathscr{S}$ in p, we have to show that $\dim \mathcal{O}_{p,Y\cap S_J}=\dim \mathcal{O}_{p,Y}-\operatorname{codim} S_J$. Now the local equations $x_1, \cdots, x_r$ of the varieties $S_1, \cdots, S_r$ form a regular sequence in $\mathcal{O}_{p,Y}$ and $\mathcal{O}_{P,Y\cap S_J}=\mathcal{O}_{P,Y/(x_1,\cdots x_r)}$ hence we get our claim.

Remark. i) If A is a local ring, $x_1, x_2, \cdots, x_k$ elements in its maximal ideal $J=(x_1, \cdots, x_k)$, we have that $x_1, x_2, \cdots, x_k$ is a regular sequence if and only if the form ring:

$$\bigoplus_{S=0}^{\infty} J^s/J^{s+1}$$

is the polynomial ring over A/J generated by the classes $\bar{x}_i\in J/J^2$.

ii) If A is a local ring, I an ideal, $x_1, \cdots, x_k$ a regular sequence in A and in A/I we have that for each $s\geqslant 0$:

$$I \cap (x_1, \cdots, x_k)^s = I \cdot (x_1, \cdots, x_k)^s.$$

Proof. i) is well known. For ii) remark first of all that if $u \in I \cap (x_1, \cdots, x_k)^s$ writing $u = f(x_1, \cdots, x_k)$ considering the form ring of $x_1, \cdots, x_k$ in A/I we see that $u = g(x_1, \cdots, x_k) + g_1(x_1, \cdots, x_k)$ where $g \in I \cdot (x_1, \cdots, x_k)^s$ and g_1 is homogeneous of degree $s+1$. Working by induction we see that the class of u modulo $I \cdot (x_1, \cdots, x_k)^s$ lies in all the powers of the ideal generated by $x_1, \cdots, x_k$. Since we are in a local ring this implies that w is zero modulo $I \cdot (x_1, \cdots, x_k)^s$, which is our claim.

Fact (Chow's moving lemma). If we assume X to be quasi projective then any cycle Y is rationally equivalent to a cycle $Y' = \sum m_i [Y_i]$ such that each Y_i intersects $\mathscr{S}$ properly.

Proof. We apply the "moving lemma" (cf. [45]) to Y and the cycle $Z = \sum_J [S_J]$.

4.2. We want to analyze now the following set up: $(\boldsymbol{X}, \mathscr{S})$ is a regular configuration; $Y \subseteq \boldsymbol{X}$ is a subscheme; $(\boldsymbol{X}_J, \mathscr{S}_J)$ the blow up of $(\boldsymbol{X}, \mathscr{S})$ along a coordinate subvariety S_J, Y_J the proper transform of Y in $\boldsymbol{X}_J$.

We wish to prove

Proposition. *If Y is transversal to $\mathscr{S}$ then*

i) $Y_J = \pi_J^{-1}(Y)$

ii) *Y_J is transversal to $\mathscr{S}_J$.*

Before entering into the proof of this proposition we want to recall the basic facts about blow ups and proper transforms. Let X be a smooth variety, $W \subset X$ a smooth subvariety and let X_W be the blow up of X along W, $\pi\colon X_W \to X$ the projection. Recall first of all that given an open set $U \subset X$, we have that $\pi^{-1}(U) = U_{W \cap U}$. This allows us to reduce the study of blow ups to the case in which X is affine and W is defined as the zero fiber of a smooth map $X \xrightarrow{\varphi} A^k$, of coordinates $x_1, \cdots, x_k$. Let $X^0 = X - W$. With the same coordinates $x_1, \cdots, x_k$ but thought as homogeneous coordinates we can define a map $\bar{\varphi}\colon X^0 \to \mathbf{P}^{k-1}$ and $X_W \subset X \times \mathbf{P}^{k-1}$ equals the closure of the graph of $\bar{\varphi}$. We cover $\mathbf{P}^{k-1}$ with the standard open sets $U_1, \cdots, U_k$ where U_i is the set where $x_i \neq 0$. Correspondingly we have an affine cover of X_W by open sets $V_i = (X \times U_i) \cap X_W$. The projection $\pi_{|V_i}$ is birational and one can identify $\mathcal{O}(V_i)$ with the subring of $K(X)$, $\mathcal{O}(X)[x_1/x_i, \cdots, x_k/x_i]$. One can also characterize $\mathcal{O}(V_i)$ as the subring of $K(X)$ consisting of rational functions f such that there exists an exponent $m \geqslant 0$ with $x_i^m f \in (x_1, \cdots, x_k)^m \subset \mathcal{O}(X)$. Given a subscheme $Y \subset X$ the proper transform $Y_W \subset X_W$ is defined as $\overline{\pi^{-1}(Y \cap X^0)}$. If we are in the case as before and we look at $Y_W \cap V_i$ we can determine its ideal I_i as follows; Let $I \subseteq \mathcal{O}(X)$ be the ideal of Y, $I_i = I[1/x_i] \cap \mathcal{O}(V_i)$.

Suppose $(X, \mathcal{S})$ is a regular configuration with X affine, $\mathcal{S} = \{S_1, \cdots, S_n\}$, S_i with equation x_i and let $S_J = S_1 \cap S_2 \cap \cdots \cap S_k$. We want to analyze the regular configuration $(X_J, \mathcal{S}_J)$.

$\mathcal{S}_J$ consists of the hypersurfaces $\bar{S}_1, \cdots, \bar{S}_n$ which are the proper transforms of $S_1, \cdots, S_n$ and of the exceptional divisor $\pi^{-1}(S_J) = \bar{S}_{n+1}$. We analyze locally in V_i. We have $\bar{S}_i \cap V_i = \phi$ and the local equation of $\bar{S}_{n+1}$ is x_i, the local equations of the $\bar{S}_j$'s, $j \leqslant k$, $j \neq i$ are x_j/x_i while the ones of the $\bar{S}_j$'s, $k < j \leqslant n$ are x_j.

Lemma. *Let A be a local ring $x_1, \cdots, x_k$ a regular sequence in*

$$A,\ B = A\left[\frac{x_1}{x_i}, \cdots, \frac{x_k}{x_i}\right] \subseteq A\left[\frac{1}{x_i}\right].$$

We have

$$B\Big/\left(\frac{x_1}{x_i}\right) = A/(x_1)\left[\frac{x_2}{x_i}, \cdots, \frac{x_k}{x_i}\right] \subseteq A/(x_1)\left[\frac{1}{x_i}\right].$$

Proof. We have a morphism

$$B \longrightarrow A\left[\frac{1}{x_i}\right] \longrightarrow A/(x_1)\left[\frac{1}{x_i}\right]$$

whose image is $A/(x_1)[x_2/x_i, \cdots, x_k/x_i]$. We must show that its kernel is the ideal generated by x_1/x_i. Let $b \in B$ be in the kernel, i.e. $b \in (x_1)[1/x_i]$; since b is also in B there is an exponent $s \geqslant 0$ such that $x_i^s b \in (x_1, \cdots, x_k)^s \cap (x_1)$. Therefore writing $x_i^s b = x_1 f(x_1, \cdots, x_k) + g(x_2, \cdots, x_k)$, f homogeneous of degree $s-1$, g homogeneous of degree s we see that $g(x_2, \cdots, x_k) \in (x_1)$.

Since $(x_2, \cdots, x_k)^s \cap (x_1) = (x_1) \cdot (x_2, \cdots, x_k)^s$ by the remark 4.1 ii) we have in particular that $x_i^s b = x_1 h(x_1, \cdots, x_k)$ with h homogeneous of degree $s-1$ so

$$b = \frac{x_1}{x_i} \frac{h(x_1, \cdots, x_k)}{x_i^{s-1}} \in \frac{x_1}{x_i} B.$$

Corollary. i) $\dfrac{x_1}{x_i}, \dfrac{x_2}{x_i}, \cdots, \dfrac{x_{i-1}}{x_i}, \dfrac{x_{i+1}}{x_i}, \cdots, \dfrac{x_k}{x_i}, x_i, x_{k+1}, \cdots, x_n$ *is a regular sequence in B.*

ii) *Any permutation of the previous elements is still a regular sequence.*

iii) *If $m \in$ Spec B and we consider the elements among*

$$\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_k}{x_i}, x_i, x_{k+1}, \cdots, x_n$$

which lie in m, they form a regular sequence in the local ring B_m.

Proof. i) We only have to show that x_1/x_i is a non zero divisor in B and then proceed by induction. This is clear since x_1 is not a zero divisor in $A[1/x_i]$.

ii) We want to show now that any permutation of the previous sequence is still regular. We reason by induction. If the sequence starts with x_h/x_i, some h, we can apply the previous analysis and work by induction.

If it starts with $x_j, j \geqslant k+1$ we see easily that x_j is not a zero divisor in B and

$$B/(x_j)=A/(x_j)\left[\frac{x_1}{x_j}, \frac{x_2}{x_j}, \cdots, \frac{x_k}{x_i}\right]$$

and again induction applies.

The only case left is when we start with x_i. The second element y of the sequence will either be one of the (x_n/x_i)'s or one of the x_j's, $j \geqslant k+1$.

Since x_i is not a zero divisor in $B \subseteq A[1/x_i]$ and by induction y, $x_i, \cdots$ is a regular sequence so is $x_i, y, \cdots$.

iii) This follows easily from ii).

Proof of the Proposition. i) Let $p \in \boldsymbol{X}$. Set $\boldsymbol{X}_p=\operatorname{Spec}(\mathcal{O}_{p,\boldsymbol{X}})$. It is sufficient to show that for each $p \in \boldsymbol{X}$ we $\boldsymbol{X}_p \times_{\boldsymbol{X}} \pi^{-1}(Y)=\boldsymbol{X}_p \times_{\boldsymbol{X}} Y_J$.

This statement is clear when $p \notin S_J$. Otherwise we can reduce to the following case: $\boldsymbol{X}$ is affine the hypersurfaces $\{S_i\}_{i=1}^n$ have equations $\{x_i\}_{i=1}^n$, $S_J=S_1 \cap \cdots \cap S_r$ and the morphism $\boldsymbol{X} \to A^r$ of coordinates $x_1, \cdots, x_r$ is smooth.

Let $V_1, \cdots, V_r \subset \boldsymbol{X}_J$ be the open sets previously defined.

We want to check that $\boldsymbol{X}_p \times_{\boldsymbol{X}}(V_i \cap \pi^{-1}(Y))=\boldsymbol{X}_p \times_{\boldsymbol{X}}(V_i \cap Y_J)$.

Thus if $I_p \subset \mathcal{O}_{p,\boldsymbol{X}}$ is the ideal of Y in p we must show

$$I_p \mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right]=I_p\left[\frac{1}{x_i}\right] \cap \mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right],$$
$$\text{for each } i=1,2, \cdots, r.$$

Clearly the left hand side is contained in the right hand side. So let

$$f \in I_p\left[\frac{1}{x_i}\right] \cap \mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right],$$

we must show that

$$f \in I_p \mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right].$$

We know that

$$\mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right] = \{h \in K(X) \mid x_i^s h \in (x_1, \cdots, x_r)^s \mathcal{O}_{p,X} \text{ for some } s \geqslant 0\}.$$

Thus we can say that $x_i^s f \in I_p \cap (x_1, \cdots, x_r)^s \mathcal{O}_{p,X}$ for a suitable s. But from Remark 4.1, ii) we have $I_p \cap (x_1, \cdots, x_r)\mathcal{O}_{p,X} = I_p \cdot (x_1, \cdots, x_r)^s \mathcal{O}_{p,X}$ hence

$$f \in I_p \frac{(x_1, \cdots, x_r)^s}{x_i^s} \mathcal{O}_{p,X} \subseteq I_p \mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right].$$

ii) Reasoning locally we reduce to show that

$$\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_r}{x_i}, x_i, x_{r+1}, \cdots, x_n$$

is a regular sequence in $B/\bar{I}$ with

$$B = \mathcal{O}_{p,X}\left[\frac{x_1}{x_i}, \cdot \cdot, \frac{x_r}{x_i}\right], \qquad \bar{I} = I_p B.$$

We claim that

$$B/\bar{I} = \mathcal{O}_{p,X}/I_p\left[\frac{x_1}{x_i}, \cdots, \frac{x_r}{x_i}\right].$$

By Corollary 4.2 this will clearly imply ii). On the other hand our claim is an obvious consequence of i).

Remark. Let $(X, \mathcal{S})$ be a regular configuration Y a subscheme, $(X', \mathcal{S}')$ obtained from $(X, \mathcal{S})$ be a sequence of coordinate blow ups, π: $X' \to X$ the corresponding morphism, $Y' \subseteq X'$ the proper transform of Y in X'. If Y is transversal to $\mathcal{S}$ in a point p then Y' is transversal to $\mathcal{S}'$ in every point $x \in \pi^{-1}(p)$. This is clear from Corollary 4.2 part iii).

4.3.

Corollary. Let $(X, \mathcal{S})$ be a regular configuration $(X_J, \mathcal{S}_J)$ the blow up along a coordinate subvariety S_J, $\pi\colon X_J\to X$ the projection, then:

i) If $S_i\cap S_J=\phi$, $\pi^{-1}(S_i)\cong S_i$.

ii) Assume $S_J=S_1\cap S_2\cap\cdots\cap S_k$ let $\bar S_{k+1}$ be the exceptional divisor. For every $i=1, 2, \cdots, k$ the map π induces an isomorphism of $\bar S_1\cap\bar S_2\cap\cdots\cap\bar S_{i-1}\cap\bar S_{i+1}\cap\cdots\cap\bar S_k=\Sigma_i$ onto $S_{J-\{i\}}$, mapping $\Sigma_i\cap\bar S_{k+1}$ onto S_J.

More generally $\Sigma_i\cap(\bar S_{i_r})$ maps isomorphically to $S_{J-\{i\}}\cap(\cap\pi(\bar S_{i_k}))$.

iii) Let $Y\subset X$ be a subscheme $Y_J\subseteq X_J$ its proper transform. Under the above isomorphism $Y_J\cap\Sigma_i$ maps to a subscheme of $Y\cap S_{J-\{i\}}$.

Proof. i) is clear.

The remaining statements being local in X we can use the same assumptions and notations of 4.2. We have remarked there that $V_j\cap\bar S_j=\phi$, therefore since $\Sigma_i\subseteq\bar S_j$, $j\neq i$, we have $\Sigma_i\subseteq V_i$. In V_i the equation on $\bar S_j$ is x_j/x_i while the equation of $\bar S_{n+1}$ is x_i so both ii) and iii) follow immediately from Lemma 4.2.

4.4.

Definition. Let $(X, \mathcal{S})$ be a regular configuration,

$$\mathcal{S}=\{S_i\}_{i=1}^n.$$

If $x\in X$, we set:

i) level $(x)=\#\{i\mid x\in S_i\}$

ii) $S_x=\{\bigcap_i S_i\mid x\in S_i\}$

iii) $\mathring{S}_x=\{y\in S_x\mid \text{level}\,(y)=\text{level}\,(x)\}$.

Proposition. *Let (X_J, S_J) be a coordinate blow up, $\pi\colon X_J\to X$ the projection;*

i) *If $x\in X_J$, level $\pi(x)\geqslant$ level (x)*

ii) *If level $\pi(x)>$level (x) we have*

$$\text{codim}\,\pi(S_x)>\text{codim}\,S_x$$

iii) *If level $\pi(x)=$level (x), π induces an isomorphism between $\mathring{S}_x$ and $\mathring{S}_{\pi(x)}$.*

Proof. Set $S_J=S_1\cap S_2\cap\cdots\cap S_k$. Let $\bar S_i$ be the proper transform of S_i and $\bar S_{k+1}$ the exceptional division.

Let $x\in X_J$, $S_x=\bar S_{i_1}\cap\bar S_{i_2}\cap\cdots\cap\bar S_{i_r}$.

If $x\notin\bar S_{k+1}$ we restrict to $X_J-\bar S_{k+1}$ which maps isomorphically to $X-S_J$ and the claim is clear.

Assume now $\bar S_{i_r}=\bar S_{k+1}$, we have 2 cases: either $x\notin\bigcup_i(\bar S_1\cap\bar S_2\cap\cdots\cap\bar S_{i-1}\cap\bar S_{i+1}\cap\cdots\cap\bar S_k)$ or $x\in\bar S_1\cap\bar S_2\cap\cdots\cap\bar S_{i-1}\cap\bar S_{i+1}\cap\cdots\cap\bar S_k$ for some i, for instance $x\in\bar S_1\cap\bar S_2\cap\cdots\cap\bar S_{k-1}$.

In the first case $\pi(x) \in \pi(S_x) \subset S_1 \cap S_2 \cap \cdots \cap S_k \cap S_{i_1} \cap \cdots \cap S_{i_{r-1}}$, in the second case Proposition 4.3 implies that, since $S_x \subseteq \bar{S}_1 \cap \bar{S}_2 \cap \cdots \cap \bar{S}_{k-1}$ the map π is an isomorphism between S_x and $\pi(S_x)$. Again from 4.3 we have $\pi(S_x) = S_1 \cap S_2 \cap \cdots \cap S_k \cap S_{i_1} \cap \cdots \cap S_{i_{r-1}}$.

Remark. Set $\ell = \max_{x \in X}\{\text{level}\,(x)\}$. Notice that for any blow up $(X_J, \mathscr{S}_J)$ we have $\ell = \max\{\text{level}\,(x)\}$. This and the above proposition imply that any codimension ℓ coordinate subvariety of X_J maps isomorphically to its image in X which is also a coordinate subvariety.

4.5.

Lemma. *Let $(X, \mathscr{S})$ be a regular configuration. $Y \subseteq X$ a subscheme not transversal to $\mathscr{S}$ in a point p. Let $\mathcal{O}_p$ be the local ring of p in X; $S_1, \cdots, S_k$ the hypersurfaces of $\mathscr{S}$ passing through p; $x_1, \cdots, x_k \in \mathcal{O}_p$ local equations for the S_i's and $I \subseteq \mathcal{O}_p$ the ideal of Y in p.*

There exists a polynomial $f(x_1, \cdots, x_k) = \sum_i \alpha_i M_i + \sum_j \beta_j \bar{M}_i$ with coefficients in $\mathcal{O}_p$ such that:

i) $f(x_1, \cdots, x_k) \equiv 0 \bmod I$

ii) $\alpha_i \not\equiv 0 \bmod (I, x_1, \cdots, x_k)$

iii) *each monomial $\bar{M}_j$ is a proper multiple of a monomial M_i.*

Proof. Since Y is not transversal to X in p the elements $\bar{x}_i \in \mathcal{O}_{p/I}$ are not a regular sequence, thus the associated form ring is not a polynomial ring. This means that we have a homogeneous polynomial $g_0(x_1, \cdots, x_k)$, of degree s, with coefficients $\not\equiv 0 \bmod (I, x_1, \cdots, x_k)$ which equals a homogeneous polynomial $g^{(0)}(x_1, \cdots, x_k)$, of degree $s+1$, modulo I. We start writing the relation $g_0 - g^{(0)} = 0 \bmod I$. We split $-g^{(0)} = g_1 + g^{(1)}$ where in $g^{(1)}$ we collect all those monomials with coefficient $\equiv 0 \bmod (I, x_1, \cdots, x_k)$. Each such coefficient β can thus be written

$$\beta = \sum \lambda_i x_i + \mu, \qquad \mu \in I.$$

Substituting we obtain a new relation

$$0 \equiv g_0 + g_1 + g^{(2)} \bmod I$$

in which g_0, g_1 have degree s, $s+1$ and coefficients non zero modulo $(I, x_1, \cdots, x_h)$ while $g^{(2)}$ has degree $s+2$. We can continue in this way obtaining after h steps

$$0 \equiv g_0 + g_1 + \cdots + g_h + g^{(h+1)} \bmod I$$

with the g_i's homogeneous of degree $s+i$ and with coefficients non zero modulo $(I, x_1, \cdots, x_k)$ while $g^{(h+1)}$ is homogeneous of degrees $S+h+1$.

We claim that for some h we will have that all monomials appearing in $g^{(h+1)}$ are divisible by some of the monomials appearing in $g_0, g_1, \cdots, g_h$. This follows clearly from the fact that the subsemigroups of the semigroups of monomials satisfy the ascending chain condition. We apply this to the sequence of semigroups $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_h$, Σ_i being generated by all the monomials in $g_0, g_1, \cdots, g_h$.

4.6.

Lemma. *Let $(X, \mathscr{S})$ be a regular configuration, $S=\{\mathscr{S}_i\}_{i=1}^h$, Y a subscheme of X not transversal to $\mathscr{S}$ in a point $p \in \bigcap_{i=1}^h S_i$.*

There exists a regular configuration $(X', \mathscr{S}')$ obtained from X by a sequence of blow ups along codimension 2 coordinate subvarieties (briefly a 2 blow up) such that if $(X'', \mathscr{S}'')$ is also obtained from X by a sequence of blow ups and dominates X', setting Y'' to be the proper transform of Y in X'' the intersection of Y'' with any coordinate variety S''_J of X'' of codimension h maps, under the isomorphism of S''_J with $\bigcap_{i=1}^h S_i$, to a proper subscheme of $Y \cap (\bigcap_{i=1}^h S_i)$.

Proof. Using the analysis of Section 3 we associate to $(X, \mathscr{S})$ a torus T of dimension h with an embedding A^h. For every smooth torus embedding Z prover over A^h we have a regular configuration $(X_Z, \mathscr{S}_Z)$ proper over $(X, \mathscr{S})$. Z is obtained from A^h by a sequence of blow ups along codimension 2 orbit closures if and only if $(X_Z, \mathscr{S}_Z)$ is a 2 blow up. The coordinate subvarieties of X_Z of codimension h correspond to the T fixpoints of Z.

Furthermore let $p \in S_1 \cap S_2 \cap \cdots \cap S_h$ and $x_1, \cdots, x_h$ be local equations of the S_i's in p. Choose a coordinate subvariety $\bar{S}_J$ of codimension h in X_Z ($\bar{S}_J$ maps isomorphically to $\bigcap_{i=1}^h S_i$), let $\bar{p} \in \bar{S}_J$ be the unique point mapping to p, then the local equations $y_1, \cdots, y_h$ of the coordinate hypersurfaces passing through $\bar{p}$ can be chosen to be fractional monomials in the x_i's, the x_i's are actual monomials in the y_i's.

If $\bar{x}_1, \cdots, \bar{x}_h$ are the coordinates of the torus embedding A^h and $\bar{y}_1, \cdots \bar{y}_h$ the ones of the affine torus embedding with fixpoint corresponding to $\bar{S}_J$ we have that the expressions of the x_i's as monomials in the y_j's are the same as the ones of the $\bar{x}_i$'s in terms of the $\bar{y}_j$'s.

Also if $f \in \mathcal{O}_{\bar{p},X_Z} \supseteq \mathcal{O}_{p,X}$ we have that if there is a monomial M in the y_j's such that Mf vanishes on Y then f vanishes on the proper transform of Y.

Now let I be the ideal of Y in $\mathcal{O}_{p,X}$. Let

$$If(x_1, \cdots, x_h)=\sum_{i=0}^{N} \alpha_i M_i + \sum_j \beta_j \tilde{M}_j, \qquad \alpha_i \notin (I, x_1, \cdots, x_h)$$

be an element as in Lemma 4.3.

Let Z_0 be the torus embedding defined as the closure of the graph of the monoidal transformation $A^h \to \mathbf{P}^N$ of coordinates $\bar{M}_i$ (cf. 2.6), $\bar{M}_i$ being formally obtained from M_i substituting the x_j's with the $\bar{x}_j$'s. Let Z be a torus embedding obtained from A^h by a sequence of blow ups of codimension 2 orbit closures which dominates Z_0. Let q be a T fixpoint in Z and $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_h$ the coordinates of the corresponding torus embedding. From 2.6 it follows that if we substitute in the $\bar{M}_i$'s the elements $\bar{x}_j$'s by their expressions in the elements $\bar{y}_j$'s we always obtain that one of the resulting monomials divides properly all the remaining ones.

Consider X_Z, $\bar{p} \in \bar{S}_J$, $y_1, \cdots, y_h$ as before.

By the above remarks we again have that substituting in the M_i's the elements x_j's by their expressions in the y_j's we obtain that one of the resulting monomials divides properly all the remaining ones. Since the monomials $\tilde{M}_j$ are proper multiples of the M_i's the same conclusion holds if we perform the same operation on them.

We go back now to $f(x_1, \cdots, x_h)$ and substitute the x_i's by their monomial expression in the y_j's obtaining a polynomial $f'(y_1, \cdots, y_h)=N \cdot h(y_1 \cdots y_h)$ where N is a monomial in the y_j's while $h=\alpha_i+h'(y_1, \cdots, y_h)$, α_i one of the previously considered coefficients and h' without constant term. It follows that h belongs to the ideal I' which is the ideal in $\bar{p}$ of the proper transform Y' of Y and that $\alpha_i \in (I', y_1, \cdots, y_h)$ hence it vanishes on the subscheme $Y' \cap S'_J$, S'_J the coordinate variety of equations $y_1=0, \cdots, y_h=0$. Since $\alpha_i \notin I$ we have the claim for any such blow up. Clearly any torus embedding Z' which dominates Z has the same properties, hence if we set $X'=X_Z$ we have satisfied our claim.

4.7. We are now ready to state and prove one of our main results.

Theorem. *Let $(X, \mathcal{S})$ be a regular configuration $Y \subseteq X$ a subscheme. There exists a 2-blow up $(X', \mathcal{S}')$ of $(X, \mathcal{S})$ such that the proper transform, Y', of Y in X' is transversal to $\mathcal{S}'$.*

Proof. We shall prove, by decreasing induction on k, the following statement: there exists $(X', \mathcal{S}')$ as above such that Y' is transversal to $\mathcal{S}'$ in all points of level $\geqslant k$.

The statement is clear if $k > \max_{x \in X}$ level (x). Since Y' is by definition the closure in X' of $Y \cap (X - \bigcup S_i)$ $(X - \bigcup S_i = \bigcup S'_j)$ it is also clear that Y' is transversal to $\mathcal{S}'$ for any X' in the points of level 0 and 1.

So we may assume Y transversal to $\mathcal{S}$ in all points of level $\geqslant k+1$ and $k \geqslant 2$.

If $(X', \mathcal{S}')$ is a two blow up of $(X, \mathcal{S})$, $\pi: X' \to X$ denote the projection, for any point $x \in X'$ with level $\pi(x) \geqslant k+1$ we have by Remark 4.2

that Y' is transversal to $\mathscr{S}'$ in x. In particular Proposition 4.4 implies that Y' is transversal to $\mathscr{S}'$ in all points of level $\geqslant k+1$. Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_h$ be the coordinate subvarieties in X of codimension k containing a point in which Y is not transversal to $\mathscr{S}$. From Lemma 4.6 and Proposition 4.4 it follows that for each $j=1, 2, \cdots, h$ we can find a 2 blow up $(X_j, \mathscr{S}_j)$ with the following property: Given a regular configuration $(X', \mathscr{S}')$ obtained from $(X, \mathscr{S})$ by successive blow ups dominating X_j, if $\pi: X' \to X$ denotes the projection, a coordinate subvariety Θ of X' of codimension k such that π maps $\mathring{\Theta}$ isomorphically onto $\mathring{\Sigma}_j$, the intersection $Y' \cap \mathring{\Theta}$ is identified to a proper subscheme of $Y \cap \mathring{\Sigma}_j$. Using Proposition 2.4 we can find an $(X', \mathscr{S}')$ which satisfies the previous property for all j's and which is a 2 blow up of $(X, \mathscr{S})$. By Proposition 4.4 given any coordinate subvariety Θ in X' of codimension k either Y' is transversal to $\mathscr{S}'$ in all points of Θ or $\mathring{\Theta}$ maps isomorphically onto one of the $\mathring{\Sigma}_j$'s, and $Y' \cap \mathring{\Theta}$ is identified to a proper subscheme of $Y \cap \mathring{\Sigma}_j$.

We list the coordinate subvarieties $\Theta_1, \cdots, \Theta_h$, of codimension k where Y' is not transversal to $\mathscr{S}'$.

If $h=0$ we are done otherwise we repeat the construction. We deduce that either we find a regular configuration with the required properties or we find an infinite sequence $\cdots (X^{(2)}, \mathscr{S}_2) \xrightarrow{\pi_2} (X^{(1)}, \mathscr{S}_1) \xrightarrow{\pi_1} (X, \mathscr{S})$ satisfying the following conditions: If $\Theta \subseteq X^{(j)}$ is a coordinate variety of codimension k in which the proper transform $Y^{(j)}$ is not transversal to $\mathscr{S}^{(j)}$, π_j maps $\mathring{\Theta}$ isomorphically to its image, $\pi(\Theta)$ is a coordinate variety of codimension k in $X^{(j-1)}$ where $Y^{(j-1)}$ is not transversal to $\mathscr{S}^{(j-1)}$ and $Y^{(j)} \cap \mathring{\Theta}$ is identified with a proper subscheme of $Y^{(j-1)} \cap \pi(\mathring{\Theta})$.

If for each j we consider all such Θ's we have a finite set A_j and a natural map $\bar{\pi}_j$: $A_j \to A_{j-1}$. We have thus a projective system of finite sets, its inverse limit is therefore non empty. This means that we can select for each j a coordinate variety $\Theta_j \subseteq X^{(j)}$ of codimension k such that $\pi_j: \mathring{\Theta}_j \to \mathring{\Theta}_{j-1}$ is an isomorphism and $Y^{(j)} \cap \mathring{\Theta}_j$ maps to a proper subscheme of $Y^{(j-1)} \cap \mathring{\Theta}_{j-1}$. This is clearly a contradiction to the Noetherian property satisfied by subschemes.

§ 5. Symmetric varieties

5.1. From now on we will work for simplicity over an algebraically closed field k of characteristic zero. As remarked in [5] the extension to general characteristic ($\neq 2$) should not be hard and in many cases it is clear. Let G be a semisimple algebraic group of adjoint type defined over k, $\sigma: G \to G$ an automorphism of order 2, $H=G^\sigma$ the fixpoints of σ. In [5] we have given a natural compactification X of the symmetric variety

G/H. We recall briefly its construction and main properties. Let G be the simply connected cover of G, σ lifts to an automorphism of order 2 of $\tilde{G}$ and we set $\tilde{H}=\tilde{G}^{\sigma}$. Fix in G a maximal anisotropic torus T^1 (i.e. $\sigma(t)=t^{-1}$, for $t \in T^1$) and $T \supseteq T^1$ a maximal torus, necessarily σ stable, $\mathfrak{t}=\mathrm{Lie}\, T$, $\mathfrak{t}_1=\mathrm{Lie}\, T^1$. Let $\Phi \subseteq \mathfrak{t}^*$ denote the root system, Φ is σ stable and decomposes $\Phi=\Phi_0 \cup \Phi_1$ where $\Phi_0=\{\alpha \in \Phi \mid \alpha^{\sigma}=\alpha\}$ and Φ_1 the complement; each element $\alpha \in \Phi_1$ restricts to a non zero linear form on $\mathfrak{t}_1$; the induced set is a root system $\bar{\Phi}$ not necessarily reduced called the set of restricted roots. Set W^1 to be the Weyl group of this root system.

One can choose the positive roots Φ^+ in such a way that $(\Phi_1^+)^{\sigma}=\Phi_1^-$.

Let B be the corresponding Borel subgroup and U its unipotent radical, $\Delta=\Delta_0 \cup \Delta_1$ the set of simple roots.

Notice that the choice of the positive roots Φ^+ induces a choice of positive roots Φ^+ in the restricted root system and that the set Δ_1 maps onto the simple roots $\bar{\Delta}_1 \subseteq \bar{\Phi}^+$.

Let $D \subseteq \mathfrak{t}^*$ be the dominant weights, we denote, for $\lambda \in D$, by V_λ the irreducible representation of highest weight λ. It is well known that $\dim V_\lambda^H \leqslant 1$ (for a proof see [5]) and there exist $h=\dim T_1$ dominant weights $\lambda_1, \cdots, \lambda_h$ such that $V_\lambda^H \neq \{0\}$ if and only if $\lambda=\sum_{i=1}^h m_i \lambda_i$, $m_i \geqslant 0$. For each $i=1, \cdots, h$, let $\mathbf{P}(V_{\lambda_i})$ be the projective space of lines in V_{λ_i} and $p_i \in \mathbf{P}(V_{\lambda_i})$ the line $V_{\lambda_i}^H$. Notice that the action of $\tilde{G}$ on each $\mathbf{P}(V_{\lambda_i})$ factors through G.

We set $X \subseteq \prod_{i=1}^h \mathbf{P}(V_{\lambda_i})$ equal to the closure of the G-orbit of the point $p=(p_1, \cdots, p_h)$.

One has that X is a smooth G variety, the open G orbit, $G \cdot p$ is isomorphic to G/H, $X-G/H$ is a union of h smooth divisors $S_1, \cdots, S_h$ meeting transversally and each orbit closure in X is the transversal intersection of some S_i's.

Furthermore $S_1 \cap S_2 \cap \cdots \cap S_h$ is the unique closed orbit in X, isomorphic to G/Q; Q the parabolic subgroup associated to the set Δ_0.

It is clear, from the previous description, that $\mathscr{S}=\{S_1, S_2, \cdots, S_h\}$ is a regular configuration in X, we can therefore apply to it the theory developed in Section 3.

We fix a torus $R=G_m^h$ and consider the vector bundle $\mathscr{V}=\bigoplus \mathcal{O}(S_i)$ on X, which is equipped with an R action and a section $s\colon X \to \mathscr{V}$.

In our case the associated torus embedding, called Y in 3.1, is A^h and the fiber bundle A equals $\mathscr{V}$. If P denotes, as in 3.1, the principal torus bundle associated to $\mathscr{V}$ for every torus embedding Z of R mapping to A^h we have defined a variety $X_Z \subseteq P \times_R Z$.

Proposition. i) *The action of $\tilde{G}$ on X lifts to a linear action on $\mathscr{V}$.*

ii) *$\tilde{G}$ commutes with R on $\mathscr{V}$.*

iii) *The section s in* $\tilde{G}$ *equivariant.*

iv) $\tilde{G}$ *acts on* $P\times_R Z$ *and* X_Z *is* $\tilde{G}$ *stable.*

v) *The action of* $\tilde{G}$ *on* X_Z *factors through G and, if* Z_1 *dominates* Z_2, *the induced map* $X_{Z_1}\to X_{Z_2}$ *is G equivariant.*

vi) *The G orbits of* X_Z *are in* 1–1 *correspondence with the R orbits of Z.*

Proof. i) and ii) follow immediately from 8.1 and 8.2 in [5]. ii) is trivial. iv) is clear from iii) and the description of X_Z.

v) follows from the fact that X_Z contains a dense open orbit isomorphic to G/H and from the fact that the map $P\times_R Z_1\to P\times_R Z_2$ is clearly $\tilde{G}$ equivariant.

vi) requires a detailed proof. Let us consider as in [5], 2.3 the standard open set $V\subseteq X$. V is B^- stable and contains the point p. The closure in V, of the T^1 orbit of p, is the smooth torus embedding A^h of coordinates $(t^{-2\bar{\alpha}_1}, \cdots, t^{-2\bar{\alpha}_h})$, $\bar{\alpha}_i$ the restricted simple roots, of the torus $\bar{T}^1=T^1/T^1\cap H$; furthermore $V\simeq U^-\times A^h$ in a U^- equivariant way, where U^- is the unipotent radical of the parabolic opposite to Q.

The regular configuration of X, restricted to V, is formed by the hypersurfaces $U^-\times H_i$, H_i the hyperplane of A^n where the i^{th} coordinate equals zero.

Consider the open set $V_Z\subseteq X_Z$ described in the following equivalent ways: V_Z is obtained from V and the torus embedding Z using the regular configuration of V, otherwise V_Z is the preimage of V under the natural map $X_Z\to X$.

Since $V=U^-\times A^h$ and its regular configuration is $U^-\times H_i$, i.e. comes from the regular configuration $\{H_i, \cdots, H_h\}$ of A^h, it is clear that we have a canonical isomorphism $V_Z\simeq U^-\times(A^h)_Z$ compatible with the actions of U^- and T^1.

Since $\bar{T}^1$ is identified to the open set of A^h where the h coordinates are non zero, we can canonically identify $\bar{T}^1$ with $G_m^h=R$. Using this identification the vector bundle on A^h associated to the regular configuration is $A^h\times A^h$ and the section is the diagonal. Hence we easily see that $(A^h)_Z$ can be identified with Z thought as a torus embedding of $\bar{T}^1$.

Since every G orbit of X intersects A^h it follows that every G orbit of X_Z intersects $(A^h)_Z=Z$. This intersection is T^1 stable hence a union of T^1 orbits. In fact we claim that it is exactly one T^1 orbit, this will establish completely our proposition. To prove this fact we remark first of all that in a torus embedding the closure of an orbit is the intersection of the closures of codimension 1 orbits which contain the orbit itself.

If now $\mathcal{O}$ is the closure of a codimension 1 orbit of Z, $U^-\times\mathcal{O}$ is an irreducible component of the complement of $U^-\times\bar{T}^1$ in $V_Z=U^-\times Z$.

Clearly $V_Z \cap G/H = U^- \times \bar{T}^1$ since the same is true for V. Since G is connected it follows that $U^- \times \mathcal{O}$ is the intersection of V_Z with an irreducible G stable component of the complement of G/H in X_Z. This analysis shows that the closures of T^1 orbits of codimension 1 in Z are the intersection of the closures of G orbits of codimension 1 in X_Z with Z. From this the rest follows from the previous remarks.

5.2. We want to show now that the varieties X_Z, defined in the previous paragraph, exhaust the class of all equivariant embeddings of G/H which map into X.

Let us consider therefore the class of all such equivariant embeddings. We think of this class as a category (in fact a partially ordered set) by consideration of the G equivariant mappings. Since any such variety contains G/H as a dense open set and we assume all maps to be the identity on G/H given two embeddings $G/H \hookrightarrow Y_1$, $i=1, 2$, there is at most one map between Y_1 and Y_2. We can in any case perform always the fibre product $Y_1 \times_X Y_2$ which is a new embedding of G/H and clearly there exists a map from Y_1 to Y_2 if and only if the canonical projection of $Y_1 \times_X Y_2$ to Y_1 is an isomorphism.

For every embedding $G/H \hookrightarrow Y$ over X, i.e.

$$\begin{tikzcd} & Y \arrow[d, "\pi_Y"] \\ G/H \arrow[ur, hook] \arrow[r, hook] & X \end{tikzcd}$$

is a commutative G equivariant diagram, consider $\pi_Y^{-1}(V)$ and since $V = U^- \times A^h$ consider also $\pi_Y^{-1}(A^h)$.

Clearly $\pi_Y^{-1}(V)$ is $U^- \times T^1$ stable and $\pi_Y^{-1}(A^h)$ is T^1 stable. Of course since Y contains again G/H we have that $\pi_Y^{-1}(V \cap G/H)$ maps under Y isomorphically to $V \cap G/H$. Let us indicate as before by $p \in G/H \cap V$ the point $(0, (1, 1, \cdots, 1))$.

Theorem. i) *$\pi_Y^{-1}(A^h)$ is the closure in $\pi_Y^{-1}(V)$ of the orbit of p under T^1.*

ii) *$\pi_Y^{-1}(V) \simeq U^- \times \pi_Y^{-1}(A^h)$ in a $U^- \times T^1$ equivariant way.*

iii) *The map $Y \to \pi_Y^{-1}(A^h)$ is a equivalence between the category of embeddings of G/H over X and the category of embeddings of $\bar{T}^1$ over A^h.*

Proof. Let us consider the composed map φ: $\pi_Y^{-1}(V) \to V \simeq U^- \times A^h \to U^-$.

For every point $x \in \pi_Y^{-1}(V)$ set $\psi(x) = \varphi(x)^{-1} \cdot x$. Set N to be the closure of the orbit $T^1 \cdot p$ in $\pi_Y^{-1}(V)$, clearly $N \subseteq \pi_Y^{-1}(A^h)$ is an embedding of the torus $\bar{T}^1$ mapping to A^h. We claim that $\psi(x) \in N$ for every $x \in \pi_Y^{-1}(V)$, in fact $\pi_Y^{-1}(V)$ is an open set of Y which is an irreducible variety so $\pi_Y^{-1}(V)$ is irreducible and $G/H \cap \pi_Y^{-1}(V)$ is dense in it.

Clearly ψ restricted to $G/H \cap \pi_Y^{-1}(V)$ map into the orbit $T^1 \cdot p$ and so ψ maps $\pi_Y^{-1}(V)$ into N. Consider now the two maps:

i) $U^- \times N \to \pi_Y^{-1}(V)$ given by the action of U^- on Y

ii) $\pi_Y^{-1}(V) \to U^- \times N$ given by $x \to (\varphi(x), \psi(x))$.

Clearly on $\pi_Y^{-1}(V) \cap G/H$ and $U^- \times \bar{T}^1$ these maps are one the inverse of the other hence in fact they are both isomorphisms. This proves at once ii) and that $N = \pi_Y^{-1}(A^h)$ hence i).

The last claim follows now from the previous analysis, if Y is given as before and $Z = \pi_Y^{-1}(A^h)$ we must show that $Y \simeq X_Z$. We perform the fiber product $Y \times_X X_Z$ and we must show that the projection of this to Y is an isomorphism. Since all maps are G equivariant we can restrict to the preimages of V which are described as $U^- xZ$ in both cases and thus on such a preimage it is clear that the projection is isomorphic, completing the claim.

5.3. One may take a slightly different point of view, given an embedding Y of G/H we can consider the closure of $\bar{T}^1$ in Y. We obtain in this way a torus embedding over which the Weyl group W^1 of the symmetric variety acts. In particular for the variety X which is complete we have a complete torus embedding over which the Weyl group acts and which contains the affine open set A^h. Since the polyhedron associated to A^h is exactly the fundamental Weyl chamber and the Weyl group acts simply transitively over the Weyl chambers, which cover the whole vector space generated by the weights, we have:

Theorem. i) *The closure of $\bar{T}^1$ in X is the torus embedding associated to the r.p.p.d. formed by Weyl chambers.*

ii) *There is a* 1–1 *correspondence between equivariant embeddings of G/H lying over X and W^1 invariant r.p.p.d.'s made of polyhedral cones contained in Weyl chambers or their faces.*

Proof. i) has been proved by the previous remarks and ii) follows from Theorem 5.2 and the action of W^1.

§ 6. The intersection ring of G/H

6.1. Let us consider in this section a general homogeneous space $M = G/H$. Let us recall that if Y_1, Y_2 are irreducible subvarieties of M we have, by Kleiman's transversality theorem [11], that $Y_1 \cap gY_2$ is a proper intersection with multiplicity 1 in each component for g belonging to a non empty open set of G.

We need a small generalization of this result which is completely straightforward. Let us take an irreducible variety X over which G acts with a finite number of orbits. Let us assume G connected. Let Z denote the cycle $\Sigma\overline{\mathcal{O}}_i$ where $\mathcal{O}_i$ runs over all the non open orbits of G in X.

Proposition. *If Y_1, Y_2 are irreducible varieties which have proper intersection with Z then there is a non empty open set $U\subseteq G$ such that:*

i) *$gY_1\cap Y_2$ is proper with multiplicity 1 in each component, for every $g\in U$.*

ii) *$gY_1\cap Y_2$ has proper intersection with Z for $g\in U$.*

Proof. Since the intersection of Y_i with Z is proper ($i=1, 2$) we have that if $Y_i\cap\mathcal{O}_j\neq\phi$, $\mathcal{O}_j$ any orbit, then

$$\operatorname{codim}_{\mathcal{O}_j} Y_i\cap\mathcal{O}_j=\operatorname{codim}_X Y_i.$$

We apply now Kleiman's transversality theorem on each orbit $\mathcal{O}_j$ and thus we can find a unique open set $U\subset G$ such that for each j the intersection of $Y_2\cap\mathcal{O}_j$ and $g(Y_1\cap\mathcal{O}_j)$ is proper (as subvarieties of $\mathcal{O}_j$) and with the multiplicity 1 property. Since clearly $g(Y_1\cap\mathcal{O}_j)=gY_1\cap\mathcal{O}_j$ we have $gY_1\cap Y_2=\bigcup_j(gY_1\cap Y_2\cap\mathcal{O}_j)=\bigcup_j\{(gY_1\cap{}_j)\cap\mathcal{O}(Y_2\cap\mathcal{O}_j)\}$. If $\mathcal{O}_0$ denotes the unique open orbit in X we clearly have that $\dim gY_1\cap Y_2=\dim gY_1\cap Y_2\cap\mathcal{O}_0$ and for any other orbit $\mathcal{O}_j$ $\dim gY_1\cap Y_2\cap\mathcal{O}_j<\dim gY_1\cap Y_2$, by the assumptions on the properness of the various intersections. Thus if $W_1,\cdots,W_k$ denote the components of $gY_1\cap Y_2\cap\mathcal{O}_0$ it follows that $gY_1\cap Y_2=\sum_{i=1}^k\overline{W}_i$, $\overline{W}_i$ the closure in M, and this is also the intersection as cycles. Moreover $\operatorname{codim}_{0_j}gY_1\cap Y_2\cap\mathcal{O}_j=\operatorname{codim}_{0_j}Y_1\cap\mathcal{O}_j+\operatorname{codim}_{0_j}Y_2\cap\mathcal{O}_j$ $\operatorname{codim}Y_1+\operatorname{codim}Y_2=\operatorname{codim}gY_1\cap Y_2$ hence $gY_1\cap Y_2$ has proper intersection with the cycle $Z=\Sigma\overline{\mathcal{O}}_j$.

We have with the same notations the following

Corollary. *If Y_1, $Y_2\subseteq M$ are irreducible varieties of complementary codimension then for g in a non empty open set of G we have $gY_1\cap Y_2\subseteq\mathcal{O}_0$ and is formed of simple points.*

As a consequence, if M is complete and non singular, one can compute the number of points given by the previous corollary by cohomology. One has that each Y_i has a fundamental homology class, denoting $[Y_i]$ the dual class we have that: the number of points of intersection $gY_1\cap Y_2$ equals the evaluation against the class of a point, of the cup product $[Y_1]$ $\cup[Y_2]$.

6.2. Let us consider again a homogeneous space $M=G/H$ of dimension n and two subvarieties Y_1, Y_2 with $\operatorname{codim}Y_1+\operatorname{codim}Y_2=\dim M=n$.

If we assume G to be connected it is easily seen, by the proof of Kleiman's transversality theorem, that for g in a non empty open set of G the number of points of intersection $gY_1 \cap Y_2$ is not only finite but also constant. We may thus define (Y_1, Y_2) to be the previous number of intersections.

We can clearly extend by linearity this definition to arbitrary cycles of complementary codimension. Thus if $\mathscr{Z}^r(M)$ denotes the group of cycles of codimension k and (a, b) denotes the previously defined pairing between $\mathscr{Z}^r(M)$ and $\mathscr{Z}^{n-r}(M)$ we can set $\mathscr{B}^r(M)=\{a \in \mathscr{Z}^r(M) \mid (a, b)=0$ for all $b \in \mathscr{Z}^{n-r}(M)\}$.

Set now $C^r(M)=\mathscr{Z}^r(M)/\mathscr{B}^r(M)$ and $C^*(M)=\oplus_{r=0}^n C^r(M)$.

$C^*(M)$ is a graded abelian group and we still have a pairing $C^r(M) \times C^{n-r}(M) \to Z$ for every $r \leqslant n$ which is non degenerate in the sense that $(a, b)= 0$ for every b implies $a=0$. Furthermore $C^0(M)=C^n(M)=Z$.

If Y is a cycle on M we will denote by $\{Y\}$ its class in $C^*(M)$.

In a way $C^*(M)$ contains the information about some enumerative problems. In order to really contain all the informations on enumerative problems we should give to $C^*(M)$ a ring structure.

One could use Kleiman's transversality theorem as follows: If Y_1, Y_2 are irreducible varieties we know that the intersection $gY_1 \cap Y_2$ is proper for g in a non empty open set; if we could show that in a possibly smaller set the class $\{gY_1 \cap Y_2\}$ is constant in $C^*(M)$ and depends only on $\{Y_1\}$ and $\{Y_2\}$ we could then define the intersection product in $C^*(M)$. This is not true in general cf. introduction, in the next section we will show that in the case of a symmetric variety the previous construction can in fact be performed and moreover the ring $C^*(M)$ can be identified to the direct limit of the Chow rings (equivalently the cohomology rings) of the equivariant compactifications of M.

Remark. The group $C^*(M)$ depends strictly on the action of G on M and has no intrinsic meaning. For instance we can consider the n dimensional affine space A^n, $n \geqslant 2$, as a homogeneous space over the affine group or over the group of translations. The groups of cycles $\mathscr{B}^r(M)$ depend then on the group chosen and we obtain two different $C^*(M)$.

6.3. In this section we go back to symmetric varieties and prove our main theorem.

The notations G, H, X, Z, X_Z etc. are as in 5.1. The theory given in Section 1 shows that a cofinal family, in the set $\{X_Z\}$ ordered as in 5.1, is formed by the 2 blow ups $X' \to X$.

We want to collect first of all some facts on the varieties X_Z in the case Z smooth and proper over A^h. This is the case in which X_Z is smooth and complete.

Since we have seen that X_Z has a finite number of G orbits (5.1), we have from Proposition 7.2 of [5] that X_Z has a finite number of T fix points, T a maximal torus of G. We may thus apply the theorem of Bialynicki-Birula [1], [2] and we have, (cf. also [3] for the properties of the Chow ring).

Proposition. i) *X_Z has a paving by affine spaces.*

ii) *The Chow ring of X_Z is isomorphic to the cohomology ring (doubling the degrees), the closures of the affine spaces paving X_Z are a basis of the Chow ring.*

We define now $\mathscr{H}^*(G/H)=\varinjlim H^*(X_Z)=\varinjlim A(X_Z)$. The limit being taken either on the class of all complete X_Z or equivalently on the smooth ones or on the 2 blow ups which are cofinal.

Theorem. i) *There is a canonical isomorphism of graded vector spaces φ: $\mathscr{H}^*(G/H)\rightrightarrows C^*(G/H)$.*

ii) *Given two cycles Y_1, Y_2 in G/H the class in $C^*(G/H)$, of the intersection $gY_1\cap Y_2$ is constant for g in a non empty open set of G.*

We have $\varphi^{-1}(\{gY_1\cap Y_2\})=\varphi^{-1}(\{Y_1\})\cup\varphi^{-1}(\{Y_2\})$.

iii) *Given $a\in C^r(G/H)$, $b\in C^{n-r}(G/H)$ the value (a, b) of the pairing is equal to the evaluation of $\varphi^{-1}(a)\cup\varphi^{-1}(b)$ against the class of a point.*

Proof. We start defining a map from the directed set $A^*(X_Z)$ to $C^*(G/H)$.

If $a\in A^r(X_Z)$ we can represent it, by Chow's moving lemma, by a cycle $\Sigma n_i Y_i$ has proper intersection with the regular configuration of X_Z. We would like to define $\varphi(a)=\Sigma n_i\{Y_i\cap G/H\}$. We should show first of all that this map is well defined. So let us choose a cycle $\Sigma n_i' Y_i'$ rationally equivalent to $\Sigma n_i Y_i$ and still has proper intersection with the regular configuration of X_Z. We must show that $\Sigma n_i\{Y_i\cap G/H\}=\Sigma n_i'\{Y_i'\cap G/H\}$, in order to do this, given a cycle D in G/H of complementary codimension we must show that:

$$(D,\ \Sigma n_i(Y_i\cap G/H))=(D,\ \Sigma n_j'(Y_j'\cap G/H)).$$

By Theorem 4.2 we can find a blow up $\pi\colon X'\to X_Z$ where the closure $\bar{D}$ of D has proper intersection with the regular configuration. By 4.2 and the basic facts on the Chow ring the cycles $\Sigma n_i\pi^{-1}(Y_i)$ and $\Sigma n_i'\pi^{-1}(Y_i')$ are rationally equivalent and represent $\pi^*(a)$ (cf. [3]). Therefore if $[\bar{D}]$ is the class in $A^*(X')$ of $\bar{D}$ we have that the evaluation of $\bar{D}\cup\pi^*(a)$ against the class of a point equals $(D, \Sigma n_i Y_i)$ and also $(D, \Sigma n_i' Y_i')$ by Corollary 6.1 and the following comments. We have thus a morphism $\varphi_Z\colon A^*(X_Z)\to C^*(G/H)$ and clearly, again by 4.2, this is a compatible family which gives a map $\varphi\colon \varinjlim A^*(X_Z)\to C^*(G/H)$.

Let us show first of all that φ_Z is injective. Let $a \in A^r(X_Z)$, $a \neq 0$, since $A^*(X_Z) = H^*(X_Z)$ we can find a $b \in A^{n-r}(X_Z)$ with $a \cap b \neq 0$.

Setting $a \cap b = n \cdot p$, p class of a point we can use Corollary 6.1 and its consequences and see that $n = (\varphi(a), \varphi(b)) \neq 0$ and so $\varphi(a) \neq 0$.

This analysis proves also the claim iii).

The claim ii) is an immediate consequence of Proposition 6.1. Finally the surjectivity of φ is clearly a consequence of Theorem 4.7 and the definition of φ.

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