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Reverse Exchangeability and Extreme Order Statistics

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Abstract: For a bivariate random vector (X, Y), symmetry conditions are presented that yield stochastic orderings among $|X|$, $|Y|$, $|\max(X, Y)|$, and $|\min(X, Y)|$. Partial extensions of these results for multivariate random vectors $(X_1, \ldots, X_n)$ are also given.

1. Introduction

Jiang (2009) introduced a new estimator of value-at-risk (among other risk and performance measures) for investment funds with short performance histories. In deriving its large sample variance, Jiang made use of the following identity (with some rearrangement; see Jiang (2009, Pg. 106, Eq. (3.6.2.4))):

$$\Phi_2(x, x; \rho) - \Phi_2(-x, -x; \rho) = \Phi(x) - \Phi(-x), \quad \forall x \geq 0, \tag{1}$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution and $\Phi_2(\cdot, \cdot; \rho)$ is the cdf of the standard bivariate normal distribution with correlation ρ.

The result (1) was somewhat unexpected because the left hand side is seemingly dependent on ρ. Note that the left hand side is in fact the cdf of $|\max(X, Y)|$, while the right hand side is the cdf of $|X|$, where (X, Y) is distributed as the standard bivariate normal distribution with correlation ρ. Hence (1) implies that

$$|\max(X, Y)| \stackrel{d}{=} |X|. \tag{2}$$

It is natural to wonder whether this simple but elegant result extends to bivariate distributions other than the standard bivariate normal distribution. Theorem 1 shows that it does hold for a broad range of bivariate distributions that are *reverse exchangeable*.

Next we consider multivariate distributions. For any sequence $X_1, X_2, \ldots$ of random variables, clearly $\max(X_1, \ldots, X_n)$ is nondecreasing in n, but this need not be true for $|\max(X_1, \ldots, X_n)|$: simply consider a non-random sequence that begins

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with $-1, 0$. Furthermore, as illustrated by Example 8, (2) need not hold even for multivariate distributions with strong symmetries. In Theorems 2 and 5, however, it is shown that $|\max(X_1, \ldots, X_n)|$ is stochastically nondecreasing in n under either of two fairly non-restrictive multivariate extensions of reverse exchangeability.

A series of examples are presented that illustrate the general results.

2. Reverse Exchangeability for Bivariate Distributions

Definition 1. The bivariate random vector (X, Y) is called *reverse exchangeable (RE)* if $(X, Y) \stackrel{d}{=} (-Y, -X)$, that is, (X, Y) and $(-Y, -X)$ are identically distributed.

Reverse exchangeability simply means that the joint distribution of (X, Y) is symmetric about the line $y = -x$. Recognizing this allows us to state the condition in terms of simple reflection. Imagine rotating the plane clockwise by 45°, so the symmetry line $y = -x$ becomes the vertical axis. The point (X, Y) is rotated to

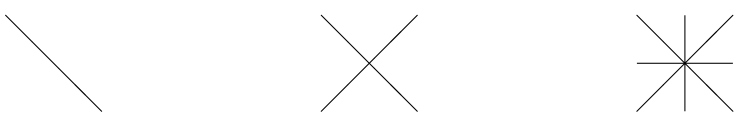
$$(U,\, V) := \left(\frac{X + Y}{\sqrt{2}}, \frac{X - Y}{\sqrt{2}} \right).$$

Then we have the following result:[1]

Proposition 1. *(X, Y) is RE if and only if the conditional distributions of U and $-U$ given V are the same, i.e.,*

$$(U \mid V = v) \stackrel{d}{=} (-U \mid V = v), \quad \text{for a.e. } v \in (-\infty, \infty). \tag{3}$$

Reverse exchangeability is a rather weak condition. For instance, if X, Y are iid (independent and identically distributed) and $X \stackrel{d}{=} -X$ then (X, Y) is RE, but the converse is not true. A condition weaker than iid but still sufficient for RE is that the distribution of (X, Y) be ESCI, that is, exchangeable (E) and sign-change-invariant (SCI).[2] Clearly ESCI is strictly stronger than RE since ESCI also implies symmetry about the line $y = x$, as well as symmetry about both coordinate axes. It is too strong for our purposes, however, since it is not satisfied by the class of standard bivariate elliptical distributions (i.e. with location parameter $(0, 0)$ and identical marginals) with nonzero correlation. Examples of interest include the standard bivariate normal and bivariate t distributions.



(a) RE (b) ERE (c) ESCI

Fig 1: Three bivariate symmetry conditions.

There is a symmetry condition intermediate between RE and ESCI, namely that (X, Y) is both exchangeable (E) and reverse exchangeable (RE), designated by

[1]Condition (3) can be stated as $(U, V) \stackrel{d}{=} (-U, V)$, but we use conditional distributions since this allows for a natural generalization – see Definition 2.

[2]The ESCI condition is equivalent to group-invariance under the dihedral group generated by all permutations and sign-changes of coordinates. See Eaton (1982, 1987); Eaton and Perlman (1977) for discussions of group-invariance.

ERE, i.e., it is symmetric about the line $y = -x$ and the line $y = x$. See Figure 1 for a comparison of the three symmetry conditions. All standard bivariate elliptical distributions are ERE, while any such distribution re-centered at any point on the line $y = -x$ except the origin satisfies RE but not ERE.

Example 1. A class of bivariate distributions that is ERE but not ESCI arises from sampling without replacement from a finite set A of real numbers that is symmetric about 0, i.e., $A = -A$. If (X, Y) is such a sample from any finite set A with $|A| \geq 2$, then (X, Y) is exchangeable since

$$\Pr[X = a, Y = b] = \frac{1}{|A|} \cdot \frac{1}{|A| - 1}, \ \forall\, a, b \in A, \ a \neq b. \tag{4}$$

If in addition $A = -A$ then (X, Y) is RE:

$$\begin{aligned} \Pr[-Y = a, -X = b] &\equiv \Pr[-X = a, -Y = b] \\ &= \Pr[X = a, Y = b] \end{aligned}$$

by exchangeability and symmetry. Thus (X, Y) is ERE, but it is not ESCI: for any nonzero $a \in A$,

$$\Pr[X = -a, Y = a] = \frac{1}{|A|} \cdot \frac{1}{|A| - 1} \neq 0 = \Pr[X = a, Y = a].$$

□

Our first result states that if (X, Y) is RE, its absolute marginal distributions are identical to those of its extreme order statistics:

Theorem 1. *If (X, Y) is reverse exchangeable, then*

$$|\max(X, Y)| \stackrel{d}{=} |\min(X, Y)| \stackrel{d}{=} |X| \stackrel{d}{=} |Y| \tag{5}$$

Theorem 1 follows directly from Proposition 2, which holds under weaker RE conditions.

Definition 2. The bivariate random vector (X, Y) is called *upper (lower) reverse exchangeable*, designated by *URE (LRE)*, if the conditional distributions of U and $-U$ given $V = v > 0$ $(v < 0)$ are the same, i.e.,

$$(U \mid V = v) \stackrel{d}{=} (-U \mid V = v), \quad \text{for a.e. } v > 0 \ (v < 0).$$

Clearly RE $\implies$ URE and LRE. The converse need not be true if $\Pr[V = 0] > 0$, i.e. if $\Pr[X = Y] > 0$, since neither URE nor LRE ensures that $U \stackrel{d}{=} -U \mid V = 0$.

For any $x \geq 0$, define the events (see Figure 2)

$$N_x := \{|X| \leq x < Y\}, \tag{6}$$

$$S_x := \{|X| \leq x < -Y\}, \tag{7}$$

$$E_x := \{|Y| \leq x < X\}, \tag{8}$$

$$W_x := \{|Y| \leq x < -X\}, \tag{9}$$

$$C_x := \{|X| \leq x, \ |Y| \leq x\}. \tag{10}$$

For any random variable Z, let F_Z denote its cdf. Clearly

$$F_{|X|}(x) = \Pr[N_x] + \Pr[C_x] + \Pr[S_x], \tag{11}$$
$$F_{|Y|}(x) = \Pr[W_x] + \Pr[C_x] + \Pr[E_x], \tag{12}$$
$$F_{|\max(X,Y)|}(x) = \Pr[W_x] + \Pr[C_x] + \Pr[S_x], \tag{13}$$
$$F_{|\min(X,Y)|}(x) = \Pr[N_x] + \Pr[C_x] + \Pr[E_x]. \tag{14}$$

Therefore,

$$F_{|X|}(x) - F_{|\max(X,Y)|}(x) = \Pr[N_x] - \Pr[W_x] = F_{|\min(X,Y)|}(x) - F_{|Y|}(x), \tag{15}$$
$$F_{|X|}(x) - F_{|\min(X,Y)|}(x) = \Pr[S_x] - \Pr[E_x] = F_{|\max(X,Y)|}(x) - F_{|Y|}(x). \tag{16}$$

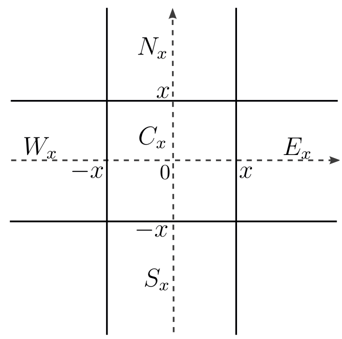


Fig 2: The union of the two closed strips $\{|X| \leq x\}$ and $\{|Y| \leq x\}$. The regions N_x, S_x, E_x, W_x, and C_x are disjoint.

Proposition 2.

(*i*) (X, Y) URE $\implies |\max(X,Y)| \overset{d}{=} |X|$ *and* $|\min(X,Y)| \overset{d}{=} |Y|$;

(*ii*) (X, Y) LRE $\implies |\max(X,Y)| \overset{d}{=} |Y|$ *and* $|\min(X,Y)| \overset{d}{=} |X|$;

(*iii*) (X, Y) URE *and* LRE $\implies |\max(X,Y)| \overset{d}{=} |\min(X,Y)| \overset{d}{=} |X| \overset{d}{=} |Y|$.

Proof. Since (X, Y) URE $\Rightarrow \Pr[N_x] = \Pr[W_x]$ and (X, Y) LRE $\Rightarrow \Pr[S_x] = \Pr[E_x]$, the results follow from (15) and (16) □

Example 2. If X, Y are iid standard normal random variables, then (X, Y) is RE. Thus if $M = \min(X, Y)$ or $M = \max(X, Y)$, then Theorem 1 implies $|M| \overset{d}{=} |X| \overset{d}{=} |Y|$, hence

$$M^2 \overset{d}{=} X^2 \overset{d}{=} Y^2 \sim \chi_1^2. \tag{17}$$

This result appeared in Casella and Berger (2002, Exercise 5.22). □

This example can be extended by relaxing normality and/or relaxing independence:

Example 3. If X, Y are iid whose common distribution is symmetric about 0, then clearly (X, Y) is ESCI, hence RE. For $M = \max(X, Y)$, Theorem 1 implies that $|M| \overset{d}{=} |X|$. This can be verified directly from the iid assumption, as follows.

For any $x \geq 0$ let $u = \Pr[X > x]$. Then

$$\begin{aligned}
\Pr[|M| \leq x] &= \Pr[M \leq x] - \Pr[M < -x] \\
&= \left(\Pr[X \leq x]\right)^2 - \left(\Pr[X < -x]\right)^2 \\
&= (1-u)^2 - u^2 \\
&= (1-u) - u \\
&= \Pr[|X| \leq x].
\end{aligned}$$

Therefore $|M| \stackrel{d}{=} |X|$. A similar proof holds if $M = \min(X, Y)$. □

Example 4. (Example 2 extended). Suppose that

$$(X, Y) \sim N_2\left((\mu, -\mu), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right),$$

the bivariate normal distribution with means μ and $-\mu$ ($-\infty < \mu < \infty$), variances 1, and correlation $\rho \in (-1, 1)$. Then (X, Y) is not ESCI but is RE, so Theorem 1 implies that for $M = \max(X, Y)$ or $\min(X, Y)$,

$$M^2 \stackrel{d}{=} X^2 \stackrel{d}{=} Y^2 \sim \chi_1^2(\mu^2), \tag{18}$$

the noncentral chi-square distribution with noncentrality parameter μ^2, extending (17). Note that this result does not depend on the value of ρ.

For $\mu = 0$, (18) reduces to (17) as in Example 2, and is equivalent to (1). It seems difficult to verify (1) directly in this case. □

Example 5. (Example 4 extended). Suppose that (X, Y) has a bivariate elliptical pdf on $\mathbb{R}^2$ given by

$$f(x, y) = |\Sigma|^{-1/2} g\left[(x - \mu,\ y + \mu)\, \Sigma^{-1} (x - \mu,\ y + \mu)'\right],$$

where

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Then (X, Y) is RE so $M^2 \stackrel{d}{=} X^2 \stackrel{d}{=} Y^2$ for all $\rho \in (-1, 1)$. □

Example 6. (Example 1 continued). Suppose that X, Y represent two random draws without replacement from a finite set A of real numbers that is symmetric about 0, i.e., $A = -A$. As noted in (4), (X, Y) is RE, so $|\max(X, Y)| \stackrel{d}{=} |X|$ by Theorem 1. If $0 \notin A$ then

$$\Pr[|X| = a] = \frac{2}{|A|}, \quad \text{for } a \in A,\ a > 0,$$

while if $0 \in A$ then

$$\Pr[|X| = a] = \begin{cases} \frac{1}{|A|}, & a = 0, \\ \frac{2}{|A|}, & a \in A,\ a > 0, \end{cases}$$

so these are the distributions of $|\max(X, Y)|$ (and of $|\min(X, Y)|$) as well. □

There is an obvious relation between bivariate RE and bivariate E:

Proposition 3. *(X, Y) is reverse exchangeable if and only if $(X, -Y)$ is exchangeable.*

Thus Theorem 1 has the following corollary:

Corollary 1. *If* (X,Y) *is exchangeable then*

$$|\max(X,-Y)| \stackrel{d}{=} |\min(X,-Y)| \stackrel{d}{=} |X| \stackrel{d}{=} |Y|.$$

Example 7. Sign-change invariance is not sufficient for the conclusion of Theorem 1 to hold. Suppose that $(X,Y)=(1,0)$ and $(-1,0)$, each with probability 1/2. Then $|X|\equiv 1$ while $|\max(X,Y)|=0$ and 1 each with probability 1/2, so (5) fails, even though (X,Y) is SCI. □

3. Reverse Exchangeability for Multivariate Distributions

It is natural to ask if Theorem 1 extends to three or more variables. That is, is

$$|\max(X_1,\dots,X_{n-1})| \stackrel{d}{=} |\max(X_1,\dots,X_n)|$$

for $n\geq 3$ under a general symmetry condition?

The short answer to this question is "no", as seen by the following simple example:

Example 8. Consider the random vector $\mathbf{X}_n\equiv(X_1,\dots,X_n)$ with (discrete) probability distribution specified by

$$\Pr[\mathbf{X}_n=\mathbf{e}_i]=\Pr[\mathbf{X}_n=-\mathbf{e}_i]=\frac{1}{2n},\quad i=1,\dots,n,$$

where $\mathbf{e}_i$ denotes the ith coordinate unit vector $(0,\dots,0,1_i,0,\dots,0)$ in $\mathbb{R}^n$. Clearly $\mathbf{X}_n$ satisfies the strong symmetry condition ESCI. However,

$$\Pr[|X_1|=j]=\begin{cases}1-\frac{1}{n}, & j=0,\\ \frac{1}{n}, & j=1,\end{cases}$$

while for $l=2,\dots,n$,

$$\Pr[|\max(X_1,\dots,X_l)|=j]=\begin{cases}1-\frac{l}{2n}, & j=0,\\ \frac{l}{2n}, & j=1,\end{cases}$$

so that

(19)
$$|X_1| \stackrel{d}{=} |\max(X_1,X_2)| <_{\mathrm{st}} |\max(X_1,X_2,X_3)| <_{\mathrm{st}} \cdots <_{\mathrm{st}} |\max(X_1,\dots,X_n)|.$$

Here $U <_{\mathrm{st}} V$ indicates that U is strictly stochastically less than V, that is, $F_U(x)\geq F_V(x)$ for all x with strict inequality for at least one x. □

This example shows that Theorem 1 does not extend to three or more dimensions. However, we shall show in Theorems 2, 3, and 5 that stochastic inequalities like those in (19) do hold under multivariate extensions of reverse exchangeability.

Definition 3. The random vector or sequence $(X_1,\dots,X_n)$ $(n\leq\infty)$ is said to be *stochastically increasing in absolute maximum (= SIAMX)* if

(20) $$|\max(X_1,\cdots,X_{l-1})| \leq_{\mathrm{st}} |\max(X_1,\dots,X_l)|,\quad \text{for } l=2,\dots,n,$$

where $U\leq_{\mathrm{st}} V$ means that U is stochastically less than V, i.e. $F_U(x)\geq F_V(x)$ for all x. It is *stochastically increasing in absolute minimum (= SIAMN)* if (20) holds with max replaced by min. It is *strictly SIAMX (= SSIAMX)* or *strictly SIAMN (= SSIAMN)* if the stochastic inequalities are strict. It is designated *SIAMX* or SSIAMX** if the stochastic inequalities hold for $l = 3, \ldots, n$ but for $l = 2$ the stochastic inequality is replaced by $|X_1| \stackrel{d}{=} |\max(X_1, X_2)|$ (e.g. see (19)). It is designated as *SIAMN* or SSIAMN** if, similarly,$|X_1| \stackrel{d}{=} |\min(X_1, X_2)|$.

Definition 4. The random vector $(X_1, \ldots, X_n)$ is said to be *RE*(k, l) for indices $1 \le k < l \le n$ if its distribution is unchanged when (X_k, X_l) is replaced by $(-X_l, -X_k)$, i.e.,

$$(X_1, \ldots, X_k, \ldots, X_l, \ldots, X_n) \stackrel{d}{=} (X_1, \ldots, -X_l, \ldots, -X_k, \ldots, X_n). \tag{21}$$

Also, $(X_1, \ldots, X_n)$ is called *RE*(n) if it is RE(k, n) for some $k < n$.

Proposition 4. *(i) If $(X_1, \ldots, X_n)$ is RE(k, l) then for $m = k$ and for $m = l$,*

$$|\max(X_i \mid 1 \le i \le n,\ i \ne m)| \le_{\text{st}} |\max(X_i \mid 1 \le i \le n)|, \tag{22}$$

$$|\min(X_i \mid 1 \le i \le n,\ i \ne m)| \le_{\text{st}} |\min(X_i \mid 1 \le i \le n)|. \tag{23}$$

(ii) Strict stochastic inequality holds in (22), respectively, in (23), if

$$\Pr[X_m > \max(|X_i| \mid 1 \le i \le n,\ i \ne m)] > 0, \text{ respectively,} \tag{24}$$

$$\Pr[X_m < -\max(|X_i| \mid 1 \le i \le n,\ i \ne m)] > 0. \tag{25}$$

Proof. (i) Without loss of generality take $(k, l) = (1, n)$ and $m = l = n$, so (21), (22), and (23) become

$$(X_1, X_2 \ldots, X_{n-1}, X_n) \stackrel{d}{=} (-X_n, X_2 \ldots, X_{n-1}, -X_1), \tag{26}$$

$$|\max(X_1, \ldots, X_{n-1})| \le_{\text{st}} |\max(X_1, \ldots, X_n)|, \tag{27}$$

$$|\min(X_1, \ldots, X_{n-1})| \le_{\text{st}} |\min(X_1, \ldots, X_n)|, \tag{28}$$

respectively. For $x \ge 0$ define the event

$$\Omega_n(x) := \{|\max(X_1, \ldots, X_n)| \le x\} = \{-x \le \max(X_1, \ldots, X_n) \le x\}.$$

To prove (27) we need to show that

$$\Pr[\Omega_n(x)] \le \Pr[\Omega_{n-1}(x)]. \tag{29}$$

For any subset $D \subseteq N := \{1, \ldots, n\}$, define the event

$$\mathcal{T}_n(D) \equiv \mathcal{T}_n(D; x) := \{X_i < -x\ \forall i \in D\} \cap \{|X_i| \le x\ \forall i \notin D\}.$$

Note that the events $\mathcal{T}_n(D)$ are disjoint for $D \subseteq N$. Then

$$\begin{aligned}
\Omega_n(x) &= \bigcup_{D \subset N} \mathcal{T}_n(D) \\
&= \left(\bigcup_{D \subset N,\, n \in D} \mathcal{T}_n(D) \right) \cup \left(\bigcup_{D \subset N,\, n \notin D} \mathcal{T}_n(D) \right) \\
&= \left(\bigcup_{D \subset N \setminus \{n\}} \mathcal{T}_n(D \cup \{n\}) \right) \cup \left(\bigcup_{D \subseteq N \setminus \{n\}} \mathcal{T}_n(D) \right) \\
&= \left(\bigcup_{D \subset N \setminus \{n\}} \Big(\mathcal{T}_n(D \cup \{n\}) \cup \mathcal{T}_n(D) \Big) \right) \cup \Big(\mathcal{T}_n(N \setminus \{n\}) \Big). \qquad (30)
\end{aligned}$$

For any $D \subseteq N\backslash\{n\}$ define

$$\widetilde{\mathcal{T}}_n(D) \equiv \widetilde{\mathcal{T}}_n(D;x) := \{X_i < -x \ \forall i \in D\} \cap \{|X_i| \leq x \ \forall i \notin D,\ i \neq n\} \cap \{X_n > x\},$$

also a family of disjoint events. Note too that $\mathcal{T}_n(D) \cap \widetilde{\mathcal{T}}_n(D') = \emptyset$ for any D, D'. If $D \subset N\backslash\{n\}$, it is straightforward to verify that

$$\mathcal{T}_{n-1}(D) = \mathcal{T}_n(D \cup \{n\}) \cup \mathcal{T}_n(D) \cup \widetilde{\mathcal{T}}_n(D),$$

a union of three disjoint events. Thus

$$\begin{aligned} &\Omega_{n-1}(x) \\ =\ & \bigcup_{D \subset N\backslash\{n\}} \mathcal{T}_{n-1}(D) \\ =\ & \bigcup_{D \subset N\backslash\{n\}} \Big(\mathcal{T}_n(D \cup \{n\}) \cup \mathcal{T}_n(D) \cup \widetilde{\mathcal{T}}_n(D)\Big) \\ =\ & \left(\bigcup_{D \subset N\backslash\{n\}} \Big(\mathcal{T}_n(D \cup \{n\}) \cup \mathcal{T}_n(D)\Big)\right) \cup \left(\bigcup_{D \subset N\backslash\{n\}} \widetilde{\mathcal{T}}_n(D)\right), \end{aligned} \tag{31}$$

where all the events involving $\mathcal{T}_n$ and $\widetilde{\mathcal{T}}_n$ are mutually disjoint. But the RE$(1,n)$ condition (21) implies that

$$\Pr[\mathcal{T}_n(N\backslash\{n\})] = \Pr[\widetilde{\mathcal{T}}_n(N\backslash\{1,n\})] \leq \Pr\left[\bigcup_{D \subset N\backslash\{n\}} \widetilde{\mathcal{T}}_n(D)\right], \tag{32}$$

which, together with (30) and (31), yields (29) and thence (27).

Now (28) follows from (27) because

$$(X_1, \ldots, X_n) \text{ is RE}(k,l) \iff (-X_1, \ldots, -X_n) \text{ is RE}(k,l) \quad \text{and} \tag{33}$$

$$|\min(X_1, \ldots, X_n)| = |\max(-X_1, \ldots, -X_n)|. \tag{34}$$

(ii) Because the events $\widetilde{\mathcal{T}}_n(D)$ are disjoint, it follows from (32) that strict inequality holds in (29) iff

$$\Pr\left[\widetilde{\mathcal{T}}_n(D;x)\right] > 0, \quad \text{for some } D \subset N\backslash\{n\},\ D \neq N\backslash\{1,n\}. \tag{35}$$

In particular, set $D = \emptyset$ to see that (35) holds if

$$\Pr[|X_i| \leq x,\ \forall\ i = 1, \ldots, n-1,\ X_n > x] > 0. \tag{36}$$

Thus a sufficient condition for strict stochastic inequality to hold in (27) is that (36) hold for at least one x, which is equivalent[3] to the condition that

$$\Pr[X_n > \max(|X_1|, \ldots, |X_{n-1}|)] > 0,$$

thus confirming (24). By (34), it follows that a sufficient condition for strict stochastic inequality to hold in (28) is that

$$\Pr[X_n < -\max(|X_1|, \ldots, |X_{n-1}|)] > 0,$$

thereby confirming (25) □

[3]Since $\{X_n > \max(|X_1|, \ldots, |X_{n-1}|)\} = \cup(\{X_n > x \geq \max(|X_1|, \ldots, |X_{n-1}|)\} \mid x \in \mathbb{Q})$.

Remark 1. The distribution of $|\max(X_i \mid 1 \le i \le n,\ i \ne m)|$ in Proposition 4 is not necessarily the same for $m = k$ and $m = l$. With $n = 3$, $k = 1$, and $l = 2$, consider the random vector (X_1, X_2, X_3) that assigns probability $1/4$ to each of the four points $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 1)$, and $(1, 0, 1)$. Then this distribution is RE$(1, 2)$ but

$$\Pr[|\max(X_1, X_3)| = r] = \begin{cases} \frac{1}{2}, & r = 0 \\ \frac{1}{2}, & r = 1, \end{cases}$$

$$\Pr[|\max(X_2, X_3)| = r] = \begin{cases} \frac{1}{4}, & r = 0 \\ \frac{3}{4}, & r = 1. \end{cases}$$

The same is true for $|\min(X_i \mid 1 \le i \le n,\ i \ne m)|$. □

Theorem 1 and Proposition 4 yield the following multivariate result:

Theorem 2. *Let the random vector $(X_1, \ldots, X_n)$ be such that $(X_1, \ldots, X_l)$ is RE(l) for each $l = 2, \ldots, n$. Then $(X_1, \ldots, X_n)$ is SIAMX* and SIAMN*. It is SSIAMX* or SSIAMN* if*

$$\Pr[X_l > \max(|X_1|, \ldots, |X_{l-1}|)] > 0, \quad l = 3, \ldots, n, \tag{37}$$

$$\text{or} \quad \Pr[X_l < -\max(|X_1|, \ldots, |X_{l-1}|)] > 0, \quad l = 3, \ldots, n, \tag{38}$$

respectively.

It is easy to see that the discrete multivariate distribution in Example 8 satisfies condition (37), thereby confirming the strict stochastic inequalities in (19). (The same holds true if max is replaced by min in (19).)

Remark 2. For (X_1, X_2), RE(2) is simply RE, which is weaker than ESCI as noted before. For $(X_1, \ldots, X_n)$ with $n \ge 3$, the conjunction of RE(2), ..., RE(n) in Theorem 2 is weaker than ESCI in general. Consider, for example, an infinite sequence $X_1, X_2, X_3, \ldots$ of iid but non-symmetric rvs (random variables). For any $n \ge 2$, $(-X_1, X_2, X_3, \ldots, X_n)$ is RE(n) but not ESCI. □

Example 9. [Example 3 continued] If $X_1, ..., X_n$ are iid random variables whose common distribution is symmetric about 0, then $(X_1, ..., X_n)$ is ESCI hence RE(n) for every $n \ge 2$. Here the conclusions of Theorem 2 can be verified directly: To show that $(X_1, \ldots, X_n)$ is SIAMX*, for any $x \ge 0$ set $u_x = \Pr[X_i > x] \le \frac{1}{2}$. Then as in Example 3,

$$\Pr[\,|\max(X_1, ..., X_n)| \le x] = (1 - u_x)^n - u_x^n,$$

which is decreasing in n since

$$\begin{array}{rcl} (1-u_x)^{n-1} - u_x^{n-1} & \ge & (1-u_x)^n - u_x^n \\ & \Updownarrow & \\ (1-u_x)^{n-1} u_x & \ge & u_x^{n-1}(1-u_x) \\ & \Updownarrow & \\ \left(\dfrac{1-u_x}{u_x}\right)^{n-1} & \ge & \dfrac{1-u_x}{u_x}. \end{array}$$

The last inequality holds since $(1 - u_x)/u_x \ge 1$.

This inequality is strict if $n \geq 3$ and $0 < u_x < \frac{1}{2}$, i.e., if $\Pr[|X_i| \leq x] > 0$. Thus for such x, $\Pr[\,|\max(X_1, ..., X_n)| \leq x]$ is strictly decreasing in n for $n \geq 2$. A necessary and sufficient condition for this to hold for at least one $x \geq 0$, and therefore for $(X_1, \ldots, X_n)$ to be SSIAMX*, is that the distribution of $|X_i|$ be non-degenerate. Note that this condition is equivalent to both (37) and (38) in this example, so under this condition, $(X_1, \ldots, X_n)$ is SSIAMN* as well. □

For independent random variables, however, the requirements of identical distributions and symmetry in Example 9 are not necessary for RE(n) to hold:

Example 10. Let $(X_1, X_2, \ldots)$ be an infinite sequence of independent random variables such that

(39) $$X_1 \overset{d}{=} -X_2 \overset{d}{=} X_3 \overset{d}{=} -X_4 \overset{d}{=} X_5 \overset{d}{=} \cdots .$$

Then for each $n \geq 2$, $(X_1, ..., X_n)$ is RE(n) with $k = n - 1$ (or $n-3,\ n-5, \ldots$). Thus Theorem 2 implies that $(X_1, X_2, \ldots)$ is SIAMX* and SIAMN*. If in addition both (37) and (38) hold, then by Proposition 4(ii), $(X_1, X_2, \ldots)$ is SSIAMX* and SSIAMN*. (These results can again be verified directly, as in Example 9.)

In fact, the same conclusions holds if (39) is weakened to the condition

(40) $$X_i \overset{d}{=} \epsilon_i X_1,\ i = 2, 3, \ldots,$$

where

$$\epsilon_2 = -1,\ \epsilon_i = \pm 1, \text{ for } i \geq 3.$$

Now $(X_1, ..., X_n)$ is RE(n) with either $k = 1$ or $k = 2$. □

We now present an example where it seems difficult to circumvent Theorem 2. Such examples arise when $X_1, X_2, \ldots$ are not independent. (Also see Examples 17 and 18.)

Example 11. Consider a Gaussian sequence $(X_1, X_2, \ldots)$ with $\mathrm{E}(X_i) = \mu_i$, $\mathrm{Var}(X_i) = \sigma^2$, and $\mathrm{Corr}(X_i, X_j) = \rho_{i,j}$. Then $(X_1, ..., X_n)$ satisfies RE(n) if and only if for some $1 \leq k(n) \leq n - 1$,

(41) $$\mu_n = -\,\mu_{k(n)},$$

(42) $$\rho_{n,j} = -\,\rho_{k(n),j},\ \ \forall\, j < n,\ j \neq k(n).$$

If these conditions hold for every $n = 2, 3, \ldots$, then Theorem 2 implies that the sequence is SIAMX* and, by Proposition 4(ii), is SSIAMX* if the Gaussian sequence is nonsingular.

Since $k(n) < n$, the functional iterates $k^{(q)}(n)$ strictly decrease with q. Let q_n be the smallest q such that $k^{(q)}(n) = 1$; note that $q_2 = 1$. Thus, if (41) holds for all $n = 2, 3, \ldots$ then $\mu_n = (-1)^{q_n}\mu_1$, so the sequence of means $\boldsymbol{\mu}_\infty := (\mu_1, \mu_2, \ldots)$ takes the form

$$\boldsymbol{\mu}_\infty = \mu \cdot \big(1, -1, (-1)^{q_3}, (-1)^{q_4}, \ldots\big)$$

for some scalar μ.

If (42) holds for all $n = 2, 3, \ldots$, the structure of the correlation matrix $\mathbf{R}_\infty := (\rho_{i,j} \mid 1 \leq i, j < \infty)$ is more complicated to describe. We present two special cases:

Case 1. $k(n) = 1$ for each $k \geq 2$: Here each $q_n = 1$ so

$$\boldsymbol{\mu}_\infty = \mu \cdot (1, -1, -1, -1, \ldots)$$

and $\mathbf{R}_\infty$ has the form

$$\mathbf{R}_\infty = \begin{pmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \cdots \\ \rho_1 & 1 & -\rho_1 & -\rho_1 & \cdots \\ \rho_2 & -\rho_1 & 1 & -\rho_2 & \cdots \\ \rho_3 & -\rho_1 & -\rho_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Case 2. $k(n) = n - 1$ for each $k \geq 2$: Here $q_n = n - 1$ so

$$\boldsymbol{\mu}_\infty = \mu \cdot (1, -1, 1, -1, 1, \dots)$$

and $\mathbf{R}_\infty$ has the form

$$\mathbf{R}_\infty = \begin{pmatrix} 1 & \rho_1 & -\rho_1 & \rho_1 & -\rho_1 & \cdots \\ \rho_1 & 1 & \rho_2 & -\rho_2 & \rho_2 & \cdots \\ -\rho_1 & \rho_2 & 1 & \rho_3 & -\rho_3 & \cdots \\ \rho_1 & -\rho_2 & \rho_3 & 1 & \rho_4 & \cdots \\ -\rho_1 & \rho_2 & -\rho_3 & \rho_4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus $(X_1, X_2, \dots)$ is SIAMX* and SIAMN*, and is SSIAMX* and SSIAMN* if the Gaussian sequence is nonsingular. □

Lastly, we have the following result for an exchangeable random vector:

Corollary 2. *If $(X_1, \dots, X_n)$ is exchangeable, then $(-X_1, X_2, \dots, X_n)$ is SIAMX* and SIAMN*.*

Proof. Exchangeability implies that $(-X_1, \dots, X_l)$ is RE$(1, l)$ for $l = 2, \dots, n$, so this result follows from Theorem 2. □

Example 12. Suppose that $X = (X_1, ..., X_n)$ represent n random draws (without replacement) from a finite set of real numbers. Since $(X_1, \dots, X_n)$ is exchangeable, it follows from Corollary 2 that $(-X_1, X_2, \dots, X_n)$ is SIAMX* and SIAMN*. □

4. Reverse Sub(Super)exchangeability

Example 13. (Example 6 extended). Suppose that $X_1, \dots, X_n$ represent n random draws without replacement from the finite symmetric set $A \subset \mathbb{R}$, where $n \leq |A|$. In Example 1 it was shown that (X_1, X_2) is RE $\equiv$ RE(2). However, $(X_1, \dots, X_l)$ is not RE(l) for $3 \leq l \leq n \wedge (|A| - 1)$: for example, if $l = 3$ and $a, b \in A$, $a, b > 0$, $a \neq b$, then (X_1, X_2, X_3) is not RE$(1, 3)$:

$$\begin{aligned} 0 = {} & \Pr[X_1 = -a,\ X_2 = -a,\ X_3 = b] \\ < {} & \Pr[X_1 = -b,\ X_2 = -a,\ X_3 = a] \\ = {} & \Pr[-X_3 = -a,\ X_2 = -a,\ -X_1 = b], \end{aligned}$$

where the strict inequality holds since $-b, -a, a$ are distinct. Similarly (X_1, X_2, X_3) is not RE$(2, 3)$, hence (X_1, X_2, X_3) is not RE(3). Thus the condition of Theorem 2 is not satisfied. Nonetheless, $(X_1, \dots, X_n)$ is SSIAMX and SSIAMN in this example. □

To establish this fact we introduce the notions of *reverse subexchangeability* and *reverse superexchangeability*, weaker conditions than reverse exchangeability. For simplicity we shall restrict attention to random vectors $(X_1, \dots, X_n)$ whose distributions are determined by $f(x_1, \dots, x_n)$, which is either a discrete probability mass function (pmf) or a probability density function (pdf) w.r.to Lebesgue measure. We begin with the bivariate case.

Definition 5. The bivariate random vector (X, Y) is called *upper (lower) reverse subexchangeable*, denoted by UR_{E} (LR_{E}), if

$$f(x,y) \geq f(-y,-x), \quad \text{for } |x| < y \quad (|y| < x). \tag{43}$$

The rv (X, Y) is called *upper (lower) reverse superexchangeable*, denoted by UR^{E} (LR^{E}), if

$$f(x,y) \leq f(-y,-x), \quad \text{for } |x| < y \quad (|y| < x). \tag{44}$$

Proposition 5.

$$\begin{array}{lll} (i) & (X,Y)\ \mathrm{UR_E} & \Longrightarrow |X| \leq_{\mathrm{st}} |\max(X,Y)| \text{ and } |\min(X,Y)| \leq_{\mathrm{st}} |Y|; \\ (ii) & (X,Y)\ \mathrm{LR_E} & \Longrightarrow |Y| \leq_{\mathrm{st}} |\max(X,Y)| \text{ and } |\min(X,Y)| \leq_{\mathrm{st}} |X|; \\ (iii) & (X,Y)\ \mathrm{UR^E} & \Longrightarrow |X| \geq_{\mathrm{st}} |\max(X,Y)| \text{ and } |\min(X,Y)| \geq_{\mathrm{st}} |Y|; \\ (iv) & (X,Y)\ \mathrm{LR^E} & \Longrightarrow |Y| \geq_{\mathrm{st}} |\max(X,Y)| \text{ and } |\min(X,Y)| \geq_{\mathrm{st}} |X|. \end{array}$$

The stochastic inequalities in (i) (resp., (iii)) are strict if and only if

$$\Pr[\,|X| < Y\,] \;>\; (<)\; \Pr[\,X < -|Y|\,]. \tag{45}$$

Likewise, the stochastic inequalities in (ii) (resp., (iv)) are strict if and only if

$$\Pr[\,|Y| < X\,] \;>\; (<)\; \Pr[\,Y < -|X|\,]. \tag{46}$$

Proof. Because (X, Y) $\mathrm{UR_E} \Rightarrow \Pr[N_x] \geq \Pr[W_x]$ and (X, Y) $\mathrm{LR_E} \Rightarrow \Pr[S_x] \leq \Pr[E_x]$, (i) and (ii) follow from (15) and (16) respectively. Parts (iii) and (iv) follow similarly with the inequalities reversed.

To establish strict stochastic inequality in (i), define $N : \{|X| < Y\}$ and, for any measurable $A \subseteq N$, define

$$\tilde{A} := \{(-y,-x) \mid (x,y) \in A\},$$

the reflection of A across the line $y = -x$. Note that $\tilde{N} = \{X < -|Y|\} =: W$ and $\tilde{N}_x = W_x$ for $x \geq 0$ (recall (6) and (9)).

For any measurable subset $A \subseteq N$, define

$$\sigma(A) := \Pr[A] - \Pr[\tilde{A}],$$

so that (recall (6) and (9)),

$$\sigma(N) = \Pr[N] - \Pr[W] = \Pr[\,|X| < Y\,] - \Pr[\,X < -|Y|\,], \tag{47}$$

$$\sigma(N_x) = \Pr[N_x] - \Pr[W_x]. \tag{48}$$

Clearly σ is a countably additive set function. Since (X, Y) is $\mathrm{UR_E}$, we have that

$$\sigma(A) \geq 0, \quad \forall \text{ measurable } A \subseteq \Omega,$$

so σ is a nonnegative measure. Thus, because N is the countable union

$$N = \bigcup (N_x \mid x \geq 0,\ x \text{ rational}),$$

it follows that

$$\sigma(N) > 0 \iff \sigma(N_x) > 0, \text{ for at least one rational } x \geq 0.$$

The result now follows from (47), (48), and (15).

Cases (ii), (iii), and (iv) are treated similarly. ☐

Example 14. (Example 5 extended). Suppose that (X, Y) has a bivariate elliptical pdf on $\mathbb{R}^2$ given by

$$f(x,y) = |\Sigma|^{-1/2} g\left[(x-\mu,\, y-\nu)\, \Sigma^{-1} (x-\mu,\, y-\nu)'\right],$$

where

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

$-1 < \rho < 1$. Assume that g is *nonincreasing and strictly positive* on $[0, \infty)$. (This includes the case where $(X,Y) \sim N_2((\mu,\nu), \Sigma))$. After some algebra we find that

$$(x+y)(\mu+\nu) \geq 0 \implies f(x,y) \geq f(-y,-x),$$

regardless of the value of ρ, so

$$\mu + \nu > 0 \implies (X,Y) \text{ is } \mathrm{UR_E} \text{ and } \mathrm{LR_E}. \tag{49}$$

Furthermore $(X-\mu, Y-\nu)$ is RE, so

$$\begin{aligned} \Pr[\,X < -|Y|\,] &= \Pr[\,(X-\mu) + \mu < -|(Y-\nu)+\nu|\,] \\ &= \Pr[\,-(Y-\nu) + \mu < -|-(X-\mu)+\nu|\,] \\ &= \Pr[\,Y - (\mu+v) > |X - (\mu+\nu)|\,] \\ &= \Pr[\,Y > X,\ X+Y > 2(\mu+\nu)\,]. \end{aligned}$$

Thus, if $\mu + \nu > 0$ then

$$\begin{aligned} &\Pr[\,|X| < Y\,] - \Pr[\,X < -|Y|\,] \\ =&\Pr[\,Y > X,\ Y > -X\,] - \Pr[\,Y > X,\ X+Y > 2(\mu+\nu)\,] \\ =&\Pr[\,Y > X,\ X+Y > 0\,] - \Pr[\,Y > X,\ X+Y > 2(\mu+\nu)\,] \\ =&\Pr[\,Y > X,\ 0 < X+Y \leq 2(\mu+\nu)\,], \end{aligned}$$

which is strictly positive since g is strictly positive on $[0, \infty)$. It follows from Proposition 5 that

$$\mu + \nu > 0 \implies |\min(X,Y)| <_{\mathrm{st}} \begin{matrix} |X| \\ |Y| \end{matrix} <_{\mathrm{st}} |\max(X,Y)|. \tag{50}$$

Similarly,

$$\mu + \nu < 0 \implies (X,Y) \text{ is } \mathrm{UR^E} \text{ and } \mathrm{LR^E} \tag{51}$$

$$\implies |\max(X,Y)| <_{\mathrm{st}} \begin{matrix} |X| \\ |Y| \end{matrix} <_{\mathrm{st}} |\min(X,Y)|. \tag{52}$$

(Note that if $\mu + \nu = 0$ then (X,Y) is RE so Example 5 applies, hence these stochastic inequalities become stochastic equalities.) ☐

Example 15. Suppose that (X, Y) has joint pmf or pdf given by

$$f(x, y) = g(x)\, h(y),$$

where in addition, g and h are symmetric about 0, i.e., $g(x) = g(-x)$ and $h(y) = h(-y)$. Thus X and Y are independent and SCI, but neither E nor RE if $g \neq h$. Here

$$f(x, y) \geq f(-y, -x) \iff g(|x|)h(|y|) \geq g(|y|)h(|x|), \tag{53}$$

$$\iff f_{|X|}(|x|)f_{|Y|}(|y|) \geq f_{|X|}(|y|)f_{|Y|}(|x|), \tag{54}$$

where $f_{|X|}$ and $f_{|Y|}$ denote the pmfs or pdfs of $|X|$ and $|Y|$, respectively.

Now specialize to the case where $f_{|X|} = f_{\theta'}$ and $f_{|Y|} = f_{\theta''}$ are pmfs or pdfs in a one-parameter family $\{f_\theta\}$ of pmfs or pdfs on $[0, \infty)$ with *strictly monotone-increasing likelihood ratio*. Then for $\theta' < \theta''$, the inequalities (53)-(54) hold whenever $0 < x < y$ while the opposite inequalities hold whenever $0 < y < x$. Thus (X, Y) is $\mathrm{UR_E}$ and $\mathrm{LR^E}$ if $\theta' < \theta''$. Furthermore by symmetry

$$\Pr[\,|X| < Y] = \Pr[Y < -|X|\,] = \frac{1}{2}\Pr[\,|X| < |Y|\,],$$

$$\Pr[\,X < -|Y|\,] = \Pr[\,|Y| < X] = \frac{1}{2}\Pr[\,|Y| < |X|\,],$$

and

$$\Pr[\,|X| < |Y|\,] > \Pr[\,|Y| < |X|\,]$$

by the strict monotone likelihood ratio assumption for $|X|$ and $|Y|$. Thus by Proposition 5 (also note $(X, Y) \stackrel{d}{=} (-X, -Y)$),

$$|X| <_{\mathrm{st}} |\min(X, Y)| \stackrel{d}{=} |\min(-X, -Y)| \stackrel{d}{=} |\max(X, Y)| <_{\mathrm{st}} |Y|. \tag{55}$$

The scale-parameter families $\{N(0, \theta) \mid \theta > 0\}$ and $\{C(0, \theta) \mid \theta > 0\}$ of centered normal and Cauchy pdfs satisfy the assumptions of this example, hence satisfy (55) when $0 < \theta' < \theta''$. □

Example 16. (Example 5 extended). Suppose that (X, Y) has a centered bivariate elliptical pdf on $\mathbb{R}^2$ given by

$$\begin{aligned} f(x, y) &= |\Sigma|^{-1/2} g\left[(x,\, y)\, \Sigma^{-1} (x,\, y)'\right] \\ &= |\Sigma|^{-1/2} g\left[d(\sigma, \tau, \rho)(x^2\tau^2 - \rho\sigma\tau xy + y^2\sigma^2)\right], \end{aligned}$$

where

$$\Sigma = \begin{pmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{pmatrix}$$

is positive definite and $d(\cdot, \cdot, \cdot) > 0$. Assume that g is *nonincreasing* on $[0, \infty)$. (This includes the case $(X, Y) \sim N_2((0, 0), \Sigma))$. Then

$$(y^2 - x^2)(\tau^2 - \sigma^2) \geq 0 \implies f(x, y) \geq f(-y, -x),$$

regardless of the value of ρ, so

$$\tau^2 > \sigma^2 \implies (X, Y) \text{ is } \mathrm{UR_E} \text{ and } \mathrm{LR^E}. \tag{56}$$

Furthermore, $(\tau X, \sigma Y)$ is RE, so

$$\begin{aligned}\Pr[\,X < -|Y|\,] &= \Pr[\,-\sigma Y/\tau < -|\tau X/\sigma|\,]\\ &= \Pr[\,Y > (\tau/\sigma)^2|X|\,].\end{aligned}$$

Thus if $\tau^2 > \sigma^2$ then

$$\Pr[\,|X| < Y\,] - \Pr[\,X < -|Y|\,] = \Pr[\,(\tau/\sigma)^2|X| \geq Y > |X|\,],$$

which is strictly positive for any bivariate elliptical distribution. It follows from Proposition 5 that (also note $(X,Y) \overset{d}{=} (-X,-Y)$)

(57)

$$\tau^2 > \sigma^2 \implies |X| <_{\mathrm{st}} |\min(X,Y)| \overset{d}{=} |\min(-X,-Y)| \overset{d}{=} |\max(X,Y)| <_{\mathrm{st}} |Y|,$$

regardless of the value of ρ. Similarly,

$$\tau^2 < \sigma^2 \implies |Y| <_{\mathrm{st}} |\min(X,Y)| \overset{d}{=} |\max(X,Y)| <_{\mathrm{st}} |X|. \tag{58}$$

□

We now turn to the multivariate case; take $n \geq 3$ for the remainder of this section.

Definition 6. The random vector $(X_1, \ldots, X_n)$ is said to be $UR_{\mathrm{E}}(k,l)$ for indices $1 \leq k < l \leq n$ if $f(x_1, \ldots, x_n)$ decreases when (x_k, x_l) is replaced by $(-x_l, -x_k)$, i.e.,

$$f(x_1, \ldots, x_k, \ldots, x_l, \ldots, x_n) \geq f(x_1, \ldots, -x_l, \ldots, -x_k, \ldots, x_n), \tag{59}$$

whenever $|x_k| < x_l$ and $x_i < -|x_k|$ for all $i \neq k, l$. It is $LR_{\mathrm{E}}(k,l)$ if (59) holds whenever $|x_l| < x_k$ and $x_i < -|x_l|$ for all $i \neq k, l$. The rv $(X_1, \ldots, X_n)$ is called $UR_{\mathrm{E}}(l)$ *(*$LR_{\mathrm{E}}(l)$*)* if it is $\mathrm{UR}_{\mathrm{E}}(k,l)$ $(\mathrm{LR}_{\mathrm{E}}(k,l))$ for some $k \neq l$. Then $UR^{\mathrm{E}}(k,l)$, $LR^{\mathrm{E}}(k,l)$, $UR^{\mathrm{E}}(n)$, *and* $LR^{\mathrm{E}}(n)$*)* are defined like their counterparts but with the inequality reversed in (59).

Proposition 6. *For* $1 \leq k < l \leq n$,

$$\begin{aligned}
&(i) && (X_1, \ldots, X_n)\ \mathrm{UR}_{\mathrm{E}}(k,l) && \implies |\max(X_i \mid i \neq l)| \leq_{\mathrm{st}} |\max(X_1, \ldots, X_n)|;\\
&(ii) && (X_1, \ldots, X_n)\ \mathrm{LR}_{\mathrm{E}}(k,l) && \implies |\max(X_i \mid i \neq k)| \leq_{\mathrm{st}} |\max(X_1, \ldots, X_n)|;\\
&(iii) && (X_1, \ldots, X_n)\ \mathrm{UR}^{\mathrm{E}}(k,l) && \implies |\min(X_i \mid i \neq k)| \leq_{\mathrm{st}} |\min(X_1, \ldots, X_n)|;\\
&(iv) && (X_1, \ldots, X_n)\ \mathrm{LR}^{\mathrm{E}}(k,l) && \implies |\min(X_i \mid i \neq l)| \leq_{\mathrm{st}} |\min(X_1, \ldots, X_n)|.
\end{aligned}$$

The stochastic inequalities in (i)–(iv) become strict under the four conditions

$$\Pr[X_l > \max(|X_i| \mid 1 \leq i \leq n,\ i \neq l)] > 0, \tag{60}$$
$$\Pr[X_k > \max(|X_i| \mid 1 \leq i \leq n,\ i \neq k)] > 0, \tag{61}$$
$$\Pr[X_k < -\max(|X_i| \mid 1 \leq i \leq n,\ i \neq k)] > 0, \tag{62}$$
$$\Pr[X_l < -\max(|X_i| \mid 1 \leq i \leq n,\ i \neq l)] > 0, \tag{63}$$

respectively.

Proof. The proof of (i) is identical to the proof of (22) in Proposition 4, except that in (32) the equality (=) is replaced by inequality (≤), which is justified by (59). The implications (ii)-(iv) are established in similar fashion. An argument similar to that used for Proposition 4(ii) verifies the conditions for strict stochastic inequality. □

Proposition6 yields the following theorem:

Theorem 3. *If* $(X_1,\dots,X_l)$ *is* $UR_E(l)$ *for* $l=2,\dots,n$, *then* $(X_1,\dots,X_n)$ *is SIAMX. If in addition*

$$\Pr[X_2>|X_1|\,]>\Pr[X_1<-|X_2|\,] \tag{64}$$

$$\text{and}\quad \Pr[X_l>\max(|X_1|,\dots,|X_{l-1}|)]>0,\quad \text{for } l=3,\dots,n, \tag{65}$$

then $(X_1,\dots,X_n)$ *is SSIAMX. If* $(X_1,\dots,X_l)$ *is* $LR^E(l)$ *for* $l=2,\dots,n$, *then* $(X_1,\dots,X_n)$ *is SIAMN. If in addition*

$$\Pr[\,|X_2|<X_1]<\Pr[X_2<-|X_1|\,] \tag{66}$$

$$\text{and}\quad \Pr[X_l<-\min(|X_1|,\dots,|X_{l-1}|)]>0,\quad \text{for } l=3\dots,n, \tag{67}$$

then $(X_1,\dots,X_n)$ *is SSIAMN.*

Example 17. (Example 13 continued). Let $X_1,\dots,X_n$ be n random draws without replacement from the finite symmetric set $A\subset\mathbb{R}$, with $n\le|A|$. It was seen in Example 13 that $(X_1,\dots,X_l)$ need not be $\mathrm{RE}(l)$ for $3\le l\le n$. However, it is $\mathrm{UR_E}(l)$, which is seen as follows:

The pmf $f(x_1,\dots,x_l)$ takes the value $c:=1/(|A|(|A|-1)\cdots(|A|-l+1))$ on the range

$$\tilde{A}^l:=\{(x_1,\dots,x_l)\mid x_1,\dots,x_l\in A \text{ and are mutually distinct}\}, \tag{68}$$

and is 0 for $(x_1,x_2,\dots,x_l)\notin\tilde{A}^l$. Thus, to verify via (59) that $(X_1,\dots,X_l)$ is $\mathrm{RE}(1,l)$ it suffices to show that

$$(x_1,x_2,\dots,x_l)\in\tilde{A}^l$$

whenever

$$(-x_l,x_2,\dots,x_{l-1},-x_1)\in\tilde{A}^l$$

and $|x_1|<x_l$ and $x_i<-|x_1|$ for all $i\ne k,l$. These two strict inequalities imply that x_1,x_i,x_l are mutually distinct for all $i=2,\dots,l-1$, while $x_2,\dots,x_{l-1}$ are distinct by (68). The assertion follows since $-x_i\in A\implies x_i\in A$ by the symmetry of A.

It follows from Theorem 3 that $(X_1,\dots,X_n)$ is SIAMX and, by symmetry, is SIAMN. Furthermore, conditions (65) and (67) hold unless $l=n=|A|$, so $(X_1,\dots,X_n)$ is SSIAMX and SSIAMN, except possibly for the case $l=n$ when $n=|A|$. □

Example 18. (Example 5 extended). Let $(X_1,\dots,X_n)$ have a centered multivariate elliptical distribution with pdf of the form

$$f(x_1,\dots,x_n)=|\Sigma|^{-1/2}g\left[(x_1,\dots,x_n)\,\Sigma^{-1}(x_1,\dots,x_n)'\right],$$

where $\Sigma\equiv\{\sigma_{ij}\}$ is positive definite with *intraclass correlation structure*: $\sigma_{ii}=\sigma^2$ and $\sigma_{ij}=\sigma^2\rho$ for all $1\le i\ne j\le n$, where $-1/(n-1)<\rho<1$. Then f has the form

$$f(x_1,\dots,x_n)=|\Sigma|^{-1/2}g\left[c(\sigma,\rho)\cdot\sum_{1\le i\le n}x_i^2-\rho\,d(\sigma,\rho)\cdot\sum_{1\le i<j\le n}x_ix_j\right],$$

where $c(\cdot,\cdot) > 0$ and $d(\cdot,\cdot) > 0$. Assume that g is *nonincreasing* on $[0,\infty)$. Then for $\rho \neq 0$,

$$\begin{aligned} &\rho\,(x_1 + x_n)(x_2 + \cdots + x_{n-1}) \geq 0 \\ \implies &f(x_1, \ldots, x_n) \geq f(-x_n, x_2, \ldots, x_{n-1}, -x_1), \end{aligned}$$

hence

$$(X_1, \ldots, X_n) \text{ is } \begin{cases} \mathrm{UR_E}(1,n), & \text{if } \rho < 0, \\ \mathrm{LR^E}(1,n), & \text{if } \rho > 0. \end{cases} \tag{69}$$

Since (X_1, X_2) is RE for all ρ, it follows from Theorems 1 and 3 that $(X_1, \ldots, X_n)$ is SSIAMX* when $\rho < 0$ and is SSIAMN* when $\rho > 0$. However, $(X_1, \ldots, X_n) \stackrel{d}{=} (-X_1, \ldots, -X_n)$ for all values of ρ, so $|\max(X_1, \ldots, X_n)| \stackrel{d}{=} |\min(X_1, \ldots, X_n)|$. Thus $(X_1, \ldots, X_n)$ is SSIAMX* and SSIAMN* for all ρ.

Note that $(X_1, \ldots, X_n)$ is also exchangeable, so by Corollary 2 and Theorem 3, $(-X_1, X_2, \ldots, X_n)$ is SIAMX* and SIAMN*. □

5. Results for Independent Symmetric Random Variables.

As noted earlier, if $X_1, \ldots, X_n$ are iid and symmetric about 0 then $(X_1, \ldots, X_n)$ is ESCI, hence $\mathrm{RE}(k,l)$ for all $1 \leq k < l \leq n$, so the conclusions of Theorem 2 hold. Under independence, however, stochastic comparisons for the extreme order statistics can be obtained under a weaker assumption than identical distributions, namely stochastic ordering. This is illustrated by the following bivariate result:

Theorem 4. *Suppose that X and Y are independent and symmetric about 0. If $|X| \leq_{\mathrm{st}} |Y|$ then*

$$|X| \leq_{\mathrm{st}} |\min(X,Y)| \stackrel{d}{=} |\max(X,Y)| \leq_{\mathrm{st}} |Y|. \tag{70}$$

These stochastic inequalities are strict iff $|X| <_{\mathrm{st}} |Y|$.

Proof. We may either pattern the proof after that of Proposition 2 or use this direct approach:

If F and G are the cdfs of X and Y, respectively, then the cdfs of $|X|$ and $|Y|$ are $2F-1$ and $2G-1$ on $[0,\infty)$. Furthermore, the cdf of $|\max(X,Y)|$ is

$$\begin{aligned} FG - (1-F)(1-G) =& F + G - 1 \\ =& \frac{1}{2}[(2F-1) + (2G-1)], \end{aligned}$$

the average of $2F-1$ and $2G-1$. But $|X| \leq_{\mathrm{st}} |Y|$ implies that $2F-1 \geq 2G-1$, hence this average lies in the interval $[2G-1,\, 2F-1]$, and strictly inside this interval for some x iff $|X| <_{\mathrm{st}} |Y|$. This yields the stochastic inequalities in (70) and the statement regarding strict stochastic inequality. The equality in (70) follows by symmetry. □

Note that the stochastic ordering assumption $|X| \leq_{\mathrm{st}} |Y|$ in Theorem 4 is weaker than the monotone likelihood ratio assumption in Example 15. For example, let $|X|$ take the values 0 and 1 with probability 1/2 each, and let $|Y|$ take the values 0, 1, and 2 with probabilities 1/2, 1/4, and 1/4 respectively.

In three or more dimensions we have the following partial complement to Proposition 6 and Theorem 3:

Theorem 5. *Let $X_1, \dots, X_n$ be independent symmetric random variables. If $|X_k| \le_{\text{st}} |X_l|$ for a pair (k, l) with $1 \le k < l \le n$, then*

$$|\max(X_i \mid 1 \le i \le n,\ i \ne l)| \le_{\text{st}} |\max(X_i \mid 1 \le i \le n)|, \tag{71}$$

$$|\min(X_i \mid 1 \le i \le n,\ i \ne l)| \le_{\text{st}} |\min(X_i \mid 1 \le i \le n)|. \tag{72}$$

If $|X_k| <_{\text{st}} |X_l|$, the stochastic inequalities in (71) and (72) are strict.

Therefore, if $|X_1| \le_{\text{st}} \cdots \le_{\text{st}} |X_n|$ then $(X_1, \dots, X_n)$ is SIAMX and SIAMN. If $|X_1| <_{\text{st}} \cdots <_{\text{st}} |X_n|$, then $(X_1, \dots, X_n)$ is SSIAMX and SSIAMN.

Proof. Without loss of generality take $k = 1$ and $l = n$. Set $F_i(x) = \Pr[X_i \le x]$, the cdf of X_i, and set $\bar{F}_i = 1 - F_i$. Similarly, let G_i denote the cdf of $|X_i|$ and $\bar{G}_i = 1 - G_i$. By symmetry, for $i = 1, \dots, n$ and $x \ge 0$ we have

$$\bar{F}_i(x) = \frac{1}{2}\bar{G}_i(x). \tag{73}$$

If we let H_n denote the cdf of $|\max(X_i \mid 1 \le i \le n)|$, then by independence,

$$H_n(x) = \prod_{i=1}^{n} F_i(x) - \prod_{i=1}^{n} \bar{F}_i(x), \quad \text{for } x \ge 0. \tag{74}$$

Therefore, by (74) and (73),

$$\begin{aligned}
&H_n(x) - H_{n-1}(x) \\
&= \bar{F}_1(x)F_n(x)\prod_{i=2}^{n-1}\bar{F}_i(x) - F_1(x)\bar{F}_n(x)\prod_{i=2}^{n-1}F_i(x) \\
&= \frac{1}{4}\left[\bar{G}_1(x)(2 - \bar{G}_n(x))\prod_{i=2}^{n-1}\bar{F}_i(x) - (2 - \bar{G}_1(x))\bar{G}_n(x)\prod_{i=2}^{n-1}F_i(x)\right].
\end{aligned}$$

Since $\bar{F}_i(x) \le F_i(x)$ for $i = 2, \dots, n-1$ and $|X_1| \le_{\text{st}} |X_n| \implies \bar{G}_1(x) \le \bar{G}_n(x)$, it follows that $H_n(x) \le H_{n-1}(x)$ for $x \ge 0$, which confirms (71).

If $|X_1| <_{\text{st}} |X_n|$ then $\bar{G}_1(x) < \bar{G}_n(x)$ for some $x \ge 0$. Since $F_i(x) \ge \frac{1}{2} > 0$ for $i = 2, \dots, n-1$, it follows that $H_n(x) < H_{n-1}(x)$ for this x, hence the stochastic inequality in (71) is strict. The remaining assertions are straightforward. □

Example 19. Let $X_1, \dots, X_n$ be independent symmetric random variables, and let $|X_i|$ $(i = 1, \dots, n)$ take the values $0, 1, \dots, i$ with probability $\frac{1}{2}, \frac{1}{2i}, \dots, \frac{1}{2i}$, respectively. Then it is clear that $|X_1| <_{\text{st}} \cdots <_{\text{st}} |X_n|$. It follows from Theorem 5 that $(X_1, \dots, X_n)$ is SSIAMX and SSIAMN. □

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