Advanced Studies in Pure Mathematics 5, 1985
Foliations
pp. 195–210

Characteristic Classes for Families of Foliations

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Introduction

In recent studies of foliations, the Gelfand-Fuks cohomology $H^*(a_q)$ (a_q: the Lie algebra of formal vector fields of q-variables) plays an important role as characteristic classes (cf. [3, 5] for example). For one parameter families of foliations, Gelfand, Feigin and Fuks ([8]) show that the derivatives of these classes, with respect to the parameter, are induced from a universal homomorphism var: $H^*(a_q) \to H^{*-1}(a_q; a_q')$ ($H^*(a_q; a_q')$: the continuous cohomology of a_q associated to the adjoint action on a_q' the dual space of a_q) by interpreting $H^*(a_q; a_q')$ as the characteristic classes of the first jets of the variations of foliations. From another point of view, an element of $H^*(a_q; a_q')$ can be considered also as a characteristic class of families of foliations on a manifold M with values in the $H^*(M, \boldsymbol{R})$-valued one form on the parameter spaces.

In this paper, we embed the homomorphism var into a complex $\{H^*(a_q; S^*a_q'), \text{var}\}$ which is related to the $H^*(M, \boldsymbol{R})$-valued de Rham complex on the parameter spaces of families of foliations on M via a characteristic homomorphism. This complex appears as the E_1-term of the spectral sequence $E(a_q)$ associated with the Weil algebra $W^*(a_q)$ with a natural filtration. The Weil algebras of infinite dimensional Lie algebras are treated in [11]. The E_2-term then gives us potential tools to investigate the homotopy structure of the space of foliations on a manifold, although it is a difficult problem to check whether these actually work or not.

The basic concepts we use in the construction of the characteristic homomorphisms are the variation bicomplex and the Vinogradov's spectral sequence associated to the differential equation of integrability condition for plane fields (cf. [14, 15]). We note that generally the E_1-term of the Vinogradov's spectral sequence of a differential equation on M is related to the $H^*(M, \boldsymbol{R})$-valued "de Rham complex" on the solution space (cf. [14]).

In Section 1, we recall some basic facts about Weil algebra of a_q. In Section 2, we construct the bundle $F_q^\infty(M) \to M$ of infinite jets of local

Received September 14, 1983.
* Partially supported by Sakkokai Foundation.

foliations of codimension q on M and show that its de Rham complex has a natural bicomplex structure, which is the variation bicomplex mentioned above. One of the two standard filtrations then induces the Vinogradov's spectral sequence. We give its interpretation in Section 2.5. In Section 3, we construct the characteristic homomorphisms and obtain the main Theorems (3.5.3), (3.5.6) and (3.7.3). We give some remarks in Section 4.

§ 1. The Weil algebra of a_q

In this section, we recall basic facts about the Weil algebra of the topological Lie algebra a_q of all the formal vector fields of q-variables. For more details, see [11].

1.1. Let a_q be the topological Lie algebra of all the formal vector fields of q variables endowed with the weakest topology for which all the maps, assigning each vector fields the partial derivatives of its coefficients, are continuous.

Let a'_q be the dual space of a_q. This is an a_q-module by the action $X.f(Y)=-f([X, Y])$ ($f \in a'_q$, $X, Y \in a_q$). This induces a_q-modules $S^k a'_q$, the k-th symmetric product of a'_q.

Let $C^k(a_q)=C^k(a_q; \boldsymbol{R})$ be the space of all the continuous k-multilinear anti-symmetric maps from $a_q \times \cdots \times a_q$ (k-times) to the field of real numbers $\boldsymbol{R}$ for positive k and $C^0(a_q)=\boldsymbol{R}$. Put $C^k(a_q; S^t a'_q)=C^k(a_q)\otimes S^t a'_q$ $(k, t \geqq 0)$.

Regard an element f of $C^k(a_q; S^t a'_q)$ as a multilinear anti-symmetric map from $a_q \times \cdots \times a_q$ (k times) to $S^t a'_q$ and define

$$df(X_1, \cdots, X_{k+1})=\sum(-1)^{i+1}X_i.f(X_1, \cdots, \check{X}_i, \cdots, X_{k+1})$$
$$+\sum(-1)^{i+j}f([X_i, X_j], \cdots, \check{X}_i, \cdots, \check{X}_j, \cdots)$$

($X_i \in a_q$) for positive k and by $df(X)=X.f$ for $f \in C^0=S^t a'_q$ ($X \in a'_q$). Then it is easy to see $df \in C^{k+1}(a_q; S^t a'_q)$. Thus we obtain a complex $\{C^*(a_q; S^t a'_q), d\}$ for each $t \geqq 0$.

On the other hand we can also regard in the obvious way an element f of $C^k(a_q; S^t a'_q)$ as a multilinear map from $a_q \times \cdots \times a_q$ ($k+t$ times) to $\boldsymbol{R}$, anti-symmetric in the first k arguments and symmetric in the last t arguments. Then for positive k we can define $\operatorname{var}(f) \in C^{k-1}(a_q; S^{t+1}a'_q)$ by

$$(\operatorname{var}(f))(X_1, \cdots, X_{k-1}; Y_1, \cdots, Y_{t+1})$$
$$=\sum_{i=1}^{t+1} f(Y_i, X_1, \cdots, X_{k-1}; Y_1, \cdots, \check{Y}_i, \cdots, Y_{t+1})$$

for X_i, $Y_j \in a_q$. Obviously $\mathrm{var} \circ \mathrm{var} = 0$. Moreover it is easy to show var commutes with d.

1.2. Put $W^{s,t}(a_q) = C^{t-s}(a_q; S^s a_q')$ for $t \geqq s$ and $=(0)$ for $t < s$. Then var and d induce homomorphisms δ and ∂ of degrees $(1, 0)$ and $(0, 1)$ respectively of the bigraded module $W^{*,*}(a_q) = \bigoplus W^{s,t}(a_q)$. Moreover $\delta^2 = \partial^2 = 0$, $\delta\partial = \partial\delta$. Thus we obtain a bicomplex $\{W^{*,*}(a_q), \delta, \partial\}$. Its total complex is denoted by $\{W^*(a_q), d\}$, which is *the Weil algebra of* a_q. We regard this as a filtered differential graded algebra by the filtration $F^i W^*(a_q) = \bigoplus_{i' \geqq i} W^{i',*}(a_q)$. The associated spectral sequence will be denoted by $E(a_q) = \{E_r^{s,t}(a_q), d_r\}$.

Remark (1.2.1). A differential graded algebra (d.g.a. for short) is an anti-commutative graded algebra $A = \bigoplus_{i \geqq 0} A^i$ with 1 endowed with an endomorphism d of degree one such that $d^2 = 0$, $d(ab) = (da)b + (-1)^k a db$ $(k = \deg a)$. A filtered d.g.a. is a d.g.a. A with a decreasing filtration $\{F^i\}$ with $F^0 = A$ and $F^i . F^j \subset F^{i+j}$. For a filtered d.g.a. (A, F), we denote by $E(A, F) = \{E_r^{s,t}(A, F), d_r\}$ the associated spectral sequence, which is obviously multiplicative and convergent.

Just as in the finite-dimensional case, we have

Theorem (1.2.2) ([11]). *The d.g.a. $W^*(a_q)$ is acyclic, i.e., $H^i(W^*(a_q)) \cong \boldsymbol{R}$ $(i=0)$, (0) $(i \neq 0)$.*

By definition we have

Proposition (1.2.3). $E_1^{s,t}(a_q) \cong H^{t-s}(a_q; S^s a_q')$ $(t \geqq s)$.

Remark (1.2.4). Note that our bidegree (and hence the filtration of $W^*(a_q)$) is different from the usual one (cf. [10] for example), which gives $W^{s,t}$ the bidegree $(2s, t-s)$.

1.3. Let (A, F) be a filtered d.g.a. Suppose there is a graded algebra map $g\colon C^*(a_q) \to A$ such that the composition g_0 of g and the natural projection $C^*(a_q) \to A/F^1 A$ is a d.g.a. map. We call such g *an algebraic a_q-connection.* Since $C^*(a_q)$ is multiplicatively generated by $C^1(a_q)$, we can prove just as in the finite-dimensional case the following

Proposition (1.3.1). *An algebraic a_q-connection on A induces a filtered d.g.a. map $\bar{g}\colon (W^*(a_q), F) \to (A, F)$ uniquely determined by the condition $\bar{g} | W^{0,*}(a_q) = g$.*

Moreover the homotopy class of $\bar{g}$ is uniquely determined by g_0. More precisely

Lemma (1.3.2). *Suppose two algebraic a_q-connections g' and g'' on A induce the same $g_0'=g_0''\colon C^*(a_q)\to A/F^1A$. Then $\bar{g}'$ and $\bar{g}''$ are homotopic as filtered d.g.a. maps.*

Remark (1.3.3). We call two filtered d.g.a. maps f_0 and f_1 from (A, F) to (B, F) *homotopic* if there is a filtered d.g.a. map $f\colon A\to B'$ such that $i_j\circ f=f_j$ $(j=0, 1)$, where (B', F) is a filtered d.g.a. and $i_j\colon B'\to B$ $(j=0, 1)$ filtered d.g.a. maps such that i_0 and i_1 induce the same spectral sequence map $E_r(B', F)\to E_r(B, F)$ for $r\geqq 1$. If f_0 and f_1 are homotopic, then obviously they induce the same spectral sequence map from $E_r(A, F)$ to $E_r(B, F)$ $(r\geqq 1)$.

1.4. We say that a Lie algebra b acts on a d.g.a. A or A is a b-d.g.a., if, for each $X\in b$, there are two operators L_X and i_X of degrees 0 and -1 respectively such that $L_X=i_Xd+di_X$, $i_{[X,Y]}=[L_X, i_Y]$ and $L_{[X,Y]}=[L_X, L_Y]$ for all $X, Y\in b$.

For example, $W^*(a_q)$ is an a_q-d.g.a. in a canonical way (cf. [11]). Hence each subalgebra of a_q, e.g. the subalgebra $gl(q, \boldsymbol{R})$ of linear vector fields, acts on $W^*(a_q)$.

Suppose b is the Lie algebra of a Lie group B. A d.g.a. A is called a B-d.g.a. if A is a b-d.g.a., B acts smoothly on A as d.g.a. automorphisms and its differentiated action is given by $X\mapsto L_X$ $(X\in b)$.

For example, $W^*(a_q)$ is a $GL(n, \boldsymbol{R})$-d.g.a. in a natural way, when $gl(q, \boldsymbol{R})$ acts on $W^*(a_q)$ as above. Hence $W^*(a_q)$ is also an $O(n)$-d.g.a.

For a B-d.g.a. A, we denote by A_B the d.g. subalgebra consisting of B-basic elements, i.e., of such elements $f\in A$ as $i_Xf=0(\forall X\in b)$ and $g.f=f(\forall g\in B)$.

We denote by $W^*(a_q, O(q))=W^*(a_q)_{O(q)}$, which is a filtered d.g.a. by the induced filtration. The associated spectral sequence will be denoted by $E(a_q, O(q))$. Note that by definition we have

Proposition (1.4.1). $E_1^{s,t}(a_q, O(q))\cong H^{t-s}(a_q, O(q); S^sa_q')$ $(t\geqq s)$, *where $H^*(a_q, O(q); S^sa_q')$ is the cohomology of $C^*(a_q; S^sa_q')_{O(q)}$.*

It is easy to show the following

Proposition (1.4.2). *Suppose (A, F) is a filtered $O(q)$-d.g.a. Let g be an $O(q)$-equivariant algebraic a_q-connection on A. Then the induced filtered d.g.a. map $\bar{g}$ is also $O(q)$-equivariant. In particular, $\bar{g}$ induces a filtered d.g.a. map $(W^*(a_q, O(q)), F)\to(A_{O(q)}, F)$.*

§ 2. The Vinogradov's spectral sequence of foliations

We construct for a manifold M, the infinite jet bundle $F_q^\infty(M)$ of local foliations of codimension q, and the Vinogradov's spectral sequence $E(M, q)$ for the integrability condition of plane fields of codimension q.

2.1. First we recall some results of [1].

Let M be a C^∞ manifold of dimension n. Let $S^\infty(M)\to M$ be the infinite frame bundle of M, i.e., the space of all the infinite jets at 0 of local diffeomorphisms from $(\boldsymbol{R}^n, 0)$ to M. $S^\infty(M)$ being the projective limit of finite-dimensional manifolds, we can use on $S^\infty(M)$ the usual terminologies about manifolds.

$S^\infty(M)\to M$ is a principal G_n-bundle, where G_n is the group of the infinite jets at 0 of local diffeomorphisms from $(\boldsymbol{R}^n, 0)$ to $(\boldsymbol{R}^n, 0)$. Moreover the group $\mathrm{Diff}(M)$ of diffeomorphisms acts naturally on $S^\infty(M)$, commuting with the action of G_n.

One of the basic facts about $S^\infty(M)$ is the existence of a natural global frame of the tangent bundle $TS^\infty(M)$: Let $\{\Omega^*(S^\infty(M)), d\}$ be the de Rham d.g.a. of $S^\infty(M)$ and $LS^\infty(M)$ the Lie algebra of C^∞ vector fields on $S^\infty(M)$. Then

Theorem (2.1.1) ([1]). *There is an a_n-valued differentiable one form w on $S^\infty(M)$ such that*

(i) $w_x\colon T_xS^\infty(M)\to a_n$ *is an isomorphism for all* $x \in S^\infty(M)$,

(ii) $dw+(1/2)[w, w]=0$, *where* $[w, w]$ *is an a_n-valued two form defined by* $[w, w]\,(X, Y)=2[w(X), w(Y)]$ $((X, Y \in LS^\infty(M))$,

(iii) $R_g^*w=\mathrm{Ad}(g^{-1})w$ $(\forall g \in G_n)$, *where* $R_g(x)=xg$ $(x \in S^\infty(M), g \in G_n)$ *and* Ad *denotes the "adjoint action" of G_n on a_n*,

(iv) $\bar{f}^*w=w$ $(\forall f \in \mathrm{Diff}(M))$, *where $\bar{f}$ is the diffeomorphism of $S^\infty(M)$ induced by f.*

This w is in fact constructed from a Lie algebra homomorphism $\rho\colon a_n\to LS^\infty(M)$ by $w(\rho(X))=X (X \in a_n)$. $\rho(a_n)$ is then a global frame of the tangent bundle $TS^\infty(M)$.

2.2. Fix positive integers p and q satisfying $p+q=n$. Fix a linear coordinate $(x, y)=(x^1, \cdots, x^p, y^1, \cdots, y^q)$ on $\boldsymbol{R}^n$. Let $G_{n,q}$ be the subgroup of G_n consisting of the elements represented by $f\colon (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$ such that

$$(y^i \circ f)(x, y)=(y^i \circ f)(0, y) \quad (1\leqq i\leqq q),$$

i.e., f preserves the foliation $F_{n,q}$ defined by $y^i=c^i$ $(1\leqq i\leqq q)$ for constants (c^i). Define

$$F_q^\infty(M)=S^\infty(M)/G_{n,q}.$$

We can regard $\pi\colon F_q^\infty(M)\to M$ as the infinite jet bundle of local foliations. In fact, let F be a C^∞ foliation of codimension q defined on an open subset V. For $x\in V$, choose a local chart $(U, h\colon U\to \boldsymbol{R}^n)$ such that $h(x)=0$, $U\subset V$ and $F|U$ is $h^{-1}(F_{n,q})$. Define then

$$j_x^\infty F=[h^{-1}].\,G_{n,q}\in F_q^\infty(M).$$

Then $j_x^\infty F$ is independent of the choice of the local charts. Thus we have

Proposition (2.2.1). *For an open subset V of M, there is a natural map $j^\infty\colon \mathrm{Fol}_q(V)\to\Gamma(F_q^\infty(V))$. Here $\mathrm{Fol}_q(V)$ denotes the space of C^∞ foliations of codimension q, and $\Gamma(F_q^\infty(V))$ denotes the space of C^∞ cross-sections of $F_q^\infty(V)\to V$.*

2.3. We construct now a flat connection on $\pi\colon F_q^\infty(M)\to M$ which makes $j^\infty F$'s flat sections.

Let $a_{n,q}$ be the subalgebra of a_n consisting of those vector fields preserving the foliation $F_{n,q}$, i.e.,

$$a_{n,q}=\{\textstyle\sum f_i(x,y)\partial/\partial x^i+\sum g_j(y)\partial/\partial y^j;\, f_i\in \boldsymbol{R}[[x,y]],\, g_j\in\boldsymbol{R}[[y]]\}.$$

Let H^∞ be the subbundle of $TS^\infty(M)$ spanned by $\rho(a_{n,q})$.

Define $H=\pi_{SF*}H^\infty\subset TF_q^\infty(M)$, $\pi_{SF}\colon S^\infty(M)\to F_q^\infty(M)$ being the natural projection. Note that the tangent bundle $V(\pi_{SF})$ of the fibers of π_{SF} is spanned by $\rho(a_{n,q}^0)$, where $a_{n,q}^0=a_{n,q}\cap a_n^0$ (a_n^0: the subalgebra of a_n consisting of those vector fields whose coefficients have zero constant terms). Hence the rank of $H^\infty/V(\pi_{SF})$ is n. Thus we have

Lemma (2.3.1). *H is a C^∞ subbundle of $TF_q^\infty(M)$ of rank n.*

Since $a_{n,q}$ is a subalgebra of a_n, we have $[\Gamma H^\infty, \Gamma H^\infty]\subset\Gamma H^\infty$. Moreover $V(\pi_{SF})\subset H^\infty$ implies $H^\infty=(\pi_{SF*})^{-1}H$. These imply obviously $[\Gamma H,\Gamma H]\subset\Gamma H$.

Let F be a C^∞ foliation of codimension q on an open subset $V\subset M$. Then obviously

$$T((j^\infty F)(V))\subset H.$$

Thus we have

Proposition (2.3.2). *There is a flat connection on $\pi\colon F_q^\infty(M)\to M$ in the following sense. There is a C^∞ subbundle H of $TF_q^\infty(M)$ of rank n such that*

(i) *for all $x\in F_q^\infty(M)$, $(\pi_*)_x\colon H_x\to T_{\pi(x)}M$ is an isomorphism,*
(ii) *$[\Gamma H,\Gamma H]\subset\Gamma H$.*

Moreover the infinite jet extensions of local foliations of codimension q are flat with respect to this connection, i.e., *their graphs are tangent to H.*

2.4. Let $V=V(\pi)$, the tangent bundle of the fibers of π. Then $TF_q^\infty(M)\cong V\oplus H$. This induces the splitting of cotangent bundle

$$T'F_q^\infty(M)\cong V'\oplus H',$$

and the isomorphism

$$\Omega^k F_q(M)\cong\bigoplus_{s+t=k}\Omega^{s,t}(M,q),$$

where $\Omega^{s,t}(M,q)=\Gamma(\bigwedge^s V'\otimes\bigwedge^t H')$.

Since H is flat, the exterior differentiation decomposes as

$$d=\delta+(-1)^s\partial$$

on $\Omega^{s,t}(M,q)$, where δ and ∂ have bidegrees $(1,0)$ and $(0,1)$ respectively. $d^2=0$ implies then $\delta^2=\partial^2=0$ and $\delta\partial=\partial\delta$. Thus we obtain a bicomplex

$$\Omega^{*,*}(M,q)=\{\oplus\Omega^{s,t}(M,q),\delta,\partial\},$$

which we call *the variation bicomplex associated to the integrability condition of plane fields of codimension q on M.*

The underlying bicomplex structure makes the de Rham d.g.a. a filtered d.g.a. by the filtration

$$(*)\qquad F^s\Omega^*F_q^\infty(M)=\bigoplus_{s'\geq s}\Omega^{s',*}(M,q).$$

We denote the induced spectral sequence by $E(M,q)=\{E_r^{s,t}(M,q),d_r\}$. Note that $\mathrm{Diff}(M)$ acts on $F_q^\infty(M)$ preserving H, and hence on $(\Omega^*(F_q^\infty(M)),F)$ and on $E(M,q)$.

Remark (2.4.1). The other filtration $'F^t=\bigoplus_{t'\geq t}\Omega^{*,t'}$ induces the usual Serre spectral sequence for the fibration π.

Remark (2.4.2). Note that generally an integrable subbundle H of the tangent bundle TX of a C^∞ manifold X gives a filtered d.g.a. structure on the de Rham d.g.a. $\Omega^*(X)$ by the filtration $F^i=\Gamma(\bigwedge^i(H^\perp)).\,\Omega^*(X)$, where $H^\perp=(TX/H)'\subset T'X$. The filtration introduced by (*) coincides with this one associated with the integrable subbundle H of $TF_q^\infty(M)$. Note further that $E_0^{0,*}(\Omega^*(X),F)\cong\Gamma(\bigwedge^*H')$ and hence $\Gamma(\bigwedge^*H')$ becomes a d.g.a., which we denote by $\{\Omega^*(H),d_H\}$.

2.5. The spectral sequence $E(M,q)$ gives us a "de Rham theory" on $\mathrm{Fol}_q(M)$ in the following sense.

Let X be a C^∞ manifold and $\sigma: X \to \mathrm{Fol}_q(M)$ be a map. σ is called C^∞ if its evaluation map $\bar{\sigma}: X \times M \to F_q^\infty(M)$ defined by $\bar{\sigma}(x, m) = j_m^\infty(\sigma(x))$ $(x \in X, m \in M)$ is C^∞.

When σ is C^∞, it is easy to see that

$$\bar{\sigma}^*: \Omega^* F_q^\infty(M) \longrightarrow \Omega^*(X \times M)$$

preserves the underlying bicomplex structures, i.e.,

$$\bar{\sigma}^*(\Omega^{s,t}(M, q)) \subset \Omega^{s,t}(X \times M),$$

where $\Omega^{s,t}(X \times M) = \Gamma(\pi_X^* \wedge^s T'X \otimes \pi_M^* \wedge^t T'M)$, π_X and π_M being the natural projections from $X \times M$ to X and M respectively (cf. [14]).

Let F be the filtration on $\Omega^*(X \times M)$ defined by $F^i = \bigoplus_{i' \geqq i} \Omega^{i',*}(X \times M)$. Then $(\Omega^*(X \times M), F)$ is a filtered d.g.a. map. Thus σ induces a spectral sequence map

$$\sigma^* = \{\sigma_r^*\}: E(M, q) \longrightarrow E(\Omega^*(X \times M), F).$$

Since $\{E_1^{*,t}, d_1\} \cong \{\Omega^*(X) \otimes H^t(M, \boldsymbol{R}), d_X \otimes \mathrm{id}\}$ as cochain complexes and $E_2^{s,t} \cong H^s(X, \boldsymbol{R}) \otimes H^t(M, \boldsymbol{R})$ as vector spaces, we have cochain complex maps

$$\sigma_1^{*,t}: \{E_1^{*,t}(M, q), d_1\} \longrightarrow \{\Omega^*(X) \otimes H^t(M, \boldsymbol{R}), d_X \otimes \mathrm{id}\}$$

and linear maps

$$\sigma_2^{s,t}: E_2^{s,t}(M, q) \longrightarrow H^s(X, \boldsymbol{R}) \otimes H^t(M, \boldsymbol{R}).$$

Thus $E_1^{*,t}(M, q)$ can be considered as the $H^t(M, \boldsymbol{R})$-valued de Rham complex on $\mathrm{Fol}_q(M)$.

The assignment $\sigma \mapsto \sigma^*$ is functorial: For a C^∞ map $f: Y \to X$, we have $(\sigma \circ f)^* = f^* \circ \sigma^*$, where $f^*: E(\Omega^*(X \times M), F) \to E(\Omega^*(Y \times M), F)$ is induced naturally from f.

By this fact, $E_2(M, q)$ provides us potential tools to investigate the homotopy structure of $\mathrm{Fol}_q(M)$. For example, suppose there is a nonzero element h of $E_2^{s,t}(M, q)$. Then h induces a map from the C^∞ homotopy class of C^∞ maps $S^s \to \mathrm{Fol}_q(M)$ to $H^t(M, \boldsymbol{R})$, hence, h might measure "$\pi_s(\mathrm{Fol}_q(M))$".

2.6. In order to treat the foliations with trivial normal bundles, we introduce the frame bundle $BF_q^\infty(M)$ of the universal normal bundle.

Let $G_{n,q}^1$ be the subgroup of $G_{n,q}$ consisting of those represented by f such that $(\partial y^i \circ f / \partial y^j(0))_{1 \leqq i, j \leqq q}$ is the identity matrix. Define then $BF_q^\infty(M) = S^\infty(M)/G_{n,q}^1$. Since $G_{n,q}/G_{n,q}^1 \cong GL(q, \boldsymbol{R})$, the natural projection π_{BF}:

$BF_q^\infty(M) \to F_q^\infty(M)$ is a principal $GL(q, \boldsymbol{R})$-bundle.

Define a subbundle of $TBF_q^\infty(M)$ by $H_B = (\pi_{BF*})^{-1}H$, which is obviously integrable. Hence by Remark (2.4.2), we obtain a filtration in the d.g.a. $\Omega^*(BF_q^\infty(M))$ and a d.g.a. $\Omega^*(H_B) \cong F^0/F^1$. Note that these are $GL(q, \boldsymbol{R})$-d.g.a. 's.

Denote $\bar{E}(M, q) = E(\Omega^*(BF_q^\infty(M)), F)$. This has similar meaning for the foliations with trivial normal bundles as $E(M, q)$ does for foliations (c.f. § 2.4): Suppose $\sigma: X \to \mathrm{Fol}_q(M)$ is a C^∞ map such that the normal bundle of $\sigma(x)$ is given a trivialization which depends smoothly on $x \in X$. Then $\bar{\sigma}: X \times M \to F_q^\infty(M)$ has a lifting to a C^∞ map $\tilde{\sigma}: X \times M \to BF_q^\infty(M)$ such that $\pi_{BF} \circ \tilde{\sigma} = \bar{\sigma}$. It is easy to see that $\tilde{\sigma}$ induces a filtered d.g.a. map $\tilde{\sigma}^*: (\Omega^*(BF_q^\infty(M)), F) \to (\Omega^*(X \times M), F)$, which is uniquely determined up to homotopies.

Hence we obtain canonical homomorphisms:

$$\tilde{\sigma}_1^{*,t}: \{\bar{E}_1(M, q), d_1\} \longrightarrow \{\Omega^*(X) \otimes H^t(M, \boldsymbol{R}), d_X\},$$
$$\tilde{\sigma}_2^{s,t}: \bar{E}_2^{s,t}(M, q) \longrightarrow H^s(X, \boldsymbol{R}) \otimes H^t(M, \boldsymbol{R}).$$

Finally note that $\mathrm{Diff}(M)$ acts on $BF_q^*(M)$ preserving H_B, and hence on $(\Omega^*(BF_q^\infty(M)), F)$ and $\bar{E}(M, q)$.

§ 3. Characteristic homomorphism

We construct filtered d.g.a. maps

$$\lambda: (W^*(a_q), F) \longrightarrow (\Omega^*(BF_q^\infty(M)), F),$$
$$\chi: (W^*(a_q, O(q)), F) \longrightarrow (\Omega^*(F_q^\infty(M)), F),$$

canonically defined up to homotopies. Thus we obtain canonical characteristic homomorphisms $\lambda_r: E_r(a_q) \to \bar{E}_r(M, q)$ and $\chi_r: E_r(a_q, O(q)) \to E(M, q)$ for $r \geqq 1$.

3.1. First we introduce the bundle $P_q^\infty(M) \to M$ of the infinite jets of local submersions of rank q.

Let $H_{n,q}$ be the subgroup of $G_{n,q}$ consisting of those represented by f with $y^i \circ f = y^i$ $(1 \leqq i \leqq q)$. Define then $P_q^\infty(M) = S^\infty(M)/H_{n,q}$. Then just as in Section 2.2, $P_q^\infty(M) \to M$ can be considered as the bundle of the infinite jets of local submersions on M of rank q.

The action of $\mathrm{Diff}(M)$ on $S^\infty(M)$ induces that on $P_q^\infty(M)$.

Since $H_{n,q} \subset G_{n,q}$, we have a natural projection $\pi_{PF}: P_q^\infty(M) \to F_q^\infty(M)$. This is a principal G_q-bundle, since there is an isomorphism $G_{n,q}/H_{n,q} \cong G_q$, induced by the homomorphism $u: G_{n,q} \to G_q$ defined by $u([f]) = [(y) \mapsto (y \circ f)(0, y)]$.

3.2. Define a subbundle of $TP_q^\infty(M)$ by $H_P=(\pi_{PF*})^{-1}H$. Then H_P is integrable and we obtain a filtered d.g.a. $(\Omega^*(P_q^\infty(M)), F)$ and a d.g.a. $\{\Omega^*(H_P), d_P\}=F^0/F^1$ (cf. Remark (2.4.2)).

Note that $\Omega^*(H_P)$ is an a_q^0-d.g.a. and G_q-d.g.a., since the action of G_q on $P_q^\infty(M)$ preserves H_P and the filtration, and a_q^0 can be naturally considered as the Lie algebra of G_q. Thus $\Omega^*(H_P)$ is also a $GL(q, \boldsymbol{R})$-d.g.a. by virtue of the natural homomorphism $GL(q, \boldsymbol{R})\subset G_q$.

Proposition (3.2.1). *There is an $\eta\in\Omega^1(H_P)\otimes a_q$ such that*

(i) *for all $x\in P_q^\infty(M)$, $\eta_x\colon (H_P)_x\to a_q$ is surjective,*

(ii) $d_P\eta+(1/2)[\eta, \eta]=0$,

(iii) $R_g^*\eta=\mathrm{Ad}(g^{-1})\eta$ $(\forall g\in G_q)$,

(iv) $\bar{f}^*\eta=\eta$ *for all $f\in \mathrm{Diff}(M)$, where $\bar{f}$ is the diffeomorphism of $P_q^\infty(M)$ induced by f.*

Proof. Let $\pi_{SP}\colon S^\infty(M)\to P_q^\infty(M)$ be the natural projection. Since $\pi_{PF}\circ\pi_{SP}=\pi_{SF}$, $(\pi_{SP*})^{-1}H_P$ equals $(\pi_{SF*})^{-1}H=H^\infty$ and is generated by $\rho(a_{n,q})$. Thus for every $x\in P_q^\infty(M)$, $(H_P)_x$ is spanned by $X_y=(\pi_{SP*})_y(\rho(X)_y)$ $(X\in a_{n,q})$, where $y\in\pi_{SP}^{-1}(x)$ is fixed. Define then $\eta(X_y)=\bar{X}\in a_q$, where $\bar{X}=\sum g_j(y)\partial/\partial y^j$ for $X=\sum f_i(x, y)\partial/\partial x^i+\sum g_j(y)\partial/\partial y^j$. Since $V(\pi_{SP})$ is spanned by $\{\sum f_i(x, y)\partial/\partial x^i\}$, η is well-defined.

The other assertions are easily proved. q.e.d.

Corollary (3.2.2). *There is a $GL(q, \boldsymbol{R})$-d.g.a. map $\lambda_P\colon C^*(a_q)\to\Omega^*(H_P)$ such that $\bar{f}^*\circ\lambda_P=\lambda_P$ for all $f\in\mathrm{Diff}(M)$.*

3.3. Since $H_{n,q}$ is a subgroup of $G_{n,q}^1$, there is a natural projection $\pi_{PB}\colon P_q^\infty(M)\to BF_q^\infty(M)$.

Lemma (3.3.1). (i) *There is a $GL(q, \boldsymbol{R})$-equivariant C^∞ section s of π_{PB}.*

(ii) *If there are two such sections s_0 and s_1, then there is a $GL(q, \boldsymbol{R})$-equivariant section S of $P_q^\infty(M)\times[0, 1]\to BF_q^\infty(M)\times[0, 1]$ such that $S\,|\,BF_q^\infty(M)\times\{i\}=s_i$ $(i=0, 1)$.*

Proof. Note that the fibering π_{PB} is isomorphic to the one induced by π_{BF} from $f\colon P_q^\infty(M)/GL(q, \boldsymbol{R})\to F_q^\infty(M)$. Since f is a homotopy equivalence, there is a section s' of f, determined uniquely up to homotopies. Hence there is a $GL(q, \boldsymbol{R})$-equivariant section s of π_{PB}, determined uniquely up to $GL(q, \boldsymbol{R})$-equivariant homotopies. q.e.d.

Fix now a $GL(q, \boldsymbol{R})$-equivariant section s of π_{PB}. Since $s_*(X)\in H_P$ $(X\in H_B)$, s induces a $GL(q, \boldsymbol{R})$-d.g.a. map $s^*\colon\Omega^*(H_P)\to\Omega^*(H_B)$. Thus we obtain a $GL(q, \boldsymbol{R})$-d.g.a. map

$$\lambda_B = s^* \circ \lambda_P \colon C^*(a_q) \longrightarrow \Omega^*(H_B).$$

3.4. Fix a C^∞ $GL(q, \boldsymbol{R})$-connection γ on the principal $GL(q, \boldsymbol{R})$-bundle π_{BF}. γ defines a $GL(q, \boldsymbol{R})$-equivariant decomposition

$$TBF_q^\infty(M) = V(\pi_{BF}) \oplus H_\gamma,$$

where $H_\gamma \cong \pi_{BF}^* TF_q^\infty(M)$. Since $TF_q^\infty(M) = V \oplus H$, we obtain a $GL(q, \boldsymbol{R})$-equivariant decomposition

$$TBF_q^\infty(M) = V(\pi_{BF}) \oplus V_\gamma \oplus \bar{H}_\gamma,$$

such that $V_\gamma \cong \pi_{BF}^* V$ and $\bar{H}_\gamma \cong \pi_{BF}^* H$. By definition $H_B = V(\pi_{BF}) \oplus \bar{H}_\gamma$, whence γ induces a $GL(q, \boldsymbol{R})$-equivariant projection $\mu_\gamma \colon TBF_q^\infty(M) \to H_B$, which in turn induces a $GL(q, \boldsymbol{R})$-equivariant graded algebra map $\mu_\gamma^* \colon \Omega^*(H_B) \to \Omega^*(BF_q^\infty(M))$.

Define a graded algebra map $C^*(a_q) \to \Omega^*(BF_q^\infty(M))$ by $\lambda_\gamma = \mu_\gamma^* \circ \lambda_B$. We have $(\lambda_\gamma)_0 = \lambda_B$, when F^0/F^1 is naturally identified with $\Omega^*(H_B)$. Thus we obtain a $GL(q, \boldsymbol{R})$-equivariant algebraic a_q-connection on $\Omega^*(BF_q^\infty(M))$. By Proposition (1.4.2), λ_γ induces a filtered $GL(q, \boldsymbol{R})$-d.g.a. map $\lambda = \bar{\lambda}_\gamma \colon (W^*(a_q), F) \to (\Omega^*(BF_q^\infty(M)), F)$ (cf. Diagrams 1 and 2).

$$\begin{tikzcd}
S^\infty(M) \arrow[drr, "\pi_{SP}"] \arrow[ddr, "\pi_{SB}"] \arrow[dddr, "\pi_{SF}"'] & & \\
& & P_q^\infty(M) \arrow[dl, "\pi_{PB}"'] \arrow[ddl, "\pi_{PF}"] \\
& BF_q^\infty(M) \arrow[d, "\pi_{BF}"] & \\
& F_q^\infty(M) \arrow[d, "\pi"] & \\
& M &
\end{tikzcd}$$

Diagram 1.

$$\begin{tikzcd}
& & \Omega^*(H_P) \arrow[d, "s^*"] \\
C^*(a_q) \arrow[urr, "\lambda_P"] \arrow[rr, "\lambda_B"] \arrow[drr, "\lambda_\gamma"] \arrow[d] & & \Omega^*(H_B) \arrow[d, "\mu_\gamma^*"] \\
W^*(a_q) \arrow[rr, "\lambda"] & & \Omega^*(BF_q^\infty(M)) \\
W^*(a_q, O(q)) \arrow[u, hook] \arrow[rr, "\lambda'"] \arrow[drr, "\chi"'] & & \Omega^*(BF_q^\infty(M)/O(q)) \arrow[u, hook] \arrow[d, "t^*"] \\
& & \Omega^*(F_q^\infty(M))
\end{tikzcd}$$

Diagram 2.

3.5. By Lemmas (1.3.2) and (3.3.1), we can show without difficulty the following

Lemma (3.5.1). *The homotopy class of λ is independent of the choices of s and γ.*

Furthermore, Lemmas (1.3.2), (3.3.1) and (3.2.2) imply

Lemma (3.5.2). *For all $f \in \mathrm{Diff}(M)$, $\bar{f}^* \circ \lambda$ is homotopic to λ.*

Thus, we have proved

Theorem (3.5.3). *Let M be a C^∞ manifold of dimension n and q a positive integer less than n. Then there is a filtered d.g.a. map λ: $(W^*(a_q), F) \to (\Omega^*(BF_q^\infty(M)), F)$ determined uniquely up to homotopies. Moreover, for all $f \in \mathrm{Diff}(M)$, $\bar{f}^* \circ \lambda$ is homotopic to λ.*

Corollary (3.5.4). *There is a canonical spectral sequence map*

$$\lambda_r\colon E_r(a_q) \longrightarrow \bar{E}_r(M, q)$$

for $r \geqq 1$, such that $\mathrm{Im}\,\lambda_r \subset \bar{E}_r(M, q)^{\mathrm{Diff}(M)}$.

By Proposition (1.2.3), we have

Corollary (3.5.5). *There is a canonical map*

$$H^t(a_q; S^s a_q') \longrightarrow \bar{E}_1^{s, s+t}(M, q)^{\mathrm{Diff}(M)}.$$

By the interpretations given in Section 2.6, we obtain

Theorem (3.5.6). *Let M be a C^∞ manifold of dimension n and q a positive integer less than n. Let $\sigma\colon X \to \mathrm{Fol}_q(M)$ be a C^∞ map such that the normal bundles of $\sigma(x)$'s are trivialized smoothly with respect to $x \in X$. Then there is a canonical homomorphism*

$$\sigma_r^*\colon E_r(a_q) \longrightarrow E_r(\Omega^*(X \times M), F)$$

for $r \geqq 1$. Moreover $(\tilde{f} \circ \sigma_r)^* = \sigma_r^*$ for all $f \in \mathrm{Diff}(M)$, where $\tilde{f}\colon \mathrm{Fol}_q(M) \to \mathrm{Fol}_q(M)$ is the natural map induced by f.

In particular, there exist canonical maps

$$\sigma_1^*\colon H^t(a_q; S^s a_q') \longrightarrow \Omega^s(X) \otimes H^{t+s}(M, \boldsymbol{R})$$

such that $\sigma_1^* \circ \mathrm{var} = d_X \circ \sigma_1^*$ and

$$\sigma_2^*\colon E_2^{s,t}(a_q) \longrightarrow H^s(X, \boldsymbol{R}) \otimes H^t(M, \boldsymbol{R}).$$

3.6. Restricting λ to the $O(q)$-basic elements, we obtain a filtered d.g.a. map

$$\lambda'\colon (W^*(a_q, O(q)), F) \longrightarrow (\Omega^*(BF_q^\infty(M)/O(q)), F').$$

Here F' is defined by the integrable subbundle $(f_*)^{-1}H$ of $T(BF_q^\infty(M)/O(q))$, where $f\colon BF_q^\infty(M)/O(q) \to F_q^\infty(M)$ is the natural projection.

Since f is a homotopy equivalence, it has a C^∞ section t determined uniquely up to homotopies. Obviously $t^*: \Omega^*(BF_q^\infty(M)/O(q)) \to \Omega^*(F_q^\infty(M))$ preserves the filtrations. Hence we have constructed a filtered d.g.a. map

$$\chi = t^* \circ \chi' : (W^*(a_q, O(q)), F) \longrightarrow (\Omega^*(F_q^\infty(M)), F).$$

3.7. Just as in Section 3.5, we have

Lemma (3.7.1). *The homotopy class of χ is independent of the choices of s, γ and t.*

Lemma (3.7.2). *For all $f \in$ Diff(M), $\bar{f}^* \circ \chi$ is homotopic to χ.*

Thus we have proved

Theorem (3.7.3). *Let M be a C^∞ manifold of dimension n and q a positive integer less than n. Then there is a filtered d.g.a. map*

$$\chi : (W^*(a_q, O(q)), F) \longrightarrow (\Omega^*(F_q^\infty(M)), F)$$

determined uniquely up to homotopies. Moreover, for all $f \in$ Diff(M), $\bar{f}^* \circ \chi$ is homotopic to χ.

This implies similar results as (3.5.4-6), which we omit.

§ 4. Remarks

4.1. Some problems naturally arise from our results.

Problem (4.1.1). *Calculate $E_r(a_q)$ and $E_r(a_q, O(q))$.*

$E_q^{0,*}(a_q) = H^*(a_q)$ and $E_1^{0,*}(a_q, O(q))$ were calculated by [7], and $E_1^{1,*}(a_q) = H^{*-1}(a; a_q')$ and $E_1^{1,*}(a_q, O(q)) = H^{*-1}(a_q, O(q); a_q')$ by [8]. We tabulate for $q=2$ these results:

Dimension of $E_1^{s,t}(a_2)$:

$s \backslash t$	0	1	2	3	4	5	6	7	8	9
0	1					2		1	2	
1						2	2		2	2

Dimension of $E_2^{s,t}(a_2)$:

$s \backslash t$	0	1	2	3	4	5	6	7	8	9
0	1							1		

Here the blanks mean zeros. Note that since by Theorem (1.2.2) $E_\infty^7=0$, we must have $E_2^{i,7-i}\neq 0$ for some $1\leqq i\leqq 3$.

For $q=1$, we can easily calculate $E_1^{s,t}$ $(s\leqq 2)$:

Dimension of $E_1^{s,t}(a_1)$:

$s \backslash t$	0	1	2	3	4	5	6
0	1			1			
1				1	1		
2					1	1	

Dimension of $E_2^{s,t}(a_1)$:

$s \backslash t$	0	1	2	3	4	5	6
0	1						
1							

For $s\geqq 2$, the computations of $E_1^{s,*}(a_q)=H^{*-1}(a_q;S^s a_q')$ seem difficult. However the following isomorphism might make the calculation easier:

$$H^*(a_q;S^s a_q')\cong H^*(LR^q;\hat{S}^s LR^{q\prime}),$$

which can be proved by the Bott's arguments (cf. [2]). The right hand side might be calculated by the method of [13].

Problem (4.1.2). *For a non-zero element ω of $E_2^{s,t}(a_q)$, find an explicit formula making it possible to evaluate $\sigma_2^{s,t}(\omega)$ for $\sigma\colon X\to \mathrm{Fol}_q(M)$.*

Problem (4.1.3). *For a non-zero element ω of $E_2^{s,t}(a_q)$, find a C^∞ map $\sigma\colon X\to \mathrm{Fol}_q(M)$ such that $\sigma_2^{s,t}(\omega)\neq 0$.*

This problem seems difficult, since even for $s=0$, the answer to it is not known. However there is a program by Fuks to show it (cf. [6]).

4.2. Let G be a finite-dimensional Lie group and g its Lie algebra. Then $E(g, H)$ has similar meaning as Theorem (3.5.6) for flat G-bundles on M, where H is an appropriate subgroup of G. This is already remarked in [10, § 4.50].

4.3. Let N be a C^∞ manifold of dimension q and LN the topological Lie algebra of all the C^∞ vector fields on N. Then the spectral sequence

$E(LN)$ associated to the "continuous" Weil algebra $W^*(LN)$ has similar interpretation as Theorem (3.5.6) for foliated trivialized N-bundles on M.

$E_1^{0,*}(LN)=H^*(LN)$ was in principle calculated by [4, 9], although the actual calculation is quite difficult (cf. [12]).

$E_1^{1,*}(LN)=H^{*-1}(LN; LN')$ is in principle calculated in [13]:

$$H^*(LN; LN')\cong H^*(LN, C_0^\infty(N))\otimes H^*(a_q^0, gl(q, \boldsymbol{R}); \wedge^q T'\otimes T'),$$

where $C_0^\infty(N)$ is the topological LN-algebra of C^∞ functions on N with compact supports and $T=\boldsymbol{R}^q$ is the standard $gl(q, \boldsymbol{R})$-module, regarded also as an a_q^0-module by the natural Lie algebra homomorphism $a_q^0\to gl(q, \boldsymbol{R})$. We note that $H^*(a_q^0, gl(q, \boldsymbol{R}); \wedge^q T'\otimes T')\cong H^{*+q}(a_q, gl(a, \boldsymbol{R}); a_q')$ is calculated in [8] and $H^*(LN, C_0^\infty(N))$ can be calculated in principle once we know the homotopy type of the d.g.a. $C^*(LN)$ (cf. [13]).

When $N=S^1$, we have

$$E_1^{0,*}\cong \wedge(\alpha, \beta),$$
$$E_1^{1,*}\cong \wedge(\omega, \theta, \xi)\otimes \boldsymbol{R}.\psi,$$

with $|\omega|=|\theta|=|\psi|=1$, $|\alpha|=|\xi|=2$, $|\beta|=3$ ($|x|=\deg x$) and $\wedge(x_1, x_2, \cdots)$ denotes the free anti-commutative graded algebra generated by $x_1, x_2, \cdots$ (cf. [13]). Moreover we have $d_1\alpha=2\psi$, $d_1\beta=2\omega\psi$, whence $E_2^{0,*}=\boldsymbol{R}$ $(*=0)$, 0 $(*\neq 0)$. However, this follows trivially from the results of S. Morita (unpublished) which assert that elements of $H^*(LS^1)$ are independent and varies as characteristic classes of foliated trivialized S^1-bundles.

4.4. For the subalgebras of a_q, the spectral sequences of their Weil algebras have similar meaning. For example, let $a_q^{\boldsymbol{C}}$ be the subalgebra of a_{2q} formed by formal holomorphic vector fields on $\boldsymbol{C}^q=\boldsymbol{R}^{2q}$. Then we can interprete $E(a_q^{\boldsymbol{C}}, U(q))$ as characteristic classes of families of complex analytic foliations of codimension q. Elements of $\bigoplus_{t\leqq 2q}E_2^{*,t}(a_q^{\boldsymbol{C}}, U(q))$ may give us tools to study the homotopy structure of "moduli spaces" of complex structures of dimension q.

4.5. The d.g. subalgebra $\Omega^*(F_q^\infty(M))^{\mathrm{Diff}(M)}$ is isomorphic to $C^*(a_n, a_{n,q}^0; \boldsymbol{R})$. Thus the spectral sequence associated to this filtered d.g.a. with the induced filtration can also be interpreted as characteristic classes for families of foliations of codimension q.

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