Advanced Studies in Pure Mathematics 5, 1985
Foliations
pp. 159–167

# A Relation between the Topological Invariance of the Godbillon-Vey Invariant and the Differentiability of Anosov Foliations

**Yoshihiko Mitsumatsu**

## § 1. Introduction and the statement of the result

In this paper, we study the Anosov's unstable foliations of the geodesic flows of negatively curved surfaces to show some relation between the topological invariance of the Godbillon-Vey invariant of codimension one $C^2$-foliations and the differentiability of such foliations. From the definition, the Godbillon-Vey invariant is invariant under foliation preserving $C^2$-diffeomorphisms. Moreover Raby proved in his recent preprint [13] that it is in fact $C^1$-invariant, i.e., foliation preserving $C^1$-diffeomorphisms between codimension one $C^2$-foliations preserve their Godbillon-Vey invariants. On the other hand, due to one of the recent theorems of Tsuboi [14] (which asserts $H_2(B\overline{\mathrm{Diff}}{}^1_c\boldsymbol{R}^1; \boldsymbol{Z})=0$), we have little possibility that the Godbillon-Vey invariant could be defined for $C^1$-foliations. Therefore the problem of the topological invariance arises naturally. For example, Duminy's vanishing theorem [5] shows that a certain topological condition is sufficient for the vanishing of the Godbillon-Vey invariant. It seems that this increases the importance of our problem.

In this paper we treat foliations only on closed oriented 3-manifolds, so that we study the Godbillon-Vey number which is the value of the Godbillon-Vey invariant on the fundamental class of the manifold.

The first example which has a non-trivial Godbillon-Vey number is the Anosov's unstable foliation of the geodesic flow on the unit tangent circle bundle of a closed surface with a hyperbolic structure. The Godbillon-Vey number of this foliation is equal to $8\pi^2(1-g)=4\pi^2\chi$ where $g$ is the genus of the surface and $\chi$ is the euler characteristic. We can regard the above foliation as a foliated $S^1$-bundle over the surface, so that the number can be calculated by using Thurston's formula ([3], [4]) which represents the Godbillon-Vey invariant as a 2-cocycle of the group $\mathrm{Diff}^2_+S^1$ of all orientation preserving $C^2$-diffeomorphisms. We can also calculate this from the definition of unstable foliations by using the structural

Received October 12, 1983.

equation of the Riemannian connection associated with the Riemannian metric of the surface. We try to deform this foliation and to vary the Godbillon-Vey number, though Heitsch's theory on the deformation of smooth foliations [7] and on the variation of the exotic classes [8] suggests that it is not so easy. To do that, we deform the Riemannian metric of the surface keeping its curvature negative. By classical theorem of Cartan, E.-Hadamard and Anosov (see [1]), if the curvature is negative, the geodesic flow is an Anosov flow so that there exists the unstable foliation. On the other hand, from the well-known existence theorem of isothermal coordinates, we can see that the space of smooth Riemannian metrics with negative curvature is connected. Therefore if we combine this fact with Anosov's structural stability theorem of Anosov flows, it turns out that every such foliations are mutually $C^0$-conjugate, i.e., there exists a leaf-preserving homeomorphism between any two of them. These homomorphisms are smooth if they are restricted to a single leaf. So we want to use these mutually $C^0$-conjugate foliations for the study of the topological invariance of the Godbillon-Vey invariant. However, $C^r$-section theorem [10] of Hirsch-Pugh guarantees nothing but these foliations to be of class $C^1$. (Of course they are smooth along the leaves and in the case of negative constant curvature they are obviously real analytic.) It is not known whether there exist $C^2$-unstable foliations associated with the metrics of non-constant curvature. Our main task in this paper is to compute the Godbillon-Vey numbers of these foliations assuming that they are of class $C^2$, and for that purpose, we look into the theorem of Cartan-Hadamard more explicitly in this case. As a result, we obtain the following main result of this paper.

**Theorem.** *Either of the following two statements occurs.*

1) *The Godbillon-Vey invariant does not have the property of topological invariance.*

2) *The only unstable foliations of the geodesic flows of negatively curved surfaces which are of class $C^2$ are those of hyperbolic surfaces.*

The author has no idea which one actually occurs and in fact it may happen that both are true.

For the definition of the Godbillon-Vey invariant, see [6]. [2] and [9] are good references for the generality of foliations and their characteristic classes. Anosov flows and Anosov foliations are investigated in detail in [1] and [10] deals with $C^r$-section theorem as well as other related problems.

The author would like to state his hearty gratitude to Professor S. Morita for his continuous encouragement.

## § 2. Proof of Theorem

In this section we describe the unstable foliations of the geodesic flows of negatively curved surfaces in such a way that we can see the theorem of Cartan-Hadamard in an explicit way, and calculate their Godbillon-Vey numbers assuming that the foliations are of class $C^2$.

Let $\Sigma_g$ be a closed oriented smooth Riemannian 2-manifold with negative curvature and $S^1\Sigma_g$ be its unit tangent circle bundle. Naturally $S^1\Sigma_g$ is identified with the principal $SO(2; \boldsymbol{R})$-bundle associated with the tangent bundle $T\Sigma_g$ of $\Sigma_g$. Thus $S^1\Sigma_g$ carries the Riemannian connection 1-form $\omega$ ($\omega$ is originally an $so(2)$-valued 1-form but of course we can naturally consider it as an $\boldsymbol{R}$-valued 1-form) and $\boldsymbol{R}^2$-valued canonical 1-form $\theta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$. As is well-known, $\langle \omega, \theta_1, \theta_2 \rangle$ is a global ortho-normal oriented framing of $A^1(S^1\Sigma_g)$. We take the dual basis $\langle w, \xi_1, \xi_2 \rangle$ to $\langle \omega, \theta_1, \theta_2 \rangle$. Then, $w$ is tangent along the fibers of $\pi: S^1\Sigma_g \to \Sigma_g$, $\xi_1$ and $\xi_2$ span the horizontal space with respect to the connection and $\xi_1$ generates the geodesic flow. The structure equations are as follows.

$$d\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = -\begin{pmatrix}0, & -\omega\\ \omega, & 0\end{pmatrix} \wedge \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \tag{2.1}$$

$$d\omega = -k \circ \pi \theta_1 \wedge \theta_2, \tag{2.2}$$

where $k$ is the sectional curvature.

Let $\gamma: \boldsymbol{R} \to \Sigma_g$ be a geodesic of unit velocity and $\tilde{\gamma}: \boldsymbol{R} \to S^1\Sigma_g$ its tautological lift so that $\tilde{\gamma}$ is an orbit of the geodesic flow and satisfies

$$\tilde{\gamma}(t) = \frac{d}{dt}\gamma(t).$$

We take the velocity vector field $V(t) = (d/dt)\gamma(t)$ and the normal vector field $\nu(t)$ along $\gamma$ such that $\langle V(t), \nu(t) \rangle$ is an ortho-normal oriented basis of $T_{\gamma(t)}\Sigma_g$, in other words, $\langle \xi_1, \xi_2 \rangle_{\tilde{\gamma}(t)}$ is the horizontal lift of $\langle V(t), \nu(t) \rangle_{\gamma(t)}$. Recall that $V$ and $\nu$ are parallel along $\gamma$, i.e., $\nabla_V V = 0$ and $\nabla_V \nu = 0$. Then a vector field $J(t)$ along $\gamma(t)$ is a Jacobi field iff $J(t)$ satisfies the following Jacobi equation;

$$\nabla_V \nabla_V J + R(J, V)V = 0, \tag{2.3}$$

where $R$ is the curvature tensor and $\nabla$ is the covariant derivative. For the precise definition and properties of them, see [11] and [12]. Using $V(t)$ and $\nu(t)$ we express $J(t)$ as $J(t) = F(t)V(t) + G(t)\nu(t)$, where $F(t)$ and $G(t)$ are smooth functions along $\gamma$ and the Jacobi equation (2.3) is expressed in terms of $F(t)$ and $G(t)$ as

$$\frac{d^2F(t)}{dt^2}=0, \quad \frac{d^2G(t)}{dt^2}+k(\gamma(t))G(t)=0. \tag{2.4}$$

Since the equation on $F(t)$ is concerning only with re-parametrization of geodesics, we may assume $F(t)\equiv 0$, and the equation on $G(t)$ is essential. As the leaves of the unstable foliations are filled up with parallel lifts of geodesics, we want to solve the above equation on $S^1\Sigma_g$ along every orbit of the geodesic flow and to find a special solution which determines the unstable direction. On $S^1\Sigma_g$ the equation is expressed as

$$\xi_1^2 G+(k\circ\pi)G=0. \tag{2.5}$$

**Proposition.** *For any smooth function $K(t)$ on $\boldsymbol{R}$ such that there exist negative constants $\alpha$ and $\beta$ satisfying*

$$\alpha<K(t)<\beta<0,$$

*the second order linear ordinary differential equation*

$$\frac{d^2G(t)}{dt^2}+K(t)G(t)=0$$

*has two families $\tilde{S}_+$ and $\tilde{S}_-$ in its 2-dimensional solution space $\tilde{S}$ as is shown below.*

(1) $\tilde{S}_+=\{G(t);\ G(t)\in\tilde{S},\ G(t)>0,\ G'(t)>0$ *for any* $t\in\boldsymbol{R}\}$
$\tilde{S}_-=\{G(t);\ G(t)\in\tilde{S},\ G(t)>0,\ G'(t)<0$ *for any* $t\in\boldsymbol{R}\}$

(2) $\tilde{S}_+$ *and* $\tilde{S}_-$ *are open half-lines with their end points at the origin in* $\tilde{S}$ *and are linearly independent, so that* $\tilde{S}_+$ *and* $\tilde{S}_-$ *span the whole solution space* $\tilde{S}$.

**Remark.** We apply this proposition taking $k(\gamma(t))$ for $K(t)$. In the case of negative constant curvature $k(\gamma(t))\equiv -1$, we have special solutions $G_+(t)=\exp(t)$ and $G_-(t)=\exp(t)$ which belong to $\tilde{S}_+$ and $\tilde{S}_-$ respectively, so that the proposition is trivially true in this case.

*Proof.* It is enough to show the following three statements.

(3) $\tilde{S}_+\neq\emptyset$, $\tilde{S}_-\neq\emptyset$.

(4) $\tilde{S}_+$ and $\tilde{S}_-$ are closed under the multiplications by positive real constants.

(5) Any two elements of $\tilde{S}_+$ are linearly dependent and the same is true for $\tilde{S}_-$.

Since the conditions we have to consider are invariant under the multiplications by positive reals and we do not take care of the trivial solution, (4) is trivial and we may projectify the deleted solution space $\tilde{S}-\{0\}$ with respect to multiplications by positive reals and make it into a circle $S^1$. In short we want to show that each of $S_+$ and $S_-$ which are the projectified images of $\tilde{S}_+$ and $\tilde{S}_-$ in $S^1$ consists of a single point in the projectified solution space $S^1$. We set

$$\tilde{P}=\{G(t)\in\tilde{S};\ G(t)\not\equiv 0,\ G(t)\geqq 0 \text{ for any } t\in \boldsymbol{R}\},$$
$$P=\{\text{the class of } G(t) \text{ in } S^1;\ G(t)\in\tilde{P}\},$$

and for any $s\in\boldsymbol{R}$,

$$\tilde{P}(s)=\{G(t)\in\tilde{P};\ G'(t)\geqq 0 \text{ for any } t\geqq s\},$$
$$P(s)=\{\text{the class of } G(t) \text{ in } S^1;\ G(t)\in\tilde{P}(s)\},$$

then $\tilde{P}$ is a closed convex cone from which the vertex has been removed and $P$ is a closed interval in $S^1$ and in particular they are not vacuous. It is easy to see that $P(s)$ is also a closed interval in $P$ and $P(s_1)\subset P(s_2)$ for any $s_1\leqq s_2$. Therefore we can conclude that

$$S_+=\bigcap_{s\in\boldsymbol{R}} P(s)=\lim_{s\to-\infty} P(s)$$

is not vacuous from the compactness of each $P(s)$ and the finite intersection property of the family $\{P(s);\ s\in\boldsymbol{R}\}$ and the similar statement is true for $S_-$. We have shown (3) and (4), and (5) follows also from an easy elementary argument using the constants $\alpha$ and $\beta$. Q.E.D.

For any solution of (2.3), we set

$$H(t)=\frac{G'(t)}{G(t)},$$

which is more essential in this problem than $G(t)$. The Jacobi equation (2.4) and (2.5) are written in terms of $H$ as

$$\frac{d}{dt}H(t)+H(t)^2+k(\gamma(t))=0, \tag{2.6}$$

$$\xi_1 H+H^2+k\circ\pi=0. \tag{2.7}$$

Then we can naturally identify the projectified solution space $S^1$ of (2.3) with the double covering of the solution space of (2.6). We set $H_\pm=G'_\pm/G_\pm$, where $H_+$ and $H_-$ correspond to $S_+$ and $S_-$ respectively through the above identification. The very important point here is that although there exist no global solutions to (2.4) on $S^1\Sigma_g$ (of course, we have 2-dimensional solution space along each geodesic), we have two solutions $H_+$ and $H_-$ which are the only global solutions to (2.7) on $S^1\Sigma_g$. Remark that in the hyperbolic case $H_\pm \equiv \pm 1$.

Let $J_\pm$ be the Jacobi field along a geodesics $\gamma$ corresponding to $G_\pm$. The deformation of $\gamma$ by geodesics along $J_+$ defines the unstable direction in the normal to the flow which together with the flow direction spans the tangent space of the unstable foliation $\mathscr{F}^u$. At any point $\gamma'(0) \in S^1\Sigma_g$, in the first order approximation, we may take $G_+(\gamma(0))$ to be the distance between $\gamma(0)$ and another geodesic deformed along $J_+$ and may take $G'_+(\gamma(0))$ to be the difference of the direction of the two geodesics. Thus the tangent space $T_v\mathscr{F}^u$ to the unstable foliation at $v=\gamma'(0) \in S^1\Sigma_g$ is spanned by $\xi_1$ and $G'_+(x)w+G_+(x)\xi_2$ where $x=\gamma(0)$. Normalizing the second by $G_+(x)$, we have

$$T_x\mathscr{F}^u = \langle \xi_1, H_+w+\xi_2 \rangle,$$

so that a 1-form $\omega_0=\omega-H_+\theta_2$ defines $\mathscr{F}^u$. We can only guarantee that $H_+$ is of class $C^1$ (see [10]), but from now on we assume that it is of class $C^2$ and calculate the Godbillon-Vey number of $\mathscr{F}^u$. The following dual expression of the structure equations (2.1) and (2.2) is useful to compute the exterior differentials of forms,

$$[\xi_1, \xi_2]=kw, \quad [\xi_2, w]=\xi_1, \quad [w, \xi_1]=\xi_2. \tag{2.8}$$

From (2.1)–(2.8) we have

$$\begin{aligned}
&d\omega_0=\omega_0\wedge\omega_1,\\
&\omega_1=H_+\theta_1-(wH_+)\theta_2,\\
&d\omega_1=2wH_+\omega\wedge\theta_1+(H_+-w^2H_+)\omega\wedge\theta_2-(\xi_2H_++\xi_1wH_+)\theta_1\wedge\theta_2,\\
&\omega_1\wedge d\omega_1=\{-H_+^2+H_+(w^2H_+)-2(wH_+)^2\}\omega\wedge\theta_1\wedge\theta_2,
\end{aligned}$$

and integrating the last form, we get the Godbillon-Vey number

$$G.V.(\mathscr{F}^u)=\int_{S^1\Sigma_g}\omega_1\wedge d\omega_1.$$

We set

$$\begin{aligned}
&I_1=-H_+^2\omega\wedge\theta_1\wedge\theta_2,\\
&I_2=\{H_+(w^2H_+)-2(wH_+)^2\}\omega\wedge\theta_1\wedge\theta_2,\\
&I_1+I_2=\omega_1\wedge d\omega_1,
\end{aligned}$$

and we can see easily that the 3-form $I_1$ is congruent to the 3-form $(k\circ\pi)\omega\wedge\theta_1\wedge\theta_2$ modulo exact forms. As $\theta_1\wedge\theta_2$ is just the pullback of the volume form of $\Sigma_g$,

$$\int_{S^1\Sigma_g} I_1=\int_{S^1\Sigma_g} k\circ\pi\omega\wedge\theta_1\wedge\theta_2$$
$$=2\pi\int_{\Sigma_g} k\cdot\text{the volume form of } \Sigma_g.$$

By the classical Gauss-Bonnet formula, we have

$$\int_{S^1\Sigma_g} I_1=4\pi^2\chi(\Sigma_g)=8\pi^2(1-g).$$

**Remark.** In the hyperbolic case $H_+\equiv 1$, so that $I_2\equiv 0$ and we obtain

$$G.V.(\mathscr{F}^u)=8\pi^2(1-g).$$

The variation of the Godbillon-Vey number with respect to the perturbation of the metric appears as the integral of $I_2$.

Now we compute the integration of $I_2$. From (2.1),

$$d(\theta_1\wedge\theta_2)=0,$$

and

$$w^2(H_+^2)=w\{2H_+(wH_+)\}=2\{H_+(w^2H_+)+(wH_+)^2\}.$$

Thus we obtain

$$\begin{aligned}&H_+(w^2H_+)\omega\wedge\theta_1\wedge\theta_2\\&\quad=-w(H_+^2)\omega\wedge\theta_1\wedge\theta_2+\frac{1}{2}w^2(H_+^2)w\wedge\theta_1\wedge\theta_2\\&\quad=-w(H_+^2)\omega\wedge\theta_1\wedge\theta_2+\frac{1}{2}d(w(H_+^2)\theta_1\wedge\theta_2),\end{aligned}$$

so that

$$\int_{S^1\Sigma_g} I_2=\int_{S^1\Sigma_g} -3(wH_+)^2\omega\wedge\theta_1\wedge\theta_2$$

and finally

$$G.V.(\mathscr{F}^u)=8\pi^2(1-g)-3\int_{S^1\Sigma_g}(wH_+)^2\omega\wedge\theta_1\wedge\theta_2. \tag{2.9}$$

**Lemma.** *If $H_+$ is constant along the fibres,* i.e., $wH_+\equiv 0$, *then the metric on $\Sigma_g$ has a constant negative curvature.*

*Proof.* For any geodesic $\gamma(t)$, let $-\gamma(t)$ be the geodesic $\gamma(-t)$ namely the reverse of $\gamma$. Then clearly the solution $H_+$ along $-\gamma$ is nothing but $-H_-$ along $\gamma$, so that, combined with the condition of this lemma,

$$H_+=-H_- \quad \text{on } S^1\Sigma_g, \tag{2.10}$$

and we get the following by putting (2.10) into (2.7).

$$\xi_1 H_+=\xi_1 H_-. \tag{2.11}$$

On the other hand, from (2.10) trivially we have

$$\xi_1 H_+=-\xi_1 H_-. \tag{2.12}$$

Therefore we have $\xi_1 H_+=0$, i.e., $H_+$ is constant along the fibres and also along the orbits of the geodesic flow. Since any two points in $S^1\Sigma_g$ can be joined by a path consisting of three arcs such that one of them is on a single orbit of the geodesic flow and each of the other two is on a single fibre, $H_+$ is constant on $S^1\Sigma_g$. Thus again from (2.7), $k\circ\pi$ turns out to be constant. Q.E.D.

From this lemma and (2.9), Theorem follows.

## References


[1] Anosov, D. V., Geodesic flow on closed Riemannian manifolds with negative curvature, Proc. Steklov Inst. Math., (1967), A.M.S. Translation (1969).

[2] Bott, R., Lectures on characteristic classes of foliations, Springer Lect. Notes in Math., **279** (1972).

[3] ——, On some formulas for the characteristic classes of group actions, Springer Lect. Notes in Math., **652** (1978), 25–61.

[4] Brooks, R., Appendix to [3] 57–61.

[5] Duminy, G., L'invariant de Godbillon-Vey d'un feuilletage se localise dans les feuilles ressort, Preprint (1982).

[6] Godbillon, C. and J. Vey, Un invariant des feuilletages de codimension 1, C. R. Acad. Sci. Paris, **A273** (1973), 92–95.

[7] Heitsch, J. L., A cohomology for foliated manifolds, Comment. Math. Helv., **50** (1975) 197–218.

[8] ——, Derivatives of secondary characteristic classes, J. Differential Geom., **13** (1978), 311–339.

[9] ——, Independent variation of secondary classes, Ann. of Math., **108** (1978), 421–460.

[10] Hirsch, M. W., C. C. Pugh and M. Shub, Invariant manifolds, Springer Lect. Notes in Math., **585** (1977).

[11] Kobayashi, S. and K. Nomizu, Foundations of differential geometry I and II, John Wiley and Sons, Inc. (1963,9).

[12] Milnor, J., Morse theory, Ann. of Math. Studies **51**, Princeton Univ. Press, (1973).
[13] Raby, G., L'invariant de Godbillon est stable par $C^1$-difféomorphisme, Preprint (1981).
[14] Tsuboi, T., On the homology of classifying spaces for foliated products, in this volume.

*Department of Mathematics*
*Faculty of Science*
*University of Tokyo*
*Tokyo 113, Japan*