A COLLECTION OF PAPERS IN MATHEMATICS AND RELATED SCIENCES,
A FESTSCHRIFT IN HONOUR OF THE LATE GALAYE DIA
EDITORS : PROFS HAMET SEYDI, GANE SAMB LO, ABOUBAKARY DIAKHABY
SPAS-EDS (SAINT-LOUIS, CALGARY, 2018), WWW.STATPAS.ORG/SPASEDS/
IN EUCLID (WWW.PROJECTEUCLID.ORG)
DOI : 10.16929/SBS/2018.100

CHAPTER 12

On some properties of the new Sine-skewed Cardioid Distribution, *by C.M.M Traoré, M. Diallo, G.S. Lo, M. Ahsanullah, L. C. Okereke*

Cherif Mamadou Moctar Traoré, USTTB & UGB, (cheriftraore75@yahoo.com)
Moumouni Diallo (moudiallo1@gmail.com).
Gane Samb Lo, UGB & AUST, (gane-samb.lo@edu.ugb.sn, gslo@aust.edu.ng, ganesamblo@ganesamblo.net)
Mouhamad Ahsanullah, RU, (ahsan@rider.edu)
Lois Chinwendu Okereke, AUST, (lokereke@aust.edu.ng)

Abstract. The new Sine Skewed Cardioid (ssc) distribution very newly introduced by Ahsanullah (2018) is further studied here(See Full Abstract).

Keywords. Sine Skewed Cardioid Distribution; Extreme Value Domain; Moment Estimator; Sample generation; Tails
AMS 2010 Mathematics Subject Classification. 62G20; 62G32.

Cite the chapter as :
Traoré C.M.M., Diallo M., Lo G.S., M. Ashanallah and Okereke L.C. (2018). On some properties of the new Sine-skewed Cardioid Distribution . In *A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia* (Editors : Seydi H., Lo G.S. and Diakhaby A.). Spas Editions, Euclid Series Book, (Doi : 10.16929/sbs/2018.100). pp. 221 –243.
Doi : 10.16929/sbs/2018.100-02-09

A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia. **Traoré C.M.M., Diallo M., Lo G.S., M. Ahsanullah and Okereke L.C. (2018) On some properties of the new Sine-skewed Cardioid Distribution.** Pages 221 — 243.

The authors.
Cherif Mamadou Moctar Traoré (cheriftraore75@yahoo.com)
Université des Sciences, des Technique et des Technologies, Bamako, Mali
Moumouni Diallo (moudiallo1@gmail.com).
Université des Scences Economiques et de Gestion (USEG), Bamako, Mali.
Gane Samb Lo. (gane-samb.lo@edu.ugb.sn, gslo@aust.edu.ng, ganesamblo@ganesamblo.net)
LERSTAD, Gaston Berger University, Saint-Louis, Sénégal, AUST - African University of Sciences and Technology, Abuja, Nigeria
Mouhamad Ahsanullah. (ahsan@rider.edu)
Department of Management Sciences. Rider University. Lawrenceville, New Jersey, USA
Lois Chinwendu Okereke (lokereke@aust.edu.ng). AUST - African University of Sciences and Technology, Abuja, Nigeria

Full Abstract. The new Sine Skewed Cardioid (ssc) distribution been just introduced and characterized by Ahsanullah (2018). Here, we study the asymptotic properties of its tails by determining its extreme value domain, the characteristic function, the moments and likelihood estimators of the two parameters, the asymptotic normality of the moments estimators and the random generation of data from the *ssc* distribution. Finally, we proceed to a simulation study to show the performance of the random generation method and the quality of the moments estimation of the parameters.

1. Introduction

Ahsanullah (2018) introduced the new distribution with probability distribution function (*pdf*)

$$f(x) = \frac{1}{2\pi}(1 + \lambda \sin x)(1 + \rho \cos x)1_{[-\pi,\pi]},$$

associated with the parameters $(\lambda, \rho) \in [-1, 1]^2$ and named as the sine-skewed cardioid distribution .

In this note, we will state a number properties of that new law. In particular, we are going asymptotic properties of its tails by determining its extreme value domain, the moments and likelihood estimators of the two parameters, the asymptotic normality of the moments estimators and the random generation of data from the *ssc* distribution. But, before we proceed, we recall for Ahsanullah (2018) that the cumulative distribution function (cdf) is

$$F(x) = \frac{1}{2} + \frac{1}{2\pi}\left(x - \lambda(\cos x + 1) + \rho \sin x - \frac{\lambda\rho}{4}(\cos 2x - 1)\right), \ x \in [-\pi, \pi].$$

To prepare studying the upper tail (in a neighborhood of π) and the lower tail (in a neighborhood of $-\pi$), we may write, respectively, for $x \in [-\pi, \pi]$,

$$F(x) = 1 - \frac{1}{2\pi}\left((\pi - x) - \lambda(\cos x + 1) + \rho \sin(\pi - x) - \frac{\lambda\rho}{4}(\cos 2x - 1)\right) \tag{1.1}$$

and

$$F(x) = \frac{1}{2\pi}\left((x + \pi) - \lambda(\cos x + 1) - \rho \sin(\pi + x) - \frac{\lambda\rho}{4}(\cos 2x - 1)\right). \tag{1.2}$$

We will have to deal with some statistical properties. So, we suppose that we have a sequence X, X_1, X_2, $\cdots$, etc. of real-valued random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P}$ with *cdf* F, then supported by $[-\pi, \pi]$. We define the sequence of empirical maxima and the minim

$$X_{1,n} = \min(X_1, \cdots, X_n) \ and \ X_{n,n} = \max(X_1, \cdots, X_n), \ n \geq 1$$

and the empirical moments

$$\overline{X}_n = \frac{1}{n}\sum_{1 \leq j \leq n} X_j \ , \ S_n^2 = \frac{1}{n-1}\sum_{1 \leq j \leq n} \left(X_j - \overline{X}_n\right)^2, \ n \geq 2$$

and the non-centered second moment

$$m_{2,n} = \frac{1}{n}\sum_{1 \leq j \leq n} X_j^2.$$

Some graphical illustrations of the *pdf* for some values of the parameters (λ, ρ) are also given by Ahsanullah (2018) in Figure 2

The rest of the paper is organized as follows. Section 2 is devoted a study of the extreme behavior the the tails of the *ssc cdf*. In Section 3, we deal with the moments and likelihood estimation of the parameters λ and ρ and

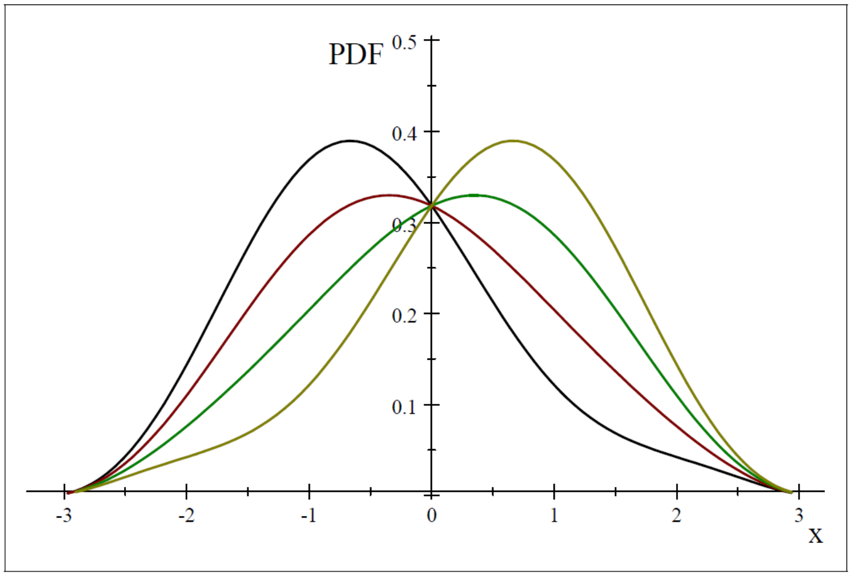

FIGURE 1. PDF of $f_{sc}(x,1,\lambda)$,Black-$\lambda=-0.6$, Red- $\lambda=-0.2$, Green-$\lambda=0.2$ and Brown-$\lambda=0.6$

determine the asymptotic laws of the moments estimators. In Section 4, we provide the characteristic function. Finally, in Section 5, we propose a method of generating samples from the *ssc* model. Simulation studies are undertaken to test the generation algorithm and next to test the performance of the moments estimators. VB6 Subroutines for the generation methods are given in the appendix. The paper ends by a conclusion section.

2. Asymptotic Properties of the tails

THEOREM 31. *Define the constants*

$$\begin{aligned}2\pi\alpha_1 &= 1+\rho,\ 2\pi\alpha_2=\frac{\lambda}{2}\left(1-\rho\right),\ 2\pi\alpha_3=-\frac{\rho}{6}\\ 2\pi\alpha_4 &= \frac{\lambda}{24}\left(4\rho-1\right),\ 2\pi\alpha_5=\frac{\rho}{120},\ 2\pi\alpha_6=\frac{\lambda}{126}\left(1-16\rho\right).\end{aligned}$$

We have, as $x\to\pi$,

$$\frac{\alpha_5}{\alpha_6(\pi-x)}\left(\frac{\alpha_4}{\alpha_5(\pi-x)}\left(\frac{\alpha_3}{\alpha_4(\pi-x)}\left(\frac{\alpha_3}{\alpha_2(\pi-x)}\left(\frac{\alpha_1}{\alpha_2(\pi-x)}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left(\frac{1-F(x)}{\alpha_1(\pi-x)}\right)-1\right)-1\right),-1\right)-1\right)-1=O(\pi-x)$$

and as $x \to -\pi$,

$$\frac{\alpha_5}{\alpha_6(\pi+x)}\left(\frac{\alpha_4}{\alpha_5(\pi+x)}\left(\frac{\alpha_3}{\alpha_4(\pi+x)}\left(\frac{\alpha_3}{\alpha_2(\pi+x)}\left(\frac{\alpha_1}{\alpha_2(\pi+x)}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left(\frac{F(x)}{\alpha_1(\pi+x)}\right)-1\right)-1\right)-1\right)-1\right)-1=O(\pi+x).$$

Remark : Such an expansion is limited to an order 6. But it might be given an any order $k \geq 2$.

Proof of Theorem 2. We only give elements for the proof of the first. We use the following elementary expansions

$$\begin{aligned}
\sin(\pi-x) &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}(\pi-x)^{2k+1}\\
\cos x+1 &= \cos x-\cos\pi=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k)!}(\pi-x)^{2k}\\
\cos 2x-1 &= \cos 2x-\cos 2\pi=\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}}{(2k)!}(x-\pi)^{2k}.
\end{aligned}$$

We consider the limited expansion at order 6 at π to have

$$\begin{aligned}
1-F(x) &= \alpha_1(\pi-x)+\alpha_2(\pi-x)^2+\alpha_3(\pi-x)^3+2\pi\alpha_4(\pi-x)^4\\
&\quad+\alpha_5(\pi-x)^5+\alpha_6(\pi-x)^6+O((\pi-x)^7.
\end{aligned}$$

This justifies the expansion at $+\pi$. The situation is the same at $-\pi$ from Formula . The reason is that $+\pi$ and $-\pi$ play the same roles in the developments and the two terms $-\rho\sin(\pi+\theta)$ and $\rho\sin(\pi-\theta)$ are expanded in the same way with respect to $\pi-x$ and $\pi+x$ respectively. $\square$

From Theorem 2, we directly get the extreme law of X and $Y=1/(\pi+X)$.

THEOREM 32. *We have the following properties*

(a) $F^{\star}(x) = F(\pi - \frac{1}{x}), x > 0$ *belong to the Frechet extreme value Domain* $D(H_1)$ *that is*

$$\forall \gamma > 0, \quad \lim_{x \rightarrow +\infty} \frac{1 - F^{\star}(\gamma x)}{1 - F^{\star}(x)} = \gamma^{-1}$$

and the second order condition

$$\forall \gamma > 0, \quad \lim_{x \rightarrow +\infty} \frac{1}{S(x)} \left(\frac{1 - F^{\star}(\gamma x)}{1 - F^{\star}(x)} - \gamma^{-1} \right) = 0$$

for $S(x) = \alpha_2 x / \alpha_1$, $x > 0$.

b) As a consequence F *is in the Weibull extreme value Domain* $D(H_{-1})$ *and we have*

$$n(1 + \rho) \left(X_{n,n} - \pi \right) \rightsquigarrow H_{-1}, \ as \ n \rightarrow \infty.$$

(c) To have the extreme lower law is found by using $X_{1,n} = -(-X)_{n,n}$ *and* $(-X)_{n,n}$ *and* $X_{n,n}$ *have the same limit in type, that is*

$$n(1 + \rho) \left((-X)_{n,n} - \pi \right) \rightsquigarrow H_{-1}, \ as \ n \rightarrow \infty.$$

Proof of Theorem 32. Point (a) is a direct consequence of Theorem at the first order. By Theorem 8 in Lo (2018b), we have that $F \in D(H_1)$ if and only if $F^{*} \in D(H_{-1})$. So (b) holds from (a). Furthermore, by Proposition 8 in Lo (2018b), we also have

$$\frac{X_{n,n} - uep(F)}{uep(F) - F^{-1}(1 - 1/n))} \rightsquigarrow H_{-1}, \ as \ n \rightarrow \infty. \tag{2.1}$$

Now from the expansions in Theorem , we have for any $-1 < u < 1$

$$\begin{aligned} & 1 - F(x) = u \\ \Leftrightarrow \quad & \alpha_1(\pi - x) + \alpha_2(\pi - x)^2 + ... + \alpha_6(\pi - x)^6 + O((\pi - x)^7). \end{aligned}$$

We get that, as $x \rightarrow +\pi$,

$$(\pi - x) \sim u/\alpha_1,$$

and

$$x = F^{-1}(1-u) = \pi - \frac{1}{\alpha_1}u + \frac{\alpha_2}{\alpha_1^3}u^2(1+\varepsilon_2(1)) + \cdots + \frac{\alpha_6}{\alpha_1^7}u^6(1+\varepsilon_6(1)) + O(u^7),$$

where the function $\varepsilon_h(u)$ go to zero as $u \downarrow 0$. In particular, we have

$$\pi - F^{-1}(1-u) \sim \frac{1}{\alpha_1}u = \frac{u}{1+\rho}$$

and, as $n \to +\infty$

$$\pi - F^{-1}(1-1/n) \sim \frac{1}{\alpha_1} = \frac{1}{n(1+\rho)},$$

which combined with Formula 2.1 concludes Point (b) of the proof.

Point (c) is based the *cdf* of $-X$, which is $F^{\perp}(x) = 1 - F(-x)$, for $-\pi \leq x \leq \pi$. And the expansion of $1 - F^{\perp}(x)$ gives the same expansion as in Formula at $+\pi$. We get the same conclusion as for X. ■

Interesting Pedagogical Example. As we can find in Lo (2018b), page 133, a criterion of belonging of F to $D(H_1)$, when $uep(F) = \pi$ is that F admits a derivative in a right neighborhood of π and that

$$\lim_{x\to\pi} \frac{(\pi - x)F'(x)}{1-F(x)} = 1. \tag{2.2}$$

It is also stated that the condition 2.2 holds if $F \in D(H_1)$ and F' is ultimately non-increasing as $x \nearrow \pi$. Here, the limit holds for all values of λ and ρ in $]-1,1[$ although F' is ultimately not non-increasing for some values of λ and ρ for example, for $\lambda = -0.8$ and $\rho = -0.4$, for which F' is increasing as shown in Figure 2

With theses values, if the practitioner concludes that $F \in D(H_1)$, he is wrong in the use of the rule but his conclusion is correct by coincidence. So it is important to check the ultimate decreasingness of F', which is used in the proof of the rule.

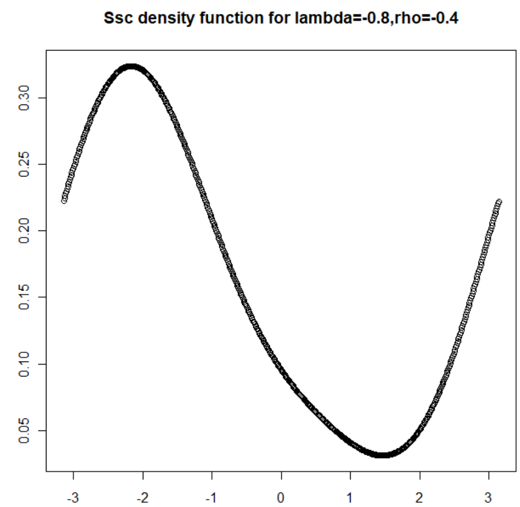

FIGURE 2. Ssc pdf for $\lambda = -0.8$ and $\rho = -0.4$

3. Parameters estimation

3.1. Moments estimation.

A. Point Estimation.

The kth moment of X is given by

$$\mathbb{E}X^k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^k(1+\lambda\sin x+\rho\cos x+(\lambda\rho/2)\sin 2x)\ dx.$$

From the following facts

$$\int_{-\pi}^{\pi} x\cos x\ dx = 0,\quad \int_{-\pi}^{\pi} x\sin x\ dx = 2\pi,\quad \int_{-\pi}^{\pi} x\cos x\sin x\ dx = -\pi/2$$
$$\int_{-\pi}^{\pi} x^2\cos x\ dx = -4\pi,\quad \int_{-\pi}^{\pi} x^2\sin x\ dx = 0,\quad \int_{-\pi}^{\pi} x^2\sin x\cos x\ dx = 0,$$

we get

$$m = \mathbb{E}X = \lambda(1-\rho/8),\ m_2 = \mathbb{E}X^2 = \pi^2/3 - 2\rho.$$

The moments estimators are solutions of the system of two equations : $m = \overline{X}_n$ and $m_2 = m_{2,n}$. We immediately have, for $n \geq 1$,

$$\hat{\rho}_n = (\pi^2/3 - m_{2,n})/2 \ and \ \hat{\lambda}_n = \frac{4\overline{X}_n}{4 - \hat{\rho}_n} = \frac{8\overline{X}_n}{8 - \pi^2/3 + m_{2,n}}. \tag{3.1}$$

We may need to check such estimation by a simulation study. We will do this in Section 4 where we propose a simple method for generating data for the *ssc* distribution. For now we want to do more on the moment problem. In the appendix, we study the integrals, for $n \geq 0$,

$$I_n = \int_{-\pi}^{\pi} x^n \cos x \ dx, \ \ J_n = \int_{-\pi}^{\pi} x^n \sin x \ dx, \ \ and \ H_n = \int_{-\pi}^{\pi} x^n \sin 2x \ dx.$$

We established the following recurrence formula

$$I_0 = 0, \forall n \geq 2, I_{2n} = -4n\pi^{2n-1} - 2n(2n-1)I_{2(n-1)} \ and \ \forall n \geq 0, I_{2n+1} = 0, , \tag{3.2}$$

$$\forall n \geq 0, J_{2n} = 0 \ and \ \forall n \geq 1, J_{2n+1} = 2\pi^{2n+1} - 2n(2n+1)J_{2(n-1)+1}, \tag{3.3}$$

and

$$\forall n \geq 0, H_{2n} = 0 \ and \ \forall n \geq 1, H_{2n+1} = -\frac{\pi^{2n+1}}{2} - \frac{n(2n+1)H_{2(n-1)+1}}{2}. \tag{3.4}$$

and for the needs of that paper, we have computed

$$\begin{array}{l} I_1 = 0, \ I_2 = (2\pi)(-2\pi), \ I_3 = 0, \ I_4 = 4(2\pi)(6 - \pi^2)) \\ J_1 = 2\pi, \ J_2 = 0, \ J_3 = (2\pi)(\pi^2 - 6), \ J_4 = 0 \\ H_1 = (2\pi)(-1/4), \ H_2 = 0, H_3 = (2\pi)((3 - 2\pi^2)/8), \ H_4 = 0. \end{array}$$

With such facts, we easily have

$$\begin{aligned} \mathbb{E}X^3 &= \lambda(\pi^2 - 6) + \frac{\lambda\rho}{8}(3 - 2\pi^2) \\ \mathbb{E}X^4 &= \frac{\pi^4}{5} - 4\rho(\pi^2 - 6). \end{aligned} \tag{3.5}$$

We will see how we need these parameters for the asymptotic laws of the moment estimators.

4. The characteristic function

PROPOSITION 2. *The characteristic function (fc) of X is given, for $t \notin \{-2,-1,0,1,2\}$, bt*

$$\psi(t)=\frac{1}{\pi}\left(\frac{1}{t}+\frac{\lambda}{i(t^2-1)}-\frac{\rho t}{t^2-1}-\frac{\lambda\rho}{i(t^2-4)}\right)sin(\pi t)$$

and is extended to values in $\{-2,-1,0,1,2\}$ by continuity of the fc.

Proof. We may write for $t \in \mathbb{R}$ and $x \in [-\pi,\pi]$,

$$\begin{aligned}
e^{it}f(x) &= \frac{1}{2\pi}e^{itx}\left[1+\frac{\lambda}{2i}\left(e^{ix}-e^{-ix}\right)+\frac{\rho}{2}\left(e^{ix}+e^{-ix}\right)+\frac{\lambda\rho}{4}\left(e^{i2x}-e^{-i2x}\right)\right]\\
&= \frac{1}{2\pi}\left[e^{itx}+\frac{\lambda}{2i}\left(e^{ix(t+1)}-e^{ix(t-1)}\right)+\frac{\rho}{2}\left(e^{ix(t+1)}+e^{ix(t-1)}\right)\right.\\
&+ \left.\frac{\lambda\rho}{4}\left(e^{ix(t+2)}-e^{-ix(t-2)}\right)\right].
\end{aligned}$$

Integrating from $-\pi$ to $+\pi$ leads to the announced results. □

B. Asymptotic Normality of the moment estimators and Statistical tests.

The moments estimators are treated by using the function empirical process defined, for any $n \geq 1$ and $h \in L^2(\mathbb{P}_X)$, by

$$\mathbb{G}_n(h)=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(h(X_j)-\mathbb{E}(h(X_j))),$$

as explained in Lo (2016) and Lo (2018a). For a more general source, the book by van der Vaart and Wellner (1996) is one the best in the field but we no need the great artillery provided there. We have the following result.

THEOREM 33. *we have the asymptotic laws of the moment estimators, as $n \rightarrow +\infty$,*

$$\left(\sqrt{n}(\hat{\rho}_n-\rho),\sqrt{n}(\hat{\lambda}_n-\lambda)\right) \rightsquigarrow \mathcal{N}_2\left(0,\Sigma\right),$$

with

$$\Sigma_{11} = \frac{\mathbb{V}ar(X^2)}{4}, \quad \Sigma_{11} = \frac{\mathbb{V}ar(X + \lambda X^2)}{\mu^2}$$

and

$$\Sigma_{12} = \Sigma_{21} = -\frac{\mathbb{C}ov(X^2, X + \lambda X^2)}{2\mu}.$$

Remark. As anticipated in Formula 3.5, the asymptotic laws need the first four moments of X.

Proof. Let us set $h_\ell(x) = x^\ell$ for $x \in \mathbb{R}$. By applying the methods in Lo (2016), we get

$$\sqrt{n}(\hat{\rho}_n - \rho) = \mathbb{G}_n(-h_2/2) + o_{\mathbb{P}(1)}.$$

Next, we have

$$\hat{\lambda}_n = \frac{8\left(m + \frac{1}{\sqrt{n}}\mathbb{G}_n(h_1)\right)}{8 - \pi^2/3 + m_2 + \frac{1}{\sqrt{n}}\mathbb{G}_n(h_2)}.$$

By Lemma 2 in Lo (2016), we get that for $\mu = 8 - \pi^2/3 + m_2$,

$$\sqrt{n}(\hat{\lambda}_n - \lambda) = \mathbb{G}_n((h_1 + \lambda h_2)/\mu) + o_{\mathbb{P}(1)}.$$

By using the functional Brownian stochastic process $\mathbb{G}$, which is the weak limit of $\mathbb{G}_n$ and defined by the variance-covariance function

$$\Gamma(h,k) = \int_{\mathbb{R}} (h(x) - \mathbb{E}h(X))(k(x) - \mathbb{E}k(X))d\mathbb{P}_X(x),$$

where $(h,k) \in L^2(\mathbb{P}_X)^2$, we get that

$$\left(\sqrt{n}(\hat{\rho}_n - \rho), \sqrt{n}(\hat{\lambda}_n - \lambda)\right) \rightsquigarrow \mathcal{N}_2(0, \Sigma),$$

with

$$\Sigma_{11} = \frac{\mathbb{V}ar(h_2(X^2)}{4}, \quad \Sigma_{11} = \frac{\mathbb{V}ar(h_1(X) + \lambda h_2(X)}{\mu^2}$$

and

$$\Sigma_{12} = \Sigma_{21} = -\frac{\mathbb{C}ov(h_2(X), h_1(X) + \lambda h_2(X))}{2\mu}. \blacksquare$$

4.1. Maximum Likelihood Estimators. we are going to see that the *ML*-estimators are not defined here. We begin the remark that the first three linear differential operators in (λ, ρ) of f are

$$\begin{aligned}
&2\pi Df(u,v) = \sin x(1+\rho\cos x)u + \cos x(1+\lambda\sin x)v,\\
&2\pi D^2 f(u,v) = (\cos x \sin x)uv, \ (u,v)\in\mathbb{R},\\
&D^3 f(u,v) = 0.
\end{aligned}$$

By applying the Taylor-Lagrange-Cauchy formula (see Valiron (1946), page 233) : for $(\lambda, \lambda_0, \rho, \rho_0) \in [-1,1]^4$, for $|\theta_i| < 1$, $i = 1, 2$

$$\begin{aligned}
f(x,\lambda,\rho) &= f(x,\lambda_0,\rho_0) + Df(\lambda-\lambda_0, \rho-\rho_0) + \frac{1}{2}D^2 f(\lambda-\lambda_0,\rho-\rho_0)\\
&+ D^2 f(\theta_1(\lambda-\lambda_0), \theta_2(\rho-\rho_0)),
\end{aligned}$$

and we get

$$\begin{aligned}
f(x,\lambda,\rho) &= f(x,\lambda_0,\rho_0) + \sin x(1+\rho\cos x)(\lambda-\lambda_0) + \cos x(1+\lambda\sin x)(\rho-\rho_0)\\
&+ (\cos x\sin x)(\lambda-\lambda_0)(\rho-\rho_0)
\end{aligned}$$

First, for $x \notin \{\mp\pi, \pm\pi/2\}$, the zeros of f are $-1/\sin x$ and $-1/\cos x$ which do not belong to $[-1,1]$. Even on $\mathbb{R}$, not requiring that f be non-negative to make it a probability density function, Formula which becomes at any critical point (λ_0, ρ_0) of this C^1-function (in λ and in ρ),

$$f(x,\lambda,\rho) = (\cos x\sin x)(\lambda-\lambda_0)(\rho-\rho_0).$$

So, for a fixed $x \notin \{\mp\pi, \pm\pi/2\}$, there can be an extremum point (λ_0, ρ_0) for the likelihood function.

5. Generation

Since the *cdf* F is explicitly known and is strictly increasing and continuous, we may use the dichotomous algorithm to find the inverse of F. It works as follows. Given $v \in [0,1]$, to find x such that $F(x) = v$, we fix the number of decimals of the solution x denoted *nbrDec*. In the Appendix, beginning by page 240, the VB 6 code for computing the *cdf* F is given in page 242, the Dichotomous algorithm is described in page 240 and implemented in VB6 in 240. Finally, the VB 6 subroutine which generates a sample of an arbitrary size is given in 242.

A. Numerical test of the computer programs.

For different values for λ and ρ in $]-1,1[$, samples of size $n = 1000$ are generated and the comparison between the exact means and the second moments as given in Formua are compared with the sample counterparts. Table 1 demonstrates the quality of the generation.

B. Moment estimation.

Based on the generation techniques as introduced above, we also report interesting performances for the estimation of the parameter when the sizes varies in Table 2. The Mean Absolute Error (MAE) and the Mean Square-root Quadratic Error (MSQE) are reported both for λ and ρ. These simulations have been for $\lambda = \rho = -0.9$.
Figure 3 shows how both errors decreases to zero.

6. Conclusion

This is an immediate contribution of the study of the Sine Skewed Caedioid distribution. Further deeper properties will be addressed later.

λ	ρ	mean (E)	mean (M)	2nd moment (E)	2nd moment (M)	quotient (Q)
0.9	-0.9	1.1025	1.2322	5.0898	5.0281	1.0122
0.9	-0.6	1.035	1.0641	4.4898	4.4796	1.0022
0.9	-0.3	0.9675	0.9396	3.8898	3.8987	0.9977
0.9	0.1	0.8775	0.8796	3.0898	3.2221	0.9589
0.9	0.4	0.81	0.8048	2.4898	2.5953	0.9593
0.9	0.7	0.7425	0.7536	1.8898	1.8690	1.0111
0.9	0.9	0.6975	0.7077	1.4898	1.4810	1.0059
-0.9	0.9	-0.6975	-0.6972	1.4898	1.5574	0.9566
-0.9	0.7	-0.7425	-0.7773	1.8898	1.8280	1.0338
-0.9	0.4	-0.81	-0.8218	2.4898	2.2965	1.0841
-0.9	0.1	-0.8775	-0.9273	3.0898	3.2964	0.9373
-0.9	-0.3	-0.9675	-1.0273	3.8898	3.9613	0.9819
-0.9	-0.6	-1.035	-0.9654	4.4898	4.4845	1.0011
-0.9	-0.9	-1.1025	-1.1300	5.0898	5.1902	0.9806

TABLE 1. Legend : (E) : Exact, (M) empirical, (Q) : quotient exact second moment to empirical second moment

Appendix.

I. Integral Computations.

Let us define, for $n \geq 0$,

Error n	10	50	100	200	300	400	500	750	1000
MAE-λ	41.17	18.44	11.88	10.41	7.39	5.93	5.43	5.08	3.91
SMQE-λ	51.97	22.49	15.04	12.07	9.81	7.74	6.78	6.17	5.20
MAE-ρ	29.83	15.20	10.05	8.51	4.97	5.56	4.59	3.56	3.14
SMQE-ρ	37.44	18.10	13.10	10.09	6.49	7.17	5.66	4.41	2.89

TABLE 2. Errors given in 100 multiples

$$I_n = \int_{-\pi}^{\pi} x^n \cos x \; dx, \; \int_{-\pi}^{\pi} x^n \sin x \; dx, \; \int_{-\pi}^{\pi} x^n \cos 2x \; dx \; and \; \int_{-\pi}^{\pi} x^n \sin 2x \; dx.$$

A - Computation of I_n. For $n = 0$,

$$I_0 = \int_{-\pi}^{\pi} \cos x \; dx = \int_{-\pi}^{\pi} d(\sin x) = 0$$

and for $n = 1$,

$$I_1 = \int_{-\pi}^{\pi} xd(\sin x) = [x \sin x]_{\pi}^{-\pi} - \int_{-\pi}^{\pi} -d(\cos x) = 0$$

and for $n = 2$

$$\begin{aligned} I_2 &= \int_{-\pi}^{\pi} x^2 d(\sin x) =]x^2 \sin x[_{\pi}^{-\pi} - 2\int_{-\pi}^{\pi} -2xd(\cos x) \\ &= \left(4[x \cos x]_{\pi}^{-\pi} - \int_{-\pi}^{\pi} \cos x)\right) \\ &= 4[x \cos x]_{\pi}^{-\pi} = -4\pi. \end{aligned}$$

Now, the general guess is that $I_{2n+1} = 0$ for $n \geq 0$. Since this already holds $n = 0$, let us remark that for $n \geq 1$,

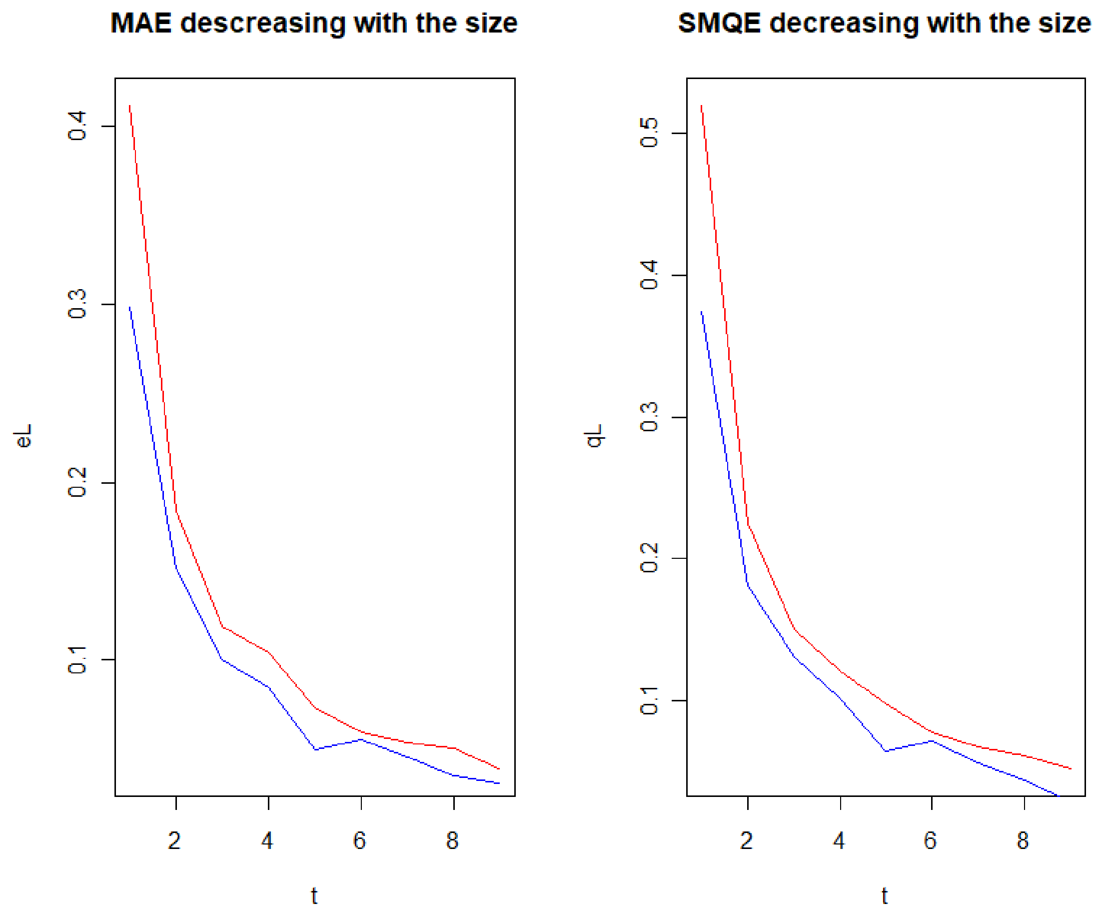

FIGURE 3. Red : related to ρ. Blue : errors related to ρ

$$\begin{aligned} I_{2n+1} &= \int_{-\pi}^{\pi} x^{2n+1} d(\sin x) =]x^{2n+1}\sin x[_{\pi}^{-\pi} - (2n+1)\int_{-\pi}^{\pi} -x^{2n} d(\cos x) \\ &= (2n+1)\int_{-\pi}^{\pi} x^{2n} d(\cos x) = (2n+1)[x^{2n}\cos x]_{\pi}^{-\pi} - (2n)(2n+1)\int_{-\pi}^{\pi} -x^{2n-1}\cos x\ dx \\ &= -(2n)(2n+1)I_{2(n-1)+1}. \end{aligned}$$

By an descendent induction, we will have $I_{2n+1} = C_n I_1 = 0$. Now, we have to find I_{2n}, for $n \geq 1$, which is

$$\begin{aligned}
I_{2n} &= \int_{-\pi}^{\pi} x^{2n} d(\sin x) =]x^{2n}\sin x[_{\pi}^{-\pi} - (2n)\int_{-\pi}^{\pi}(-x^{2n-1})d(\cos x)\\
&= (2n)\int_{-\pi}^{\pi} x^{2n-1}d(\cos x) = [x^{2n-1}\cos x]_{\pi}^{-\pi} - (2n)(2n-1)\int_{-\pi}^{\pi} -x^{2(n-1)}\cos x\ dx\\
&= -4n\pi^{2n-1} - 2n(2n-1)I_{2(n-1)}.
\end{aligned}$$

For example, we find again $I_2 = -4\pi$ for $n = 1$.

B - Computation of J_n. For $n = 0$, we have

$$J_0 = \int_{-\pi}^{\pi} \sin x\ dx = \int_{-\pi}^{\pi} d(-\cos x) = 0$$

and for $n = 1$,

$$J_1 = \int_{-\pi}^{\pi} xd(-\cos x) = -[x\cos x]_{\pi}^{-\pi} + \int_{-\pi}^{\pi} -d(\sin x) = 2\pi.$$

Our guess is that $J_{2n} = 0$ for $n \geq 0$ and we have for $n \geq 1$,

$$\begin{aligned}
J_{2n} &= \int_{-\pi}^{\pi} x^{2n} d(-\cos x) =]x^{2n}\cos x[_{\pi}^{-\pi} + (2n)\int_{-\pi}^{\pi} x^{2n-1}d(\sin x)\\
&= (2n)\int_{-\pi}^{\pi} x^{2n-1}d(\sin x) = (2n)[x^{2n-1}\sin x]_{\pi}^{-\pi} - (2n)(2n-1)\int_{-\pi}^{\pi} x^{2(n-1)}\sin x\ dx\\
&= -(2n)(2n-1)J_{(n-1)}.
\end{aligned}$$

By an descendent induction, we will have $J_{2n} = C_n J_0 = 0$. Now, he have to find J_{2n+1}, for $n \geq 1$, which is

$$\begin{aligned}
J_{2n+1} &= \int_{-\pi}^{\pi} x^{2n+1} d(-\cos x) = -]x^{2n+1}\cos x[_{\pi}^{-\pi} + (2n+1)\int_{-\pi}^{\pi} x^{2n}d(\sin x)\\
&= 2\pi^{2n+1} + (2n+1)[x^{2n}\sin x]_{\pi}^{-\pi} - (2n)(2n+1)\int_{-\pi}^{\pi} x^{2(n-1)+1}\sin x\ dx\\
&= 2\pi^{2n+1} - 2n(2n+1)J_{2(n-1)+1}.
\end{aligned}$$

C - Computation of H_n. For $n = 0$, we have

$$H_0 = \int_{-\pi}^{\pi} \sin x d(\sin x) = (1/2)\int_{-\pi}^{\pi} d(\sin^2 x) = 0,$$

and for $n = 1$,

$$\begin{aligned} H_1 &= (1/2)\int_{-\pi}^{\pi} x\sin 2x \ dx = -(1/4)\int_{-\pi}^{\pi} xd(\cos 2x) \\ &= -(1/4)\left([x\cos 2x]_{\pi}^{-\pi} - \int_{-\pi}^{\pi}(1/2)d(\sin 2x)\right) \\ &= -\pi/2. \end{aligned}$$

We still guess that $H_{2n}=0$ for $n \geq 0$ and find, for $n \geq 0$,

$$\begin{aligned} H_{2n} &= (1/2)\int_{-\pi}^{\pi} x^{2n}\sin 2x \ dx = -(1/4)\int_{-\pi}^{\pi} x^{2n}d(\cos 2x) \\ &= -(1/4)\left([x^{2n}\cos 2x]_{\pi}^{-\pi} - (1/2)(2n)\int_{-\pi}^{\pi} x^{2n-1}d(\sin 2x)\right) \\ &= (1/4)n\int_{-\pi}^{\pi} x^{2n-1}d(\sin 2x) = (1/4)n\left([x^{2n-1}\sin 2x]_{\pi}^{-\pi} - (2n-1)\int_{-\pi}^{\pi} x^{2(n-1)}\sin 2x \ dx\right) \\ &= -(1/4)n(2n-1)H_{2(n-1)}, \end{aligned}$$

and we conclude that $H_{2n}=0$ for all $n \geq 0$, based on that $H_0 = 0$ and the unveiled descendent induction formula above. Nor for all $n \geq 0$,

$$\begin{aligned} H_{2n+1} &= (1/2)\int_{-\pi}^{\pi} x^{2n+1}\sin 2x \ dx = -(1/4)\int_{-\pi}^{\pi} x^{2n+1}d(\cos 2x) \\ &= -(1/4)\left([x^{2n+1}\cos 2x]_{\pi}^{-\pi} - (1/2)(2n+1)\int_{-\pi}^{\pi} x^{2n}d(\sin 2x)\right) \\ &= -(1/4)\left((2\pi^{2n+1}) - (1/2)(2n+1)\left([x^{2n}\sin 2x]_{\pi}^{-\pi} - (2n)\int_{-\pi}^{\pi} x^{2n-1}\sin 2x)\right)\right) \\ &= -(1/2)\pi^{2n+1} - (1/4)(1/2)(2n+1)(2n)\int_{-\pi}^{\pi} x^{2n-1}\sin 2x) \\ &= -(1/2)\pi^{2n+1} - (1/2)n(2n+1)H_{2(n-1)+1} \end{aligned}$$

We conclude

(6.1) $\quad I_0 = 0, \forall n \geq 2, I_{2n} = -4n\pi^{2n-1} - 2n(2n-1)I_{2(n-1)} \ and \ \forall n \geq 0, I_{2n+1} = 0, ,$

(6.2) $\quad \forall n \geq 0, J_{2n} = 0 \ and \ \forall n \geq 1, J_{2n+1} = 2\pi^{2n+1} - 2n(2n+1)J_{2(n-1)+1},$

and

$$\text{(6.3)} \qquad \forall n \geq 0, H_{2n} = 0 \; and \; \forall n \geq 1, H_{2n+1} = -\frac{\pi^{2n+1}}{2} - \frac{n(2n+1)H_{2(n-1)+1}}{2}.$$

For the needs of this paper, we have

$$\begin{array}{l}
I_1 = 0, \; I_2 = (2\pi)(-2\pi), \; I_3 = 0, \; I_4 = 4(2\pi)(6-\pi^2), \\
\; I_5 = 0, \; I_6 = -6(2\pi)(\pi^4 - 20\pi^2 + 120), \; I_7 = 0, \; I_8 = -8(2\pi)(\pi^6 - 42\pi^4 + 840\pi^2 - 5040) \\
J_1 = 2\pi, \; J_2 = 0, \; J_3 = (2\pi)(\pi^2 - 6), \; J_4 = 0, \\
J_5 = 2\pi(\pi^4 - 20\pi^2 + 120), \; J_6 = 0, \; J_7 = 2\pi(\pi^6 - 42\pi^4 + 840\pi^2 - 5040), \; J_8 = 0, \\
H_1 = (2\pi)(-1/4), \; H_2 = 0, H_3 = (2\pi)((3 - 2\pi^2)/8), \; H_0 = 0, \\
H_5 = -2\pi\dfrac{(2\pi^4 - 10\pi^2 + 15)}{8}, \; H_6 = 0, \; H_7 = -2\pi\dfrac{(4\pi^6 - 42\pi^4 + 210\pi^2 - 315)}{16}.
\end{array}$$

$$\begin{aligned}
\mathbb{E}X^5 &= \lambda(\pi^4 - 20\pi^2 + 120) - \frac{\lambda\rho}{16}(2\pi^4 - 10\pi^2 + 15) \\
\mathbb{E}X^6 &= \frac{\pi^6}{7} - 6\rho(\pi^4 - 20\pi^2 + 120) \\
\mathbb{E}X^7 &= \lambda(\pi^6 - 42\pi^4 + 840\pi^2 - 5040) - \frac{\lambda\rho}{32}(4\pi^6 - 42\pi^4 + 210\pi^2 + 315) \\
\mathbb{E}X^8 &= \frac{\pi^8}{9} - 8\rho(\pi^6 - 42\pi^4 + 840\pi^2 - 5040).
\end{aligned}$$

II. Samples generations.

Description of the dichotomous algorithm.

```
0. Fix the count index of the decimals countDec to −1 : countDec=−1.

1. Fix x=1 and compute F(1).

2. IF F(1)=v. Take x=1. Stop

3. IF F(1)<v, then

3a. Fix x=1 and set h=1.
3b. Progressively increment x by h and test
and compare F(x) and v.
3bA. IF F(x)=v. x is the searched number.Stop
3bB. IF F(x)<v, Goto 3bA and continue
the incrementation.
3bC. IF F(x)>v, then
Decrement by h : x=x−h
Increment countDec
If countDec<nbrDec
take h=h/10
Goto 3b
If countDec=nbrDec. Stop

3. IF F(1)>v, then

4a. Fix x=1 and set h=0.1
4b. Progressively decrement x by h : x=x−h and test and compare
F(x) and v.
4bA. IF F(x)=v. x is the searched number.Stop
4bB. IF F(x)>v, Goto 4bA and continue
the decrementation
4bC. IF F(x)<v, then
increment by h : x=x−h
Increment countDec
If countDec<nbrDec−1
take h=h/10
Goto 4b
If countDec=nbrDec. Stop
```

Implementation of the dichotomous algorithm in Visual Basic 6.

```
Function sscinv(v As Double, lambda As Double, rho As Double, nbrDec) As Double
```

```
Dim dich As Double, count As Integer, h As Double
Dim countDec


sscinv = 1
dich = ssc(sscinv, lambda, rho)

If (dich = v) Then
Exit Function
End If

If (dich < v) Then
h = 1
count = 0
countDec = -1
While ((countDec < nbrDec) And (count < 2000))
count = count + 1
sscinv = sscinv + h
If (ssc(sscinv, lambda, rho) = v) Then
Exit Function
End If

If (ssc(sscinv, lambda, rho) > v) Then
countDec = countDec + 1
sscinv = sscinv - h
h = h / 10
End If
Wend
End If

If (dich > v) Then
h = 0.1
count = 0
countDec = 0
While ((countDec < nbrDec) And (count < 2000))
count = count + 1
sscinv = sscinv - h
If (ssc(sscinv, lambda, rho) = v) Then
Exit Function
End If

If (ssc(sscinv, lambda, rho) < v) Then
countDec = countDec + 1
```

```
sscinv = sscinv + h
h = h / 10
End If

Wend
End If
End Function
```

VB6 Subroutine for cumputing the *cdf* F.

```
Private Function ssc(x As Double, lambda As Double, rho As Double) As Double
Dim pi As Double

pi = 4 * Atn(1)
ssc = (1 / 2) + (1 / (2 * pi)) * (x - (lambda * (Cos(x) + 1)) + rho * Sin(x)
- (((lambda * rho) / 4) * (Cos(2 * x) - 1)))

End Function
```

rssc : VB6 Subroutine for generating samples from the ssc distribution, which the sample in the file *ssc100*.

```
Sub rssc(tail As Integer, lambda As Double, rho As Double, nbrDec As Integer)
Dim ii As Integer, cheminD As String

cheminD = "G:\backup\DataGslo\gslo\TveSimulation\ssc\"
Open cheminD & "ssc100.txt" For Output As #1

Randomize Timer
For ii = 1 To tail
Print #1, sscinv(Rnd(1), lambda, rho, nbrDec)
Next
Close #1
End Sub
```

Bibliography

Ahsanullah M. (2018). Sine-Skewed Cardioid Distribution. In *A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia* (Editors : Seydi H., Lo G.S. and Diakhaby A.). Spas Editions, Euclid Series Book. Doi : 10.16929/sbs/2018.100-10-01

Lo G.S.(2018). Weak Convergence (IIIA) - A handbook of Asymptotic Representations of Statistics in the Functional Empirical process and Applications. Arxiv :

Lo G.S.(2016). How to use the functional empirical process for deriving asymptotic laws for functions of the sample. ArXiv:1607.02745.

Lo G.S.(2018). Weak Convergence (IIA) - Functional and Random Aspects of Univariate Theory of Extreme Value Theory. Arxiv : 1810.01625.

Valiron, G.(1946). *Théorie des Functions*. Masson. Paris.

van der Vaart A. W. and Wellner J. A.(1996). *Weak Convergence and Empirical Processes With Applications to Statistics*. Springer, New-York.