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The Orbital Decomposition of Some Prehomogeneous Vector Spaces

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Introduction

Let G' be a semi-simple algebraic group, ρ': $G' \to \mathrm{GL}(V)$ its finite-dimensional rational representation, all defined over $\mathbf{C}$. Then, we have $\rho' = \rho_1 \oplus \cdots \oplus \rho_k$, $V = V_1 \oplus \cdots \oplus V_k$ where ρ_i: $G' \to \mathrm{GL}(V_i)$ is an irreducible representation for $i = 1, \cdots, k$. Put $G = \mathrm{GL}(1)^k \times G'$ and let ρ be the composition of ρ' and the scalar multiplications $\mathrm{GL}(1)^k$ on each irreducible components. In [7], the classification of such triplets (G, ρ, V) which admit only a finite number of orbits has been discussed. To complete this classification, one must give the orbital decomposition of some spaces, which will be done in this paper. We give the orbital decomposition of the following spaces. We use the same notations as in [7] (See Definition 1.10 in [7]).

(1.1) $\overset{G_2}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$, (1.2) $\overset{1}{\circ}\overset{\Lambda}{\text{———}}\overset{\mathrm{Spin}(7)}{\circ}\overset{\Lambda'}{\text{———}}\overset{2}{\circ}$, (1.3) $\overset{\mathrm{Spin}(7)}{\circ}\overset{\Lambda}{\text{———}}\overset{2}{\circ}$,

(1.4) $\overset{\mathrm{Spin}(7)}{\circ}\overset{\Lambda}{\text{———}}\overset{3}{\circ}$, (1.5) $\overset{1}{\circ}\overset{\Lambda}{\text{———}}\overset{\mathrm{Spin}(8)}{\circ}\overset{\Lambda'}{\text{———}}\overset{3}{\circ}$,

(1.6) $\overset{1}{\circ}\overset{\Lambda}{\text{———}}\overset{\mathrm{Spin}(8)}{\circ}\overset{\Lambda'}{\text{———}}\overset{2}{\circ}$, (1.7) $\overset{1}{\circ}\overset{\Lambda}{\text{———}}\overset{\mathrm{Spin}(10)}{\circ}\overset{\Lambda'}{\text{———}}\overset{3}{\circ}$

(1.8) $\overset{\mathrm{Sp}(n)}{\circ}\text{———}\overset{\mathrm{SO}(3)}{\circ}$, (1.9) $\overset{1}{\circ}\overset{\Lambda_1}{\text{———}}\overset{\mathrm{Sp}(2)}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$,

(2.1) $\overset{5}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$, (2.2) $\overset{6}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$, (2.3) $\overset{7}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$,

(2.4) $\overset{1}{\circ}\text{———}\overset{7}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$, (2.5) $\overset{1}{\circ}\text{———}\overset{5}{\circ}\overset{\Lambda_2}{\text{———}}\overset{2}{\circ}$, (2.6) $\overset{2}{\circ}\overset{\Lambda_2}{\text{———}}\overset{5}{\circ}\text{———}\overset{2}{\circ}$,

(2.7) $\overset{4}{\circ}\overset{\Lambda_2}{\text{———}}\overset{n}{\circ}$ $(n=3, 4)$, (2.8) $\overset{1}{\circ}\text{———}\overset{4}{\circ}\overset{\Lambda_2}{\text{———}}\overset{n}{\circ}$ $(n=3, 4)$,

(2.9) $\overset{1}{\circ}\overset{\Lambda_3}{\text{———}}\overset{7}{\circ}\text{———}\overset{2}{\circ}$,

(3.1) $\overset{1}{\circ}\text{———}\overset{n}{\circ}\overset{\Lambda_2}{\text{———}}\overset{1}{\circ}$, (3.2) $\overset{1}{\circ}\text{———}\overset{6}{\circ}\overset{\Lambda_3}{\text{———}}\overset{1}{\circ}$, (3.3) $\overset{1}{\circ}\text{———}\overset{7}{\circ}\overset{\Lambda_3}{\text{———}}\overset{1}{\circ}$,

(3.4) $\overset{1}{\circ}\text{———}\overset{\mathrm{Sp}(3)}{\circ}\overset{\Lambda_3}{\text{———}}\overset{1}{\circ}$, (3.5) $\overset{1}{\circ}\overset{\Lambda}{\text{———}}\overset{\mathrm{Spin}(10)}{\circ}\overset{\Lambda'}{\text{———}}\overset{1}{\circ}$,

(3.6) $\overset{1}{\circ}\overset{\Lambda}{\text{———}}\overset{\mathrm{Spin}(12)}{\circ}\overset{\Lambda'}{\text{———}}\overset{1}{\circ}$,

where Λ (resp. Λ') denotes the (half-)spin (resp. vector) representation of Spin(n),

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One can check that $\overset{1}{\circ}-\overset{4\Lambda_2}{\circ}-\!\!-\overset{3}{\circ}-\overset{1}{\circ}$ and $\overset{1}{\circ}-\overset{4\Lambda_2}{\circ}-\!\!-\overset{4}{\circ}-\overset{2}{\circ}$ have also only a finite number of orbits by calculation using (2.8) (See [3]). Thus, we have complete the classification of such spaces. Now let G be any reductive algebraic group and $\rho\colon G\to \mathrm{GL}(V)$ its finite-dimensional rational representation all defined over **C**. If V decomposes into a finite union of G-orbits, then the restriction of ρ to the semi-simple part $[G, G]$ of G should appear in our classification. In this sense, modulo scalar multiplications, we have decided the representation of reductive algebraic groups with finitely many orbits. Since it is too long, we give the proof only to some of the cases.

§ 1.

First we consider the triplet $((G_2)\times \mathrm{GL}(2),\ \Lambda_2\otimes\Lambda_1,\ V(7)\otimes V(2))$. The Lie algebra $(\mathfrak{g}_2)$ of (G_2) is given by (1.8), p. 20 in [8], i.e.,

$$\left[\begin{array}{c|c|c} 0 & 2\,{}^tz & 2\,{}^ty \\ \hline y & X & Z \\ \hline z & Y & -{}^tX \end{array}\right],\quad X=\begin{bmatrix} \lambda_1 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_2 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_3 \end{bmatrix},\quad Y=\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix},$$

$$Z=\begin{bmatrix} 0 & f & -e \\ -f & 0 & d \\ e & -d & 0 \end{bmatrix},$$

${}^ty=(a, b, c)$, ${}^tz=(d, e, f)$, $\lambda_1+\lambda_2+\lambda_3=0$, which gives Λ_2. The representation space is identified with $V=V(7)\oplus V(7)$.

Proposition 1.1. *The triplet* $((G_2)\times \mathrm{GL}(2),\ \Lambda_2\otimes\Lambda_1,\ V(7)\otimes V(2))$ *has the following eight orbits.*

	Representative Points	*Codimension*
(1)	(e_2, e_5)	0
(2)	(e_2, e_3+e_6)	1
(3)	(e_2, e_3)	3
(4)	(e_1, e_5)	3
(5)	(e_2, e_6)	5
(6)	$(e_1, 0)$	6
(7)	$(e_2, 0)$	7
(8)	$(0, 0)$	14

Proof. Let $\tilde{x}=(x, y)$ be a representative of one of the orbits of $V=$

$V(7)\oplus V(7)$. The triplet $(GL(1)\times(G_2), \Lambda_2, V(7))$ has three orbits represented by $e_1, e_2, 0$ (See [1]). Therefore we may assume that $x=0, e_2, e_1$. For example, the action of $(a(\lambda))$ means the action of $\exp \lambda\tilde{A}$ in (G_2) where $\tilde{A}$ is an element of $(\mathfrak{g}_2)$ such that all its components are zero except for a and $a=1$. If $x=0$, then we have also $y=e_1, e_2, 0$, i.e., (6), (7), (8). Let $y={}^t(y_1, y_2, \cdots, y_7)$.

The case for $x=e_2$. Assume that $y_5\neq 0$. Then by the action of $e, f, a, \lambda_{12}, \lambda_{13}$ and GL(2), we have $y=e_2$, i.e., (1). Assume that $y_5=0$. We may assume that $y_2=0$ by the action of GL(2). If $y_3\neq 0$ or $y_4\neq 0$, then we may assume that $y_3=1, y_4=0$ by the action of $\lambda_{23}, \lambda_{32}$ and GL(2). By the action of e and a, we have $y_1=y_7=0$. If $y_6=0$, then we have (3), and if $y_6\neq 0$, then we have $y_6=1$ by the action of λ_2 and GL(2), and hence we have (2). If $y_3=y_4=0$, then we may assume that $y_7=0$ by the action of $\lambda_{23}, \lambda_{32}$. If $y_1=0$, then we have (5), and if $y_1\neq 0$, then we have $y_6=0$ by the action of e, and hence $y=e_1$, i.e., (e_2, e_1). By the action of e, b and GL(2), $(e_2, e_1)\sim(3)$.

The case for $x=e_1$. We may assume that $y_1=0$ by the action of GL(2). Assume that one of y_2, y_3 and y_4 is not zero. Then we may assume that $y_2\neq 0$ by the action of λ_{12} and λ_{13}. By the action of $\lambda_{21}, \lambda_{31}$ and λ_2, we have $y_3=y_4=0$ and $y_2=1$. If $y_5\neq 0$, then we have $y_6=y_7=0$ and $y_5=1$ by the action of $\lambda_{12}, \lambda_{13}, \lambda_2$ and GL(2), and hence we have $y=e_2+e_5$, i.e., (e_1, e_2+e_5). By the action of $a(1)$, GL(2), $d(-(1/2))$ and GL(2), we have $(e_1, e_2+e_5)\sim(1)$. If $y_5=0$, then we may assume that $y_7=0$ by the action of $\lambda_{23}, \lambda_{32}$. If $y_6=0$, then we have (e_1, e_2), and if $y_6\neq 0$, then we have $y_6=1$ by the action of λ_3 and GL(2), and hence we have (e_1, e_2+e_6). By the action of λ_2 and GL(2), $e(1)$, $b(-(1/2))$, $\lambda_{32}(-2)$, $\lambda_{23}(1/2)$ and GL(2), we have $(2)\sim(e_1, e_2+e_6)$. Assume that $y_2=y_3=y_4=0$. We may assume that one of y_i $(i=5, 6, 7)$ is not zero. Also we may assume that $y_5\neq 0$ by the action of λ_{21} and λ_{31}. Then we have $y_6=y_7=0$, $y_5=1$ by the action of $\lambda_{12}, \lambda_{13}$ and λ_2, and hence we have $y=e_5$, i.e., (4). Q.E.D.

Remark. Put $K=\left[\begin{array}{c|c|c} -2 & 0 & 0 \\ \hline 0 & 0 & I_3 \\ \hline 0 & I_3 & 0 \end{array}\right]$. Then, for $X\in M(7, 2)=V(7)\oplus V(7)$, rank ${}^tXKX$ is invariant. Since its value of (3) (resp. (4)) is 0 (resp. 1), (3) and (4) are different from each other (See [4]).

Next we consider the triplet $(GL(1)^2\times \mathrm{Spin}(7)\times SL(2)$, spin rep. $\otimes 1$ + vector rep. $\otimes\Lambda_1$, $V(8)+V(7)\otimes V(2))$. The representation space is identified with $V=V(8)\oplus M(7, 2)$ where $V(8)$ is spanned by $1, e_ie_j (1\leq i<j\leq 4)$, $e_1e_2e_3e_4$ (See p. 110–112 in [8]). The action $\rho=\rho_1\otimes 1+\rho_2\otimes\Lambda_1$ is given by $\rho(g)\tilde{x}=(\alpha\rho_1(g_1)X, \beta\rho_2(g_1)Y{}^tg_2)$ for $g=(\alpha, \beta, g_1, g_2)\in GL(1)^2\times\mathrm{Spin}(7)\times SL(2)$, $\tilde{x}=(X, Y)\in V=V(8)\oplus M(7, 2)$ where ρ_1 (resp. ρ_2) denotes the spin

(resp. vector) representation of Spin (7) on $V(8)$ (resp. $V(7)$) (See (5.31), p. 115 in [8]).

Proposition 1.2. *The triplet* $(\mathrm{GL}(1)^2\times\mathrm{Spin}(7)\times\mathrm{SL}(2)$, *spin rep.* $\otimes 1$ $+$ *vector rep.* $\otimes\Lambda_1$, $V(8)+V(7)\otimes V(2))$ *has the following thirty orbits.*

	Representative Points	*Codimension*
(1)	$(1+e_1e_2e_3e_4, (e_2, e_5))$	0
(2)	$(1+e_1e_2e_3e_4, (e_2, e_3+e_6))$	1
(3)	$(1+e_1e_2e_3e_4, (e_2, e_3))$	3
(4)	$(1+e_1e_2e_3e_4, (e_1, e_5))$	3
(5)	$(1+e_1e_2e_3e_4, (e_2, e_6))$	5
(6)	$(1+e_1e_2e_3e_4, (e_1, 0))$	6
(7)	$(1+e_1e_2e_3e_4, (e_2, 0))$	7
(8)	$(1+e_1e_2e_3e_4, (0, 0))$	14
(9)	$(1, (e_2+e_5, e_3+e_6))$	1
(10)	$(1, (e_2+e_5, e_3))$	2
(11)	$(1, (e_2, e_1+e_5))$	3
(12)	$(1, (e_2+e_5, e_1))$	3
(13)	$(1, (e_2, e_3))$	4
(14)	$(1, (e_2, e_5))$	4
(15)	$(1, (e_2, e_1))$	4
(16)	$(1, (e_2+e_5, e_6))$	5
(17)	$(1, (e_2, e_6))$	6
(18)	$(1, (e_1, e_5))$	7
(19)	$(1, (e_2+e_5, 0))$	7
(20)	$(1, (e_2, 0))$	8
(21)	$(1, (e_5, e_6))$	9
(22)	$(1, (e_1, 0))$	10
(23)	$(1, (e_5, 0))$	11
(24)	$(1, (0, 0))$	15
(25)	$(0, (e_2, e_5))$	8
(26)	$(0, (e_1, e_2))$	9
(27)	$(0, (e_2, e_3))$	11
(28)	$(0, (e_1, 0))$	14
(29)	$(0, (e_2, 0))$	15
(30)	$(0, (0, 0))$	22

Proof. The triplet $(\mathrm{GL}(1)\times\mathrm{Spin}(7)$, spin rep., $V(8))$ has three orbits represented by $1+e_1e_2e_3e_4$, 1, 0 (See [2]). The case for $(1+e_1e_2e_3e_4, Y)$. The isotropy subalgebra at $1+e_1e_2e_3e_4$ is $(\mathfrak{g}_2)$. Hence we obtain (1)~(8) from Proposition 1.1. The case for $(1, Y)$. The isotropy subalgebra at 1 is given as follows by transform $\begin{pmatrix}1234567\\4567123\end{pmatrix}$.

$$\left\{\left[\begin{array}{c|c|c} -{}^tA & 2b & C \\ \hline 0 & 0 & -{}^tb \\ \hline 0 & 0 & A \end{array}\right];\quad b=\begin{pmatrix}c_{14}\\ c_{24}\\ c_{34}\end{pmatrix},\quad {}^tC=-C\right\}$$

In this case, $Y=\left(\frac{Y_1}{Y_2}\right)$, $Y_1={}^t\begin{pmatrix}x_5x_6x_7x_1\\ y_5y_6y_7y_1\end{pmatrix}$, $Y_2={}^t\begin{pmatrix}x_2\ x_3\ x_4\\ y_2\ y_3\ y_4\end{pmatrix}$. Then $SL(3)\times GL(2)$ acts on Y_2 by $\Lambda_1\otimes\Lambda_1$, and hence $Y_2\sim E_{11}+E_{22}$, E_{11}, 0. Assume that $Y_2=E_{11}+E_{22}$. Then we may assume that $x_1=x_7=y_1=y_7=0$. By the action of c_{12}, we may assume that $Y_1'=\begin{pmatrix}x_5y_5\\ x_6y_6\end{pmatrix}$ is a symmetric matrix, and hence $GL(2)$ acts on Y_1' by $2\Lambda_1$. Hence we have $Y_1'\sim E_{11}+E_{22}$, E_{11}, 0, i.e., (9), (10), (13). Assume that $Y_2=E_{11}$. Then by the action of c_{12}, c_{13} and c_{14}, $x_1=x_6=x_7=0$. If $y_5\neq 0$, then $y_6=y_7=0$ and $x_5=0$, and hence we have (11), (14). If $y_5=0$ and $y_1\neq 0$, then, by the action of $2c_{24}$ and $2c_{34}$, $y_6=y_7=0$, and hence we have (12), (15). If $y_5=y_1=0$, then $GL(2)$ acts on $y=(y_6, y_7)$ by Λ_1, and hence $y\sim(1, 0)$ or $(0, 0)$. Hence we have (16), (17) and (19), (20). Assume that $Y_2=0$. Then we have (18), (21), (22), (23), (24). Finally the case for $(0, Y)$. Then we have (25)~(30), because the triplet $(SO(7)\times GL(2), \Lambda_1\otimes\Lambda_1, V(7)\otimes V(2))$ has six orbits (See [9]). Q.E.D.

Remark. There is a possibility that (11) and (12) belong to the same orbits.

Next we consider the triplet $(\mathrm{Spin}(7)\times GL(2)$, spin rep. $\otimes\Lambda_1$, $V(8)\otimes V(2))$. The representation space $V(8)\otimes V(2)$ is identified with $V=V(8)\oplus V(8)$ where $V(8)$ is spanned by 1, e_ie_j $(1\leqq i<j\leqq 4)$, $e_1e_2e_3e_4$ (See p. 110-112 in [8]). The action $\rho=\rho_1\otimes\Lambda_1$ is given by $\rho(g)x=(\rho_1(g_1)X, \rho_1(g_1)Y){}^tg_2$ for $g=(g_1, g_2)\in\mathrm{Spin}(7)\times GL(2)$, $x=(X, Y)\in V=V(8)\oplus V(8)$ where ρ_1 denotes the spin representation of Spin (7) on $V(8)$. If $\lambda\in\mathbf{C}^\times$, for any index i satisfying $1\leqq i\leqq 3$, we put $S_i(\lambda)=\lambda^{-1}+(\lambda-\lambda^{-1})e_if_i$. Then $S_i(\lambda)$ is an element of Spin (7). For any two distinct indices i, j satisfying $1\leqq i, j\leqq 7$, $i, j\neq 4$, $i\neq j+4$, $j\neq i+4$, we put $S_{ij}(\lambda)=1+\lambda e_ie_j=\exp(\lambda e_ie_j)$ where $e_k=f_{k-4}$ for $5\leqq k\leqq 7$. For any index i satisfying $1\leqq i\leqq 7$, $i\neq 4$, we put $S_{4i}(\lambda)=1+\lambda e_i(e_4+e_8)$, $S_{i4}(\lambda)=1+\lambda(e_4+e_8)e_i$ where $e_k=f_{k-4}$ for $5\leqq k\leqq 7$. Then S_{ij} $(i\neq j+4, j\neq i+4)$ is an element of Spin (7) satisfying $S_{ij}(\lambda)S_{ji}(\lambda)=1$ (See [8]). The triplet $(GL(1)\times\mathrm{Spin}(7)$, spin rep., $V(8))$ has three orbits represented by $1+e_1e_2e_3e_4$, 1, 0 (See [2]).

Proposition 1.3. *The triplet* (Spin (7)$\times$GL (2), *spin rep.* $\otimes\Lambda_1$, $V(8)\otimes V(2)$) *has the following seven orbits.*

	Representative Points	*Codimension*
(1)	$(1, e_1e_2e_3e_4)$	0
(2)	$(1, e_1e_2+e_3e_4)$	1
(3)	$(1, e_1e_2)$	3
(4)	$(1, e_1e_4)$	5
(5)	$(1+e_1e_2e_3e_4, 0)$	7
(6)	$(1, 0)$	8
(7)	$(0, 0)$	16

Proof. Let $\tilde{x}=(x, y)$ be a representative of one of the orbits of $V=V(8)\oplus V(8)$. Then we may assume that $x=0$, 1, or $1+e_1e_2e_3e_4$. If $x=0$, then we have also $y=1+e_1e_2e_3e_4$, 1, 0, and by the action of GL(2), we have (5), (6), (7). The case for $x=1$. We may put $y=y_0\cdot 1+\sum_{1\leqq i<j\leqq 3}y_{ij}e_ie_j+(\sum_{i=1}^{3}y_ie_i+y_4e_1e_2e_3)e_4\neq 0$. We may assume that $y_0=y_{13}=y_{23}=y_2=0$ by the action of GL (2), S_{36}, S_{25} and so on. If $y_4\neq 0$, then by the action of S_{47}, GL (2), S_{67}, S_{56} and GL (2), we have (1). Assume that $y_4=0$. If $y_3=0$ and $y_{12}\neq 0$, then by the action of S_{46} and GL (2), we have (3). If $y_3=y_{12}=0$, then we have (4). If $y_3\neq 0$, then we may assume that $y_1=0$, $y_3=1$ by the action of S_{17} and GL (2). If $y_{12}\neq 0$, then by the action of $S_1(\lambda)$ and λI_2 with $\lambda^2y_{12}=1$, we have (2). If $y_{12}=0$, then we have (4) by the action of $S_{17}(1)$ and $S_{35}(-1)$. Finally the case for $x=1+e_1e_2e_3e_4$. We may assume that $y_{13}=y_{23}=y_2=y_0=0$ as above, i.e., $y=y_{12}e_1e_2+(y_1e_1+y_3e_3+y_4e_1e_2e_3)e_4$. Assume that $y_3\neq 0$. Then we may assume that $y_1=0$ by the action of S_{17}. By the action of $S_{47}(\lambda)$, $S_{12}(\lambda)$ with $\lambda=-(y_4/2y_3)$ and GL (2), we may also assume that $y_4=0$, and by the action of GL (2), we may assume that $y=y_{12}e_1e_2+e_3e_4$. By the action of $S_{12}(\sqrt{y_{12}})$, $S_{47}(\sqrt{y_{12}})$ and GL (2), $y=e_3e_4+2\sqrt{y_{12}}e_1e_2e_3e_4$. If $y_{12}\neq 0$, then by the action of $S_{56}(1/2\sqrt{y_{12}})$, $S_{43}(1/2\sqrt{y_{12}})$ and GL (2), we have (1). If $y_{12}=0$, i.e., $y=e_3e_4$, then by the action of $S_{47}(1)$, $S_{12}(1)$, $S_{47}(1)$, $S_{43}(1)$, GL (2) and so on, we have (2). Assume that $y_3=0$. If $y_4\neq 0$, then by the action of $S_{47}(\lambda)$, $S_{12}(\lambda)$ with $\lambda y_4=y_{12}$ and $S_{67}(\mu)$, $S_{41}(\mu)$ with $\mu y_4=y_1$ and GL (2), we have (1). If $y_4=0$ and $x_{12}\neq 0$, then by the action of $S_{46}(y_1/y_{12})$, $S_{13}(-(y_1/y_{12}))$ and GL(2), we have $y=e_1e_2$. By the action of $S_{56}(-1)$, $S_{12}(-1)$, GL(2) and so on, we have (2). If $y_4=y_{12}=0$, then by $S_{35}(1/y_1)$, $S_{17}(-y_1)$, we have $y=e_3e_4$, i.e., (2). Q.E.D.

Next we consider the triplet (Spin (7)$\times$GL (3), spin rep. $\otimes\Lambda_1$, $V(8)\otimes V(3)$). The representation space $V(8)\otimes V(3)$ is identified with $V=V(8)\oplus V(8)\oplus V(8)$ where $V(8)$ is spanned by 1, e_ie_j $(1\leqq i<j\leqq 4)$, $e_1e_2e_3e_4$.

Proposition 1.4. *The triplet* (Spin (7)$\times$GL (3), *spin rep.* $\otimes\Lambda_1$, $V(8)\otimes V(3)$) *has the following fourteen orbits.*

	Representative Points	*Codimension*
(1)	$(1, e_1e_2e_3e_4, e_1e_2+e_3e_4)$	0
(2)	$(1, e_1e_2e_3e_4, e_1e_2+e_1e_4)$	1
(3)	$(1, e_1e_2, e_3e_4)$	3
(4)	$(1, e_1e_2, e_1e_3+e_2e_4)$	3
(5)	$(1, e_1e_4, e_1e_2+e_3e_4)$	5
(6)	$(1, e_1e_2, e_1e_3)$	6
(7)	$(1, e_1e_2e_3e_4, 0)$	6
(8)	$(1, e_1e_2, e_1e_4)$	7
(9)	$(1, e_1e_2+e_3e_4, 0)$	7
(10)	$(1, e_1e_2, 0)$	9
(11)	$(1, e_1e_4, 0)$	11
(12)	$(1+e_1e_2e_3e_4, 0, 0)$	14
(13)	$(1, 0, 0)$	15
(14)	$(0, 0, 0)$	24

Proof. Let $\tilde{x}=(x, y, z)$ be a representative of one of the orbits of $V=V(8)\oplus V(8)\oplus V(8)$. Then we may assume that $(x, y)=$[1] $(0, 0)$, [2] $(1, 0)$ [3] $(1+e_1e_2e_3e_4, 0)$, [4] $(1, e_1e_4)$, [5] $(1, e_1e_2)$, [6] $(1, e_1e_2+e_3e_4)$, [7] $(1, e_1e_2e_3e_4)$. In the first three cases, repeating the same argument, we obtain (7), (9), (10), (11), (12), (13), (14). First we consider the case [4]. By the action of GL(3), we may put $z=\sum_{1\leq i<j\leq 3} z_{ij}e_ie_j+(\sum_{i=2}^{3} z_ie_i+z_4e_1e_2e_3)e_4\neq 0$. The case for $z_{23}=z_4=0$. Assume that $z_{12}\neq 0$ and $z_3\neq 0$. By the action of $S_{36}(-(z_{13}/z_{12}))$, we may assume that $z=z_{12}e_1e_2+z_2e_2e_4+z_3'e_3e_4$. If $z_3'\neq 0$, then by S_{27}, S_2 and GL(3), we have $e_1e_2+e_3e_4$, i.e., (5). If $z_3'=0$, then by $S_{45}(-(z_2/z_{12}))$ and GL(3), we have e_1e_2, i.e., (8). Assume that $z_{12}\neq 0$ and $z_3=0$. By $S_{45}(-(z_2/z_{12}))$ and $S_{36}(-(z_{13}/z_{12}))$, we may assume that $z=z_{12}e_1e_2+z_3'e_3e_4$. If $z_3'\neq 0$, then we have (5). If $z_3'=0$, then we have (8). Assume that $z_{12}=0$ and $z_3\neq 0$. By $S_{27}(-(z_2/z_3))$, we may assume that $z=z_{12}'e_1e_2+z_{13}e_1e_3+z_3e_3e_4$. If $z_{12}'\neq 0$, then by S_{36}, S_2 and GL(3), we have (5). If $z_{12}'=0$ and $z_{13}\neq 0$, then by S_{45}, S_{27} and S_{36}, we have (8). If $z_{12}'=z_{13}=0$, then we have e_3e_4. By the action of $S_{45}(1)$, $S_{36}(1)$, $S_{27}(-1)$ and $S_{41}(1)$, $(8)\sim(1+e_1e_4, e_1e_4\ e_3e_4)$ and hence $(1, e_1e_4, e_3e_4)\sim(8)$. Assume that $z_{12}=z_3=0$. If $z_{13}\neq 0$ and $z_2\neq 0$, then we have (5) by $S_{27}(1/z_{13})$, $S_{45}(-z_2)$, $S_{36}(-z_{13})$, S_2 and GL(3). If $z_{13}\neq 0$ and $z_2=0$, we have (8) by $S_{27}(1/z_{13})$ and S_{35}. If $z_{13}=0$ and $z_2\neq 0$, then we have $e_3e_4\sim(8)$, by S_{36} and S_{27}. The case

for $z_{23}=0$ and $z_4\neq 0$. By the action of $S_{57}(-(z_2/z_4))$, $S_{56}(z_3/z_4)$, $S_{47}(z_{12}/z_4)$, $S_{46}(-(z_{13}/z_4))$ and GL(3), we have $e_1e_2e_3e_4$. The case for $z_{23}\neq 0$ and $z_4=0$. By $S_{17}(z_{12}/z_{23})$, $S_{16}(-(z_{13}/z_{23}))$, $S_{47}(z_2/z_{23})$, $S_{46}(-(z_3/z_{23}))$ and GL(3), we have e_2e_3. The case for $z_{23}\neq 0$ and $z_4\neq 0$. By $S_{17}(z_{12}/z_{23})$, $S_{16}(-(z_{13}/z_{23}))$, $S_{57}(-(z_2/z_4))$, $S_{56}(z_3/z_4)$, $S_{45}(z_{23}/z_4)$ and GL (3), we have $e_1e_2e_3e_4$. By the action of $S_{41}(1)$, $\begin{pmatrix}1&1&0\\&1&\\&&1\end{pmatrix}\in$ GL (3), $S_{45}(1)$ and GL(3), $(1, e_1e_4, e_2e_3)\sim(1, e_1e_4, e_1e_2e_3e_4)$. Next we consider the case [5]. By the action of GL (3), we may put $z=\sum_{i=1}^{2}z_{i3}e_ie_3+(\sum_{i=1}^{3}z_ie_i+z_4e_1e_2e_3)e_4\neq 0$. The case for $z_4\neq 0$. By the action of $S_{67}(z_1/z_4)$, $S_{57}(-(z_2/z_4))$, $S_{56}(z_3/z_4)$ and GL(3), we may assume that $z=z_{13}e_1e_3+z_{23}e_2e_3+z_4e_1e_2e_3e_4$. If $z_{13}\neq 0$ or $z_{23}\neq 0$, then we may assume that $z_{13}\neq 0$ and $z_{23}=0$ by the action of S_{16} and S_{25}. By the action of $S_{46}(-(z_{13}/z_4))$, S_2 and GL (3), we have $(1, e_1e_2, z_{13}e_1e_3+z_4e_1e_2e_3e_4)\sim(2)$. If $z_{13}=z_{23}=0$, then we have $e_1e_2e_3e_4$. By the action of $S_{12}(1)$, $S_{56}(1))$ and GL(3), we have $(3)\sim(1, e_1e_2, e_1e_2e_3e_4)$. The case for $z_4=0$. Assume that $z_{13}\neq 0$ or $z_{23}\neq 0$. By the action of S_{16}, S_{24}, we may assume that $z_{13}\neq 0$ and $z_{23}=0$. Then, by $S_{47}(z_1/z_{13})$ and GL(3), we may assume that $z_1=0$. If $z_3\neq 0$, then, by the action of $S_{27}(-(z_2/z_3))$, GL(3), S_1 and GL(3), we have $e_1e_3+e_3e_4$. By the action of $S_{43}(1)$, $\begin{pmatrix}1&0&-1\\&1&\\&&1\end{pmatrix}\in$ GL (3), $S_{13}(1)$, $S_{47}(1)$, $S_{35}(1)$, $S_{27}(-1)$, $S_{36}(1)$ and GL (3), we have $(1, e_1e_2, e_1e_3+e_3e_4)\sim(2)$. If $z_3=0$ and $z_2\neq 0$, then we have (4) by the action of S_1 and GL(3). If $z_3=z_2=0$, then we have (6). Assume that $z_{13}=z_{23}=0$. If $z_3\neq 0$, then by the action of $S_{17}(-(z_1/z_3))$, $S_{27}(-(z_2/z_3))$ and GL(3), we have (3). If $z_3=0$, then we may assume that $z_1\neq 0$ and $z_2=0$ by the action of S_{16} and S_{25}, and hence we have (8). Next we consider the case [6] By the action of GL (3), we may put $z=\sum_{1\leq i<j\leq 3}z_{ij}e_ie_j+(\sum_{i=1}^{2}z_ie_i+z_4e_1e_2e_3)e_4\neq 0$. The case for $z_4\neq 0$. By the action of $S_{27}(-(z_{23}/z_4))\circ S_{45}(z_{23}/z_4)$, $\begin{pmatrix}1&0&0\\&1&\lambda\\&&1\end{pmatrix}\in$ GL(3) with $\lambda=-(z_{13}z_{23}/z_4)$, $S_{17}(-(z_{13}/z_4))\circ S_{46}(-(z_{13}/z_4))$ and GL(3), we may assume that $z_{23}=z_{13}=0$. Then, by the action of S_{47}, S_{67}, S_{57} and GL(3), we have $e_1e_2e_3e_4$, i.e., (1). The case for $z_4=0$. Assume that $z_{13}\neq 0$ or $z_{23}\neq 0$. By the action of S_{16} and S_{25}, we may assume that $z_{13}\neq 0$ and $z_{23}=0$. Then, by the action of $S_{47}(z_1/z_{13})$, we may assume that $z=z_{12}e_1e_2+z_{13}e_1e_3+z_2e_2e_4$. If $z_{12}\neq 0$ and $z_2\neq 0$, then, by the action of $\begin{pmatrix}1&&\\&1&\\&&1/z_{12}\end{pmatrix}\in$ GL(3), $S_1(\alpha)\circ S_3(\alpha)$ with $\alpha^2(z_{13}/z_{12})=1$, $S_2(\beta)\circ S_3(\beta)$ with $\beta^2(z_2/z_{12})=1$ and GL(3), we have $(1, e_1e_2+e_3e_4, z)\sim(1, e_1e_2+e_3e_4, e_1e_2+e_1e_3+e_2e_4)$ and the codimension of this point is one. Hence this point is equivalent to (2) (See Prop. 12, p. 64 in [8]). If $z_{12}\neq 0$ and $z_2=0$, then, by

the action of $S_{12}(1)$, $\begin{pmatrix} 1 & & \\ -1 & 1 & \\ -z_{12} & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3)$, $S_{56}(1)$, $S_{13}(z_{13}/z_{12})$ and $\begin{pmatrix} 0 & 0 & -1/z_{12} \\ 1 & 0 & 0 \\ 0 & 1 & -1/z_{12} \end{pmatrix} \in \mathrm{GL}(3)$, we have $(1, e_1e_2+e_3e_4, z) \sim (1, e_1e_2, e_1e_2e_3e_4-(z_{13}/z_{12})e_1e_3)$ and hence we can reduce to the case [5]. If $z_{12}=0$ and $z_2 \neq 0$, then we have $e_1e_3+e_2e_4$ by S_1 and GL(3). By the action of $S_{36}(1)$, $\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \in \mathrm{GL}(3)$, $S_{45}(1)$, $\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & -1 \end{pmatrix} \in \mathrm{GL}(3)$, $S_{27}(-\frac{1}{2})\begin{pmatrix} 1 & & \\ & 1 & 1/2 \\ & & 1 \end{pmatrix} \in \mathrm{GL}(3)$, S_1 and GL(3), we have $(1, e_1e_2+e_3e_4, e_1e_3+e_2e_4) \sim (1, e_1e_2, e_1e_3+e_3e_4)$ and we have (2) from the case [5]. If $z_{12}=z_2=0$, then we have e_1e_3. By the action of $S_{27}(1)$, $S_{36}(-1)$, $S_{27}(1)$, S_3 and GL(3), we have $(1, e_1e_2+e_3e_4, e_1e_3) \sim (4)$. Assume that $z_{13}=z_{23}=0$. If $z_1 \neq 0$ or $z_2 \neq 0$, then we may assume that $z_1 \neq 0$ and $z_2=0$ by the action of S_{16}, S_{25}. If $z_{12} \neq 0$, then by the action of $\begin{pmatrix} 1 & & \\ & 0 & 1/z_{12} \\ & 1 & 0 \end{pmatrix} \in \mathrm{GL}(3)$, $S_{46}(z_1/z_{12})$, $\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \in \mathrm{GL}(3)$ and $S_{17}(z_1/z_{12})$, we have (3). If $z_{12}=0$, then we have e_1e_4, i.e., (5). If $z_1=z_2=0$, then we have e_1e_2, i.e., (3). Finally we consider the case [7]. By the action of GL(3), we may put $z=\sum_{1 \leqq i<j \leqq 3} z_{ij}e_ie_j+\sum_{i=1}^{3} z_ie_ie_4 \neq 0$. The case for $z_{12} \neq 0$ or $z_{13} \neq 0$ or $z_{23} \neq 0$. By the action of S_{25}, S_{35}, S_{16} and S_{17}, we may assume that $z_{23} \neq 0$ and $z_{12}=z_{13}=0$. Assume that $z_1 \neq 0$ or $z_2 \neq 0$ or $z \neq 0$. By the action of S_{35}, S_{36} and S_{27}, we may assume that $z_3 \neq 0$ and $z_2=0$. If $z_1 \neq 0$, then we have (1) by the action of $S_{35}(-(z_3/z_1))$, $S_{17}(-(z_1/z_{23}))$, $S_{35}(z_{23})$, $S_{17}(-(z_1/z_{23}))$, S_2 and GL(3). If $z_1=0$, we have (2) by the action of S_2 and GL(3), $S_{17}(1)$, $S_{35}(-1)$, S_2 and GL(3). Assume that $z_1=z_2=z_3=0$. Then we have e_2e_3. The case for $z_{12}=z_{13}=z_{23}=0$. Then we may assume that $z_1 \neq 0$ and $z_2=z_3=0$. Hence we have e_1e_4. From the case [4], $(1, e_1e_2e_3e_4, e_1e_4) \sim (1, e_1e_4, e_2e_3)$, and by the action of $S_{23}(1)$, $\begin{pmatrix} 1 & 0 & -1 \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{GL}(3)$, $S_{67}(1)$, $\begin{pmatrix} 1 & & \\ & 1 & \\ 1 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3)$, $(1, e_1e_4, e_2e_3) \sim (1, e_1e_2e_3e_4, e_2e_3)$. By the action of $S_{17}(-1)$, $S_{35}(1)$ and GL(3), $(1, e_1e_2e_3e_4, e_2e_3) \sim (1, e_1e_2, e_1e_2e_3e_4)$, and from the case [5], $(1, e_1e_2, e_1e_2e_3e_4) \sim$ (3). Q.E.D.

Remark The isotropy subgroup at (3) (resp. (4)) is locally isomorphic to $(\mathrm{GL}(1) \times \mathrm{SL}(2) \times \mathrm{SO}(2)) \cdot U(4)$ (resp. $(\mathrm{GL}(1) \times \mathrm{SL}(2)) \cdot U(5)$) and hence (3) and (4) are not equivalent.

Next we consider the triplets $(GL(1)^2\times \mathrm{Spin}(8)\times SL(m)$, half-spin rep. $\otimes 1+$vector rep. $\otimes \Lambda_1$, $V_1(8)+V_2(8)\otimes V(m))$ $(m=3, 2)$. The representation space is identified with $V=V_1(8)\oplus M(8, m)$ where $V_1(8)$ is spanned by $1, e_ie_j, e_1e_2e_3e_4$ $(1\leqq i<j\leqq 4)$ (See p. 110–112 in [8]). The action $\rho=\rho_1\otimes 1+\rho_2\otimes \Lambda_1$ is given by $\rho(g)\tilde{x}=(\alpha\rho_1(g_1)X, \beta\rho_2(g_1)Y{}^tg_2)$ for $g=(\alpha, \beta; g_1, g_2)\in GL(1)^2\times \mathrm{Spin}(8)\times SL(m)$, $\tilde{x}=(X, Y)\in V=V_1(8)\oplus M(8, m)$ where ρ_1 (resp. ρ_2) denotes the half-spin (resp. vector) representation of Spin (8) on $V_1(8)$ (resp. $V_2(8)$) (See (5.30), p. 114 in [8]). The Lie algebra of $GL(1)^2\times$ Spin (8) $\times SL(m)$ is given by

$$\left\{(\alpha, \beta)\oplus\left(\begin{array}{c|c} A & B \\ \hline C & -{}^tA \end{array}\right)\oplus(D);\ {}^tB=-B,\ {}^tC=-C,\ D=(d_{ij})\in \mathfrak{sl}(m)\right\}$$

$$(m=3, 2).$$

Proposition 1.5. *The triplet* $(GL(1)^2\times \mathrm{Spin}(8)\times SL(3)$, *half-spin rep.* $\otimes 1+$*vector rep.* $\otimes \Lambda_1$, $V(8)+V(8)\otimes V(3))$ *has the following forty-eight orbits.*

	Representative Points	*Codimension*
(1)	$(1+e_1e_2e_3e_4, (e_1, e_5, e_2-e_6))$	0
(2)	$(1+e_1e_2e_3e_4, (e_1, e_5, e_2+e_8))$	1
(3)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_6))$	3
(4)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_3+e_7))$	3
(5)	$(1+e_1e_2e_3e_4, (e_1, e_8, e_2-e_6))$	5
(6)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_3))$	6
(7)	$(1+e_1e_2e_3e_4, (e_1, e_5, 0))$	6
(8)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_8))$	7
(9)	$(1+e_1e_2e_3e_4, (e_1, e_2-e_6, 0))$	7
(10)	$(1+e_1e_2e_3e_4, (e_1, e_2, 0))$	9
(11)	$(1+e_1e_2e_3e_4, (e_1, e_8, 0))$	11
(12)	$(1+e_1e_2e_3e_4, (e_1+e_5, 0, 0))$	14
(13)	$(1+e_1e_2e_3e_4, (e_1, 0, 0))$	15
(14)	$(1+e_1e_2e_3e_4, (0, 0, 0))$	24
(15)	$(1, (e_1+e_5, e_2+e_6, e_3+e_7))$	1
(16)	$(1, (e_1+e_5, e_2+e_6, e_3))$	2
(17)	$(1, (e_1, e_2+e_6, e_5))$	3
(18)	$(1, (e_1+e_5, e_2, e_3))$	4
(19)	$(1, (e_1, e_2, e_5))$	4
(20)	$(1, (e_1+e_5, e_2+e_6, e_7))$	5

(21)	$(1, (e_1+e_5, e_2, e_7))$	6
(22)	$(1, (e_1, e_2, e_3))$	7
(23)	$(1, (e_1+e_5, e_2+e_6, 0))$	7
(24)	$(1, (e_1, e_5, e_6))$	7
(25)	$(1, (e_1+e_5, e_2, 0))$	8
(26)	$(1, (e_1, e_2, e_7))$	8
(27)	$(1, (e_1+e_5, e_6, e_7))$	9
(28)	$(1, (e_1, e_2, 0))$	10
(29)	$(1, (e_1, e_5, 0))$	10
(30)	$(1, (e_1, e_6, e_7))$	10
(31)	$(1, (e_1+e_5, e_6, 0))$	11
(32)	$(1, (e_1, e_6, 0))$	12
(33)	$(1, (e_5, e_6, e_7))$	13
(34)	$(1, (e_1+e_5, 0, 0))$	15
(35)	$(1, (e_5, e_6, 0))$	15
(36)	$(1, (e_1, 0, 0))$	16
(37)	$(1, (e_5, 0, 0))$	19
(38)	$(1, (0, 0, 0))$	25
(39)	$(0, (e_1, e_5, e_2+e_6))$	8
(40)	$(0, (e_1, e_2, e_5))$	9
(41)	$(0, (e_1, e_2+e_6, e_3))$	11
(42)	$(0, (e_1, e_2, e_3))$	14
(43)	$(0, (e_1, e_5, 0))$	14
(44)	$(0, (e_1, e_2+e_6, 0))$	15
(45)	$(0, (e_1, e_2, 0))$	17
(46)	$(0, (e_1+e_5, 0, 0))$	22
(47)	$(0, (e_1, 0, 0))$	23
(48)	$(0, (0, 0, 0))$	32

Proof. Let $\tilde{x}=(x, (y_1, y_2, y_3))$ be a representative of one of the orbits of $V=V_1(8)\oplus M(8, 3)$. Then we may assume that $x=1+e_1e_2e_3e_4$, 1, 0.

The case for $x=1+e_1e_2e_3e_4$. The isotropy subalgebra at $1+e_1e_2e_3e_4$ is the spin representation of $\mathfrak{o}(7)$, and hence, by Proposition 1.4., $(y_1, y_2, y_3) \sim (e_1, e_5, e_2-e_6)$, (e_1, e_5, e_2+e_8), (e_1, e_2, e_6), (e_1, e_2, e_3+e_7), (e_1, e_8, e_2-e_6), (e_1, e_2, e_3), $(e_1, e_5, 0)$, (e_1, e_2, e_8), $(e_1, e_2-e_6, 0)$, $(e_1, e_2, 0)$, $(e_1, e_8, 0)$, $(e_1+e_5, 0, 0)$, $(e_1, 0, 0)$, $(0, 0, 0)$, i.e., (1)$\sim$(14). The case for $x=1$. The isotropy subalgebra at 1 is given by

$$\left\{\left[\begin{array}{c|c} A & 0 \\ \hline C & -{}^tA \end{array}\right]; A \in \mathfrak{gl}(4), {}^tC=-C\right\}.$$

Put $(y_1, y_2, y_3)=[Y_2/Y_1] \in M(8,3), Y_1, Y_2 \in M(4,3)$. Then $\mathrm{SL}(4)\times\mathrm{GL}(3)$ acts on Y_2 by $\Lambda_1\otimes\Lambda_1$ and hence $Y_2\sim E_{11}+E_{22}+E_{33}, E_{11}+E_{22}, E_{11}, 0$. Assume that $Y_2=E_{11}+E_{22}+E_{33}$. By the action of $-c_{14}, -c_{24}, -c_{34}, Y_1=[Y/0], Y\in M(3)$. We may assume that Y is a symmetric matrix by the action of c_{12}, c_{13}, c_{23}. Then GL(3) acts on Y by $2\Lambda_1$ and hence $Y\sim E_{11}+E_{22}+E_{33}, E_{11}+E_{22}, E_{11}$, 0. Hence we have $(e_1+e_5, e_2+e_6, e_3+e_7)$, (e_1+e_5, e_2+e_6, e_3), (e_1+e_5, e_2, e_3), (e_1, e_2, e_3), i.e., (15), (16), (18), (22). Assume that $Y_2=E_{11}+E_{22}$. By the action of $-c_{13}, -c_{23}, -c_{14}, -c_{24}$, $Y_1=\left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline 0 & Y_{22}\end{array}\right]$, $Y_{11}\in M(2)$, $Y_{12}, Y_{22}\in M(2,1)$. Then GL(2) acts on Y_{12} by Λ_1 and hence $Y_{12}\sim {}^t(1,0), {}^t(0,0)$. If $Y_{12}={}^t(1,0)$, then $Y_{22}=0$ by a_{13}, a_{14}, and $Y_{11}=[0/Y]$, $Y\in M(1,2)$, by d_{13}, d_{23}. Then $\left(\begin{smallmatrix} * & * \\ 0 & *\end{smallmatrix}\right)$ acts on ${}^tY$ by Λ_1 and hence $Y\sim(0,1), (1,0), (0,0)$. Hence we have (e_1, e_2+e_6, e_5), (e_1+e_6, e_2, e_5), (e_1, e_2, e_5). By the action of $c_{12}(1)$, $d_{23}(-1)$, we have $(e_1+e_6, e_2, e_5)\sim(e_1, e_2, e_5)$. Hence we have (e_1, e_2+e_6, e_5), (e_1, e_2, e_5), i.e., (17), (19). If $Y_{12}={}^t(0,0)$, then we may assume that Y_{11} is a symmetric matrix by the action of c_{12}. Then GL(2) acts on Y_{11} by $2\Lambda_1$ and GL(2) acts on Y_{22} by Λ_1, and hence $Y_{11}\sim E_{11}+E_{22}, E_{11}, 0$, and $Y_{22}\sim {}^t(1,0), {}^t(0,0)$. Hence we have (e_1+e_5, e_2+e_6, e_7), (e_1+e_5, e_2, e_7), (e_1, e_2, e_7), $(e_1+e_5, e_2+e_6\ 0)$, $(e_1+e_5, e_2, 0)$, $(e_1, e_2, 0)$ i.e., (20), (21), (26), (23), (25), (28). Assume that $Y_2=E_{11}$. By the action of $-c_{12}, -c_{13}, -c_{14}$, $Y_1=\left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline 0 & Y_{22}\end{array}\right]$, $Y_{12}\in M(1,2)$, $Y_{22}\in M(3,2)$. Then GL(2) acts on ${}^tY_{12}$ by Λ_1 and hence $Y_{12}\sim(1,0), (0,0)$. If $Y_{12}=(1,0)$, then $Y_{11}=0$ by d_{12} and $Y_{22}=[0\ Y]$, $Y\in M(3,1)$, by the action of $-a_{12}, -a_{13}, -a_{14}$. Then GL(3) acts on Y by Λ_1 and hence $Y\sim {}^t(1,0,0)$, ${}^t(0,0,0)$. Hence we have (e_1, e_5, e_6), $(e_1, e_5, 0)$, i.e., (24), (29). If $Y_{12}=(0,0)$, then $\mathrm{SL}(3)\times\mathrm{GL}(2)$ acts on Y_{22} by $\Lambda_1\otimes\Lambda_1$ and GL(1) acts on Y_{11} by Λ_1, and hence $Y_{22}\sim E_{11}+E_{22}, E_{11}, 0$ and $Y_{11}\sim 1, 0$. Hence we have (e_1+e_5, e_6, e_7), $(e_1+e_5, e_6, 0)$, $(e_1+e_5, 0, 0)$, (e_1, e_6, e_7), $(e_1, e_6, 0)$, $(e_1, 0, 0)$, i.e., (27), (31), (34), (30), (32), (36). Assume that $Y_2=0$. Then $\mathrm{SL}(4)\times\mathrm{GL}(3)$ acts on Y_1 by $\Lambda_1\otimes\Lambda_1$ and hence $Y_1\sim E_{11}+E_{22}+E_{33}, E_{11}+E_{22}, E_{11}, 0$. Hence we have (e_5, e_6, e_7), $(e_5, e_6, 0)$, $(e_5, 0, 0)$, $(0, 0, 0)$, i.e., (33), (35), (37), (38). The case for $x=0$. The isotropy subalgebra at 0 is $\mathfrak{o}(8)$, and the triplet $(\mathrm{SO}(8)\times\mathrm{GL}(3), \Lambda_1\otimes\Lambda_1, V(8)\otimes V(3))$ has the following ten orbits; (e_1, e_5, e_2+e_6), (e_1, e_2, e_5), (e_1, e_2+e_6, e_3), (e_1, e_2, e_3), $(e_1, e_5, 0)$, $(e_1, e_2+e_6, 0)$, $(e_1, e_2, 0)$, $(e_1+e_5, 0, 0)$, $(e_1, 0, 0)$, $(0, 0, 0)$. Hence we have (39)$\sim$(48). Q.E.D.

Corollary 1.6. *The triplet* $(\mathrm{GL}(1)^2\times\mathrm{Spin}(8)\times\mathrm{SL}(2)$, *half-spin rep.* $\otimes 1+$ *vector rep.* $\otimes\Lambda_1$, $V(8)+V(8)\otimes V(2))$ *has the following twenty-four orbits.*

Representative Points	*Codimension*
(1) $(1+e_1e_2e_3e_4, (e_1, e_5))$	0

(2)	$(1+e_1e_2e_3e_4, (e_1, e_2-e_6))$	1
(3)	$(1+e_1e_2e_3e_4, (e_1, e_2))$	3
(4)	$(1+e_1e_2e_3e_4, (e_1, e_8))$	5
(5)	$(1+e_1e_2e_3e_4, (e_1+e_5, 0))$	7
(6)	$(1+e_1e_2e_3e_4, (e_1, 0))$	8
(7)	$(1+e_1e_2e_4e_4, (0, 0))$	16
(8)	$(1, (e_1+e_5, e_2+e_6))$	1
(9)	$(1, (e_1+e_5, e_2))$	2
(10)	$(1, (e_1, e_2))$	4
(11)	$(1, (e_1, e_5))$	4
(12)	$(1, (e_1+e_5, e_6))$	5
(13)	$(1, (e_1, e_6))$	6
(14)	$(1, (e_1+e_5, 0))$	8
(15)	$(1, (e_5, e_6))$	9
(16)	$(1, (e_1, 0))$	9
(17)	$(1, (e_5, 0))$	12
(18)	$(1, (0, 0)$	17
(19)	$(0, (e_1, e_5))$	8
(20)	$(0, (e_1, e_2+e_6))$	9
(21)	$(0, (e_1, e_2))$	11
(22)	$(0, (e_1+e_5, 0))$	15
(23)	$(0, (e_1, 0))$	16
(24)	$(0, (0, 0))$	24

Next we consider the triplet $(GL(1)^2\times \mathrm{Spin}(10)\times SL(3)$, half-spin rep. $\otimes 1+$vector rep. $\otimes \Lambda_1$, $V(16)+V(10)\otimes V(3))$. The representation space is identified with $V=V(16)\oplus M(10, 3)$ where $V(16)$ is spanned by 1, e_ie_j $(1\leqq i<j\leqq 5)$, $e_ie_je_ke_l$ $(1\leqq i<j<k\leqq 5)$ (See p. 110–112 in [8]). The action $\rho=\rho_1\otimes 1+\rho_2\otimes\Lambda_1$ is given by $\rho(g)\tilde{x}=(\alpha\rho_1(g_1)X, \beta\rho_2(g_1)Y{}^tg_2)$ for $g=(\alpha, \beta; g_1, g_2)\in GL(1)^2\times \mathrm{Spin}(10)\times SL(3)$, $\tilde{x}=(X, Y)\in V=V(16)\oplus M(10,3)$ where ρ_1 (resp. ρ_2) denotes the half-spin (resp. vector) representation of Spin(10) on $V(16)$ (resp. $V(10)$) (See (5.38), p. 120 in [8]). The triplet $(GL(1)\times\mathrm{Spin}(10)$, half-spin rep., $V(16))$ has three orbits represented by $1+e_1e_2e_3e_4$, 1, 0 (See [2]).

Proposition 1.7. *The triplet* $(GL(1)^2\times\mathrm{Spin}(10)\times SL(3)$, *half-spin rep.* $\otimes 1+$ *vector rep.* $\otimes\Lambda_1$, $V(16)+V(10)\otimes V(3))$ *has the following seventy-seven orbits.*

	Representative Points	*Codimension*
(1)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, e_1, e_6))$	0

(2)	$(1+e_1e_2e_3e_4, (e_5, e_1, e_6))$	1
(3)	$(1+e_1e_2e_3e_4, (e_5, e_1+e_{10}, e_2+e_7))$	1
(4)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, e_1, e_2+e_7))$	2
(5)	$(1+e_1e_2e_3e_4, (e_5, e_1, e_2+e_7+e_{10}))$	2
(6)	$(1+e_1e_2e_3e_4, (e_5, e_1, e_2+e_7))$	3
(7)	$(1+e_1e_2e_3e_4, (e_5, e_1+e_{10}, e_2))$	3
(8)	$(1+e_1e_2e_3e_4, (e_1, e_6, e_2+e_7))$	3
(9)	$(1+e_1e_2e_3e_4, (e_1, e_6, e_2+e_9))$	4
(10)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, e_1,e_2))$	5
(11)	$(1+e_1e_2e_3e_4, (e_5, e_1+e_{10}, e_9))$	5
(12)	$(1+e_1e_2e_3e_4, (e_5, e_1, e_2))$	6
(13)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_7))$	6
(14)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_3+e_8))$	6
(15)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, e_1, e_9))$	7
(16)	$(1+e_1e_2e_3e_4, (e_5, e_1+e_6, e_{10}))$	7
(17)	$(1+e_1e_2e_3e_4, (e_5, e_1, e_9))$	8
(18)	$(1+e_1e_2e_3e_4, (e_5, e_1, e_{10}))$	8
(19)	$(1+e_1e_2e_3e_4, (e_1, e_9, e_2+e_7))$	8
(20)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, e_1+e_6, 0))$	8
(21)	$(1+e_1e_2e_3e_4, (e_1, e_6, e_{10}))$	9
(22)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_3))$	9
(23)	$(1+e_1e_2e_3e_4, (e_5, e_1+e_6, 0))$	9
(24)	$(1+e_1e_2e_3e_4, (e_5, e_1+e_{10}, 0))$	9
(25)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_9))$	10
(26)	$(1+e_1e_2e_3e_4, (e_1, e_2+e_7, e_{10}))$	10
(27)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, e_1, 0))$	10
(28)	$(1+e_1e_2e_3e_4, (e_1, e_6, 0))$	10
(29)	$(1+e_1e_2e_3e_4, (e_5, e_1, 0))$	11
(30)	$(1+e_1e_2e_3e_4, (e_1, e_2+e_7, 0))$	11
(31)	$(1+e_1e_2e_3e_4, (e_1, e_2, e_{10}))$	12
(32)	$(1+e_1e_2e_3e_4, (e_1, e_2, 0))$	13
(33)	$(1+e_1e_2e_3e_4, (e_1, e_9, e_{10}))$	14
(34)	$(1+e_1e_2e_3e_4, (e_1, e_9, 0))$	15
(35)	$(1+e_1e_2e_3e_4, (e_5, e_{10}, 0))$	16
(36)	$(1+e_1e_2e_3e_4, (e_1+e_6, e_{10}, 0))$	17
(37)	$(1+e_1e_2e_3e_4, (e_1, e_{10}, 0))$	18
(38)	$(1+e_1e_2e_3e_4, (e_5+e_{10}, 0, 0))$	18
(39)	$(1+e_1e_2e_3e_4, (e_5, 0, 0))$	19
(40)	$(1+e_1e_2e_3e_4, (e_1+e_6, 0, 0))$	19

(41)	$(1+e_1e_2e_3e_4, (e_1, 0, 0))$	20
(42)	$(1+e_1e_2e_3e_4, (e_{10}, 0, 0))$	27
(43)	$(1+e_1e_2e_3e_4, (0, 0, 0))$	30
(44)	$(1, (e_1+e_6, e_2+e_7, e_3+e_8))$	5
(45)	$(1, (e_1+e_6, e_2+e_7, e_3))$	6
(46)	$(1, (e_1+e_6, e_2, e_3))$	8
(47)	$(1, (e_1, e_2+e_7, e_6))$	8
(48)	$(1, (e_1, e_2, e_6))$	9
(49)	$(1, (e_1+e_6, e_2+e_7, e_8))$	10
(50)	$(1, (e_1+e_6, e_2, e_8))$	11
(51)	$(1, (e_1, e_2, e_3))$	11
(52)	$(1, (e_1, e_2, e_8))$	13
(53)	$(1, (e_1, e_6, e_7))$	13
(54)	$(1, (e_1+e_6, e_2+e_7, 0))$	13
(55)	$(1, (e_1+e_6, e_2, 0))$	14
(56)	$(1, (e_1+e_6, e_7, e_8))$	15
(57)	$(1, (e_1, e_7, e_8))$	16
(58)	$(1, (e_1, e_2, 0))$	16
(59)	$(1, (e_1, e_6, 0))$	17
(60)	$(1, (e_1+e_6, e_7, 0))$	18
(61)	$(1, (e_1, e_7, 0))$	19
(62)	$(1, (e_6, e_7, e_8))$	20
(63)	$(1, (e_6, e_7, 0))$	23
(64)	$(1, (e_1+e_6, 0, 0))$	23
(65)	$(1, (e_1, 0, 0))$	24
(66)	$(1, (e_6, 0, 0))$	28
(67)	$(1, (0, 0, 0))$	35
(68)	$(0, (e_1, e_6, e_2+e_7))$	16
(69)	$(0, (e_1, e_2, e_6))$	17
(70)	$(0, (e_1, e_2+e_7, e_3))$	19
(71)	$(0, (e_1, e_2, e_3))$	22
(72)	$(0, (e_1, e_6, 0))$	24
(73)	$(0, (e_1, e_2+e_7, 0))$	25
(74)	$(0, (e_1, e_2, 0))$	27
(75)	$(0, (e_1+e_6, 0, 0))$	34
(76)	$(0, (e_1, 0, 0))$	35
(77)	$(0, (0, 0, 0))$	46

Remark. There is a possibility that (4) and (5) belong to the same orbits.

Now let us consider the triplet $(GL(1)\times Sp(n)\times SO(3), \Lambda_1\otimes\Lambda_1, V(2n)\otimes V(3))$. We identify $V(2n)\otimes V(3)$ with $V=V(2n)\oplus V(2n)\oplus V(2n)$, and then the action $\Lambda_1\otimes\Lambda_1$ is given by $(x, y, z)\mapsto(\lambda g_1 x, \lambda g_1 y, \lambda g_1 z)^t g_2$ for $g=(\lambda, g_1, g_2)\in G=GL(1)\times Sp(n)\times SO(3)$ and $X=(x, y, z)\in V$. The Lie algebra $\mathfrak{g}$ of $GL(1)\times Sp(n)\times SO(3)$ is given by

$$\mathfrak{g}=\left\{(d)\oplus\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}\oplus\begin{pmatrix} 0 & a & b \\ -b & c & 0 \\ -a & 0 & -c \end{pmatrix};\quad A=(a_{ij})\in\mathfrak{gl}(n),\right.$$

$$\left. B=(b_{ij})={}^tB\in M(n),\quad C=(c_{ij})={}^tC\in M(n)\right\}.$$

For any i, j with $i\neq j$, we put $s_{ij}(\lambda)$ (resp. $t_{ij}(\lambda)$, $r_{ij}(\lambda)$) $=\exp\lambda\tilde{A}$ where $\tilde{A}=\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$ with $a_{ij}=1$ (resp. $b_{ij}=b_{ji}=1$, $c_{ij}=c_{ji}=1$), all remaining parts zero. These $s_{ij}(\lambda)$, $t_{ij}(\lambda)$, $r_{ij}(\lambda)$ are elements of the symplectic group $Sp(n)$. Also define an element $s_i(\lambda)$ of $Sp(n)$ by

$$s_i(\lambda)=\operatorname{diag}(1, \cdots, \overset{i}{\lambda}, \cdots, \overset{n+i}{\lambda^{-1}}, \cdots, 1)$$

and $t_i(\lambda)$ (resp. $r_i(\lambda)$) by $\exp\lambda\tilde{A}$ with $b_{ii}=1$ (resp. $c_{ii}=1$), all remaining parts zero. Now define elements $S_1(\lambda)$, $S_2(\lambda)$, $S_3(\lambda)$ of $SO(3)$ by

$$(x, y, z)^tS_1(\lambda)=(x+y, y, -\lambda x-\tfrac{1}{2}\lambda^2 y+z)$$
$$(x, y, z)^tS_2(\lambda)=(x+\lambda z, -\lambda x+y-\tfrac{1}{2}\lambda^2 z, z)$$
$$(x, y, z)^tS_3(\lambda)=(x, \lambda y, \lambda^{-1}z)\quad(\lambda\neq 0).$$

We identify $V(2n)$ with $\mathbf{C}^{2n}$ by a base $\{u_1, \cdots, u_{2n}\}$.

Proposition 1.8. *The triplet* $(GL(1)\times Sp(n)\times SO(3), \Lambda_1\otimes\Lambda_1, V(2n)\otimes V(3))$ $(n\geqq 3)$ *has the following ten orbits.*

	Representative Points	*Codimension*
(1)	(u_1, u_2, u_{n+2})	0
(2)	(u_1, u_2, u_{n+1})	1
(3)	(u_1, u_2, u_3)	3
(4)	$(0, u_1, u_{n+1})$	$2n-2$
(5)	$(u_1, u_{n+1}, 0)$	$2n-1$
(6)	$(0, u_1, u_2)$	$2n-1$
(7)	$(u_1, u_2, 0)$	$2n$

(8)	$(u_1, 0, 0)$	$4n-2$
(9)	$(0, 0, u_1)$	$4n-1$
(10)	$(0, 0, 0)$	$6n$

Proof. Let (x, y, z) be an element of $V=V(2n)\oplus V(2n)\oplus V(2n)$. Then we may assume that $x=0$ or u_1 by the action of $\mathrm{Sp}(n)$.

The case for $x=0$. Similarly we have $y=0$ or u_1 and if $y=0$, then we have $z=0$ or u_1, i.e., (10), (9). Now assume that $y=u_1$. Since $\mathrm{Sp}(n-1)$ acts on $\langle u_2, \cdots, u_n, u_{n+2}, \cdots, u_{2n}\rangle$, we may assume that $z=z_1u_1+z_2u_2+z_{n+1}u_{n+1}$. If $z_{n+1}\neq 0$, then we have $z_1=z_2=0$, $z_{n+1}=1$ by t_1, t_{12}, S_3 and GL(1), i.e., (4). If $z_{n+1}=0$ and $z_2\neq 0$, then, by s_{12}, S_3 and GL(1), we have $z_1=0$ and $z_2=1$, i.e., (6). If $z_{n+1}=z_2=0$ and $z_1\neq 0$, then we have $z_1=1$ by S_3 and GL(1), i.e., $(0, u_1, u_1)$. It is equivalent to (8) by $S_1(\sqrt{2})$, $S_2(1/\sqrt{2})$ and GL(1). If $z=0$, then we have $(0, u_1, 0)$ which is equivalent to (9) by $S_1(\sqrt{-2})$ and $S_2(-\sqrt{-2})$. Now consider the case for $x=u_1$. By the same argument as above, we may assume that $y=0$, u_1, u_2, or u_{n+1}. If $y=0$, then we may assume that $z=z_1u_1+z_2u_2+z_{n+1}u_{n+1}$. If $z\neq 0$, then it is reduced to the case $y\neq 0$ by S_1 and S_2. If $z=0$, then we have (8). If $y=u_1$, then we can reduce to the case for $x=0$ by the action of $S_1(-1)$. Assume that $y=u_2$. Since $\mathrm{Sp}(n-2)$ acts on $\langle u_3, \cdots, u_n, u_{n+3}, \cdots, u_{2n}\rangle$, we may assume that $z=z_1u_1+z_2u_2+z_3u_3+z_{n+1}u_{n+1}+z_{n+2}u_{n+2}$. If $z_{n+1}\neq 0$ or $z_{n+2}\neq 0$, then we have $z_1=z_2=0$ by t_1, t_2 and t_{12}. Hence, we may assume that (A) $z=z_1u_1+z_2u_2+z_3u_3$, or (B) $z=z_3u_3+z_{n+1}u_{n+1}+z_{n+2}u_{n+2}$. First consider the case (A). If $z_3\neq 0$, then we may assume that $z_1=z_2=0$, i.e., (3). If $z_3=0$, then we have $(u_1, u_2, z_1u_1+z_2u_2)\sim(u_1, u_2, (z_2-\frac{1}{2}z_1^2)u_2)$ by $S_1(z_1)$ and $S_{21}(-z_1)$. If $z_2-\frac{1}{2}z_1^2=0$, then we have (7). If $\lambda^2=z_2-\frac{1}{2}z_1^2\neq 0$, then we have (u_1, u_2, u_2) by $S_3(\lambda)$ and $s_2(1/\lambda)$. We shall see that $(u_1, u_2, u_2)\sim(0, u_1, u_2)$, i.e., (6). By $S_1(\lambda)$ and $s_{12}(\lambda^2/2)$, $(0, u_1, u_2)$ is transformed to $(\lambda u_1, u_1, u_2)$. By $S_2(1/\lambda)$ and $s_{21}(-(1/\lambda^2))$, we have $(\lambda u_1, -(1/2\lambda^2)u_2, u_2)$. By $s_1(1/\lambda)$ and taking λ satisfying $2\lambda^2=-1$, we have (u_1, u_2, u_2). Now consider the case (B). If $z_3\neq 0$, then by t_{12} or t_{23}, we have (3). If $z_3=0$ and $z_{n+1}\neq 0$, then by s_{12}, we have $z_{n+2}=0$. By changing y and z, it is reduced to the case $y=u_{n+1}$. If $z_3=z_{n+1}=0$, then we may assume that $z_{n+2}\neq 0$. By s_2, GL(1) and s_1, we have (1). Finally assume that $y=u_{n+1}$. In this case, we may assume that $z=z_1u_1+z_2u_2+z_{n+1}u_{n+1}$. By the action of $S_1(z_1)$ and r_1, we have $(u_1, u_{n+1}, \alpha u_2+\beta u_{n+1})$. By s_1, s_2, S_3 and GL(1), we have (u_1, u_{n+1}, z) with $z=u_2$, u_{n+1}, u_2+u_{n+1} or 0. By the similar argument as above, (u_1, u_{n+1}, u_{n+1}) is equivalent to (4). Since $(u_1, u_{n+1}, u_2+u_{n+1})$ belongs to the orbit of codimension zero, it is equivalent to (1). Hence we have (2), (4), (1), (5). Q.E.D.

Remark. The isotropy subgroup at (5) (resp. (6)) is locally isomorphic to $(\mathrm{GL}(1)\times\mathrm{Sp}(n-1))\cdot\mathrm{U}(1)$ (resp. $(\mathrm{GL}(1)^2\times\mathrm{Sp}(n-2))\cdot\mathrm{U}(4n-5))$ and hence, (5) and (6) are not G-equivalent, where $G=\mathrm{GL}(1)\times\mathrm{Sp}(n)\times\mathrm{SO}(3)$.

Next we consider the triplet $(\mathrm{GL}(1)^2\times\mathrm{Sp}(2)\times\mathrm{SL}(2), \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1, V(5)\otimes V(2)+V(4))$. Let $V(4)$ be the vector space spanned by u_1, u_2, u_3, u_4. Then the representation space of Λ_2 for $\mathrm{SL}(4)$ is given by $V(6)=\sum_{1\leqq i<j\leqq 4}\mathbf{C}u_i\wedge u_j$. If we restrict this representation to a subgroup $\mathrm{Sp}(2)$, then $V(6)$ decomposes into $V(5)\oplus V(1)$ where $V(5)$ is spanned by $\omega_1=u_1\wedge u_3-u_2\wedge u_4$, $\omega_2=u_1\wedge u_2$, $\omega_3=u_1\wedge u_4$, $\omega_4=u_3\wedge u_4$ and $\omega_5=u_2\wedge u_3$, and $V(1)=\mathbf{C}(u_1\wedge u_3+u_2\wedge u_4)$. Now we identify $V(5)\otimes V(2)+V(4)$ with $(V(5)\oplus V(5))\oplus V(4)$. Then the action $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1$ is given by $\rho(g)\tilde{x}=((\alpha\rho_2(g_1)x_1, \alpha\rho_2(g_1)x_2)^t g_2, \beta\rho_1(g_1)y)$ for $g=(\alpha, \beta; g_1, g_2)\in\mathrm{GL}(1)^2\times\mathrm{Sp}(2)\times\mathrm{SL}(2)$ and $\tilde{x}=((x_1, x_2), y)\in V=(V(5)\oplus V(5))\oplus V(4)$.

Proposition 1.9. *The triplet* $(\mathrm{GL}(1)^2\times\mathrm{Sp}(2)\times\mathrm{SL}(2), \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1, V(5)\otimes V(2)+V(4))$ *has the following twenty-one orbits.*

	Representative Points	*Codimension*
(1)	$((\omega_2, \omega_4), u_1+u_3)$	0
(2)	$((\omega_2, \omega_4), u_1+u_4)$	1
(3)	$((\omega_2, \omega_4), u_1)$	2
(4)	$((\omega_2, \omega_4), 0)$	4
(5)	$((\omega_1, \omega_2), u_3+u_4)$	1
(6)	$((\omega_1, \omega_2), u_3)$	2
(7)	$((\omega_1, \omega_2), u_1+u_2)$	3
(8)	$((\omega_1, \omega_2), u_1)$	4
(9)	$((\omega_1, \omega_2), 0)$	5
(10)	$((\omega_2, \omega_3), u_3)$	3
(11)	$((\omega_2, \omega_3), u_2)$	4
(12)	$((\omega_2, \omega_3), u_1)$	6
(13)	$((\omega_2, \omega_3), 0)$	7
(14)	$((\omega_1, 0), u_1+u_2)$	4
(15)	$((\omega_1, 0), u_1)$	6
(16)	$((\omega_1, 0), 0)$	8
(17)	$((\omega_2, 0), u_3)$	5
(18)	$((\omega_2, 0), u_1)$	7
(19)	$((\omega_2, 0), 0)$	9
(20)	$((0, 0), u_1)$	10
(21)	$((0, 0), 0)$	14

Proof. Let $\tilde{x}=((x_1, x_2), y)$ be an element of $V=(V(5)\oplus V(5))\oplus V(4)$.

Then we may assume that (x_1, x_2) is one of the following six points by the action of $(\mathrm{Sp}(2)\times \mathrm{GL}(2), \Lambda_2\otimes\Lambda_1)\cong(\mathrm{SO}(5)\times \mathrm{GL}(2), \Lambda_1\otimes\Lambda_1)$; [$a$] (ω_2, ω_4), [b] (ω_1, ω_2), [c] (ω_2, ω_3), [d] $(\omega_1, 0)$, [e] $(\omega_2, 0)$, [f] $(0, 0)$.

The case for [a]. The isotropy subalgebra at [a] is given by

$$\left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & -{}^tA \end{array}\right)\oplus\begin{pmatrix} -\mathrm{tr}\,A & 0 \\ 0 & \mathrm{tr}\,A \end{pmatrix};\ A\in\mathfrak{gl}(2)\right\}.$$

Then $\mathrm{GL}(1)\times \mathrm{GL}(2)$ acts on y by $\Lambda_1\otimes(\Lambda_1+\Lambda_1^*)$, and hence we have $y\sim u_1+u_3,\ u_1+u_4,\ u_1,\ u_3,\ 0$. By the action of $\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)\in \mathrm{Sp}(2)$ with $\mathrm{I}=\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$ and $\mathrm{GL}(2)$, we have $([a], u_1)\sim([a], u_3)$. Hence we have (1)$\sim$(4).

The case for [b]. The isotropy subalgebra at [b] is given by

$$\left\{\left(\begin{array}{c|c} A & B \\ \hline 0 & -A \end{array}\right)\oplus\begin{pmatrix} 0 & -2b_{12} \\ 0 & -\mathrm{tr}\,A \end{pmatrix};\ A=\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},\ B=\begin{pmatrix} b_1 & b_{12} \\ b_{12} & b_2 \end{pmatrix}\right\}.$$

If $y'=(y_3, y_4)\neq 0$, then, by the action of B, $y_1=y_2=0$ and hence $y\sim u_3+u_4$, u_3, u_4. If $y'=0$, then $y\sim u_1+u_2,\ u_1,\ u_2,\ 0$. By the action of $J_2=\left(\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right)\in \mathrm{Sp}(2)$ with $J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathrm{GL}(2)$, we have $([b], u_3)\sim([b], u_4)$ and $([b], u_1)\sim([b], u_2)$. Hence we have (5)$\sim$(9).

The case for [c]. The isotropy subalgebra at [c] is given by

$$\left\{\left(\begin{array}{c|c} A & B \\ \hline C & -{}^tA \end{array}\right)\oplus\begin{pmatrix} -a_1-a_2, & -c_2 \\ -b_2 & -a_1+a_2 \end{pmatrix};\right.$$
$$\left. A=\begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix},\quad B=\begin{pmatrix} b_1 & b_{12} \\ b_{12} & b_2 \end{pmatrix},\quad C=\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}\right\}.$$

Hence we have $y\sim u_3,\ u_2,\ u_1,\ 0$, i.e., (10) (13).

The case for [d]. The isotropy subalgebra at [d] is given by

$$\left\{\left(\begin{array}{c|c} A & B \\ \hline C & -{}^tA \end{array}\right)\oplus\begin{pmatrix} 0 & \gamma \\ 0 & \alpha \end{pmatrix};\ A=\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},\ B=\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},\ C=\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\right\}.$$

Then $\mathrm{SL}(2)\times \mathrm{SL}(2)$ acts on ${}^t(y_1, y_3; y_2, y_4)$ by $\Lambda_1\otimes 1+1\otimes\Lambda_1$. Then we have $y\sim u_1+u_2,\ u_1,\ u_2,\ 0$. By the action of $J_2\in \mathrm{Sp}(2)$ and $\mathrm{GL}(2)$, we have $([d], u_1)\sim([d], u_2)$. Hence we have (14)$\sim$(16).

The case for $[e]$. The isotropy subalgebra at $[e]$ is given by

$$\left\{\left(\begin{array}{c|c} A & B \\ \hline 0 & -{}^tA \end{array}\right)\oplus\begin{pmatrix} -\mathrm{tr}\,A & \gamma \\ 0 & \alpha \end{pmatrix};\ A\in\mathfrak{gl}(2),\ {}^tB=B\right\}.$$

Hence we have $y\sim u_3,\ u_1,\ 0$. i.e., (17) $\sim$ (19).

The case for $[f]$. Then Sp(2) acts on y by Λ_1 and hence we have $y\sim u_1,\ 0$, i.e., (20), (21). Q.E.D.

§ 2.

Let $V(n)$ be the n-dimensional vector space spanned by $u_1, \cdots, u_n$. Then $SL(n)$ acts on $V(n)$ by $\rho_1(g)(u_1, \cdots, u_n)=(u_1, \cdots, u_n)g$ for $g\in SL(n)$. Let $V_2=V(\frac{1}{2}n(n-1))$ be the vector space spanned by skew-tensors $u_i\wedge u_j$ $(1\leqq i<j\leqq n)$. Then the action $\rho_2=\Lambda_2$ of $SL(n)$ on V_2 is given by $\rho_2(g)(u_i\wedge u_j)=\rho_1(g)u_i\wedge\rho_1(g)u_j$.

First we consider the triplets $(SL(n)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V(\frac{1}{2}n(n-1)))$ $(n=5, 6, 7)$. The representation space is identified with $V=V(\frac{1}{2}n(n-1))\oplus V(\frac{1}{2}n(n-1))$ and the action $\rho=\Lambda_2\otimes\Lambda_1$ on V is given by $\rho(g)\tilde{x}=(\rho_2(g_1)X_1,\ \rho_2(g_1)X_2){}^tg_2$ for $g=(g_1, g_2)\in SL(n)\times GL(2)$, $\tilde{x}=(X_1, X_2)\in V=V(\frac{1}{2}n(n-1))\oplus V(\frac{1}{2}n(n-1))$, $(n=5, 6, 7)$.

Proposition 2.1. *The triplet* $(SL(5)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V(10)\otimes V(2))$ *has the following eight orbits.*

	Representative Points	*Codimension*
(1)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_2\wedge u_3+u_4\wedge u_5)$	0
(2)	$(u_1\wedge u_2,\ u_3\wedge u_4+u_1\wedge u_5)$	2
(3)	$(u_1\wedge u_2,\ u_3\wedge u_4)$	4
(4)	$(u_1\wedge u_2,\ u_1\wedge u_3+u_2\wedge u_4)$	5
(5)	$(u_1\wedge u_2,\ u_1\wedge u_3)$	8
(6)	$(u_1\wedge u_2+u_3\wedge u_4,\ 0)$	9
(7)	$(u_1\wedge u_2,\ 0)$	12
(8)	$(0, 0)$	20

Proposition 2.2. *The triplet* $(SL(6)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V(15)\otimes V(2))$ *has following fifteen orbits.*

	Representative Points	*Codimension*
(1)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_2+u_5\wedge u_6)$	0
(2)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_3+u_5\wedge u_6)$	1
(3)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_2\wedge u_3+u_1\wedge u_5+u_4\wedge u_6)$	2

(4)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_5+u_4\wedge u_6)$	4
(5)	$(u_1\wedge u_2,\ u_3\wedge u_4+u_5\wedge u_6)$	5
(6)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_2\wedge u_3+u_4\wedge u_5)$	5
(7)	$(u_1\wedge u_2,\ u_1\wedge u_3+u_2\wedge u_6+u_4\wedge u_5)$	6
(8)	$(u_1\wedge u_2,\ u_3\wedge u_4+u_1\wedge u_5)$	7
(9)	$(u_1\wedge u_2,\ u_3\wedge u_4)$	10
(10)	$(u_1\wedge u_2,\ u_1\wedge u_3+u_2\wedge u_4)$	11
(11)	$(u_1\wedge u_2+u_3\wedge u_4+u_5\wedge u_6,\ 0)$	14
(12)	$(u_1\wedge u_2,\ u_1\wedge u_3)$	15
(13)	$(u_1\wedge u_2+u_3\wedge u_4,\ 0)$	15
(14)	$(u_1\wedge u_2,\ 0)$	20
(15)	$(0,\ 0)$	30

Proposition 2.3. *The triplet* $(SL(7)\times GL(2),\ \Lambda_2\otimes\Lambda_1,\ V(21)\times V(2))$ *has the following twenty orbits.*

	Representative Points	*Codimension*
(1)	$(u_1\wedge u_2+u_3\wedge u_4+u_5\wedge u_6,\ u_2\wedge u_3+u_4\wedge u_5+u_6\wedge u_7)$	0
(2)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_2\wedge u_3+u_4\wedge u_6+u_5\wedge u_7)$	2
(3)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_5+u_6\wedge u_7)$	4
(4)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_5+u_4\wedge u_6+u_2\wedge u_7)$	5
(5)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_2+u_5\wedge u_6)$	6
(6)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_3+u_5\wedge u_6)$	7
(7)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_2\wedge u_3+u_1\wedge u_5+u_4\wedge u_6)$	8
(8)	$(u_1\wedge u_2+u_3\wedge u_4+u_5\wedge u_6,\ u_6\wedge u_7)$	9
(9)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_1\wedge u_5+u_4\wedge u_6)$	10
(10)	$(u_1\wedge u_2,\ u_3\wedge u_4+u_5\wedge u_6)$	11
(11)	$(u_1\wedge u_2+u_3\wedge u_4,\ u_2\wedge u_3+u_4\wedge u_5)$	12
(12)	$(u_1\wedge u_2,\ u_1\wedge u_3+u_2\wedge u_6+u_4\wedge u_5)$	12
(13)	$(u_1\wedge u_2,\ u_3\wedge u_4+u_1\wedge u_5)$	14
(14)	$(u_1\wedge u_2,\ u_3\wedge u_4)$	18
(15)	$(u_1\wedge u_2,\ u_1\wedge u_3+u_2\wedge u_4)$	19
(16)	$(u_1\wedge u_2+u_3\wedge u_4+u_5\wedge u_6,\ 0)$	20
(17)	$(u_1\wedge u_2+u_3\wedge u_4,\ 0)$	23
(18)	$(u_1\wedge u_2,\ u_1\wedge u_3)$	24
(19)	$(u_1\wedge u_2,\ 0)$	30
(20)	$(0,\ 0)$	42

Next we consider the triplet $(GL(1)^2\times SL(7)\times SL(2),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1,\ V(21)\otimes V(2)+V(7))$. Let $V(7)$ be the 7-dimensional vector space spanned by $u_1, u_2, \cdots, u_7$ and let $V(21)$ be the vector space spanned by 2-forms $u_i\wedge u_j$ $(1\leqq i<j\leqq 7)$. Then the representation space is identified with $V=(V(21)\oplus V(21))\oplus V(7)$ and the action $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1$ on V is given by $\rho(g)\tilde{x}=(\alpha(\rho_2(g_1)X_1,\ \rho_2(g_1)X_2)^t g_2,\ \beta\rho_1(g_1)Y)$ for $g=(\alpha,\beta;g_1,g_2)\in GL(1)^2\times SL(7)\times SL(2)$, $\tilde{x}=((X_1, X_2), Y)\in V=(V(21)\oplus V(21))\oplus V(7)$.

Proposition 2.4. *The triplet* $(GL(1)^2\times SL(7)\times SL(2),\ \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1,\ V(21)\otimes V(2)+V(7))$ *has the following* 104 *orbits. Here ij stands for* $u_i\wedge u_j$.

	Representative Points	*Codimension*
(1)	$((12+34+56,\ 23+45+67),\ u_3+u_5)$	0
(2)	$((12+34+56,\ 23+45+67),\ u_5)$	1
(3)	$((12+34+56,\ 23+45+67),\ u_7)$	2
(4)	$((12+34+56,\ 23+45+67),\ u_4)$	4
(5)	$((12+34+56,\ 23+45+67),\ u_2)$	5
(6)	$((12+34+56,\ 23+45+67),\ 0)$	7
(7)	$((12+34,\ 23+46+57),\ u_1+u_6)$	2
(8)	$((12+34,\ 23+46+57),\ u_6)$	3
(9)	$((12+34,\ 23+46+57),\ u_3+u_5)$	3
(10)	$((12+34,\ 23+46+57),\ u_1+u_5)$	4
(11)	$((12+34,\ 23+46+57),\ u_3)$	5
(12)	$((12+34,\ 23+46+57),\ u_2+u_5)$	5
(13)	$((12+34,\ 23+46+57),\ u_1)$	6
(14)	$((12+34,\ 23+46+57),\ u_5)$	6
(15)	$((12+34,\ 23+46+57),\ u_2)$	7
(16)	$((12+34,\ 23+46+57),\ u_4)$	8
(17)	$((12+34,\ 23+46+57),\ 0)$	9
(18)	$((12+34,\ 15+67),\ u_2+u_5)$	4
(19)	$((12+24,\ 15+67),\ u_2+u_6)$	5
(20)	$((12+34,\ 15+67),\ u_3+u_6)$	6
(21)	$((12+34,\ 15+67),\ u_2)$	7
(22)	$((12+34,\ 15+67),\ u_3)$	8
(23)	$((12+34,\ 15+67),\ u_1)$	10
(24)	$((12+34,\ 15+67),\ 0)$	11
(25)	$((12+34,\ 15+46+27),\ u_6)$	5
(26)	$((12+34,\ 15+46+27),\ u_3+u_5)$	6
(27)	$((12+34,\ 15+46+27),\ u_5)$	7
(28)	$((12+34,\ 15+46+27),\ u_3)$	8

(29)	$((12+34,\ 15+46+27),\ u_1)$	9
(30)	$((12+34,\ 15+46+27),\ u_4)$	11
(31)	$((12+34,\ 15+46+27),\ 0)$	12
(32)	$((12+34,\ 12+56),\ u_7)$	6
(33)	$((12+34,\ 12+56),\ u_1+u_3+u_5)$	7
(34)	$((12+34,\ 12+56),\ u_1+u_3)$	9
(35)	$((12+34,\ 12+56),\ u_1)$	11
(36)	$((12+34,\ 12+56),\ 0)$	13
(37)	$((12+34,\ 13+56),\ u_7)$	7
(38)	$((12+34,\ 13+56),\ u_2+u_5)$	8
(39)	$((12+34,\ 13+56),\ u_2)$	10
(40)	$((12+34,\ 13+56),\ u_1+u_5)$	10
(41)	$((12+34,\ 13+56),\ u_1)$	12
(42)	$((12+34,\ 13+56),\ u_5)$	12
(43)	$((12+34,\ 13+56),\ 0)$	14
(44)	$((12+34,\ 23+15+46),\ u_7)$	8
(45)	$((12+34,\ 23+15+46),\ u_5)$	9
(46)	$((12+34,\ 23+15+46),\ u_2)$	11
(47)	$((12+34,\ 23+15+46),\ u_1)$	13
(48)	$((12+34,\ 23+15+46),\ 0)$	15
(49)	$((12+34+56,\ 67),\ u_5)$	9
(50)	$((12+34+56,\ 67),\ u_1+u_7)$	10
(51)	$((12+34+56,\ 67),\ u_1)$	11
(52)	$((12+34+56,\ 67),\ u_7)$	14
(53)	$((12+34+56,\ 67),\ u_6)$	15
(54)	$((12+34+56,\ 67),\ 0)$	16
(55)	$((12+34,\ 15+46),\ u_7)$	10
(56)	$((12+34,\ 15+46),\ u_2+u_6)$	11
(57)	$((12+34,\ 15+46),\ u_2)$	12
(58)	$((12+34,\ 15+46),\ u_1)$	15
(59)	$((12+34,\ 15+46),\ 0)$	17
(60)	$((12,\ 34+56),\ u_7)$	11
(61)	$((12,\ 34+56),\ u_1+u_3)$	12
(62)	$((12,\ 34+56),\ u_3)$	14
(63)	$((12,\ 34+56),\ u_1)$	16
(64)	$((12,\ 34+56),\ 0)$	18
(65)	$((12+34,\ 23+45),\ u_6)$	12
(66)	$((12+34,\ 23+45),\ u_3)$	14
(67)	$((12+34,\ 23+45),\ u_1)$	15

(68)	$((12+34, 23+45), u_2)$	17
(69)	$((12+34, 23+45), 0)$	19
(70)	$((12, 13+26+45), u_7)$	12
(71)	$((12, 13+26+45), u_3)$	13
(72)	$((12, 13+26+45), u_4)$	15
(73)	$((12, 13+26+45), u_1)$	17
(74)	$((12, 13+26+45), 0)$	19
(75)	$((12, 34+15), u_6)$	14
(76)	$((12, 34+15), u_5)$	16
(77)	$((12, 34+15), u_2+u_3)$	17
(78)	$((12, 34+15), u_3)$	18
(79)	$((12, 34+15), u_2)$	19
(80)	$((12, 34+15), u_1)$	20
(81)	$((12, 34+15), 0)$	21
(82)	$((12, 34), u_5)$	18
(83)	$((12, 34), u_1+u_3)$	21
(84)	$((12, 34), u_1)$	23
(85)	$((12, 34), 0)$	25
(86)	$((12, 13+24), u_5)$	19
(87)	$((12, 13+24), u_3)$	22
(88)	$((12, 13+24), u_1)$	24
(89)	$((12, 13+24), 0)$	26
(90)	$((12+34+56, 0), u_7)$	20
(91)	$((12+34+56, 0), u_1)$	21
(92)	$((12+34+56, 0), 0)$	27
(93)	$((12+34, 0), u_5)$	23
(94)	$((12+34, 0), u_1)$	24
(95)	$((12+34, 0), 0)$	30
(96)	$((12, 13), u_4)$	24
(97)	$((12, 13), u_2)$	28
(98)	$((12, 13), u_1)$	30
(99)	$((12, 13), 0)$	31
(100)	$((12, 0), u_3)$	30
(101)	$((12, 0), u_1)$	35
(102)	$((12, 0), 0)$	37
(103)	$((0, 0), u_1)$	42
(104)	$((0, 0), 0)$	49

Next we consider the triplet $(GL(1)^2 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1, V(10) \otimes V(2) + V(5))$. Let $V(5)$ be the 5-dimensional vector spacespanned by $u_1, u_2, \cdots, u_5$ and let $V(10)$ be the vector space spanned by 2-forms $u_i \wedge u_j$ $(1 \leqq i < j \leqq 5)$. Then the representation space is identified with $V = (V(10) \oplus V(10)) \oplus V(5)$ and the action $\rho = \Lambda_2 \oplus \Lambda_1 + \Lambda_1 \otimes 1$ on V is given by $\rho(g)\tilde{x} = (\alpha(\rho_2(g_1)X_1, \rho_2(g_1)X_2)^t g_2, \beta\rho_1(g_1)Y)$ for $g = (\alpha, \beta; g_1, g_2) \in GL(1)^2 \times SL(5) \times SL(2)$, $\tilde{x} = ((X_1, X_2), Y) \in V = (V(10) \oplus V(10)) \oplus V(5)$.

Proposition 2.5. *The triplet* $(GL(1)^2 \times SL(5) \times SL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1, V(10) \otimes V(2) + V(5))$ *has the following thirty orbits. Here ij stands for* $u_i \wedge u_j$.

	Representative Points	*Codimension*
(1)	$((12+34, 23+45), u_3)$	0
(2)	$((12+34, 23+45), u_5)$	1
(3)	$((12, 34+15), u_5)$	2
(4)	$((12+34, 23+45), u_2)$	3
(5)	$((12, 34+15), u_2+u_3)$	3
(6)	$((12, 34+15), u_3)$	4
(7)	$((12, 34), u_5)$	4
(8)	$((12+34, 23+45), 0)$	5
(9)	$((12, 34+15), u_2)$	5
(10)	$((12, 34), u_1+u_3)$	5
(11)	$((12, 13+24), u_5)$	5
(12)	$((12, 34+15), u_1)$	6
(13)	$((12, 13+24), u_3)$	6
(14)	$((12, 34+15), 0)$	7
(15)	$((12, 34), u_1)$	7
(16)	$((12, 13+24), u_1)$	8
(17)	$((12, 13), u_4)$	8
(18)	$((12, 34), 0)$	9
(19)	$((12+34, 0), u_5)$	9
(20)	$((12, 13+24), 0)$	10
(21)	$((12, 13)\ u_2)$	10
(22)	$((12+34, 0), u_1)$	10
(23)	$((12, 13), u_1)$	12
(24)	$((12, 0), u_3)$	12
(25)	$((12, 13), 0)$	13

(26)	$((12+34, 0), 0)$	14
(27)	$((12, 0), u_1)$	15
(28)	$((12, 0), 0)$	17
(29)	$((0, 0), u_1)$	20
(30)	$((0, 0), 0)$	25

Next we consider the triplet $(GL(1)^2\times SL(2)\times SL(5)\times SL(2), \Lambda_1\otimes\Lambda_2\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1, V(2)\otimes V(10)+V(5)\otimes V(2))$. Let $V(5)$ be the vector space spanned by $u_1, \cdots, u_5$, and let $V(10)$ be the vector space spanned by 2-form $u_i\wedge u_j$ $(1\leqq i<j\leqq 5)$. Then the representation space is identified with $V=(V(10)\oplus V(10))\oplus(V(5)\oplus V(5)))$ and the action $\rho=\Lambda_1\otimes\Lambda_2\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1$ is given by

$$\rho(g)\tilde{x}=((\alpha\rho_2(g_2)(aX+cY), \alpha\rho_2(g_2)(bX+dY)), (\beta\rho_1(g_2)Z, \beta\rho_1(g_2)W)^t g_3)$$

for $g=\left(\alpha, \beta; \begin{pmatrix} a & b \\ c & d\end{pmatrix}, g_2, g_3\right)\in GL(1)^2\times SL(2)\times SL(5)\times SL(2)$,

$\tilde{x}=((X, Y), (Z, W))\in V.$

Proposition 2.6. *The triplet* $(GL(1)^2\times SL(2)\times SL(5)\times SL(2), \Lambda_1\otimes\Lambda_2\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1, V(2)\otimes V(10)+V(5)\otimes V(2))$ *has the following seventy-three* 73 *orbits. Here ij stands for* $u_i\wedge u_j$.

	Representative Points	*Codimension*
(1)	$((12+34, 23+45), (u_5, u_1))$	0
(2)	$((12+34, 23+45), (u_2+u_5, u_3))$	1
(3)	$((12+34, 23+45), (u_5, u_3))$	2
(4)	$((12+34, 23+45), (u_3, u_2+u_4))$	2
(5)	$((12+34, 23+45), (u_3, u_2))$	3
(6)	$((12+34, 23+45), (u_5, u_4))$	3
(7)	$((12+34, 23+45), (u_5, u_2))$	4
(8)	$((12+34, 23+45), (u_2, u_4))$	6
(9)	$((12+34, 23+45), (u_3, 0))$	4
(10)	$((12+34, 23+45), (u_5, 0))$	5
(11)	$((12+34, 23+45), (u_2, 0))$	7
(12)	$((12+34, 23+45), (0, 0))$	10
(13)	$((12, 34+15), (u_5, u_2+u_3))$	2
(14)	$((12, 34+15), (u_5, u_1+u_3))$	3
(15)	$((12, 34+15), (u_5, u_3))$	4
(16)	$((12, 34+15), (u_5, u_2))$	4
(17)	$((12, 34+15), (u_2+u_3, u_4))$	4

(18)	$((12, 34+15), (u_5, u_1))$	5
(19)	$((12, 34+15), (u_3, u_2))$	5
(20)	$((12, 34+15), (u_3, u_4))$	6
(21)	$((12, 34+15), (u_2+u_3, u_1))$	6
(22)	$((12, 34+15), (u_3, u_1))$	7
(23)	$((12, 34+15), (u_2, u_1))$	8
(24)	$((12, 34+15), (u_5, 0))$	6
(25)	$((12, 34+15), (u_2+u_3, 0))$	7
(26)	$((12, 34+15), (u_3, 0))$	8
(27)	$((12, 34+15), (u_2, 0))$	9
(28)	$((12, 34+15), (u_1, 0))$	10
(29)	$((12, 34+15), (0, 0))$	12
(30)	$((12, 34), (u_5, u_1+u_3))$	4
(31)	$((12, 34), (u_5, u_1))$	6
(32)	$((12, 34), (u_1+u_3, u_2+u_4))$	6
(33)	$((12, 34), (u_1+u_3, u_4))$	7
(34)	$((12, 34), (u_3, u_1))$	8
(35)	$((12, 34), (u_1, u_2))$	10
(36)	$((12, 34), (u_5, 0))$	8
(37)	$((12, 34), (u_1+u_3, 0))$	9
(38)	$((12, 34), (u_1, 0))$	11
(39)	$((12, 34), (0, 0))$	14
(40)	$((12, 13+24), (u_5, u_3))$	5
(41)	$((12, 13+24), (u_5, u_1))$	7
(42)	$((12, 13+24), (u_3, u_4))$	7
(43)	$((12, 13+24), (u_3, u_1))$	8
(44)	$((12, 13+24), (u_3, u_2))$	9
(45)	$((12, 13+24), (u_1, u_2))$	11
(46)	$((12, 13+24), (u_5, 0))$	9
(47)	$((12, 13+24), (u_3, 0))$	10
(48)	$((12, 13+24), (u_1, 0))$	12
(49)	$((12, 13+24), (0, 0))$	15
(50)	$((12, 13), (u_4, u_5))$	8
(51)	$((12, 13), (u_4, u_2))$	9
(52)	$((12, 13), (u_4, u_1))$	11
(53)	$((12, 13), (u_2, u_3))$	12
(54)	$((12, 13), (u_2, u_1))$	13
(55)	$((12, 13), (u_4, 0))$	12

(56)	$((12, 13), (u_2, 0))$	14
(57)	$((12, 13), (u_1, 0))$	16
(58)	$((12, 13), (0, 0))$	18
(59)	$((12+34, 0), (u_5, u_1))$	9
(60)	$((12+34, 0), (u_1, u_2))$	11
(61)	$((12+34, 0), (u_1, u_3))$	12
(62)	$((12+34, 0), (u_5, 0))$	13
(63)	$((12+34, 0), (u_1, 0))$	14
(64)	$((12+34, 0), (0, 0))$	19
(65)	$((12, 0), (u_3, u_4))$	12
(66)	$((12, 0), (u_3, u_1))$	14
(67)	$((12, 0), (u_1, u_2))$	18
(68)	$((12, 0), (u_3, 0))$	16
(69)	$((12, 0), (u_1, 0))$	19
(70)	$((12, 0), (0, 0))$	22
(71)	$((0, 0), (u_1, u_2))$	20
(72)	$((0, 0), (u_1, 0))$	24
(73)	$((0, 0), (0, 0))$	30

Next we consider the triplets $(SL(4)\times GL(m), \Lambda_2\otimes\Lambda_1, V(6)\otimes V(m))$ $(m=3, 4)$. Let $V(6)$ be the vector space spanned by 2-forms $u_i\wedge u_j$ $(1\leqq i<j\leqq 4)$. Then the representation space is identified with $V=\overbrace{V(6)\oplus\cdots\oplus V(6)}^{m}$ and the action $\rho=\Lambda_2\otimes\Lambda_1$ on V is given by $\rho(g)\tilde{x}=(\rho_2(g_1)X_1, \cdots, \rho_2(g_1)X_m)^t g_2$ for $g=(g_1, g_2)\in SL(4)\times GL(m)$, $\tilde{x}=(X_1, \cdots, X_m)\in V$, $(m=3, 4)$.

Proposition 2.7.
(I) *The triplet* $(SL(4)\times GL(3), \Lambda_2\otimes\Lambda_1, V(6)\otimes V(3))$ *has the following eleven orbits.*

	Representative Points	*Codimension*
(1)	$(u_1\wedge u_3+u_2\wedge u_4, u_1\wedge u_2, u_3\wedge u_4)$	0
(2)	$(u_1\wedge u_3, u_2\wedge u_4, u_1\wedge u_2)$	1
(3)	$(u_1\wedge u_4+u_2\wedge u_3, u_1\wedge u_2, u_1\wedge u_3)$	3
(4)	$(u_1\wedge u_2, u_3\wedge u_4, 0)$	4
(5)	$(u_1\wedge u_3+u_2\wedge u_4, u_1\wedge u_2, 0)$	5
(6)	$(u_1\wedge u_2, u_1\wedge u_3, u_1\wedge u_4)$	6
(7)	$(u_1\wedge u_2, u_1\wedge u_3, u_2\wedge u_3)$	6

(8)	$(u_1\wedge u_2, u_1\wedge u_3, 0)$	7
(9)	$(u_1\wedge u_2+u_3\wedge u_4, 0, 0)$	10
(10)	$(u_1\wedge u_2, 0, 0)$	11
(11)	$(0, 0, 0)$	18

(II) *The triplet* $(SL(4)\times GL(4), \Lambda_2\otimes\Lambda_1, V(6)\otimes V(4))$ *has the following fourteen orbits.*

	Representative Points	*Codimension*
(1)	$(u_1\wedge u_3, u_1\wedge u_4, -u_2\wedge u_4, u_2\wedge u_3)$	0
(2)	$(u_1\wedge u_3-u_2\wedge u_4, u_1\wedge u_4, u_2\wedge u_3, u_1\wedge u_2)$	1
(3)	$(u_2\wedge u_3, u_1\wedge u_4, u_1\wedge u_2, u_1\wedge u_3)$	3
(4)	$(u_1\wedge u_3+u_2\wedge u_4, u_1\wedge u_2, u_3\wedge u_4, 0)$	3
(5)	$(u_1\wedge u_3, u_2\wedge u_4, u_1\wedge u_2, 0)$	4
(6)	$(u_1\wedge u_4+u_2\wedge u_3, u_1\wedge u_2, u_1\wedge u_3, 0)$	6
(7)	$(u_1\wedge u_2, u_3\wedge u_4, 0, 0)$	8
(8)	$(u_1\wedge u_2, u_1\wedge u_3, u_1\wedge u_4, 0)$	9
(9)	$(u_1\wedge u_2, u_1\wedge u_3, u_2\wedge u_3, 0)$	9
(10)	$(u_1\wedge u_3+u_2\wedge u_4, u_1\wedge u_2, 0, 0)$	9
(11)	$(u_1\wedge u_2, u_1\wedge u_3, 0, 0)$	11
(12)	$(u_1\wedge u_2+u_3\wedge u_4, 0, 0, 0)$	15
(13)	$(u_1\wedge u_2, 0, 0, 0)$	16
(14)	$(0, 0, 0, 0)$	24

Next we consider the triplets $(GL(1)^2\times SL(4)\times SL(m), \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1, V(6)\otimes V(m)+V(4))$ $(m=3, 4)$. Let $V(4)$ be the 4-dimensional vector space spanned by u_1, u_2, u_3, u_4. Then the representation space is identified with $V=(\overbrace{V(6)\oplus\cdots\oplus V(6)}^{m})\oplus V(4)$ and the action $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1$ on V is given by $\rho(g)\tilde{x}=((\alpha\rho_2(g_1)X_1, \cdots, \alpha\rho_2(g_1)X_m)^t g_2, \beta\rho_1(g_1)Y)$ for $g=(\alpha, \beta; g_1, g_2)\in GL(1)^2\times SL(4)\times SL(m)$, $\tilde{x}=((X_1, \cdots, X_m), Y)\in V$, $(m=3, 4)$.

Proposition 2.8.
(I) *The triplet* $(GL(1)^2\times SL(4)\times SL(3), \Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1, V(6)\otimes V(3)+V(4))$ *has the following thirty-five orbits. Here ij stands for* $u_i\wedge u_j$.

	Representative Points	*Codimension*
(1)	$((13+24, 12, 34), u_1+u_3)$	0
(2)	$((13+24, 12, 34), u_1)$	1
(3)	$((13, 24, 12), u_3+u_4)$	1

(4)	$((13, 24, 12), u_3)$	2
(5)	$((13, 24, 12), u_1+u_2)$	3
(6)	$((14+23, 12, 13), u_4)$	3
(7)	$((13+24, 12, 34), 0)$	4
(8)	$((13, 24, 12), u_1)$	4
(9)	$((14+23, 12, 13), u_2)$	4
(10)	$((12, 34, 0), u_1+u_3)$	4
(11)	$((13, 24, 12), 0)$	5
(12)	$((13+24, 12, 0), u_3)$	5
(13)	$((14+23, 12, 13), u_1)$	6
(14)	$((12, 13, 23), u_4)$	6
(15)	$((12, 34, 0), u_1)$	6
(16)	$((12, 13, 14), u_2)$	6
(17)	$((14+23, 12, 13), 0)$	7
(18)	$((12, 13, 23), u_1)$	7
(19)	$((13+24, 12, 0), u_1)$	7
(20)	$((12, 13, 0), u_4)$	7
(21)	$((12, 34, 0), 0)$	8
(22)	$((12, 13, 0), u_2)$	8
(23)	$((13+24, 12, 0), 0)$	9
(24)	$((12, 13, 14)), u_1)$	9
(25)	$((12, 13, 14), 0)$	10
(26)	$((12, 13, 23), 0)$	10
(27)	$((12, 13, 0), u_1)$	10
(28)	$((12+34, 0, 0), u_1)$	10
(29)	$((12, 13, 0), 0)$	11
(30)	$((12, 0, 0), u_3)$	11
(31)	$((12, 0, 0), u_1)$	13
(32)	$((12+34, 0, 0), 0)$	14
(33)	$((12, 0, 0), 0)$	15
(34)	$((0, 0, 0), u_1)$	18
(35)	$((0, 0, 0), 0)$	22

(II) *The triplet* $(GL(1)^2 \times SL(4) \times SL(4), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1, V(6) \otimes V(4) + V(4))$ *has the following forty-five orbits. Here ij stands for* $u_i \wedge u_j$.

	Representative Points	*Codimension*
(1)	$((13, 14, -24, 23), u_1+u_3)$	0
(2)	$((13, 14, -24, 23), u_1)$	2

(3)	$((13, 14, -24, 23), 0)$	4
(4)	$((13-24, 14, 23, 12), u_3)$	1
(5)	$((13-24, 14, 23, 12), u_1)$	3
(6)	$((13-24, 14, 23, 12), 0)$	5
(7)	$((23, 14, 12, 13), u_4)$	3
(8)	$((23, 14, 12, 13), u_2)$	4
(9)	$((23, 14, 12, 13), u_1)$	6
(10)	$((23, 14, 12, 13), 0)$	7
(11)	$((13+24, 12, 34, 0), u_1+u_3)$	3
(12)	$((13+24, 12, 34, 0), u_1)$	4
(13)	$((13+24, 12, 34, 0), 0)$	7
(14)	$((13, 24, 12, 0), u_3+u_4)$	4
(15)	$((13, 24, 12, 0), u_3)$	5
(16)	$((13, 24, 12, 0), u_1+u_2)$	6
(17)	$((13, 24, 12, 0), u_1)$	7
(18)	$((13, 24, 12, 0), 0)$	8
(19)	$((14+23, 12, 13, 0), u_4)$	6
(20)	$((14+23, 12, 13, 0), u_2)$	7
(21)	$((14+23, 12, 13, 0), u_1)$	9
(22)	$((14+23, 12, 13, 0), 0)$	10
(23)	$((12, 34, 0, 0), u_1+u_3)$	8
(24)	$((12, 34, 0, 0), u_1)$	10
(25)	$((12, 34, 0, 0), 0)$	12
(26)	$((12, 13, 14, 0), u_2)$	9
(27)	$((12, 13, 14, 0), u_1)$	12
(28)	$((12, 13, 14, 0), 0)$	13
(29)	$((12, 13, 23, 0), u_4)$	9
(30)	$((12, 13, 23, 0), u_1)$	10
(31)	$((12, 13, 23, 0), 0)$	13
(32)	$((13+24, 12, 0, 0), u_3)$	9
(33)	$((13+24, 12, 0, 0), u_1)$	11
(34)	$((13+24, 12, 0, 0), 0)$	13
(35)	$((12, 13, 0, 0), u_4)$	11
(36)	$((12, 13, 0, 0), u_2)$	12
(37)	$((12, 13, 0, 0), u_1)$	14
(38)	$((12, 13, 0, 0), 0)$	15
(39)	$((12+34, 0, 0, 0), u_1)$	15
(40)	$((12+34, 0, 0, 0), 0)$	19

(41)	$((12, 0, 0, 0), u_3)$	16
(42)	$((12, 0, 0, 0), u_1)$	18
(43)	$((12, 0, 0, 0), 0)$	20
(44)	$((0, 0, 0, 0), u_1)$	24
(45)	$((0, 0, 0, 0), 0)$	28

Next we consider the triplet $(GL(1)^2\times SL(7)\times SL(2), \Lambda_3\otimes 1+\Lambda_1\otimes\Lambda_1, V(35)+V(7)\otimes V(2))$. Let $V(7)$ be the 7-dimensional vector space spanned by $u_1, \cdots, u_7$ and let $V(35)$ be the vector space spanned by skew-tensors $u_i\wedge u_j\wedge u_k$ $(1\leqq i<j<k\leqq 7)$. Then the action $\rho_3=\Lambda_3$ of SL(7) on $V(35)$ is given by $\rho_3(g)\,(u_i\wedge u_j\wedge u_k)=\rho_1(g)u_i\wedge\rho_1(g)u_j\wedge\rho_1(g)u_k$. The representation space $V(35)+V(7)\otimes V(2)$ is identified with $V(35)\oplus(V(7)\oplus V(7))$ and the action $\rho=\Lambda_3\otimes 1+\Lambda_1\otimes\Lambda_1$ is given by $\rho(g)\tilde{x}=(\alpha\rho_3(g_1)X, (\beta\rho_1(g_1)Y_1, \beta\rho_1(g_1)Y_2)^t g_2)$ for $g=(\alpha, \beta; g_1, g_2)\in GL(1)^2\times SL(7)\times SL(2)$, $\tilde{x}=(X, (Y_1, Y_2))\in V=V(35)\oplus(V(7)\oplus V(7))$.

Proposition 2.9. *The triplet* $(GL(1)^2\times SL(7)\times SL(2), \Lambda_3\otimes 1+\Lambda_1\otimes\Lambda_1, V(35)+V(7)\otimes V(2))$ *has the following* 102 *orbits. Here* ijk *stands for* $u_i\wedge u_j\wedge u_k$.

	Representative Points	*Codimension*
(1)	$(234+567+1(25+36+47), (u_2, u_5))$	0
(2)	$(234+567+1(25+36+47), (u_2, u_3+u_6))$	1
(3)	$(234+567+1(25+36+47), (u_2, u_3))$	3
(4)	$(234+567+1(25+36+47), (u_1, u_5))$	3
(5)	$(234+567+1(25+36+47), (u_2, u_6))$	5
(6)	$(234+567+1(25+36+47), (u_1, 0))$	6
(7)	$(234+567+1(25+36+47), (u_2, 0))$	7
(8)	$(234+567+1(25+36+47), (0, 0)$	14
(9)	$(235+346+1(27-45), (u_6, u_7))$	1
(10)	$(235+346+1(27-45), (u_5, u_1+u_6))$	2
(11)	$(235+346+1(27-45), (u_5, u_1+u_4))$	3
(12)	$(235+346+1(27-45), (u_5, u_6))$	4
(13)	$(235+346+1(27-45), (u_5, u_1+u_3))$	4
(14)	$(235+346+1(27-45), (u_6, u_1+u_4))$	4
(15)	$(235+346+1(27-45), (u_5, u_1))$	5
(16)	$(235+346+1(27-45), (u_6, u_3))$	5
(17)	$(235+346+1(27-45), (u_6, u_1))$	6
(18)	$(235+346+1(27-45), (u_1, u_4))$	7

(19)	$(235+346+1(27-45), (u_5, 0))$	7
(20)	$(235+346+1(27-45), (u_1+u_4, u_2))$	8
(21)	$(235+346+1(27-45), (u_6, 0))$	8
(22)	$(235+346+1(27-45), (u_1, u_2))$	10
(23)	$(235+346+1(27-45), (u_1, u_3))$	10
(24)	$(235+346+1(27-45), (u_1+u_4, 0)$	10
(25)	$(235+346+1(27-45), (u_1, 0))$	11
(26)	$(235+346+1(27-45), (0, 0))$	15
(27)	$(134+256+127, (u_7, u_3+u_5))$	4
(28)	$(134+256+127, (u_7, u_2+u_3))$	6
(29)	$(134+256+127, (u_3+u_5, u_4+u_6))$	6
(30)	$(134+256+127, (u_7, u_3))$	7
(31)	$(134+256+127, (u_3+u_5, u_4))$	7
(32)	$(134+256+127, (u_7, u_1+u_2))$	8
(33)	$(134+256+127, (u_5, u_3))$	8
(34)	$(134+256+127, (u_7, u_1))$	9
(35)	$(134+257+127, (u_3+u_5, u_1+u_2))$	9
(36)	$(134+256+127, (u_3+u_5, u_1))$	10
(37)	$(134+256+127, (u_3, u_4))$	10
(38)	$(134+256+127, (u_7, 0))$	10
(39)	$(134+256+127, (u_3, u_1+u_2))$	11
(40)	$(134+256+127, (u_3+u_5, 0))$	11
(41)	$(134+256+127, (u_3, u_1))$	12
(42)	$(134+256+127, (u_3, u_2))$	12
(43)	$(134+256+127, (u_3, 0))$	13
(44)	$(134+256+127, (u_1+u_2))$	14
(45)	$(134+256+127, (u_1+u_2, 0))$	15
(46)	$(134+256+127, (u_1, 0))$	16
(47)	$(134+256+127, (0, 0))$	18
(48)	$(234+1(25+36+47), (u_5, u_6))$	7
(49)	$(234+1(25+36+47), (u_5, u_2))$	9
(50)	$(234+1(25+36+47), (u_5, u_3))$	10
(51)	$(234+1(25+36+47), (u_5, u_1))$	12
(52)	$(234+1(25+36+47), (u_2, u_3))$	13
(53)	$(234+1(25+36+47), (u_5, 0))$	13
(54)	$(234+1(25+36+47), (u_2, u_1))$	15
(55)	$(234+1(25+36+47), (u_2, 0))$	16
(56)	$(234+1(25+36+47), (u_1, 0))$	19

(57)	$(234+1(25+36+47), (0, 0))$	21
(58)	$(123+456, (u_7, u_1+u_4))$	9
(59)	$(123+456, (u_1+u_4, u_2+u_5))$	11
(60)	$(123+456, (u_7, u_1))$	12
(61)	$(123+456, (u_1+u_4, u_2))$	13
(62)	$(123+456, (u_4, u_1))$	15
(63)	$(123+456, (u_7, 0))$	15
(64)	$(123+456, (u_1+u_4, 0))$	16
(65)	$(123+456, (u_1, u_2))$	17
(66)	$(123+456, (u_1, 0))$	19
(67)	$(123+456, (0, 0))$	23
(68)	$(126-135+234, (u_7, u_4))$	10
(69)	$(126-135+234, (u_4, u_5))$	12
(70)	$(126-135+234, (u_7, u_1))$	13
(71)	$(126-135+234, (u_4, u_2))$	14
(72)	$(126-135+234, (u_4, u_1))$	16
(73)	$(126-135+234, (u_7, 0))$	16
(74)	$(126-135+234, (u_4, 0))$	17
(75)	$(126-135+234, (u_1, u_2))$	18
(76)	$(126-135+234, (u_1, 0))$	20
(77)	$(126-135+234, (0, 0))$	24
(78)	$(1(25+36+47), (u_2, u_5))$	14
(79)	$(1(25+36+47), (u_2, u_3))$	15
(80)	$(1(25+36+47), (u_2, u_1))$	19
(81)	$(1(25+36+47), (u_2, 0))$	20
(82)	$(1(25+36+47), (u_1, 0))$	26
(83)	$(1(25+36+47), (0, 0))$	28
(84)	$(1(24+35), (u_6, u_7))$	15
(85)	$(1(24+35), (u_6, u_2))$	16
(86)	$(1(24+35), (u_2, u_4))$	19
(87)	$(1(24+35), (u_6, u_1))$	20
(88)	$(1(24+35), (u_2, u_3))$	20
(89)	$(1(24+35), (u_6, 0))$	21
(90)	$(1(24+35), (u_2, u_1))$	22
(91)	$(1(24+35), (u_2, 0))$	23
(92)	$(1(24+35), (u_1, 0))$	27
(93)	$(1(24+35), (0, 0))$	29
(94)	$(123, (u_4, u_5))$	22

(95)	$(123, (u_4, u_1))$	25
(96)	$(123, (u_4, 0))$	28
(97)	$(123, (u_1, u_2))$	30
(98)	$(123, (u_1, 0))$	32
(99)	$(123, (0, 0))$	36
(100)	$(0, (u_1, u_2))$	35
(101)	$(0, (u_1, 0))$	41
(102)	$(0, (0, 0))$	49

§ 3.

In this section, we shall deal with the simple case. Define an element e_i of $V(n)=\mathbf{C}^n$ by $e_i={}^t(0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$ for $i=1, \cdots, n$.

Now consider the triplet $(\mathrm{GL}(1)^2\times \mathrm{SL}(n),\ 2\Lambda_1\oplus\Lambda_1^*,\ V(\frac{1}{2}n(n+1))\oplus V(n)^*)$. We have

$$V(\tfrac{1}{2}n(n+1))=\{X\in M(n);\ {}^tX=X\} \quad \text{and} \quad (X, x)\longmapsto(\alpha AX{}^tA,\ \beta{}^tA^{-1}x)$$

for $(X, x)\in V$, and $(\alpha, \beta; A)\in \mathrm{GL}(1)^2\times\mathrm{SL}(n)$. Hence, $r_1=$ rank X, $r_2=$ rank x, $r_3=$rank Xx, $r_4=$rank ${}^txXx$ are invariants. Put $I_k'=\begin{pmatrix} I_k & 0 \\ 0 & 0\end{pmatrix}\in M(n)$ where I_k is the identity matrix of degree k.

Proposition 3.1. *The triplet* $(\mathrm{GL}(1)^2\times\mathrm{SL}(n),\ 2\Lambda_1\oplus\Lambda_1^*,\ V(\frac{1}{2}n(n+1))\oplus V(n)^*)$ *has* $4n$ *orbits.*

	Representatives	r_1	r_2	r_3	r_4	
(1)	(I_k', e_1)	k	1	1	1	$(1\leqq k\leqq n)$
(2)	$(I_k', e_1+\sqrt{-1}e_2)$	k	1	1	0	$(2\leqq k\leqq n)$
(3)	(I_k', e_n)	k	1	0	0	$(0\leqq k\leqq n-1)$
(4)	$(I_k', 0)$	k	0	0	0	$(0\leqq k\leqq n)$

Proof. The isotropy subalgebra $\mathfrak{g}_{I_k'}$ at I_k' is given by $\left\{\left(\begin{array}{c|c} A & C \\ \hline 0 & B\end{array}\right);\ A\in \mathrm{O}(k)\right\}$. Consider the action $\begin{pmatrix} x_1 \\ x_2\end{pmatrix}\mapsto\left(\begin{array}{c|c} A & 0 \\ \hline -{}^tC & -{}^tB\end{array}\right)\begin{pmatrix} x_1 \\ \hline x_2\end{pmatrix}$ where $x_1\in\mathbf{C}^k$ and $x_2\in\mathbf{C}^{n-k}$. If $k=0$, then $x=x_2=0$ or e_n, i.e., (4) (3). If $k=1$, then $x_1=0$ or 1. If $x_1=0$, then $x_2={}^t(0\cdots 01)$ or 0, i.e., (3) (4). If $x_1=1$, then $x_2=0$ by C, i.e. (1). If $k=n$, then $x=e_1$, $e_1+\sqrt{-1}e_2$, or 0 from the orbital decomposition of $(\mathrm{GO}(n), \Lambda_1)$ (See p. 168 in [9]), i.e., (1) (2) (4). If $2\leqq k\leqq n-1$, then $x_1=0$, $e_1+\sqrt{-1}e_2$ or e_1. If $x_1=0$, then $x_2={}^t(0\cdots 01)$ or 0, i.e., (3) (4). If $x_1 \neq 0$, then $x_2=0$ by C, i.e., (2) (1). Q.E.D.

Now consider the triplet $(GL(1)^2 \times SL(6),\ \Lambda_3 \oplus \Lambda_1,\ V(20) \oplus V(6))$. Let $e_1, \cdots, e_6$ be a basis of $V(6)$. Then we have $X=(x, y) \in V(20) \oplus V(6)$ with $x=\sum x_{ijk} e_i \wedge e_j \wedge e_k$ $(1 \leqq i<j<k \leqq 6)$ and $y=\sum y_l e_l$. Put $\hat{x}=\sum_i^6 z_i(\partial x/\partial e_i)$, $x \wedge \hat{x}=\sum_{i,j}^6 \psi_{ij}(x) z_i v_j$, $e_j \wedge v_j=\omega(=e_1 \wedge \cdots \wedge e_6)$ where z_i are indeterminants. Then we have $\psi(\rho(g)x)=(\det g) g \psi(x) g^{-1}$ for $g \in GL(6)$, $\rho=\Lambda_3$ where $\psi(x)=\psi_{ij}(x))$ is a 6×6 matrix (See [10]). Therefore, $r_1=\operatorname{rank} x$ (as $x \in M(20, 1)$), $r_2=\operatorname{rank} y$, $r_3=\operatorname{rank} \psi(x)$ and $r_4=\operatorname{rank} \psi(x) y$ are invariants. Since $x \wedge y$ determines an element of the space $(SL(6), \Lambda_4, V(15)) \simeq (SL(6), \Lambda_2^*, V(15)^*)$, there exists a 6×6 skew-symmetric matrix $(x \wedge y)^*$ such that $(\rho(g) x \wedge g y)^*={}^t g^{-1}(x \wedge y)^* g^{-1}$ for $g \in SL(6)$. Hence $r_5=\operatorname{rank}(x \wedge y)^*$ is also an invariant.

Proposition 3.2. *The triplet* $(GL(1)^2 \times SL(6),\ \Lambda_3 \oplus \Lambda_1,\ V(20) \oplus V(6))$ *has following fifteen orbits. Here ijk stands for* $e_i \wedge e_j \wedge e_k$.

	Representatives	r_1	r_2	r_3	r_4	r_5
(1)	$(123+456,\ e_1+e_4)$	1	1	6	1	4
(2)	$(123+456,\ e_1)$	1	1	6	1	2
(3)	$(123+456,\ 0)$	1	0	6	0	0
(4)	$(123+145+246,\ e_6)$	1	1	3	1	4
(5)	$(123+145+246,\ e_1)$	1	1	3	0	2
(6)	$(123+145+246,\ 0)$	1	0	3	0	0
(7)	$(123+145,\ e_6)$	1	1	1	1	4
(8)	$(123+145,\ e_2)$	1	1	1	0	2
(9)	$(123+145,\ e_1)$	1	1	1	0	0
(10)	$(123+145,\ 0)$	1	0	1	0	0
(11)	$(123,\ e_6)$	1	1	0	0	2
(12)	$(123,\ e_1)$	1	1	0	0	0
(13)	$(123,\ 0)$	1	0	0	0	0
(14)	$(0,\ e_1)$	0	1	0	0	0
(15)	$(0,0)$	0	0	0	0	0

Proof. It is well-known that $(GL(6), \Lambda_3, V(20))$ has five orbits represented by $x_1=123+456$, $x_2=123+145+246$, $x_3=123+145$, $x_4=123$ and $x_5=0$. One can check that

$$\psi(x_1)=\begin{pmatrix} I_3 & 0 \\ 0 & -I_3 \end{pmatrix},\quad \psi(x_2)=\begin{bmatrix} 0 & & & & & -2 \\ & & & & -2 & \\ & & & 0 & & \\ & & -2 & & & \\ & 0 & & & & \\ 0 & & & & & 0 \end{bmatrix},$$

$$\psi(x_3)=\begin{pmatrix} & & -2 \\ & .0 & \\ 0^{\cdot} & & \end{pmatrix},\ \psi(x_4)=\psi(x_5)=0.$$

If $x=x_1$, the isotropy subgroup at x_1 is given by $\left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B\end{array}\right),\ \left(\begin{array}{c|c} 0 & A \\ \hline B & 0\end{array}\right); A, B \in \mathrm{SL}(3)\right\}$ and hence $y\sim e_1+e_4$, e_1, 0, i.e., (1) (2) (3). If $x_2'=126+234-135$, the isotropy subalgebra at x_2' is given by (8.2) in p. 40 in [6], so $y\sim 0$, e_3, e_6. By the permutation $\begin{pmatrix}1 & 3 & 4\\ 4 & 1 & 3\end{pmatrix}$, if $x=x_2$, then $y\sim 0$, e_1, e_6, i.e., (3) (5) (4). If $x_3'=124+135$, the isotropy subalgebra at x_5' is given by (8.3) in p. 40 in [6]. Hence if $y_6\neq 0$ for $y={}^t(y_1,\cdots,y_6)$, we have $y_1=\cdots=y_5=0$, i.e., (7). If $y_6=0$, then Sp(2) acts on $y'={}^t(y_2,\cdots,y_5)$ and hence $y'={}^t(10\cdots 0)$ or 0. If $y'={}^t(10\cdots 0)$, then we may assume $y_1=0$, i.e., (8). If $y'=0$, then $y_1=1$ or 0, i.e., (9) (10). By the permutation (3, 4), we obtain the case when $x=x_3$. If $x=x_4$, then the isotropy subalgebra at x_4 is given by (8.4) in p. 41 in [6], we have $y'={}^t(0\ 0\ 1)$ or 0 where $y'={}^t(y_4, y_5, y_6)$. If $y'={}^t(0\ 0\ 1)$, then we have $y_1=y_2=y_3=0$ by C, i.e., (11). If $y'=0$, then $y\sim e_1$ or 0, i.e., (12) (13). If $x=0$, then $y\sim e_1$ or 0. Q.E.D.

Next consider the triplet $(\mathrm{GL}(1)^2\times\mathrm{SL}(7), \Lambda_3\oplus\Lambda_1^*, V(35)\oplus V(7)^*)$. We have $X=(x,y)\in V(35)\oplus V(7)^*$ where $x=\sum x_{ijk}e_i\wedge e_j\wedge e_k\ (1\leq i<j<k\leq 7)$ and $y=\sum_{l=1}^{7} y_l e_l$. Put $\hat{x}=\sum_l z(\partial x/\partial e_l)$ and $x\wedge\hat{x}\wedge\hat{x}=\sum_{i,j}\psi_{ij}(x)z_iz_j\omega$ where $\omega=e_1\wedge\cdots\wedge e_7$ and z_i are indeterminants. Then we have $\psi(\rho(g)x)=(\det g)g\psi(x){}^tg$ for $g\in\mathrm{GL}(7)$ and $\rho=\Lambda_3$ (See [10]). Since $(x, y)\mapsto(\rho(g)x, {}^tg^{-1}y)$, $r=\operatorname{rank}\psi(x)$, $r_1=\operatorname{rank}\psi(x)y$, $r_2=\operatorname{rank} y$, $r'=\operatorname{rank}{}^ty\psi(x)y$ and the dimension of the orbits are invariants. Here dim (resp. dim*) denotes the dimension of orbits for $\Lambda_3+\Lambda_1$ (resp. $\Lambda_3\oplus\Lambda_1^*$).

Proposition 3.3. *The triplet* $(\mathrm{GL}(1)^2\times\mathrm{SL}(7), \Lambda_3\oplus\Lambda_1^*, V(35)\oplus V(7)^*)$ *has following thirty-eight orbits.*

	Representatives	r	r_1	r_2	r'	dim	dim*
(1)	$(234+567+1(25+36+47), e_1)$	7	1	1	1	42	42
(2)	$(234+567+1(25+36+47), e_2)$	7	1	1	0	41	41
(3)	$(235+346+1(27-45), e_5)$	4	0	1	0	41	37
(4)	$(235+346+1(27-45), e_6)$	4	0	1	0	40	36
(5)	$(235+346+1(27-45), e_1+e_4)$	4	1	1	0	38	41
(6)	$(134+256+127, e_7)$	2	0	1	0	38	32
(7)	$(235+346+1(27-45), e_1)$	4	1	1	0	37	40

(8)	(134+256+127, e_3+e_5)	2	0	1	0	37	36
(9)	(234+567+1 (25+36+47), 0)	7	0	0	0	35	35
(10)	(134+256+127, e_3)	2	0	1	0	35	34
(11)	(234+1 (25+36+47), e_5)	1	0	1	0	35	31
(12)	(235+346+1 (27−45), 0)	4	0	0	0	34	34
(13)	(134+256+127, e_1+e_2)	2	1	1	0	33	38
(14)	(123+456, e_7)	0	0	1	0	33	27
(15)	(134+256+127, e_1)	2	1	1	0	32	37
(16)	(234+1 (25+36+47), e_2)	1	0	1	0	32	34
(17)	(123+456, e_1+e_4)	0	0	1	0	32	33
(18)	(126−135+234, e_7)	0	0	1	0	32	26
(19)	(126−135+234, e_4)	0	0	1	0	31	29
(20)	(134+256+127, 0)	2	0	0	0	31	31
(21)	(234+1 (25+36+47), e_1)	1	1	1	0	29	35
(22)	(123+456, e_1)	0	0	1	0	29	30
(23)	(234+1 (25+36+47), 0)	1	0	0	0	28	28
(24)	(126−135+234, e_1)	0	0	1	0	28	32
(25)	(1 (25+36+47), e_2)	1	0	1	0	28	27
(26)	(1 (24+35), e_6)	0	0	1	0	27	22
(27)	(123−456, 0)	0	0	0	0	26	26
(28)	(126−135+234, 0)	0	0	0	0	25	25
(29)	(1 (24+35), e_2)	0	0	1	0	25	26
(30)	(1 (25+36+47), e_1)	1	1	1	0	22	28
(31)	(1 (25+36+47), 0)	1	0	0	0	21	21
(32)	(1 (24+35), e_1)	0	0	1	0	21	27
(33)	(1 (24+35), 0)	0	0	0	0	20	20
(34)	(123, e_4)	0	0	1	0	20	17
(35)	(123, e_1)	0	0	1	0	16	20
(36)	(123, 0)	0	0	0	0	13	13
(37)	(0, e_1)	0	0	1	0	7	7
(38)	(0, 0)	0	0	0	0	0	0

Proof. By Proposition 1.4 in [7], it is equivalent to classify $(GL(1)^2 \times SL(7), \Lambda_3 \oplus \Lambda_1, V(35) \oplus V(7))$. The triplet $(GL(7), \Lambda_3, V(35))$ has ten orbits represented by $x_1=234+567+1(25+36+47)$, $x_2=235+346+1(27-45)$, $x_3=134+256+127$, $x_4=234+1\,(25+36+47)$, $x_5=123+456$, $x_6=126-135+234$, $x_7=1\,(25+36+47)$, $x_8=1(24+35)$, $x_9=123$, and $x_{10}=0$. [I] The case for $x=x_1$. The isotropy subalgebra at x_1 is $\mathfrak{g}_2$ (See p. 20 in [8]) and $(GL(1)\times G_2, \Lambda_2, V(7))$ has thre eorbits. Hence $y\sim e_1, e_2, 0$, i.e.,(1)(2)(9).

[II] The case for $x=x_2$. The isotropy subalgebra at x_2 is given by (10.2), p. 47 in [6]. Since GO(3) acts on $y'={}^t(y_5y_6y_7)$, we have $y'^t\sim(1\ 0\ 0)$, ${}^t(0\ 1\ 0)$ or ${}^t(0\ 0\ 0)$. If $y'\neq 0$, we have $y_1=\cdots=y_4=0$ by γ_i's and $y\sim e_5$, e_6, i.e., (3) (4). If $y'=0$, then $GL(1)\times SL(2)\times SL(2)$ acts on $y''={}^t(y_1,\cdots,y_4)$ and hence $y''\sim e_1+e_4$, e_1, 0, i.e., (5) (7) (12). [III] The case for $x=x_3$. The isotropy subalgebra at x_3 is given by (10.3) p. 47 in [6]. If $y_7\neq 0$, then we have $y\sim e_7$. Assume $y_7=0$. The group $GL(2)\times GL(2)$ acts on $y'={}^t(x_3x_4x_5x_6)$ by $\Lambda_1\otimes 1+1\otimes\Lambda_1$, so we have $y'\sim{}^t(1\ 0\ 1\ 0)$, ${}^t(1\ 0\ 0\ 0)$, ${}^t(0\ 0\ 1\ 0)$, $(0\ 0\ 0\ 0)$. If $y'\neq 0$, we may assume $y_1=y_2=0$, and hence $y\sim e_3+e_5$, e_3, e_5. If $y'=0$, then $(y_1, y_2)=(1\ 1)$, $(1\ 0)$, $(0, 1)$ or $(0, 0)$, i.e., $y\sim e_1+e_2$, e_2, e_1, 0. However, by the transformation $e_1\leftrightarrow e_2$, $e_3\leftrightarrow e_5$, $e_4\leftrightarrow e_6$, $e_7\leftrightarrow -e_7$, we have $(x_3, e_1)\sim(x_3, e_2)$ and $(x_3, e_3)\sim(x_3, e_5)$, and hence $y\sim e_7$, e_1+e_2, e_1, e_3+e_5, e_3, 0, i.e., (6) (13) (15) (8) (10) (20). [IV] The case for $x=x_4$. The isotropy subalgebra at x_4 is given by (10.4), p. 48 in [6]. Hence $y'={}^t(y_5y_6y_7)\sim{}^t(1\ 0\ 0)$ or 0. If $y'\neq 0$, then we may assume that $y_1=\cdots=y_4=0$, i.e., $y\sim e_5$. If $y'=0$, then $y''={}^t(y_2y_3y_4)\sim{}^t(1\ 0\ 0)$ or 0. If $y''\neq 0$, we may assume $y_1=0$, i.e., $y\sim e_2$. If $y''=0$, then $y_1=1$ or 0, i.e., $y\sim e_1$ or 0. Hence $y\sim e_5$, e_2, e_1, 0, i.e., (11) (16) (21) (23). [V] The case for $x=x_5$. The isotropy subalgebra at x_5 is given by (10.6), p. 49 in [6]. If $y_7\neq 0$, we have $y_1=\cdots=y_6=0$ by Z and W, i.e., $y\sim e_7$. If $y_7=0$, $SL(3)\times SL(3)$ acts on $y'={}^t(y_1, \cdots, y_6)$ by $\Lambda_1\otimes 1+1\otimes\Lambda_1$, and $y'\sim e_1+e_4$, e_1, e_4, 0. By the transform $1\leftrightarrow 4$, $2\leftrightarrow 5$, $3\leftrightarrow 6$, we have $(x_5, e_1)\sim(x_5, e_4)$, and hence $y\sim e_7$, e_1+e_4, e_1, 0, i.e., (14) (17) (22) (27). [VI] The case for $x=x_6$. The isotropy subalgebra at x_6 is given by (10.7), p. 49 in [6]. If $y_7\neq 0$, then $y\sim e_7$. If $y_7=0$, then $y'={}^t(y_4y_5y_6)\sim{}^t(1\ 0\ 0)$ or 0. If $y'\neq 0$, we have $y_1=y_2=y_3=0$ by B i.e., $y\sim e_4$. If $y'=0$, $y''={}^t(y_1y_2y_3)\sim{}^t(1\ 0\ 0)$ or 0, i.e., $y\sim e_1$ or 0. Hence $y\sim e_7$, e_4, e_1, 0, i.e., (18) (19) (24) (28). [VII] The case for $x=x_7$. The isotropy subalgebra at x_7 is given by (10.9), p. 50 in [6]. Then we have $y'={}^t(y_2, \cdots, y_7)\sim{}^t(1\ 0\cdots 0)$ or 0. If $y'\neq 0$, then $y_1=0$ by Y, i.e., $y\sim e_2$. If $y'=0$, then $y_1=1$ or 0, i.e., $y\sim e_1$, 0. Thus we obtain (25) (30) (31). [VIII] The case for $x=x_8$. The isotropy subalgebra at x_8 is given by (10.10), p. 50 in [6]. We have $y'={}^t(y_6, y_7)\sim{}^t(1, 0)$ or 0. If $y'\neq 0$, then, by U, Z, $y_1=\cdots=y_5=0$, i.e., $y\sim e_6$. If $y'=0$, then $y''={}^t(y_2, \cdots, y_5)\sim{}^t(1\ 0\ 0\ 0)$ or 0. If $y''\neq 0$, then $y_1=0$ by W, i.e., $y\sim e_2$. If $y''=0$, then $y_1=1$ or 0, i.e., $y\sim e_1$, 0. Hence we have $y\sim e_6$, e_2, e_1, 0, i.e., (26) (29) (32) (33). [IX] The case for $x=x_9$. The isotropy subalgebra at x_9 is given by (10.12) ,p. 51 in [6]. We have $y'=(y_4, \cdots, y_7)\sim{}^t(1\ 0\ 0\ 0)$ or 0. If $y'\neq 0$, then $y_1=y_2=y_3=0$ by Z, i.e., $y\sim e_4$. If $y'=0$, then $y\sim e_1$, 0. Thus we have (34) (35) (36). [X] The case for $x=x_{10}=0$. We have $y\sim e_1$, 0, i.e., (37) (38). Q.E.D.

Consider the triplet $(GL(1)^2\times Sp(3), \Lambda_3\oplus\Lambda_1, V(14)\oplus V(6))$. If we restrict the triplet (GL(6), Λ_3, $V(20)$) to Sp(3), then we have $V(20)=V(14)\oplus V(6)$ as a representation space of Sp(3), and $V(14)$ is spanned by $\omega_1=123$, $\omega_2=456$, $\omega_3=234$, $\omega_4=156$, $\omega_5=135$, $\omega_6=246$, $\omega_7=126$, $\omega_8=345$, $\omega_9=125-136$, $\omega_{10}=245-346$, $\omega_{11}=124+236$, $\omega_{12}=145+356$, $\omega_{13}=134-235$, and $\omega_{14}=146-256$ where ijk stands for $e_i\wedge e_j\wedge e_k$. The triplet (GL(1) $\times$Sp(3), Λ_3, $V(14)$) has five orbits represented by $x_1=123+456$, $x_2=126+(134-235)$, $x_3=134-235$, $x_4=123$, $x_5=0$ (See [2]).

Proposition 3.4. *The triplet* $(\mathrm{GL}(1)^2\times\mathrm{Sp}(3)$, $\Lambda_3\oplus\Lambda_1$, $V(14)\oplus V(6)$ *has following nineteen orbits.*

	Representatives	dim
(1)	$(123+456, e_1+e_4)$	20
(2)	$(123+456, e_1+e_5)$	19
(3)	$(126+(134-235), e_6)$	19
(4)	$(126+(134-235), e_5)$	18
(5)	$(123+456, e_1)$	17
(6)	$(126+(134-235), e_3)$	16
(7)	$(134-235), e_6)$	16
(8)	$(126+(134-235), e_1)$	15
(9)	$((134-235), e_1+e_2)$	15
(10)	$(123+456, 0)$	14
(11)	$(126+(134-235), 0)$	13
(12)	$((134-235), e_1)$	13
(13)	$(123, e_4)$	13
(14)	$((134-235), e_3)$	11
(15)	$((134-235), 0)$	10
(16)	$(123, e_1)$	10
(17)	$(123, 0)$	7
(18)	$(0, e_1)$	6
(19)	$(0, 0)$	0

Proof. [I] The case for $x=x_1$ with $(x, y)\in V(14)\oplus V(6)$. The isotropy subalgebra at x_1 is given by (9.2), p. 43 in [6]. Hence $y\sim e_1+e_4$, e_1+e_5, e_1, e_4, 0. However $(x_1, e_4)\sim(-123+456, e_1)$ by $\left(\begin{array}{c|c} 0 & I_3 \\ \hline -I_3 & 0 \end{array}\right)\in\mathrm{Sp}(3)$,

and by $\begin{pmatrix} I_2 & & & \\ & \sqrt{-1} & & \\ & & I_2 & \\ & & & -\sqrt{-1} \end{pmatrix}\in\mathrm{Sp}(3)$, $\sim(-\sqrt{-1}\,123-\sqrt{-1}\,456, e_1)$ $\sim(x_1, e_1)$ by $\sqrt{-1}\in\mathrm{GL}(1)$. Hence $y\sim e_1+e_4$, e_1+e_5, e_1, 0, i.e., (1) (2) (5) (10). [II] The case for $x=x_2$. The isotropy subalgebra at x_2 is given by (9.3), p. 43 in [6]. Hence GO(3) acts on $y'={}^t(y_4y_5y_6)$, and $y'\sim{}^t(0\ 0\ 1)$, ${}^t(0\ 1\ 0)$, 0. If $y'\neq 0$, then $y_1=y_2=y_3=0$ by b_{ij} and b_i, i.e., $y\sim e_6$, e_5. If $y'=0$, then GO(3) acts on $y''={}^t(y_1y_2y_3)$ and hence $y\sim e_3$, e_1, 0. Thus we obtain (3) (4) (6) (8) and (11). [III] The case for x_3. The isotropy subalgebra at x_3 is given by (9.4), p. 44 in [6], and hence if $y_6\neq 0$, then $y_1=\cdots=y_5=0$ by $\alpha\beta\gamma\delta\varepsilon$, i.e., $y\sim e_6$. Assume $y_6=0$. If $y'={}^t(y_1y_2y_4y_5)=0$, then $y_3=1$ or 0, i.e., $y\sim e_3$, 0. If $y'\neq 0$, then we may assume $y_3=0$. In this case, $SL(2)\times SL(2)$ acts on y' by $\Lambda_1\otimes 1+1\otimes\Lambda_1$, so $y'\sim{}^t(1\ 1\ 0\ 0)$, ${}^t(1\ 0\ 0\ 0)$, ${}^t(0\ 1\ 0\ 0)$. We have $(x_3, e_1)\sim(x_3, e_2)$ by

$$A=\begin{pmatrix} J' & 0\\ 0 & J'\end{pmatrix}\in \mathrm{Sp}(3)\quad\text{with}\quad J'=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Thus we obtain (7) (15) (14) (9) (12). [IV] The case for $x=x_4$. The isotropy subalgebra at x_4 is given by (9.5), p. 45 in [6]. Hence $y'={}^t(y_4y_5y_6)\sim{}^t(1\ 0\ 0)$ or 0. If $y'\neq 0$, then $y_1=y_2=y_3=0$ by B, i.e., $y\sim e_4$. If $y'=0$, then $y''={}^t(y_1y_2y_3)\sim{}^t(1\ 0\ 0)$ or 0, i.e., $y\sim e_1$, 0. Thus $y\sim e_4$, e_1, 0, i.e. (13) (16) (17). [V] The case for $x=x_5=0$. Then we have $y\sim e_1$, 0, i.e., (18), (19). Q.E.D.

Consider the triplet ($GL(1)^2\times$Spin(10), half-spin rep. $\oplus$vector rep., $V(16)\oplus V(10)$). The triplet (Spin(10), half-spin rep., $V(16)$) has 3 orbits represented by $1+e_1e_2e_3e_4$, 1, 0 (See [2]). For $(x, y)\in V(16)\oplus V(10)$, let $g(y)$ be the quadratic invariant on $V(10)$. Then $r_1=\text{rank } x$, $r_2=\text{rank } y$, $r=\text{rank }(q(y))$ are invariants.

Proposition 3.5. *The triplet* ($GL(1)^2\times$Spin(10), *half-spin rep.*$\oplus$*vector rep.*, $V(16)\oplus(V(10))$ *has thirteen orbits.*

	Representatives	r	r_1	r_2	dim
(1)	$(1+e_1e_2e_3e_4,\ e_5+e_{10})$	1	1	1	26
(2)	$(1+e_1e_2e_3e_4,\ e_1+e_6)$	1	1	1	25
(3)	$(1+e_1e_2e_3e_4,\ e_5)$	0	1	1	25
(4)	$(1+e_1e_2e_3e_4,\ e_1)$	0	1	1	24
(5)	$(1,\ e_1+e_6)$	1	1	1	21
(6)	$(1,\ e_1)$	0	1	1	20
(7)	$(1+e_1e_2e_3e_4,\ e_{10})$	0	1	1	17
(8)	$(1+e_1e_2e_3e_4,\ 0)$	0	1	0	16
(9)	$(1,\ e_6)$	0	1	1	16
(10)	$(1,\ 0)$	0	1	0	11

(11)	$(0, e_1+e_6)$	1	0	1	10
(12)	$(0, e_1)$	0	0	1	9
(13)	$(0, 0)$	0	0	0	0

Proof. (I) The case for $x=1+e_1e_2e_3e_4$. The isotropy subalgebra at x is given by (5.39), p. 121 in [8]. Hence, if $y_5\neq 0$, then all $y_i=0$ except $i=5$, 10 by $a_{15}\sim a_{45}$ and $c_{15}\sim c_{45}$. Then $y_5=1$, $y_{10}=1$ or 0, i.e., (1) (3). If $y_5=0$, then Spin (7) acts on $y'={}^t(y_1\cdots y_4y_6\cdots y_9)$ by the spin representation, hence $y'\sim{}^t(1\ 0\ 0\ 0\ 1\ 0\ 0\ 0)$, ${}^t(1\ 0\cdots 0)$, 0. If $y'\neq 0$, then $y_{10}=0$ by c_{15}. If $y'=0$, then $y_{10}=1$ or 0. Hence $y\sim e_1+e_6$, e_1, e_{10}, 0, i.e., (2) (4) (7) (8). (II) The case for $x=1$. The isotropy subalgebra at x is $\left\{\left[\begin{array}{c|c} A & 0 \\ \hline C & -{}^tA \end{array}\right]\right\}$, hence GL(5) acts on $y'={}^t(y_1\cdots y_5)$. Thus $y'\sim{}^t(1\ 0\cdots 0)$ or 0. If $y'={}^t(1\ 0\cdots 0)$, then $y_7=\cdots=y_{10}=0$ by C and $y_6=1$ or 0. If $y'=0$, then $y''={}^t(y_6\cdots y_{10})\sim{}^t(1\ 0\cdots 0)$ or 0. Thus we have (5) (6) (9) (10). (III) If $x=0$, then $y\sim e_1+e_6$, e_1, 0, i.e., (11) (12) (13). Q.E.D.

Finally consider the triplet $(GL(1)^2\times \mathrm{Spin}\ (12)$, half-spin rep.$\oplus$vector rep., $V(32)\oplus V(12))$. For $x=(x, y)\in V(32)\oplus V(12)$, let $Q(x)$ is a relative invariant of degree 4 on $V(32)$ and $q(y)$ a quadratic invariant on $V(12)$. Then $R=\mathrm{rank}\,(Q(x))$ as 1×1 matrix, $r=\mathrm{rank}\ (q(y))$, $r_1=\mathrm{rank}\ x$ as an element of $M(32, 1)$, $r_2=\mathrm{rank}\ y$ are invariants.

Proposition 3.6. *The triplet* $(GL(1)^2\times \mathrm{Spin}(12)$, *half-spin rep.*$\oplus$*vector rep.*, $V(32)\oplus V(12))$ *has following twenty-one orbits.*

	Representatives	R	r	r_1	r_2	dim
(1)	$(1+e_1e_2e_3e_4e_5e_6,\ e_1+e_7)$	1	1	1	1	44
(2)	$(1+e_1e_2e_3e_4e_5e_6,\ e_1+e_8)$	1	0	1	1	43
(3)	$(1+e_2e_3e_5e_6+e_1e_3e_4e_6,\ e_1+e_7)$	0	1	1	1	43
(4)	$(1+e_2e_3e_5e_6+e_1e_3e_4e_6,\ e_1)$	0	0	1	1	42
(5)	$(1+e_1e_2e_3e_4e_5e_6,\ e_1)$	1	0	1	1	38
(6)	$(1+e_2e_3e_5e_6+e_1e_3e_4e_6,\ e_7)$	0	0	1	1	37
(7)	$(1+e_2e_3e_5e_6,\ e_1+e_7)$	0	1	1	1	37
(8)	$(1+e_2e_3e_5e_6,\ e_1)$	0	0	1	1	36
(9)	$(1+e_2e_3e_5e_6,\ e_2+e_8)$	0	1	1	1	35
(10)	$(1+e_2e_3e_5e_6,\ e_2)$	0	0	1	1	34
(11)	$(1+e_1e_2e_3e_4e_5e_6,\ 0)$	1	0	1	0	32
(12)	$(1+e_2e_3e_5e_6+e_1e_3e_4e_6,\ 0)$	0	0	1	0	31
(13)	$(1,\ e_1+e_7)$	0	1	1	1	28
(14)	$(1+e_2e_3e_5e_6,\ e_7)$	0	0	1	1	27

(15)	$(1, e_1)$	0	0	1	1	27
(16)	$(1+e_2e_3e_5e_6, 0)$	0	0	1	0	25
(17)	$(1, e_7)$	0	0	1	1	22
(18)	$(1, 0)$	0	0	1	0	16
(19)	$(0, e_1+e_7)$	0	1	0	1	12
(20)	$(0, e_1)$	0	0	0	1	11
(21)	$(0, 0)$	0	0	0	0	0

Proof. The triplet $(GL(1)\times Spin(12)$, half-spin rep., $V(32))$ has five orbits represented by $x_1=1+e_1e_2e_3e_4e_5e_6$, $x_2=1+e_2e_3e_5e_6+e_1e_3e_4e_6$, $x_3=1+e_2e_3e_5e_6$, $x_4=1$ and $x_5=0$ (See [2]). (I) The case for $x=x_1$. The isotropy subalgebra at x_1 is given by $\left\{\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix};\ A \in \mathfrak{sl}(6)\right\}$ and hence $y\sim e_1+e_7, e_1+e_8, e_1, e_7, 0$. The isotropy subgroup of Spin(12) at x_1 has two connected components ([2]) and, by this, we have $(x_1, e_7)\sim(x_1, -e_1)$. By GL(1), $(x_1, -e_1)\sim(x_1, e_1)$. Thus we obtain (1) (2) (5) (11). (II) The case for $x=x_2$. The isotropy subalgebra at x_2 is given by (5.3), p. 28 in [6]. Since Sp(3) acts on $y'={}^t(y_1y_2y_9y_4y_5y_{12})$, we have $y'\sim{}^t(1\,0\cdots0)$ or 0. If $y'={}^t(1\,0\cdots0)$, then, by $a_{31}, a_{61}, c_{12}, c_{14}, c_{15}$, we have $y''={}^t(y_7y_8y_3y_{10}y_{11}y_6)={}^t(1\,0\cdots0)$ or 0, i.e., $y\sim e_1+e_7, e_1$. If $y'=0$, then Sp(3) acts on y'' and hence $y''={}^t(1\,0\cdots0)$ or 0, i.e., $y\sim e_7, 0$. Thus we have (3) (4) (6) (12). (III) The case for $x=x_3$. The isotropy subalgebra at x_3 is given by (5.4), p. 29 in [6]. GL(2) acts on ${}^t(y_1y_4)$ and hence ${}^t(y_1y_4)\sim{}^t(1, 0)$ or 0. If ${}^t(y_1, y_4)={}^t(1, 0)$, we have $y_2=\cdots y_6=y_8=\cdots=y_{12}=0$, $y_7=1$ or 0, by $a_{21}\sim a_{61}$, $c_{12}\sim c_{16}$, i.e., $y\sim e_1+e_7, e_1$. Assume that $y_1=y_4=0$. Then Spin (7) acts on $y'={}^t(y_2y_3y_5y_6y_8y_9y_{11}y_{12})$ by the spin representation with scalar multiplication, we have $y'\sim{}^t(1\,0\,0\,0\,1\,0\,0\,0)$, ${}^t(1\,0\cdots0)$ or 0. If $y'\neq0$, then $y_7=y_{10}=0$ by c_{12} and c_{24}, i.e., $y\sim e_2+e_8, e_2$. If $y'=0$, then GL(2) acts on ${}^t(y_7, y_{10})$ and hence ${}^t(y_7, y_{10})\sim{}^t(1\ 0)$ or 0, i.e., $y\sim e_7, 0$. Thus we obtain (7) (8) (9) (10) (14) (16). (IV) The case for $x=x_4$. The isotropy subalgebra at x_4 is given by (5.5), p. 29 in [6]. Hence $y'={}^t(y_1\cdots y_6)\sim(1\,0\cdots0)$ or 0. If $y'={}^t(1\,0\cdots0)$, then $y''={}^t(y_7\cdots y_{12})\sim{}^t(1\,0\cdots0)$ or 0 by C, i.e., $y\sim e_1+e_7, e_1$. If $y'=0$, then $y''\sim(1\,0\cdots0)$ or 0, i.e., $y\sim e_7, 0$. Thus we obtain (13) (15) (17) (18). (V) The case for $x=x_5=0$. We have $y\sim e_1+e_7, e_1, 0$, i.e., (19) (20) (21). Q.E.D.

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