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Functional Limit Laws of Strassen and Wichura type for multiple generations of branching processes

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Abstract: This paper is concerned with the study of functional limit theorems constructed from multiple generations of a supercritical branching process. The results we present include infinite dimensional functional laws of Strassen and Chung-Wichura type in the space $(C_0[0,1])^\infty$.

Contents

1. Introduction and Main Results

In the recent paper [11] we studied functional laws of large numbers and central limit theorems for multiple generations of Galton-Watson branching processes. Our interest in multiple generations of such processes, especially in the super-critical case, was motivated by their use to model the exponential growth phase in Polymerase Chain Reaction (PCR) experiments, and the need to detect the random endpoint of this growth phase. The general idea is that if one has knowledge of the limiting quantities for multiple generations, then prediction of this endpoint should become more accurate. Additional information and some relevant references on the use of branching processes in connection with PCR experiments can be found in [11] and [12].

Here we turn to more theoretical issues and seek analogues of the functional law of the iterated logarithm due to Strassen in [15], and the so-called other LIL due to Chung appearing in [3]. The functional version of Chung's result first appeared in the important paper [16], and subsequently in other settings such as [4], [9], and [10]. Hence now such results are known as Chung-Wichura laws of the iterated logarithm. Since branching processes are special triangular arrays, our analogues of these results involve only a single logarithm, and therefore are more properly

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termed functional laws of the logarithm. We adopt this terminology throughout the paper.

In order to describe our results in detail, we begin with a brief description of the branching process. Let $\{\xi_{n,j}, j \geq 1, n \geq 1\}$ denote a double array of non-negative integer valued i.i.d. random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and having probability distribution $\{p_j : j \geq 0\}$, i.e. $P(\xi_{1,1} = k) = p_k$. Then $\{Z_n : n \geq 0\}$ denotes the Galton-Watson process initiated by a single ancestor $Z_0 \equiv 1$. It is iteratively defined on $(\Omega, \mathcal{F}, P)$ for $n \geq 1$ by

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{n,j}.$$

Let $m = E(Z_1)$. It is well known that if $m > 1$ (*i.e.* the process is supercritical), then $Z_n \to \infty$ with positive probability and that the probability that the process becomes extinct, namely q, is less than one. The complement of the set $\cup_{n=1}^{\infty}\{Z_n = 0\}$ is the so called survival set, and is denoted by S. If $m > 1$, then $P(S) = 1 - q$ and $Z_n \to \infty$ a.s. on S. Also, $q = 0$ if and only if $p_0 = 0$. We assume throughout the paper that $1 < m < \infty$.

Our goal is to obtain functional limit theorems for supercritical branching processes based on $r(n)$-generations, where $1 \leq r(n) \leq n$. In particular, the integer sequence $\{r(n)\}$ may approach infinity as n goes to infinity. More precisely, let $\sigma^2 = \mathrm{Var}(Z_1) < \infty$ denote the offspring variance, on the set $\{Z_{n-1} > 0\}$ define $X_{n,Z_{n-1}}(0) = 0$, and for $0 < t \leq 1$ set

$$X_{n,Z_{n-1}}(t) = \frac{1}{\sigma\sqrt{Z_{n-1}}}\left\{\sum_{j=1}^{\lfloor tZ_{n-1}\rfloor}(\xi_{n,j} - m) + c_{n,Z_{n-1}}(t)\right\}, \tag{1.1}$$

where $c_{n,Z_{n-1}}(t) = (tZ_{n-1} - \lfloor tZ_{n-1}\rfloor)(\xi_{n,\lfloor tZ_{n-1}\rfloor+1} - m)$. On $\{Z_{n-1} = 0\}$ we define $X_{n,Z_{n-1}}(t) = 0$ for $0 \leq t \leq 1$. Then define

$$\mathbf{X}_{n,r(n)}(t) \equiv (X_{n,Z_{n-1}}(t), X_{n-1,Z_{n-2}}(t), \ldots, X_{n-r(n)+1,Z_{n-r(n)}}(t), 0, 0, \ldots). \tag{1.2}$$

In (1.2) $X_{n-k,Z_{n-k-1}}(\cdot)$ denotes an element of the space of continuous functions on [0,1] that vanish at zero, which we denote by $C_0[0,1]$, and the zeros in the previous vector are understood to be the zero function on $[0,1]$. Hence $\mathbf{X}_{n,r(n)}(\cdot)$ is an element of the infinite product space $(C_0[0,1])^\infty$. Assuming $C_0[0,1]$ has the sup-norm topology, we give $(C_0[0,1])^\infty$ the product topology, which has metric given by

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sum_{k \geq 1} \frac{1}{2^k} \frac{\|x_k - y_k\|}{1 + \|x_k - y_k\|}, \tag{1.3}$$

where $\|\cdot\|$ is the sup-norm on $C_0[0,1]$.

We now describe our results concerning the laws of the logarithm. If (M, d) is a metric space, $x \in M$, and $A \subseteq M$, then we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

If $\{x_n\}$ is a sequence in M we define the cluster set $C(\{x_n\})$ to be the set of all limit points of $\{x_n\}$.

If $AC_0[0,1]$ denotes the absolutely continuous functions f on $[0,1]$ such that $f(0)=0$, then the limit set K_∞ in $(C_0[0,1])^\infty$ for the processes $\{\mathbf{X}_{n,r(n)}\}$ properly normalized is

$$K_\infty = \left\{(f_1, f_2, \dots) \in (C_0[0,1])^\infty : f_k \in AC_0[0,1], k \geq 1, \sum_{k\geq 1} \int_0^1 (f_k'(s))^2 ds \leq 1\right\},$$

and the analogue of Strassen's law for these processes is the following theorem. Here and throughout the remainder of the paper we let $Lt = \max\{1, \log_e t\}$ for $t \geq 0$.

Theorem 1.1. *Assume $E(Z_1^2(L(Z_1))^r) < \infty$ for some $r > 4$, that $1 \leq r(n) \leq n$, and we also have $\lim_{n\to\infty} r(n) = \infty$. Then*

$$P\left(\lim_{n\to\infty} d_\infty\left(\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}, K_\infty\right) = 0\right) = 1. \tag{1.4}$$

In addition, if S denotes the survival set of the process and clustering is determined with respect to the product topology, then we have

$$P\left(C\left(\left\{\frac{\mathbf{X}_{n,r(n)}}{(2Ln)^{\frac{1}{2}}}\right\}\right) = K_\infty|S\right) = 1. \tag{1.5}$$

Remark 1.1. Looking at the first coordinate of $\mathbf{X}_{n,r(n)}$ and setting $t = 1$ implies a result of Heyde under weaker conditions than is available in [7]. Of course, a similar result holds for any finite collection of coordinates, and one also has functional results for any finite set of coodinates.

Remark 1.2. The fact that $r > 4$ in Theorem 1.1, and also in Theorem 1.2 below, results from the use of standard estimates for the Prokhorov distance in the classical invariance theorem. That these estimates are essentially best possible can be seen from [2] and also [14]. Thus an attempt at reducing $r > 4$ to , say $r > 1$, would seem to require a substantially different approach than what we use here. In particular, in the setting of functional limit theorems of high dimension, the difficulties imposed when working with partial sums from successive generations of a branching process make many typical LIL arguments along subsequences unavailable. Finally, perhaps it is worth mentioning that using the methods of this paper one can prove analogues of Theorems 1.1 and 1.2 for triangular arrays of independent random variables under a variety of conditions. For example, such results hold as long as the row lengths have length $n^{8+\delta}$, the random variables ar e identically distributed with three moments, and the rows are independent, but there are other conditions that suffice as well. The additional assumption that the rows of the triangular array have some form of independence is necessary to show that (1.5) and (1.14) hold. What is surprising is that in the supercritical branching process model no additional assumptions need be made, and although the rows are not independent, there is enough asymptotic independence when combined with the conditional Borel-Cantelli lemma to allow a proof. The log harmonic moments of Lemma 2.4, which are also consequences of the branching process model, allow the use of minimal moment assumptions. That the Prokhorov distance could be useful in proving functional laws of the logarithm was observed earlier in Theorem 4.3 of [8].

We next introduce the maximal processes used in connection to our Chung-Wichura law of the logarithm. To describe these results we need further notation.

Let $\mathcal{M}$ denote the non-decreasing functions on [0,1] into $[0,\infty]$ such that $f(0)=0$, and f is right continuous on (0,1). If $\{h_n\}\subseteq\mathcal{M}$, we say $\{h_n\}$ converges to $h\in\mathcal{M}$ if $\lim_n h_n(t)=h(t)$ for all $t\in[0,1]$ where $h(\cdot)$ is continuous into $[0,\infty]$. The limit set in Wichura's LIL is

$$\mathcal{K}_1=\Big\{h\in\mathcal{M}:\int_0^1 h^{-2}(s)ds\le 1\Big\}. \tag{1.6}$$

Furthermore, it is easy to see from classical arguments, see [6], that the convergence in $\mathcal{M}$ mentioned above can be metrized through the use of the Lévy metric on the non-decreasing functions h^* on $(-\infty,\infty)$ which are right continuous on (0,1), $h^*(0)=0$, $h^*(1)\le 1$, $h^*(t)=h^*(1)$ for $t\ge 1$, and such that $h^*(t)=h^*(0)$ for $t<0$. That is, if $\lambda(s)=s/(1+s)$ for $0\le s\le\infty$, with $\lambda(\infty)=1$, then the metric ρ on $\mathcal{M}$, which is of interest, is given by

$$\rho(h,g)=d_L(h^*,g^*), \tag{1.7}$$

where

$$h^*(s)=\lambda(h(s)),\quad 0\le s\le 1, \tag{1.8}$$

and d_L is Lévy's metric. Of course, for given $h\in\mathcal{M}$ the function h^* used in (1.7) is assumed to be such that $h^*(s)=0$ for $s<0$, $h^*(s)=h^*(1)$ for $s>1$, and given by (1.8) on [0,1]. $(\mathcal{M},\rho)$ is also separable since the subprobabilities on [0,1] are separable in Lévy's metric. We also define the maximal process related to $X_{n,Z_{n-1}}(\cdot)$ by

$$M_{n,Z_{n-1}}(t)=\sup_{0\le s\le t}|X_{n,Z_{n-1}}(s)|,\quad 0\le t\le 1. \tag{1.9}$$

We are, of course, interested in the infinite dimensional version of the maximal processes. To this end, we define the vector maximal process $\mathbf{M}_{n,r(n)}$ analogous to (1.9) as follows:

$$\mathbf{M}_{n,r(n)}(t)=(M_{n,Z_{n-1}}(t),M_{n-1,Z_{n-2}}(t),\ldots,M_{n-r_n+1,Z_{n-r(n)}}(t),0,0,\ldots). \tag{1.10}$$

The infinite dimensional Chung-Wichura limit set is as follows:

$$\mathcal{K}_\infty=\Big\{(h_1,h_2,\ldots,)\in\mathcal{M}^\infty:\sum_{k=1}^\infty\int_0^1 h_k^{-2}(s)ds\le 1\Big\}, \tag{1.11}$$

where $\mathcal{M}^\infty$ is the infinite cartesian product of $\mathcal{M}$. The topology on $\mathcal{M}^\infty$ is the product topology which is complete and separable in the topology given by the metric

$$\rho_\infty(\mathbf{f},\mathbf{g})=\sum_{k\ge 1}\frac{1}{2^k}\frac{\rho(f_k,g_k)}{1+\rho(f_k,g_k)}, \tag{1.12}$$

where $\mathbf{f}=(f_1,f_2,\ldots),\mathbf{g}=(g_1,g_2,\ldots)$ and ρ is the metric given in (1.7).

Our next result presents the functional form of the Chung-Wichura law for the vectors $\mathbf{M}_{n,r(n)}$. In all that follows in connection with the Chung-Wichura results, we'll always assume $c_2=\frac{\pi^2}{8}$. This constant results from the small ball probability estimates for Brownian motion, and enters into our considerations through the application of Lemma 2.6 in Lemmas 3.5 and 3.6 below.

Theorem 1.2. *Assume $E(Z_1^2(L(Z_1))^r) < \infty$ for some $r > 4$, that $1 \le r(n) \le n$, and we also have $\lim_{n\to\infty} r(n) = \infty$. Let S denote the survival set of the process. Then,*

$$P\Big(\lim_{n\to\infty} \rho_\infty\Big(\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n)}, \mathcal{K}_\infty\Big) = 0|S\Big) = 1. \tag{1.13}$$

Furthermore, when clustering is determined with respect to the ρ_∞-topology, then

$$P\Big(C\Big(\Big\{\sqrt{\frac{Ln}{c_2}}\mathbf{M}_{n,r(n)}\Big\}\Big) = \mathcal{K}_\infty|S\Big) = 1. \tag{1.14}$$

The proofs of these theorems follow similar lines, but the precise details differ considerably. However, since Strassen's result is typically better understood, we will only provide a proof of Theorem 1.2. The interested reader can see a complete proof of Theorem 1.1 in the preprint [12] , as well as other details.

2. Some Lemmas and Remarks

Let $\mathcal{F}_n$ denote the $\sigma-$field generated by the sequence $\{Z_0, Z_1, \dots, Z_n\}$. Let $W_n = \frac{Z_n}{m^n}$. Then it is known that $\{(W_n, \mathcal{F}_n) : n \ge 0\}$ is a non-negative martingale sequence, and an important classical result due to Kesten and Stigum is that it converges to a non-degenerate limit W if and only if $E(Z_1 \log Z_1) < \infty$, see, for example, [1], Theorem 1, page 24. Furthermore, as can be seen from [1], Corollary 4, page 36, almost surely on the survival set S we have $0 < W < \infty$. Here we assume $E(Z_1^2(L(Z_1))^r < \infty$ for $r > 4$, so the Kesten-Stigum result applies. It is useful to us as it implies the following fact which for convenience of the reader we state as a lemma. Its proof is immediate since the Kesten-Stigum result implies Z_n/Z_{n-1} converges almost surely to m on S.

Lemma 2.1. *Let $E(Z_1 L(Z_1)) < \infty$, $m = E(Z_1) > 1$, and define*

$$S_0 = \Big\{\omega : \lim_{n\to\infty} \frac{Z_n(\omega)}{Z_{n-1}(\omega)} = m\Big\}. \tag{2.1}$$

Then $P(S\Delta S_0) = 0,\ P(S_0) = 1 - q$, and $S_0 \cap S' = \emptyset$ and on S_0 the following hold: for every $1 < \beta < m$, and all $\omega \in S_0$, there is a $n_0(\omega)$ such that for all $n \ge n_0(\omega)+1$

$$Z_n(\omega) > \beta Z_{n-1}(\omega) > Z_{n-1}(\omega) \tag{2.2}$$

and

$$Z_n(\omega) \ge \max\{Z_0(\omega), \dots, Z_{n-1}(\omega)\}. \tag{2.3}$$

Thus (2.2) and (2.3) imply that for all $\omega \in S_0$ and $n \ge n_0(\omega)$,

$$Z_n(\omega) \ge \beta^{n-n_0} Z_{n_0}(\omega), \tag{2.4}$$

where $n_0 = n_0(\omega)$.

The next lemma is Theorem 3 of [1], p.41, if $p_k \ne 1$ for some $k \ge 2$, but in those cases it is trivial. Hence its proof is omitted.

Lemma 2.2. *Let $\{Z_n : n \geq 0\}$ be a supercritical Galton-Watson process with $Z_0 = 1$. Then there exists a constant $\gamma \in (0,1)$ such that*

$$\lim_{n\to\infty} P(Z_n = k)/\gamma^n = \nu_k, \tag{2.5}$$

where $0 \leq \nu_k < \infty$ for all $k \geq 1$.

In order to prove Theorem 1.2 we use probability estimates involving partial sum processes built from suitable truncations of the random variables used to define the various coordinates of $\mathbf{X}_{n,r(n)}(t)$ in (1.2) and $\mathbf{M}_{n,r(n)}(t)$ in (1.10). The next lemma is useful in our calculations involving these partial sum processes several times. In particular, when combined with the log harmonic moments of Lemma 2.4 below we are able to use conditional Borel-Cantelli arguments along with the error estimates we obtain when these probabilities are replaced by their analogues for Brownian motion. A critical tool to study these probabilities is the estimate of the Prokhorov distance between partial sum processes and Brownian motion obtained in Corollary 2 of [5]. To determine the relevant estimates for the required Brownian motion probabilities we then use some results on small deviations obtained in [4]. For the ease of the reader, the relevant results from [5] and [4] are stated in Lemmas 2.5 and 2.6 below. Finally, perhaps it is also worthwhile to mention that we need to iterate the previous combination of estimates a finite number of times. This makes the argument notationally more difficult, but otherwise follows from conditioning and properties of the branching process.

Lemma 2.3. *Suppose $\phi(t) = t^2(Lt)^r$ where $r > 0$, $t \geq 0$, and as before $Lt = \max\{1, \log_e t\}$. If $E(\phi(Z_1)) < \infty$, $m = E(Z_1)$, and $\mathcal{L}(\xi) = \mathcal{L}(Z_1)$, where ξ is independent of Z_{n-1}, then there exists a finite positive constant $c(\xi, r)$, depending only on $r > 0$ and the law $\mathcal{L}(\xi) = \mathcal{L}(Z_1)$, such that*

$$Z_{n-1}P(|\xi - m| \geq Z_{n-1}^{1/2}|Z_{n-1})I(Z_{n-1} > 0) \leq c(r,\xi)/(LZ_{n-1})^r, \tag{2.6}$$

and

$$Z_{n-1}E(|\eta/Z_{n-1}^{1/2}|^3|Z_{n-1})I(Z_{n-1} > 0) \leq c(r,\xi)/(LZ_{n-1})^r, \tag{2.7}$$

where

$$\eta = (\xi - m)I(|\xi - m| \leq Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}, \tag{2.8}$$

and

$$\mu_{n,Z_{n-1}} = E((\xi - m)I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1}). \tag{2.9}$$

Proof. Since the terms to be dominated in (2.6) and (2.7) are all zero when $Z_{n-1} = 0$, and $Lt \geq 1$ for all $t \geq 0$, the result holds in this situation. Hence it suffices to prove the result when we assume $Z_{n-1} > 0$.

To simplify notation, let $\rho = \xi - m$. Then, since $\phi(t)$ is increasing for $t \geq 0$, we have by the conditional Markov inequality that

$$Z_{n-1}P(|\rho| \geq Z_{n-1}^{1/2}|Z_{n-1})I(Z_{n-1} > 0) \leq Z_{n-1}\frac{E(\phi(|\rho|)I(|\rho| \geq Z_{n-1}^{1/2})|Z_{n-1})}{\phi(Z_{n-1}^{1/2})},$$

and hence

$$\begin{aligned} Z_{n-1}P(|\rho| \geq Z_{n-1}^{1/2}|Z_{n-1})I(Z_{n-1} > 0) &\leq E(\phi(|\rho|))/(LZ_{n-1}^{1/2})^r \\ &\leq 2^r E(\phi(|\rho|))/(LZ_{n-1})^r, \end{aligned}$$

where in the last inequality we have used that $(Lt^{1/2})^r \geq (Lt)^r/2^r$ for $t \geq 0$ and $r > 0$. Thus (2.6) holds with $c(r,\xi) \geq 2^r E(\phi(|\xi - m|))$.

To verify (2.7) observe that

$$Z_{n-1}E(|\eta/Z_{n-1}^{1/2}|^3|Z_{n-1})I(Z_{n-1} > 0) \leq Z_{n-1}^{-1/2}\{a_{1,n} + a_{2,n}\},$$

where

$$a_{1,n} = E(|\rho I(|\rho| \leq Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}|^2|\rho|I(|\rho| \leq Z_{n-1}^{1/2})|Z_{n-1}),$$

and

$$a_{2,n} = E(|\rho I(|\rho| \leq Z_{n-1}^{1/2}) - \mu_{n,Z_{n-1}}|^2|\mu_{n,Z_{n-1}}||Z_{n-1}).$$

Recalling $\mu_{n,Z_{n-1}}$ is $\sigma(Z_{n-1})$ measurable, we have

$$a_{1,n} \leq 2E(|\xi - m|^3 I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1}) + 2\mu_{n,Z_{n-1}}^2 E(|\xi - m|),$$

and we also easily see that

$$a_{2,n} \leq |\mu_{n,Z_{n-1}}|E((\xi - m)^2).$$

Thus

$$Z_{n-1}E(|\eta/Z_{n-1}^{1/2}|^3|Z_{n-1})I(Z_{n-1} > 0) \leq Z_{n-1}^{-1/2}\{a_{3,n} + a_{4,n} + a_{5,n}\},$$

where $a_{3,n} = 2E(|\xi - m|^3 I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1})$, $a_{4,n} = 2\mu_{n,Z_{n-1}}^2 E(|\xi - m|)$, and $a_{5,n} = E((\xi - m)^2)|\mu_{n,Z_{n-1}}|$. Since $|\mu_{n,Z_{n-1}}| \leq E(|\xi - m|)$, we see that

$$Z_{n-1}^{-1/2}\{a_{2,n} + a_{4,n} + a_{5,n}\} \leq c(r,\xi)/(LZ_{n-1})^r,$$

where $c(r,\xi)$ is a finite positve constant depending only on $r > 0$ and $\mathcal{L}(\xi)$.

Hence (2.7) will hold, and the lemma will be proved, if we show

$$Z_{n-1}^{-1/2}a_{3,n} = 2Z_{n-1}^{-1/2}E(|\xi - m|^3 I(|\xi - m| \leq Z_{n-1}^{1/2})|Z_{n-1}) \leq c(r,\xi)/(LZ_{n-1})^r,$$

where again $c(r,\xi)$ is a finite positve constant depending only on $r > 0$ and $\mathcal{L}(\xi)$. To verify this last inequality take $c_0 = c_0(r)$ such that $c_0 \geq e^e$ and if $t \geq c_0$, then $\log_e t - 2r\log_e(\log_e t) > (\log_e t)/2$.

If $c_0 > Z_{n-1}$, then

$$Z_{n-1}^{-1/2}a_{3,n} \leq 2c_0 E(|\xi - m|^2)/Z_{n-1}^{1/2},$$

and again (2.7) will hold for a sufficiently large constant $c(r,\xi)$. Hence it remains to consider the case where $c_0 \leq Z_{n-1}$. Thus we observe that

$$Z_{n-1}^{-1/2}a_{3,n} \leq 2(A_{1,n} + A_{2,n}),$$

where

$$\begin{aligned} A_{1,n} &= E\left(|\xi - m|^2|\xi - m|Z_{n-1}^{-1/2}I\left(0 < |\xi - m| \leq \frac{Z_{n-1}^{1/2}}{(LZ_{n-1})^r}\right)|Z_{n-1}\right) \\ &\leq \frac{E(|\xi - m|^2)}{(LZ_{n-1})^r}, \end{aligned}$$

and

$$A_{2,n} = E\Big(|\xi - m|^2|\xi - m|Z_{n-1}^{-1/2} I\Big(\frac{Z_{n-1}^{1/2}}{(LZ_{n-1})^r} \le |\xi - m| \le Z_{n-1}^{1/2}\Big)\Big|Z_{n-1}\Big)$$
$$\le \frac{E(\phi(|\xi - m|))}{\{L(\frac{Z_{n-1}^{1/2}}{(LZ_{n-1})^r})\}^r}.$$

Since $c_0 \le Z_{n-1}$, our choice of c_0 now allows us to complete the proof. □

Our next lemma provides harmonic moments for $L(Z_n)$. Its proof can be seen in [11] or [12].

Lemma 2.4. *Let $r > 1$, and assume $\{Z_n : n \ge 0\}$ is a Galton-Watson branching process with $1 < m = E(Z_1) < \infty$. Then*

$$\limsup_{n\to\infty} n^r E((LZ_n)^{-r} I(Z_n > 0))/(\log_e n)^r < \infty. \tag{2.10}$$

A statement of Corollary 2 in [5] sufficient for this paper is the following lemma. In particular, we state and use Einmahl's result for real valued random variables, whereas Corollary 2 holds for finite dimensional random vectors.

Lemma 2.5. *Let $\eta_1, \dots, \eta_n$ be independent real valued random variables with zero means and $E(\eta_j^2) = \sigma_j^2$, $1 \le j \le n$, where $\sum_{j=1}^n \sigma_j^2 = 1$. Let $S_n(0) = 0$,*

$$S_n(t) = \sum_{j=1}^k \eta_j, \quad t = \sum_{j=1}^k \sigma_j^2 = 1,\ 1 \le k \le n,$$

and define $S_n(t)$ otherwise on $[0,1]$ via piecewise linear and continuous interpolation. If W denotes Wiener measure on $C_0[0,1]$ and $2 < s < 5$, then the Prokhorov distance between W and $\mathcal{L}(S_n(\cdot))$, when we use the sup-norm distance on $C_0[0,1]$, satisfies

$$\rho(\mathcal{L}(S_n(\cdot)), W) \le c_E \left[\sum_{j=1}^n E(|\eta_j|^s)\right]^{\frac{1}{s+1}}, \tag{2.11}$$

where $c_E = c_E(s)$ depends only on s.

The small deviation probabilities for Brownian motion we use are our next lemma. It combines Propositions 2.2 and 2.4 from [4], with Proposition 2.2 slightly modified, although its proof remains the same.

Lemma 2.6. *Let $\{B(t); 0 \le t \le 1\}$ be a sample continuous Brownian motion, and $M(t) = \sup_{0\le s\le t} |B(s)|$, $0 \le t \le 1$. Fix sequences $\{t_i\}_{i=0}^r$, $\{a_i\}_{i=1}^r$, and $\{b_i\}_{i=1}^r$, where $0 = t_0 < t_1 < \cdots < t_r \le 1$, $a_i < b_i$ for $1 \le 1 \le r$, and $b_1 \le b_2 \le \cdots \le b_r$. Then*

$$\limsup_{\epsilon\to 0^+} \epsilon^2 \log P(a_i\epsilon \le M(t_i) \le b_i\epsilon,\ 1 \le i \le r) \le -\frac{\pi^2}{8}\sum_{i=1}^r (t_i - t_{i-1})/b_i^2. \tag{2.12}$$

In addition, if we assume $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_r < b_r$, then

$$\limsup_{\epsilon\to 0^+} \epsilon^2 \log P(a_i\epsilon \le M(t_i) \le b_i\epsilon,\ 1 \le i \le r) \ge -\frac{\pi^2}{8}\sum_{i=1}^r (t_i - t_{i-1})/b_i^2. \tag{2.13}$$

3. Proof of the The Chung-Wichura Functional Law

First we observe that the infinite product space $(\mathcal{M}^\infty, \rho_\infty)$ is a compact metric space. This follows since the metric ρ_∞ gives the product topology on $\mathcal{M}^\infty$, and by the definition (1.7) the space $(\mathcal{M}, \rho)$ itself is a compact metric space, i.e. $(\mathcal{M}, \rho)$ is homeomorphic to the space of sub-probabilities on $[0, 1]$ under the Lévy metric, which is a compact metric space. In order to prove Theorem 1.2 our first step is to show the limit set $\mathcal{K}^\infty$ is a compact subset of $(\mathcal{M}^\infty, \rho_\infty)$.

Lemma 3.1. *$\mathcal{K}^\infty$ is a compact subset of the space $(\mathcal{M}^\infty, \rho_\infty)$.*

Proof. Since $(\mathcal{M}^\infty, \rho_\infty)$ is compact, it suffices to show $\mathcal{K}^\infty$ is a closed subset of $(\mathcal{M}^\infty, \rho_\infty)$. Hence let $\{\mathbf{f}_n\}$ be a sequence in $\mathcal{K}^\infty$, and $\mathbf{f} \in \mathcal{M}^\infty$ such that $\mathbf{f}_n = (f_{n,1}, f_{n,2}, \ldots)$ and $\mathbf{f} = (f_1, f_2, \ldots)$. Then, since ρ_∞ gives the product topology on $\mathcal{M}^\infty$, we have $\lim_{n\to\infty} \rho_\infty(\mathbf{f}_n, \mathbf{f}) = 0$ iff $\lim_{n\to\infty} \rho(f_{n,j}, f_j) = 0$ for every $j \geq 1$. Furthermore, (1.6-7) and the classical facts regarding convergence in Levy's metric, see [6], pages 32-37, imply that $\lim_{n\to\infty} \rho(f_{n,j}, f_j) = 0$ iff $\lim_{n\to\infty} f_{n,j}(t) = f_j(t)$ for all $t \in [0, 1]$ which are continuity points of the limit function f_j. Hence it is immediate that $\lim_{n\to\infty} \rho_\infty(\mathbf{f}_n, \mathbf{f}) = 0$ implies for all $j \geq 1$ we have

$$\lim_{n\to\infty} f_{n,j}(t) = f_j(t),$$

except possibly for countably many $t \in [0, 1]$.

Hence let $\{\mathbf{f}_n\}$ be sequence in $\mathcal{K}_\infty$ with $\lim_{n\to\infty} \rho_\infty(\mathbf{f}_n, \mathbf{f}) = 0$. Then for every integer $N \geq 1$ the above implies we have that

$$\sum_{j=1}^{N} \int_0^1 f_j^{-2}(s)ds = \sum_{j=1}^{N} \int_0^1 \lim_{n\to\infty} f_{n,j}^{-2}(s)ds \leq \liminf_{n\to\infty} \sum_{j=1}^{N} \int_0^1 f_{n,j}^{-2}(s)ds \leq 1,$$

where the first inequality above is Fatou's lemma and the second because $\mathbf{f}_n \in \mathcal{K}_\infty$. Since N is arbitrary, this implies $\mathbf{f} \in \mathcal{K}_\infty$, so we have $\mathcal{K}_\infty$ is closed. Thus the lemma is proven. $\square$

Now we introduce some further notation, which will yield a useful open neighborhood base for the topological space $(\mathcal{M}, \rho)$.

Definition. If $f \in \mathcal{M}$, then we set $t_f^* = \sup\{t : 0 \leq t \leq 1, f(t) < \infty\}$ and note that $t_f^* = 1$ by default if $f(1) < \infty$. If

$$0 = t_0 < t_1 < t_2 < \cdots < t_r < t_f^* \leq t_{r+1} < \cdots < t_{r+s} \leq 1$$

is an arbitrary partition of the interval $[0, 1]$, we often will abbreviate the partition by $\mathcal{P}$ without explicitly displaying the points of the partition, or the number of points in the partition, which is also arbitrary. If $f \in \mathcal{M}$, $\alpha, \beta > 0$, and $\mathcal{P}$ is the partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_r < t_f^* \leq t_{r+1} < \cdots < t_{r+s} \leq 1$$

we define the neighborhood

$$N(f, t_1, \ldots, t_{r+s}, \alpha, \beta) = N^{(1)}(f, t_1, \ldots, t_r, \alpha) \cap N^{(2)}(f, t_{r+1}, \ldots, t_{r+s}, \beta), \tag{3.1}$$

where

$$N^{(1)}(f, t_1, \ldots, t_r, \alpha) = \{g \in \mathcal{M} : f(t_j) - \alpha < g(t_j) < f(t_j) + \alpha,\ j = 1, \ldots, r\}$$

and

$$N^{(2)}(f, t_{r+1}, \dots, t_{r+s}, \beta) = \{g \in \mathcal{M} : g(t_{r+k}) > \beta,\ k = 1, \dots, s\}.$$

When the partition $\mathcal{P}$ and α, β are understood we will sometimes simply write $N(f)$, $N(f, \mathcal{P})$, or $N(f, \mathcal{P}, \alpha, \beta)$. If the partition $\mathcal{P}$ contains only points from $[0, t_f^*)$, then we will use $N(f, \mathcal{P}, \alpha)$ to denote

$$\{g \in \mathcal{M} : f(t_j) - \alpha < g(t_j) < f(t_j) + \alpha,\ j = 1, \dots, r\}.$$

Finally, if $t_f = 1$, $f(1) < \infty$, and $t = 1$ is a continuity point of f, then we will allow $t = 1$ in partitions of the form $N(f, \mathcal{P}, \alpha)$.

Our next lemma justifies the neighborhood terminology we use for the sets $N(f)$. Since $(\mathcal{M}, \rho)$ is homeomorphic to the space of sub-probabilities on $[0, 1]$ metrized by Lévy's metric, and convergence in Lévy's metric is equivalent to pointwise convergence at all points where the limit function is continuous, the same holds for ρ convergence on $\mathcal{M}$ by definition of ρ in (1.7). Hence the following lemma is hardly surprising, and the interested reader can find a proof in the preprint [12].

Lemma 3.2. *The collection of sets $N(f, \mathcal{P}, \alpha, \beta)$, as $\mathcal{P}$ varies over all possible partitions of continuity points of f in $[0, 1]$ and we also allow $\alpha, \beta > 0$ to be arbitrary, forms an open neighborhood base at the point $f \in \mathcal{M}$. That is, given*

$$N(f, t_1, \dots, t_{r+s}, \alpha, \beta) = N^{(1)}(f, t_1, \dots, t_r, \alpha) \cap N^{(2)}(f, t_{r+1}, \dots, t_{r+s}, \beta),$$

there is an $\epsilon > 0$ and open neighborhood

$$H(f) = \{g \in \mathcal{M} : \rho(f, g) < \epsilon\}$$

such that $H(f) \subseteq N(f, t_1, t_2, \dots, t_{r+s}, \alpha, \beta)$, and for each such $H(f)$ there is an $\alpha, \beta > 0$ and a partition $\mathcal{P}$ such that $N(f, \mathcal{P}, \alpha, \beta) \subseteq H(f)$. Moreover, if $f(1) < \infty$, then the sets $N(f, \mathcal{P}, \alpha)$, as $\mathcal{P}$ varies over all finite partitions of continuity points of f and $\alpha > 0$ is arbitrary, form an open neighborhood base at f.

Remark 3.1. Once we prove the above set inclusions, the fact that the sets $N(f)$ are actually open follows immediately since the inequalities that define $N(f)$ as per (3.1) are strict inequalities.

Another elementary lemma involving the spaces $(\mathcal{M}, \rho)$ and $(\mathcal{M}^\infty, \rho_\infty)$ is as follows.

Lemma 3.3. *Let $f \in \mathcal{M}$ and for $n \geq 1$, $M > 0$, and $0 \leq t \leq 1$, define*

$$h_n(t) = \frac{n+1}{n} f(t) \quad \text{and} \quad f^{(M)}(t) = f(t) \wedge M.$$

Then

$$\rho(h_n, f) \leq 1/n \quad \text{and} \quad \rho(f^{(M)}, f) \leq 1/(M+1).$$

Moreover, if $f \in \mathcal{K}$, then

$$\int_0^1 h_n^{-2}(s) ds = (n/(n+1))^2 \int_0^1 f^{-2}(s) ds < 1.$$

Furthermore, if $\mathbf{f} = (f_1, f_2, \dots) \in \mathcal{K}_\infty$, $\mathbf{h}_n = \frac{n+1}{n}\mathbf{f}$, $\mathbf{f}^{(M)} = (f_1^{(M)}, f_2^{(M)}, \dots)$, then

$$\rho_\infty(\mathbf{h}_n, \mathbf{f}) \leq 1/n \quad \text{and} \quad \rho_\infty(\mathbf{f}^{(M)}, \mathbf{f}) \leq 1/(M+1),$$

and

$$\sum_{j=1}^\infty \int_0^1 h_n^{-2}(s) ds \leq \left(\frac{n}{n+1}\right)^2 < 1.$$

Proof. First observe that

$$\rho(h_n, f) = d_L(h_n^*, f^*) \leq \|h_n^* - f^*\|,$$

where the equality is by definition of the ρ-metric, and the inequality follows since the sup-norm dominates the Lévy metric. However, $h_n^*(t) - f^*(t) = 0$ if $t \geq t_f^*$ or $t = 0$, and for $0 < t < t_f^*$

$$h_n^*(t) - f^*(t) \leq \frac{f(t)/n}{(1 + f(t))^2} \leq 1/n.$$

Thus $\|h_n^* - f^*\| \leq 1/n$, which implies $\rho(h_n, f) \leq 1/n$ as indicated. Similarly, $f^*(t) - (f^{(M)})^*(t) = 0$ if $0 \leq f(t) \leq M$ and for $M < f(t)$

$$f^*(t) - (f^{(M)})^*(t) = \frac{f(t)}{1 + f(t)} - \frac{M}{1 + M} \leq 1 - \frac{M}{1 + M} = \frac{1}{1 + M}.$$

Thus $\rho(f^{(M)}, f) \leq 1/(M + 1)$ as indicated. The remainder of the proof is now immediate. □

To prove Theorem 1.2 we next prove a lemma which allows us to transfer estimates on $X_{n,Z_{n-1}}$ being close to B in law, to estimates on $M_{n,Zn-1}$ being close to

$$M_B(t) = \sup_{0 \leq s \leq t} |B(s)|, \quad 0 \leq t \leq 1 \tag{3.2}$$

in law.

Lemma 3.4. *Let $\Lambda : C[0, 1] \to C[0, 1]$ be defined by*

$$(\Lambda f)(t) = \sup_{0 \leq s \leq t} |f(s)|, \quad 0 \leq t \leq 1,$$

and for any Borel probability measure μ on $C[0, 1]$ define $\mu^\Lambda(A) = \mu(\Lambda^{-1}(A))$ for Borel sets A. If $\rho(\mu, \nu)$ is the Prokhorov metric for probability measures on $C[0, 1]$ when we use the sup-norm distance on $C[0, 1]$, then

$$\rho(\mu^\Lambda, \nu^\Lambda) \leq \rho(\mu, \nu). \tag{3.3}$$

Proof. Take $\delta > \rho(\mu, \nu)$ and A an arbitrary Borel subset of $C[0, 1]$. Then we have

$$\mu^\Lambda(A) = \mu(\Lambda^{-1}(A)) \leq \nu((\Lambda^{-1}(A))^\delta) + \delta \leq \nu(\Lambda^{-1}(A^\delta)) + \delta = \nu^\Lambda(A^\delta) + \delta. \tag{3.4}$$

In the above, the second inequality follows from the fact that Λ is a Lip-1 map with Lipshitz constant one, and hence $(\Lambda^{-1}(A))^\delta \subseteq \Lambda^{-1}(A^\delta)$ for every A and $\delta > 0$. That is, if $f \in (\Lambda^{-1}(A))^\delta$, then there exists $g \in \Lambda^{-1}(A)$ with $\|f - g\|_\infty < \delta$. Hence $\Lambda(g) \in A$ and since $\|\Lambda(f) - \Lambda(g)\|_\infty \leq \|f - g\|_\infty$, this implies $\Lambda(f) \in A^\delta$ and hence $f \in \Lambda^{-1}(A^\delta)$. The proof that $\|\Lambda(f) - \Lambda(g)\|_\infty \leq \|f - g\|_\infty$ follows easily from the triangle inequality. Finally (3.4) for arbitrary A and $\delta > \rho(\mu, \nu)$ implies the lemma.

Now that the various topological considerations have been established, we finish the proof of Theorem 1.2 with two additional lemmas. The first establishes (1.13), showing that almost surely on the survival set S we have convergence to $\mathcal{K}_\infty$. The second will verify (1.14), proving the cluster set fills $\mathcal{K}_\infty$ almost surely on S. □

Lemma 3.5. *Let S denote the survival set of the process and set $c_2 = \pi^2/8$. Then, under the conditions of Theorem 1.2 we have*

$$P\Big(\Big\{\lim_{n\to\infty} \rho_\infty((Ln/c_2)^{1/2}\mathbf{M}_{n,r(n)}, \mathcal{K}_\infty) = 0\Big\} \cap S\Big) = P(S). \tag{3.5}$$

Proof. To simplify notation we let $\eta_n(t) = (Ln/c_2)^{1/2} M_{n,Z_{n-1}}(t)$ for $n \geq 1$ and $0 \leq t \leq 1$. We also define for $0 \leq t \leq 1,\ n \geq 1,\ l \geq 1$ the vector valued processes

$$\hat{\boldsymbol{\eta}}_{n,l}(t) = (Ln/c_2)^{1/2}(M_{n,Z_{n-1}}(t), \ldots, M_{n-l+1,Z_{n-l}}(t)),$$

and

$$\boldsymbol{\eta}_{n,r(n)}(t) = (Ln/c_2)^{1/2}(M_{n,Z_{n-1}}(t), \ldots, M_{n-r(n)+1,Z_{n-r(n)}}(t), 0, 0, \ldots).$$

Since $(\mathcal{M}_\infty, \rho_\infty)$ is a compact metric space, it is separable. Hence $\mathcal{K}_\infty$ closed implies that (3.5) will follow if we show for every $\mathbf{f} \notin \mathcal{K}_\infty$ there exists an open set V containing $\mathbf{f}$ such that $V \cap \mathcal{K}_\infty = \phi$ and V satisfies $P(\{\boldsymbol{\eta}_{n,r(n)} \in V\ i.o.\} \cap S) = 0$.

Letting S_0 be defined as in (2.1), we have $P(S \triangle S_0) = 0$, and hence it suffices to show

$$P(\{\boldsymbol{\eta}_{n,r(n)} \in V\ i.o.\} \cap S_0) = 0. \tag{3.6}$$

for each $\mathbf{f} \notin \mathcal{K}_\infty$ and suitable open set V disjoint from $\mathcal{K}_\infty$ containing $\mathbf{f}$.

If $\mathbf{f} = (f_1, f_2 \ldots) \notin \mathcal{K}_\infty$, then $\sum_{j\geq 1} \int_0^1 f_j^{-2}(s)ds > 1$. Hence there exists an integer $l \geq 1$ and $\delta > 0$ such that

$$\sum_{j=1}^{l} \int_0^1 f_j^{-2}(s)ds > 1 + \delta.$$

Furthermore, since the $f_j's$ are nondecreasing on $[0,1]$, there exist finite partitions $\mathcal{P}_j$ of $[0, t^*_{f_j})$ consisting of continuity points of f_j and $\alpha > 0$ such that

$$\sum_{j=1}^{l} \sum_{t_k \in \mathcal{P}_j} (f_j(t_k) + 4\alpha)^{-2}(t_k - t_{k-1}) > 1 + \delta. \tag{3.7}$$

Here the reader should note that we need not take any points in the partition $\mathcal{P}_j$ which are in $[t_{f_j^*}, 1]$ since $\int_{t_{f_j^*}}^1 f_j^{-2}(s)ds = 0$. In particular, if $t_{f_j^*} = 0$ we will not form a partition, but rather define $V_j = \mathcal{M}$, to be used as indicated below. That is, if $V = \prod_{j=1}^\infty V_j$, where $V_j = N(f_j, \mathcal{P}_j, \alpha)$, or $V_j = \mathcal{M}$ should $t_{f_j^*} = 0$, for $1 \leq j \leq l$, and $V_j = \mathcal{M}$ for $j \geq l+1$, then for $\mathbf{g} = (g_1, g_2, \ldots) \in V$ we have

$$\sum_{j\geq 1} \int_0^1 g_j^{-2}(s)ds \geq \sum_{j=1}^{l} \sum_{t_k \in \mathcal{P}_j} (f_j(t_k) + \alpha)^{-2}(t_k - t_{k-1}) > 1 + \delta. \tag{3.8}$$

Of course, if $t_{f_j^*} = 0$ for some $j,\ 1 \leq j \leq l$, then those terms do not need to appear in (3.8), but to simplify the notation we write the proof as if all $t_{f_j^*} > 0$ for $j = 1, \ldots, l$.

In particular, we now have $\mathbf{f} \in V$ and $V \cap \mathcal{K}_\infty = \phi$. Furthermore, since $V_j = \mathcal{M}$ for all $j \geq l+1$ and eventually $r(n) > l$, we have

$$\{\boldsymbol{\eta}_{n,r(n)} \in V\} = \left\{\hat{\boldsymbol{\eta}}_{n,l} \in \prod_{j=0}^{l-1} V_{j+1}\right\} = \bigcap_{j=0}^{l-1} \{\eta_{n-j,Z_{n-j-1}} \in V_{j+1}\}.$$

Hence (3.6) will follow if we show

$$P\left(\left\{\hat{\boldsymbol{\eta}}_{n,l} \in \prod_{j=0}^{l-1} V_{j+1}\ i.o.\right\} \cap S_0\right) = 0. \tag{3.9}$$

Letting $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$ for $n \geq 1$, we define

$$\mathcal{G}_{n,k} = \mathcal{F}_{nl+k}, k = 0, 1, 2, \ldots, l-1,\ n \geq 0,$$

and

$$E_n = \bigcap_{j=0}^{l-1} A_{n,j,\alpha}, \tag{3.10}$$

where

$$A_{n,j,\alpha} = \{\eta_{n-j, Z_{n-j-1}} \in V_{j+1} = N(f_{j+1}, \mathcal{P}_{j+1}, \alpha)\}$$

for $j = 0, 1, 2, \ldots, l-1$. Strictly speaking these sets also involve δ through (3.7), but we supress that as our choice of α implies (3.7).

Then E_{nl+k} is $\mathcal{G}_{n,k}$ measurable and (3.9) holds by the conditional Borel-Cantelli lemma if we show that

$$\sum_{n \geq 1} P(E_{nl+k} | \mathcal{G}_{n-1,k}) < \infty \tag{3.11}$$

a.s. on S_0 for each $k = 0, 1, \ldots l-1$. That is, since $\{E_n\ i.o.\} \cap S_0$ is the event in (3.9) and

$$\{E_n\ \ i.o\} \cap S_0 \subseteq \cup_{k=0}^{l-1} \{E_{nl+k}\ \ i.o. \text{ in n}\} \cap S_0, \tag{3.12}$$

the conditional Borel-Cantelli lemma and (3.11) implies

$$P(\{E_{nl+k}\ \ i.o. \text{ in } n\} \cap S_0) = 0. \tag{3.13}$$

Hence, (3.11) holding a.s. on S_0 for $k = 0, 1, 2, \ldots l-1$ and (3.12) and (3.13) combine to prove (3.6). We will prove (3.11) for $k = 0$ and observe that the other cases are exactly the same. Furthermore, to simplify our notation we will let $\mathcal{H}_n = \mathcal{G}_{n,0} = \mathcal{F}_{nl}$ for $n = 0, 1, \ldots$. Hence, we must show that a.s. on S_0

$$\sum_{n \geq 1} P(E_{nl} | \mathcal{H}_{n-1}) < \infty.$$

To this end, notice that by Lemma 2.1 on S_0 we eventually have $Z_n > \beta^n$ for some $1 < \beta < m$. Then, since $Z_{(n-1)l}$ is $\mathcal{H}_{n-1}$ measurable, for sufficiently large n on the set S_0 we have that

$$P(E_{nl} | \mathcal{H}_{n-1}) = P(\cap_{j=0}^{l-1} A_{nl,j,\alpha} | \mathcal{H}_{n-1}) I(Z_{(n-1)l} > \beta^{(n-1)l}) \tag{3.14}$$

$$= P(\cap_{j=0}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) | \mathcal{H}_{n-1}). \tag{3.15}$$

Thus, for all n sufficiently large, on S_0 we have

$$P(E_{nl} | \mathcal{H}_{n-1}) = \theta_{n,1} + \theta_{n,2}, \tag{3.16}$$

where

$$\theta_{n,1} = E[I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) \cdot T_{n,l,\alpha,1} | \mathcal{H}_{n-1}],$$

$$\theta_{n,2} = E[I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) \cdot T_{n,l,\alpha,2} | \mathcal{H}_{n-1}],$$

$$T_{n,l,\alpha,1} = E(I(A_{nl,0,\alpha} \cap B_{n,\alpha}) | \mathcal{F}_{nl-1}),$$

$$T_{n,l,\alpha,2} = E(I(A_{nl,0,\alpha} \cap B'_{n,\alpha}) | \mathcal{F}_{nl-1}),$$

and

$$B_{n,\alpha} = \left\{ c_E \left[\frac{c(r,\xi)}{(Z_{nl-1})^r} \right]^{1/4} < \gamma_0, Z_{nl-1} > r_0(f_1, \dots f_l; \alpha) \geq 3 \right\}. \tag{3.17}$$

In the definition of $B_{n,\alpha}$ we take $\gamma_0 > 0$, c_E is the constant from Einmahl's result presented in Lemma 2.5 with $s = 3$, $c(r, \xi)$ is given as in Lemma 2.3, and $r_0(f_1, \dots f_{l+1}; \alpha)$ is such that $Z_k > r_0(f_1, \dots f_{l+1}; \alpha)$ implies $\frac{\sigma}{\sigma_k} N(f_j, \mathcal{P}_j, 2\alpha) \subseteq N(f_j, \mathcal{P}_j, \frac{5}{2}\alpha)$ for $j = 1, 2, \dots l$, where $\sigma_k^2 = H(Z_k)$ and $H(Z_k)$ is defined via the truncated variance

$$H(a) = \mathrm{Var}((\xi - m) I(|\xi - m| \leq a)).$$

Recalling $B'_{n,\alpha}$ denotes the complement of $B_{n,\alpha}$, we thus have

$$\theta_{n,2} \leq P(B'_{n,\alpha} | \mathcal{H}_{n-1})$$

and hence

$$\sum_{n \geq 1} E(\theta_{n,2}) \leq \sum_{n \geq 1} P(B'_{n,\alpha}) < \infty$$

since Lemma 2.2 implies $\sum_{n=1}^{\infty} P(Z_n \leq J) < \infty$ for all $J < \infty$. Thus we have that $\sum_{n \geq 1} \theta_{n,2}$ converges with probability 1.

We now deal with $\theta_{n,1}$. To this end, define the truncated version of the $X_{n,Z_{n-1}}$ process as follows. If $Z_{n-1} = 0$ then set $T_n(t) = 0$ for all $t \in [0, 1]$, and if $Z_{n-1} > 0$, define for $t = \frac{k}{Z_{n-1}}$ and $1 \leq k \leq Z_{n-1}$,

$$T_n(t) = (\sigma^2 Z_{n-1})^{-1/2} \sum_{j=1}^{k} ((\xi_{n,j} - m) I(|\xi_{n,j} - m| \leq \sqrt{Z_{n-1}}) - \mu_{n,Z_{n-1}}), \tag{3.18}$$

where the function is linearly interpolated for other values of t with $T_n(0) = 0$ and

$$\mu_{n,Z_{n-1}} = E((\xi - m) I(|(\xi - m)| \leq Z_{n-1}^{\frac{1}{2}} | Z_{n-1})).$$

Recalling $V_1 = N(f_1, \mathcal{P}_1, \alpha)$ and the map Λ from Lemma 3.4, we have

$$P(A_{nl,0,\alpha} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}) = P(\{\eta_{nl,Z_{nl-1}} \in V_1\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}) \leq I_n + II_n,$$

where

$$I_n = P(\{(L(nl)/c_2)^{1/2} \Lambda(T_{nl}) \in N(f_1, \mathcal{P}_1, 2\alpha)\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}), \tag{3.19}$$

and

$$II_n = P(\{\|\eta_{nl,Z_{nl-1}} - (L(nl)/c_2)^{1/2} \Lambda(T_{nl})\| > \alpha/2\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}). \tag{3.20}$$

Thus, for n sufficiently large

$$\theta_{n,1} \le E(I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) \cdot (I_n + II_n)|\mathcal{H}_{n-1}\}) \tag{3.21}$$

$$\le E(I(\cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}) \cdot I_n|\mathcal{H}_{n-1}\} + \Gamma_n,$$

where $\Gamma_n = E(I(\{Z_{(n-1)l} > \beta^{(n-1)l}\})II_n|\mathcal{H}_{n-1})$.

We first deal with the second term Γ_n. First we observe that

$$I(\{Z_{(n-1)l} > \beta^{(n-1)l}\})II_n \le \alpha_n + \beta_n,$$

where α_n denotes

$$P\left(\left(\frac{L(nl)}{c_2}\right)^{\frac{1}{2}} \sup_{1\le k\le Z_{nl-1}} \left|\sum_{j=1}^{k}(\xi_{n,j} - m)I(|\xi_{n,j} - m| > Z_{nl-1}^{\frac{1}{2}})\right| > 0|\mathcal{F}_{nl-1}\right),$$

and β_n equals

$$P\left(\left\{\left(\frac{L(nl)}{c_2}\right)^{\frac{1}{2}} Z_{nl-1}|\mu_{n,Z_{nl-1}}|/Z_{nl-1}^{\frac{1}{2}} > \alpha/2\right\} \cap B_{n,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}|\mathcal{F}_{nl-1}\right).$$

Applying Lemma 2.3 we thus have

$$\alpha_n \le Z_{nl-1}P(|\xi - m| \ge Z_{nl-1}^{\frac{1}{2}}|Z_{nl-1})I(Z_{nl-1} > 0) \tag{3.22}$$

$$\le c(r,\xi)(LZ_{nl-1})^{-r}I(Z_{nl-1} > 0). \tag{3.23}$$

Thus by the harmonic moment result in Lemma 2.4, we have a.s. on Ω that $\sum_{n\ge1} E(\alpha_n|\mathcal{H}_{n-1}) < \infty$.

We now turn to an estimate of β_n. When $Z_{nl-1} > 0$, and since $E(\xi - m) = 0$, we see with $\mu_{n,Z_{nl-1}}$ defined as in (2.9) that

$$\frac{Z_{nl-1}|\mu_{n,Z_{nl-1}}|}{Z_{nl-1}^{\frac{1}{2}}} = Z_{nl-1}^{\frac{1}{2}} \int_{-Z_{nl-1}^{\frac{1}{2}}}^{Z_{nl-1}^{\frac{1}{2}}} t\, dF_{(\xi-m)}(t)$$

$$\le \frac{\int_{Z_{nl-1}^{\frac{1}{2}}}^{\infty} t^2(Lt)^r dF_{|\xi-m|}(t)}{(LZ_{nl-1})^r} \le \frac{c}{(LZ_{nl-1})^r}$$

for some finite constant c since $E(\xi^2(L\xi)^r) < \infty$. Hence β_n is bounded above by

$$P\left(\left\{\left(\frac{L(nl)}{c_2}\right)^{\frac{1}{2}} \frac{c}{(LZ_{nl-1})^r} > \alpha/2\right\} \cap \{Z_{nl-1} \ge 3\} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}|\mathcal{F}_{nl-1}\right),$$

which implies

$$\beta_n \le P(\{3 \le Z_{nl-1} \le \exp\{u(L(nl))^{\frac{1}{2r}}\}\} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}|\mathcal{F}_{nl-1}),$$

where u is a finite positive constant depending only on α, c, r, c_2. Letting

$$H_n = \{3 \le Z_{nl-1} \le \exp\{u(L(nl))^{\frac{1}{2r}}\}$$

and $B_k = \{Z_{(n-1)l} = k\}$ for $k = 0, 1, 2, \ldots$, then since these sets are $\mathcal{F}_{nl-1}$ measurable we have

$$\beta_n \le I(H_n)I(\{Z_{(n-1)l} > \beta^{(n-1)l}\}).$$

Using the Markov property we have

$$E(\beta_n|\mathcal{H}_{n-1}) \le E(I(H_n)|Z_{(n-1)l})I(\{Z_{(n-1)l} > \beta^{(n-1)l}\}).$$

Now

$$\begin{aligned} E(I(H_n)|Z_{(n-1)l})&I(\{Z_{(n-1)l} > \beta^{(n-1)l}\}) \\ &= \sum_{k\ge 0} \frac{\int_{B_k} I(H_n)dP}{P(B_k)} I(B_k)I(\{Z_{(n-1)l} > \beta^{(n-1)l}\}) \\ &= \sum_{k\ge[\beta^{(n-1)l}]+1} I(B_k)\frac{P(\{Z_{(n-1)l} = k\}\cap H_n)}{P(B_k)} \\ &= \sum_{k\ge[\beta^{(n-1)l}]+1} I(B_k)P(H_n|Z_{(n-1)l} = k), \end{aligned}$$

and hence we have

$$E(\beta_n|\mathcal{H}_{n-1}) \le \sum_{k\ge[\beta^{(n-1)l}]+1} P\left(3 < \sum_{j=1}^{k} Z_{l-1,j} < \exp\{u(L(nl))^{\frac{1}{2r}}\}\right),$$

where $\mathcal{L}(Z_{l-1,j}) = \mathcal{L}(Z_{l-1})$ are independent for $j \ge 1$. Hence

$$E(\beta_n|\mathcal{H}_{n-1}) \le \sum_{k\ge[\beta^{(n-1)l}]+1} P\left(\exp\left\{-\sum_{j=1}^{k} Z_{l-1,j}\right\} \ge \exp\{-\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\right),$$

and Markov's inequality therefore implies

$$\begin{aligned} E(\beta_n|\mathcal{H}_{n-1}) &\le \sum_{k\ge[\beta^{(n-1)l}]+1} \exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}(E(\exp\{-Z_{l-1}\}))^k) \\ &= \sum_{k\ge[\beta^{(n-1)l}]+1} \exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\gamma^k \\ &= \exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\frac{\gamma^{[\beta^{(n-1)l}]+1}}{1-\gamma}, \end{aligned}$$

where $\gamma = E(\exp\{-Z_{l-1}\}) < 1$ since $p_0 < 1$. Since $r > 4$ we have $1/2r < 1$, and thus for large n we have

$$\exp\{\exp\{u(L(nl))^{\frac{1}{2r}}\}\}\frac{\gamma^{[\beta^{(n-1)l}]+1}}{1-\gamma} \le \gamma^{\frac{1}{2}\beta^{(n-1)l}}.$$

Thus for such n we have

$$E(\beta_n|\mathcal{H}_{n-1}) \le \gamma^{\frac{1}{2}\beta^{(n-1)l}},$$

and hence $\sum_{n\ge 1} E(\beta_n|\mathcal{H}_{n-1}) < \infty$ almost surely, which implies

$$\sum_{n\ge 1}\Gamma_n = \sum_{n\ge 1} E(\alpha_n + \beta_n|\mathcal{H}_{n-1})$$

converges with probability one on Ω.

We now turn to estimating I_n. To simplify writing, let

$$G_n = \cap_{j=1}^{l-1} A_{nl,j,\alpha} \cap \{Z_{(n-1)l} > \beta^{(n-1)l}\}. \tag{3.24}$$

Hence on S_0 with $Z_{nl-1} \geq r_0(f_1 \cdots f_l; \alpha)$, we have $\frac{\sigma}{\sigma_{nl}} N(f_1, \mathcal{P}_1, 2\alpha) \subset N(f_1, \mathcal{P}_1, \frac{5}{2}\alpha)$. Therefore, recalling I_n from (3.19) and that $B_{n,\alpha}$ is $\mathcal{F}_{nl-1}$ measurable, we have

$$I_n \leq P\bigg(\bigg\{\Lambda\bigg(\frac{\sigma}{\sigma_{nl}} T_{nl}\bigg) \in (c_2/L(nl))^{1/2} N\bigg(f_1, \mathcal{P}_1, \frac{5}{2}\alpha\bigg)\bigg\} \cap B_{n,\alpha} | \mathcal{F}_{nl-1}\bigg) \tag{3.25}$$

$$\leq P\bigg(M_B \in \bigg[(c_2/L(nl))^{1/2} N\bigg(f_1, \mathcal{P}_1, \frac{5}{2}\alpha\bigg)\bigg]^{2\rho_n}\bigg) + 2\rho_n I(B_{n,\alpha}), \tag{3.26}$$

where the last inequality follows from Lemma 3.4, M_B is as defined in (3.2), and ρ_n denotes the Prokhorov distance $\rho(\mathcal{L}(\frac{\sigma}{\sigma_{nl}} T_{nl} | Z_{nl-1}), \mathcal{L}(B))$. Furthermore, taking $s = 3$ in Einmahl's result appearing in Lemma 2.5, Lemma 2.3 implies we have

$$\rho_n = \rho\bigg(\mathcal{L}\bigg(\frac{\sigma}{\sigma_{nl}} T_{nl} | Z_{nl-1}\bigg), \mathcal{L}(B)\bigg) \leq c_E \bigg[\frac{c(r,\xi)}{(LZ_{nl-1})^r}\bigg]^{1/4} I(Z_{nl-1} > 0)$$

on $B_{n\alpha}$. Of course, $\mathcal{L}(B)$ is the law of the standard Brownian motion on $C_0[0,1]$. Now eventually on S_0 we have $Z_n > \beta^n$ for $1 < \beta < m$ and hence almost surely on S_0 eventually we have that the Prokhorov distance ρ_n is less than $1/n$. Thus $\lim_{n\to\infty} \rho_n (L(nl))^{1/2} = 0$ there, and almost surely on S_0 we have eventually in n that

$$\begin{aligned} P\bigg(M_B \in \bigg[\bigg(\frac{c_2}{L(nl)}\bigg)^{1/2} N\bigg(f_1, \mathcal{P}_1, \frac{5}{2}\alpha\bigg)\bigg]^{2\rho_n}\bigg) \\ \leq P\bigg(M_B \in \bigg(\frac{c_2}{L(nl)}\bigg)^{1/2} N(f_1, \mathcal{P}_1, 3\alpha)\bigg), \end{aligned}$$

where the probability inequality follows from simple set inclusion. Hence on S_0 we have eventually in n that

$$\theta_{n,1} \leq \psi_{n,1} + \psi_{n,2} + II_n,$$

where

$$\psi_{n,1} = E(I(G_n)|\mathcal{H}_{n-1}) P\bigg(M_B \in \bigg(\frac{c_2}{L(nl)}\bigg)^{1/2} N(f_1, \mathcal{P}_1, 3\alpha)\bigg) \tag{3.27}$$

and

$$\psi_{n,2} = c_E c(r,\xi)^{1/4} E(I(G_n \cap B_{n,\alpha})(LZ_{nl-1})^{-r/4} | \mathcal{H}_{n-1}).$$

Now,

$$\sum_{n\geq 1} E(\psi_{n,2}) \leq c_E c(r,\xi)^{1/4} \sum_{n \geq 1} E((LZ_{nl-1})^{-r/4} I(Z_{nl-1>0})) < \infty$$

by the log harmonic moment results of Lemma 2.4 and that $r > 4$. Hence the series $\sum_{n\geq 1} \psi_{n,2}$ converges with probability one, and on S_0 we have

$$P(E_{nl}|\mathcal{H}_{n-1}) \leq \psi_{n,1} + \psi_{n,3},$$

where

$$\psi_{n,3} = \theta_{n,2} + \psi_{n,2} + II_n$$

and $\theta_{n,2}$, $\psi_{n,2}$, and II_n are summable with probability one. Furthermore, since the term

$$P(M_B \in (c_2/L(nl))^{1/2} N(f_1, \mathcal{P}_1, 3\alpha))$$

in (3.27) is deterministic, we have $\sum_{n\geq 1} \psi_{n,1} < \infty$ almost surely on S_0 if

$$\sum_{n\geq 1} E(I(G_n)|\mathcal{H}_{n-1}) P(M_B \in (c_2/L(nl))^{1/2} N(f_1, \mathcal{P}_1, 3\alpha)) < \infty$$

almost surely on S_0. Hence we need to study this last series, and recalling that G_n involves one less of the sets $A_{nl,j,\alpha}$, we iterate the above argument $l-1$ more times, starting at (3.14-3.16) with subsequent analogues of $B_{n,\alpha}$, to obtain on S_0 for all sufficiently large n that

$$\begin{aligned} &P(E_{nl}|\mathcal{H}_{n-1}) \\ &\quad \leq \psi_{n,4} + \prod_{j=1}^{l} P\left(M_B \in \left(\frac{c_2}{L(nl)}\right)^{1/2} N(f_j, \mathcal{P}_j, 3\alpha)\right) I(Z_{(n-1)l} > \beta^{(n-1)l}) \end{aligned}$$

where $\sum_{n\geq 1} \psi_{n,4} < \infty$. Now we apply (2.12) in Lemma 2.6. Thus our choice of α in forming the open set V as in (3.7) implies for $\gamma > 0$ and for all sufficiently large n that

$$(3.28)\quad P(E_{nl}|\mathcal{H}_{n-1}) \leq \psi_{n,4} + \exp\{-\log_e(nl)(1-\gamma)(1+\delta)\} I(Z_{(n-1)l} > \beta^{(n-1)l}).$$

Now taking γ sufficiently small so that $(1-\gamma)(1+\delta) > 1$, we have a.s. on S_0 that

$$(3.29)\qquad \sum_{n\geq 1} P(E_{nl}|\mathcal{H}_{n-1}) < \infty.$$

The proof of the lemma now follows as indicated from (3.12), since the other $l-1$ cases are completely similar. □

Hence from this last lemma we have almost surely on the survival set S that

$$C(\{(Ln/c_2)^{1/2} \mathbf{M}_{n,r(n)}\}) \subseteq \mathcal{K}_\infty,$$

when we use the product topology on $(C_0[0,1])^\infty$. Our next lemma establishes that in this setting the cluster set $C(\{(Ln/c_2)^{1/2} \mathbf{M}_{n,r(n)}\})$ is actually $\mathcal{K}_\infty$ almost surely on S.

Lemma 3.6. *Let S denote the survival set of the process and set $c_2 = \pi^2/8$. Then, under the conditions of Theorem 1.2 we have*

$$(3.30)\qquad P(\{C(\{(Ln/c_2)^{1/2} \mathbf{M}_{n,r(n)}\}) = \mathcal{K}_\infty\} \cap S) = P(S).$$

Proof. Since the cluster set of a sequence of points in $((C_0[0,1])^\infty, \rho_\infty)$ is closed, and the topological space $((C_0[0,1])^\infty, \rho_\infty)$ is separable, it is sufficient to show that for an arbitrary point $\mathbf{f} \in \mathcal{K}_\infty$ with $\sum_{j\geq 1} \int_0^1 f_j^{-2}(s)\, ds \leq 1$, we have a.s. on S_0 that

$$(3.31)\qquad (Ln/c_2)^{1/2} \mathbf{M}_{n,r(n)} \in V \quad i.o.$$

where V is an arbitrarily small open set containing $\mathbf{f}$. Furthermore, if

$$\mathcal{K}_0 = \left\{ \mathbf{f} = (f_1, f_2, \ldots) \in \mathcal{K}_\infty : \ f_j(1) < \infty \text{ for all } j \geq 1, \sum_{j \geq 1} \int_0^1 f_j^{-2}(s)ds < 1 \right\},$$

then by Lemma 3.3 we see $\mathcal{K}_0$ is dense in $\mathcal{K}_\infty$. Hence it suffices to show that almost surely on S_0 (3.31) holds for each $\mathbf{f} \in \mathcal{K}_0$, when $V = \prod_{j=1}^\infty V_j$ is an open set containing $\mathbf{f}$ of the form in Lemma 3.5. That is, since $f_j(1) < \infty$ for all $j \geq 1$ when $\mathbf{f} = (f, f_2, \ldots) \in \mathcal{K}_0$, then by Lemma 3.2 it suffices to take l an arbitrary positive integer, partitions $\mathcal{P}_j$ of continuity points of f_j for $1 \leq j \leq l$, and a single $\alpha > 0$ arbitrarily small with

$$V_j = N(f_j, \mathcal{P}_j, \alpha), \quad 1 \leq j \leq l,$$

and $V_j \in \mathcal{M}$ for $j \geq l+1$. Moreover, by replacing $\mathbf{f} = (f_1, f_2, \ldots) \in \mathcal{K}_0$ by $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \ldots)$ where

$$\tilde{f}_j(t) = f_j(t) + \frac{\epsilon t}{2^j}, \quad 0 \leq t \leq 1,$$

and $\epsilon > 0$ is arbitrarily small, there is no loss in generality in assuming that each f_j is strictly increasing on $[0,1]$. Thus we also assume this holds for our $\mathbf{f} \in \mathcal{K}_0$.

By the conditional Borel Cantelli lemma it suffices to show that

$$\sum_{n \geq 1} P(E_{nl} | \mathcal{H}_{n-1}) = \infty \tag{3.32}$$

where E_{nl} and $\mathcal{H}_{n-1}$ are as before in Lemma 3.5 and (3.10), except now $\mathbf{f} = (f_1, f_2, \ldots) \in \mathcal{K}_0$ so we also have $\sum_{j \geq 1} \int_0^1 f_j^{-2}(s)ds < 1$, and l and α are arbitrary but fixed in our argument. Thus to verify (3.32), observe that for all n sufficiently large, on S_0

$$P(E_{nl}|\mathcal{H}_{n-1}) = E(E(I(A_{nl,0,\alpha}|\mathcal{F}_{nl-1})I(G_n)|\mathcal{H}_{n-1}) > \theta_{n,1} - \theta_{n,2},$$

where

$$\theta_{n,1} = E(I(G_n)E(I(A_{nl,0,\alpha} \cap B_{n,\alpha}|\mathcal{F}_{nl-1})|\mathcal{H}_{n-1}),$$
$$\theta_{n,2} \leq P(B'_{n,\alpha}|\mathcal{H}_{n-1}),$$

and

$$B_{n,\alpha} = \{c_E[c(r,\xi)(LZ_{nl-1})^{-r}]^{1/4} < \gamma_0, Z_{nl-1} > r_0(f_1, \ldots, f_l; \alpha) \geq 3\},$$

where c_E and $c(r,\xi)$ are defined as above. Of course, here G_n is as in (3.24) with the sets $A_{nl,j,\alpha}$ defined as before following (3.10), except that now they are defined in terms of the sets $N(f_j, \mathcal{P}_j, \alpha)$. Also, here we take $r_0(f_1, \ldots f_l; \alpha)$ such that $Z_k > r_0(f_1, \ldots f_l; \alpha)$ implies

$$\frac{\sigma}{\sigma_k} N\left(f_j, \mathcal{P}_j, \frac{3}{4}\alpha\right) \supseteq N\left(f_j, \mathcal{P}_j, \frac{\alpha}{2}\right) \tag{3.33}$$

for $j = 1, 2, \ldots, l$. Now

$$\sum_{n=1}^\infty E(\theta_{n,2}) \leq \sum_{n=1}^\infty P(B'_{n,\alpha}) < \infty,$$

by applying Lemma 2.2 as in the proof of Lemma 3.5. Thus $\sum_{n\geq 1}\theta_{n,2}<\infty$ a.s. on Ω, and recalling that $V_1=N(f_1,\mathcal{P}_1,\alpha)$, we see

$$P(A_{nl,0,\alpha}\cap B_{n,\alpha}|\mathcal{F}_{nl-1})=P(\{\eta_{nl,Z_{nl-1}}\in V_1\cap B_{n,\alpha}|\mathcal{F}_{nl-1})\geq I_n-II_n,$$

where

$$I_n=P\bigg(\bigg\{(L(nl)/c_2)^{1/2}\Lambda(T_{nl})\in N\Big(f_1,\mathcal{P}_1,\frac{3}{4}\alpha\Big)\bigg\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}\bigg),$$

and

$$II_n=P\bigg(\bigg\{\|\eta_{nl,Z_{nl-1}}-(L(nl)/c_2)^{1/2}\Lambda(T_{nl})\|>\frac{\alpha}{4}\bigg\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}\bigg).$$

Hence,

$$\begin{aligned}\theta_{n,1}&\geq E(I(G_n)(I_n-II_n)|\mathcal{H}_{n-1})\\&\geq E(I(G_n)I_n|\mathcal{H}_{n-1})-E(II_nI(Z_{(n-1)l}>\beta^{(n-1)l})|\mathcal{H}_{n-1}),\end{aligned}$$

where as in (3.20-24), we have $\sum_{n\geq 1}E(II_nI(Z_{(n-1)l}>\beta^{(n-1)l})|\mathcal{H}_{n-1})<\infty$ with probability one.

Recalling the definition of $B_{n,\alpha}$ in this setting, we see that

$$Z_{nl-1}>r_0(f_1,f_2,\ldots,f_l;\alpha)$$

implies

$$\frac{\sigma}{\sigma_k}N\Big(f_j,\mathcal{P}_j,\frac{3}{4}\alpha\Big)\supseteq N\Big(f_j,\mathcal{P}_j,\frac{\alpha}{2}\Big),$$

and we also have that

$$\begin{aligned}&P\bigg(M_B\in\Big(\frac{c_2}{L(nl)}\Big)^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{4}\Big)\bigg)I(B_{n,\alpha})\\&\quad\leq P\bigg(\bigg\{\Lambda\Big(\frac{\sigma}{\sigma_{nl}}T_{nl}\Big)\in\bigg[\Big(\frac{c_2}{L(nl)}\Big)^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{4}\Big)\bigg]^{2\rho_n}\bigg\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}\bigg)\\&\qquad+2\rho_nI(B_{n,\alpha}),\end{aligned}$$

where ρ_n denotes the Prokorov distance

$$\rho\bigg(\mathcal{L}\Big(\frac{\sigma}{\sigma_{nl}}T_{nl}|Z_{nl-1}>0\Big),\mathcal{L}(B)\bigg).$$

Furthermore, taking $s=3$ in Lemma 2.5, Lemma 2.3 implies we have on $B_{n,\alpha}$ that

$$\rho_n\leq c_E[c(r,\xi)(LZ_{nl-1})^{-r}]^{1/4}<\gamma_0.$$

In addition, $\lim_{n\to\infty}\rho_n(L(nl))^{\frac{1}{2}}=0$ on S_0. Hence if $\gamma_0<\alpha/12$ the above implies for all n sufficiently large that

$$\begin{aligned}&P\bigg(M_B\in\Big(\frac{c_2}{L(nl)}\Big)^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{4}\Big)\bigg)I(B_{n,\alpha})\\&\quad\leq P\bigg(\bigg\{\Lambda\Big(\frac{\sigma}{\sigma_{nl}}T_{nl}\Big)\in\bigg[\Big(\frac{c_2}{L(nl)}\Big)^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{4}\Big)\bigg]^{2\rho_n}\bigg\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}\bigg)\\&\qquad+2\rho_nI(B_{n,\alpha})\end{aligned}$$

$$\le P\Big(\Big\{\Lambda\Big(\frac{\sigma}{\sigma_{nl}}T_{nl}\Big)\in\Big(\frac{c_2}{L(nl)}\Big)^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{2}\Big)\Big\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}\Big)$$
$$+2\rho_n I(B_{n,\alpha})$$
$$\le P\Big(\Big\{\Lambda(T_{nl})\in\Big(\frac{c_2}{L(nl)}\Big)^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{3}{4}\alpha\Big)\Big\}\cap B_{n,\alpha}|\mathcal{F}_{nl-1}\Big)+2\rho_n I(B_{n,\alpha}),$$

where Lemma 3.4 is used in the first inequality and M_B is as defined in (3.2).

Now $\sum_{n\ge1}E(\rho_n I(B_{n,\alpha})|\mathcal{H}_{n-1})<\infty$ a.s. on Ω by the harmonic moment results of Lemma 2.4; i.e. $\rho_n<\gamma_0<\infty$ implies $Z_{nl-1}>0$. In addition, we also have a.s. on Ω that $\sum_{n\ge1}\theta_{n,2}<\infty$. Thus the above shows it suffices to verify

$$\sum_{n\ge1}P(G_n\cap B_{n,\alpha}|\mathcal{H}_{n-1})P\Big(M_B\in(c_2/L(nl))^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{4}\Big)\Big)=\infty \tag{3.34}$$

a.s. on S_0. Now,

$$P(G_n\cap B_{n,\alpha}|\mathcal{H}_{n-1})=P(G_n|\mathcal{H}_{n-1})-P(B'_{n,\alpha}|\mathcal{H}_{n-1})$$

and since $\sum_{n\ge1}P(B'_{n,\alpha}|\mathcal{H}_{n-1})<\infty$ with probability one by what we did earlier, (3.34) will follow if we show that a.s. on S_0

$$\sum_{n\ge1}P(G_n|\mathcal{H}_{n-1})P\Big(M_B\in(c_2/L(nl))^{1/2}N\Big(f_1,\mathcal{P}_1,\frac{\alpha}{4}\Big)\Big)=\infty.$$

Iterating the previous argument $l-1$ more times we see as before that since the quantities

$$P\Big(M_B\in(c_2/L(nl))^{1/2}N\Big(f_j,\mathcal{P}_j,\frac{\alpha}{4}\Big)\Big)$$

are deterministic, it suffices to show that a.s. on S_0

$$\sum_{n\ge1}\prod_{j=1}^{l}P\Big(M_B\in(c_2/L(nl))^{1/2}N\Big(f_j,\mathcal{P}_j,\frac{\alpha}{4}\Big)\Big)I(Z_{(n-1)l}>\beta^{(n-1)l})=\infty.$$

Since

$$P(\{Z_{(n-1)l}>\beta^{(n-1)l}\ \textit{eventually}\}\cap S_0)=P(S_0),$$

it is therefore sufficient to show that

$$\sum_{n\ge1}\prod_{j=1}^{l}P\Big(M_B\in(c_2/L(nl))^{1/2}N\Big(f_j,\mathcal{P}_j,\frac{\alpha}{4}\Big)\Big)=\infty. \tag{3.35}$$

Now we apply (2.13) in Lemma 2.6, which implies for $\gamma>0$ and $j=1,\dots,l$ that

$$P\Big(M_B\in(c_2/L(nl))^{1/2}N\Big(f_j,\mathcal{P}_j,\frac{\alpha}{4}\Big)\Big)$$
$$\ge\exp\Big\{-(1+\gamma)L(nl)\sum_{t_k\in\mathcal{P}_j}(t_k-t_{k-1})/\Big(f_j(t_k)+\frac{\alpha}{4}\Big)^2\Big\}$$

provided $n\ge n(\gamma)$. Since the $f_j's$ are increasing we have

$$\sum_{j=1}^{l}\Big\{\sum_{t_k\in\mathcal{P}_j}(t_k-t_{k-1})/\Big(f_j(t_k)+\frac{\alpha}{4}\Big)^2\Big\}\le\sum_{j=1}^{l}\int_0^1 f_j^{-2}(s)\,ds,$$

and since

$$\sum_{j\geq 1}\int_0^1 f_j^{-2}(s)\,ds < 1,$$

there exists $\gamma > 0$ sufficiently small so that for $n \geq \tilde{n}(\gamma)$ we have

$$\prod_{j=1}^{l} P\left(M_B \in (c_2/L(nl))^{1/2} N\left(f_j, \mathcal{P}_j, \frac{\alpha}{4}\right)\right) \geq \exp\{-Ln\}.$$

Hence (3.35) holds. This proves (3.32) and the lemma. □

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