§0. Introduction

In these notes we construct an inner model with a Woodin cardinal, and develop fine structure theory for this model. Our model is of the form $L[\vec{E}]$, where $\vec{E}$ is a coherent sequence of extenders, and our work builds upon the existing theory of such models. In particular, we rely upon the fine structure theory of $L[\vec{E}]$ models with strong cardinals, which is due to Jensen, Solovay, Dodd-Jensen, and Mitchell, and upon the theory of iteration trees and "backgrounded" $L[\vec{E}]$ models with Woodin cardinals, which is due to Martin and Steel. Our work is what results when fine structure meets iteration trees.

One of our motivations was the desire to remove the severe limitations on the theory developed in [MS] caused by its use of an external comparison process. Because of this defect, the internal theory of the model $L[\vec{E}]$ constructed in [MS] is to a large extent a mystery. For example it is open whether the $L[\vec{E}]$ of [MS] satisfies GCH. Moreover, the use of an external comparison process blocks the natural generalization to models with infinitely many Woodin cardinals of even the result [MS] does prove about $L[\vec{E}]$, that $L[\vec{E}] \models \mathrm{CH} + \mathbb{R}$ has a definable wellordering.

Our strategy for making the comparison process internal is due to Mitchell and actually predates [MS]. The strategy includes taking finely calibrated partial ultrapowers ("dropping to a mouse") at certain stages in the comparison process. Thus to define the internal comparison process and prove it succeeds one needs fine structure theory. Of course, fine structure theory requires a comparison process, but fortunately we are led not into a vicious circle, but into a benign helix: that is, an induction. The whole of what follows can be viewed as a long inductive proof that a certain construction yields a model $L[\vec{E}]$ whose levels have certain fine structural properties. Among those properties is a strong local form of GCH.

We have as a corollary that if ZFC + "There is a Woodin cardinal" is consistent, then so is ZFC + "There is a Woodin cardinal" + GCH. But our interest is not so much in this relative consistency result, which can probably be proved more easily using forcing, as in the inner model $L[\vec{E}]$ itself, and the fine structure techniques which should eventually decide many questions about $L[\vec{E}]$ and similar models containing more Woodin cardinals.

The model $L[\vec{E}]$ and its fine structure theory are likely indispensable for proving certain relative consistency statements in which the theory hypothesized consistent does not directly assert the existence of large cardinals. For example the following conjecture is widely believed to be true:

Conjecture. *If ZFC + "There is an ω_2-saturated ideal on ω_1" is consistent, so is ZFC + "there is a Woodin cardinal".*

Of course, the conjecture is really that the relative consistency is provable in Peano Arithmetic. Shelah has proven the converse relative consistency result. Mitchell ([M?]) has proved the conjecture with its conclusion weakened to "ZFC $+ \exists\kappa(o(\kappa) = \kappa^{++})$ is consistent". The present paper is a step toward extending Mitchell's arguments so as to prove the full conjecture. What we lack at the moment is a method which does not use large cardinals in V for showing that a certain $L[\vec{E}]$ type model is sufficiently iterable. This "Core model iterability" problem is one of the key open problems in the area. Its solution should lead to a proof of the conjecture, and to much more.

The notes are organized as follows. In §1 we introduce potential premice, which are structures having some of the first order properties of the levels of the model we eventually construct. Perhaps the most notable thing here is that the extender sequence $\vec{E}^{\mathcal{M}}$ of a potential premouse (ppm) $\mathcal{M}$ may contain extenders which do not measure all sets in $\mathcal{M}$. In general, an E on $\vec{E}^{\mathcal{M}}$ measures only subsets of $\operatorname{crit} E$ constructed in $\mathcal{M}$ before the stage at which E was added to $\vec{E}^{\mathcal{M}}$. This tactic, which is due to S. Baldwin and Mitchell, greatly simplifies fine structure theory.

Section §2 studies definability over potential premice. We introduce the $r\Sigma_n$ hierarchy, a slight variant on the usual Levy hierarchy. We follow Magidor and Silver in introducing Skolem terms so as to avoid proving $r\Sigma_n$ uniformization, and in working directly with $r\Sigma_n$ formulae rather than master codes and iterated $r\Sigma_1$ definability. We show that being a ppm is preserved under the appropriate embeddings. Finally, we introduce projecta, standard parameters, solidity and universality of parameters, cores, and soundness. These are standard fine structural notions, with the exception of solidity.

The analysis of §2 is not appropriate for a certain sort of ppm, the "active type III" variety. In §3 we modify it slightly so that it suits these ppm. This leads to an annoying case split in the details of many arguments, a split which we have sometimes ignored.

One important feature of the Baldwin-Mitchell tactic is that all levels of the model we build will be completely sound. Ultrapowers of sound structures can be unsound, but all proper initial segments of the ultrapower will be sound. So it suffices to consider only ppm all of whose proper initial segments are sound. These we call premice.

In §4 we define the $r\Sigma_n$ ultrapower $\operatorname{Ult}_n(\mathcal{M}, E)$ of a ppm $\mathcal{M}$ by an extender E measuring all sets in $\mathcal{M}$ and satisfying $\operatorname{crit} E < \rho_n^{\mathcal{M}}$. We prove Los' theorem and show that the canonical embedding is $r\Sigma_{n+1}$ elementary if $\mathcal{M}$ is n-sound. We show that if $\rho_{n+1}^{\mathcal{M}} \leq \operatorname{crit} E$, $\mathcal{M}$ is n-sound, and E is "close to being a member of $\mathcal{M}$", then the canonical embedding preserves the $n+1^{\text{st}}$ standard parameter, *provided this parameter is solid.* This result explains the importance of solidity.

Section §5 introduces iteration trees and n-iterability. It also proves the Dodd-

Jensen lemma on the minimality of iteration maps, which is a key tool in our work.

In Section §6 we investigate the uniqueness of wellfounded branches in iteration trees. Theorem 6.1 is a straightforward generalization of the uniqueness theorem of [MS]. Theorem 6.2 is a fine structural strengthening of theorem 6.1 which takes considerably more work to prove. Theorem 6.2 has the important consequence that all the iteration trees we care about have at most one cofinal wellfounded branch.

Section §7 proves a comparison lemma for iterable premice. The lemma is never used in what follows, but the method of proof, the comparison process, is used throughout.

In §8 we prove our main fine-structural result: the $n+1^{st}$ standard parameter of an n-sound, n-iterable premouse is $n+1$-solid and $n+1$-universal. The method of proof traces back to Dodd's proof that GCH holds in the model of [D]. The method also gives a useful condensation result, Theorem 8.2.

In §11 we finally construct (assuming there is a Woodin cardinal in V) some iterable premice. We in fact construct a model $L[\vec{E}]$ with a Woodin cardinal all of whose levels are ω-sound and ω-iterable premice. §9 and §10 are devoted to some preliminary lemmas which guarantee that the construction of §11 puts enough extenders on $\vec{E}$ that we do indeed get a model with a Woodin cardinal. Section §12 shows that the construction of §11 produces an iterable structure $L[\vec{E}]$ by associating to any iteration tree on $L[\vec{E}]$ an iteration tree on V and then using the results of [MS].

We did the work described here during 1987–1989 and wrote it up in a set of notes which has been informally circulated since October 1989. This paper is essentially identical to that set of notes. We wish to thank Kai Hauser, Mitch Rudominer and Ernest Schimmerling for reading those notes carefully and bringing errors to our attention.

Since 1989 the theory described here has advanced in several ways. In the spring of 1990, Steel found a solution to the core model iterability problem mentioned above, and with it was able to extend the work of [M?] to the level of a Woodin Cardinal [S?a]. He used this to show that if there is a saturated ideal on ω_1, together with a measurable cardinal, then there is an inner model with a Woodin cardinal. The measurable cardinal should not be necessary here and its use may indicate a weakness in the basic theory of [S?a]. As expected, other relative consistency results have come out of this work. Some of these use the weak covering lemma for the model of [S?a], which was proved in late 1990 by Mitchell [MSS?].

Schimmerling [Sch] has investigated the combinatorial set theory of the model $L[\vec{E}]$ described in this paper. He showed that $\Box_{\omega_1}$ holds in this model, and that weak $\Box_\kappa$ holds for all κ. It is open whether $L[\vec{E}]$ satisfies $\forall\kappa\,\Box_\kappa$. Schimmerling was able to combine his work on $\square_\kappa$ with the ideas of [S?a], [MSS?] and arguments of Todorcevik and Magidor in order to show that the proper forcing axiom implies that there is an inner model with a Woodin cardinal.

Finally, Steel ([S?b], [S?c]) has extended the theory presented here to models having more Woodin cardinals.