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On the Cohomology of Discriminantal Arrangements and Orlik-Solomon Algebras

Daniel C. Cohen

For Peter Orlik on the occasion of his sixtieth birthday.

Abstract.

We relate the cohomology of the Orlik-Solomon algebra of a discriminantal arrangement to the local system cohomology of the complement. The Orlik-Solomon algebra of such an arrangement (viewed as a complex) is shown to be a linear approximation of a complex arising from the fundamental group of the complement, the cohomology of which is isomorphic to that of the complement with coefficients in an arbitrary complex rank one local system. We also establish the relationship between the cohomology support loci of the complement of a discriminantal arrangement and the resonant varieties of its Orlik-Solomon algebra.

Introduction

Let $\mathcal{A}$ be an arrangement of N complex hyperplanes, and let $M(\mathcal{A})$ be its complement. For each hyperplane H of $\mathcal{A}$, let f_H be a linear polynomial with kernel H, and let λ_H be a complex number. Each point $\boldsymbol{\lambda} = (\dots, \lambda_H, \dots) \in \mathbb{C}^N$ determines an integrable connection $\nabla = d + \Omega_{\boldsymbol{\lambda}}$ on the trivial line bundle over $M(\mathcal{A})$, where $\Omega_{\boldsymbol{\lambda}} = \sum_{H \in \mathcal{A}} \lambda_H d \log f_H$, and an associated complex rank one local system $\mathcal{L}$ on $M(\mathcal{A})$. Alternatively, if $\mathbf{t} \in (\mathbb{C}^*)^N$ is the point in the complex torus corresponding to $\boldsymbol{\lambda}$, then the local system $\mathcal{L}$ is induced by the representation of the fundamental group of $M(\mathcal{A})$ which sends any meridian about $H \in \mathcal{A}$ to $t_H = \exp(-2\pi \mathrm{i} \lambda_H)$.

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Due largely to its various applications, the cohomology of $M(\mathcal{A})$ with coefficients in $\mathcal{L}$ has been the subject of considerable recent interest. These applications include representations of braid groups, generalized hypergeometric functions, and the Knizhnik-Zamolodchikov equations from conformal field theory. See, for instance, the works of Aomoto, Kita, Kohno, Schechtman, and Varchenko [1, 2, 21, 28, 30], and see Orlik and Terao [25] as a general reference for arrangements. Of particular interest in these applications are the discriminantal arrangements of [28], the complements of which may be realized as configuration spaces of ordered points in $\mathbb{C}$ punctured finitely many times. (Note that our use of the term "discriminantal" differs from that of [25].)

The local system cohomology $H^*(M(\mathcal{A});\mathcal{L})$ may be studied from a number of points of view. For instance, if $\mathcal{A}$ is real, that is, defined by real equations, the complement $M(\mathcal{A})$ is homotopy equivalent to the Salvetti complex X of $\mathcal{A}$, see [26]. In this instance, the complex X may be used in the study of local systems on $M(\mathcal{A})$. This approach is developed by Varchenko in [30], to which we also refer for discussion of the applications mentioned above, and has been pursued by Denham and Hanlon [13] in their study of the homology of the Milnor fiber of an arrangement.

If $\mathcal{A}$ is $K(\pi,1)$, that is, the complement $M(\mathcal{A})$ is a $K(\pi,1)$-space, then local systems on $M(\mathcal{A})$ may be studied from the point of view of cohomology of groups. Any representation of the fundamental group G of the complement of a $K(\pi,1)$ arrangement gives rise to a G-module L, and a local system of coefficients $\mathcal{L}$ on $M(\mathcal{A})$. Since $M(\mathcal{A})$ is a $K(\pi,1)$-space, we have $H_*(M(\mathcal{A});\mathcal{L}) = H_*(G;L)$ and $H^*(M(\mathcal{A});\mathcal{L}) = H^*(G;L)$, see for instance Brown [8]. The class of $K(\pi,1)$ arrangements includes the discriminantal arrangements noted above, as they are examples of fiber-type arrangements, well-known to be $K(\pi,1)$, see e.g. Falk and Randell [17].

For any arrangement $\mathcal{A}$, let $B(\mathcal{A})$ denote the Brieskorn algebra of $\mathcal{A}$, generated by 1 and the closed differential forms $d\log f_H$, $H \in \mathcal{A}$. As is well-known, the algebra $B(\mathcal{A})$ is isomorphic to $H^*(M(\mathcal{A});\mathbb{C})$, and to the Orlik-Solomon algebra $A(\mathcal{A})$, so is determined by the lattice of $\mathcal{A}$, see [7, 24, 25]. If $\mathcal{L}$ is a local system on $M(\mathcal{A})$ determined by "weights" $\boldsymbol{\lambda}$ which satisfy certain Aomoto non-resonance conditions, work of Esnault, Schechtman, and Viehweg [14], extended by Schechtman, Terao, and Varchenko [27], shows that $H^*(M(\mathcal{A});\mathcal{L})$ is isomorphic to the cohomology of the complex $(B(\mathcal{A}),\Omega_{\boldsymbol{\lambda}}\wedge)$. Thus for non-resonant weights, the local system cohomology may be computed by combinatorial means, using the Orlik-Solomon algebra equipped with differential $\mu(\boldsymbol{\lambda})$, given

by left-multiplication by $\omega_{\boldsymbol{\lambda}}$, the image of $\Omega_{\boldsymbol{\lambda}}$ under the isomorphism $B(\mathcal{A}) \to A(\mathcal{A})$.

For arbitrary (resonant) weights, one has

$$\dim H^k(A(\mathcal{A}), \mu(\boldsymbol{\lambda})) \leq \dim H^k(M(\mathcal{A}); \mathcal{L}) \leq \dim H^k(M(\mathcal{A}); \mathbb{C})$$

for each k. See Libgober and Yuzvinsky [23] for the first of these inequalities. The second is obtained using stratified Morse theory in [9], and resolves a question raised by Aomoto and Kita in [2]. For resonant weights, the precise relation between $H^*(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ and $H^*(M(\mathcal{A}); \mathcal{L})$ is not known.

However, recent results suggest that $H^k(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ may be viewed as a "linear approximation" of $H^k(M(\mathcal{A}); \mathcal{L})$, at least for small k. The resonant varieties, $\mathcal{R}^m_k(A(\mathcal{A})) = \{\lambda \in \mathbb{C}^N \mid \dim H^k(A(\mathcal{A}), \mu(\boldsymbol{\lambda})) \geq m\}$, of the Orlik-Solomon algebra were introduced by Falk in [16]. For $k = 1$ and any arrangement $\mathcal{A}$, it is known that $\mathcal{R}^m_1(A(\mathcal{A}))$ coincides with the tangent cone of the cohomology support locus of the complement, $\Sigma^1_m(M(\mathcal{A})) = \{\mathbf{t} \in (\mathbb{C}^*)^N \mid \dim H^1(M(\mathcal{A}); \mathcal{L}) \geq m\}$, at the point $(1, \ldots, 1)$, see [11, 22, 23]. For certain arrangements, we present further "evidence" in support of this philosophy here.

If $\mathcal{A}$ is a fiber-type arrangement, the fundamental group G of the complement $M(\mathcal{A})$ may be realized as an iterated semidirect product of free groups, and $M(\mathcal{A})$ is a $K(G, 1)$-space, see [17, 25]. For any such group, we construct a finite, free $\mathbb{Z}G$-resolution, $C_\bullet(G)$, of $\mathbb{Z}$ in [10]. This resolution may be used to compute the homology and cohomology of G with coefficients in any G-module L, or equivalently, that of $M(\mathcal{A})$ with coefficients in any local system $\mathcal{L}$. We have $H_*(M(\mathcal{A}); \mathcal{L}) = H_*(C_\bullet(G) \otimes_G L)$ and $H^*(M(\mathcal{A}); \mathcal{L}) = H^*(\mathcal{H}om_G(C_\bullet(G), L))$, see [8].

Briefly, for a fiber-type arrangement $\mathcal{A}$, the relationship between the cohomology theories $H^*(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ and $H^*(M(\mathcal{A}); \mathcal{L})$ is given by the following assertion. *For any $\boldsymbol{\lambda}$, the complex $(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ is a linear approximation of the complex $\mathcal{H}om_G(C_\bullet(G), L)$.* We prove a variant of this statement in the case where $\mathcal{A}$ is a discriminantal arrangement here. We also establish the relationship between the resonant varieties $\mathcal{R}^m_k(A(\mathcal{A}))$ and cohomology support loci $\Sigma^k_m(M(\mathcal{A}))$ of these arrangements, analogous to that mentioned above in the case $k = 1$.

The paper is organized as follows. The Orlik-Solomon algebra of a discriminantal arrangement admits a simple description, which fascilitates analysis of the differential of the complex $(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$. We carry out this analysis, which is elementary albeit delicate, in section 1, and obtain an explicit (inductive) description of the differential $\mu(\boldsymbol{\lambda})$. In section 2, we recall the construction of the resolution $C_\bullet(G)$ from [10] in the instance where G is the fundamental group of the complement of

a discriminantal arrangement, and exhibit a complex $(\mathbf{C}^\bullet, \delta^\bullet(\mathbf{t}))$ which computes the cohomology $H^*(M(\mathcal{A}), \mathcal{L})$ for an arbitrary rank one local system. We then study in section 3 a linear approximation of $(\mathbf{C}^\bullet, \delta^\bullet(\mathbf{t}))$, and relate it, for arbitrary $\boldsymbol{\lambda}$, to the complex $(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$. We conclude by realizing the resonant varieties of the Orlik-Solomon algebra of a discriminantal arrangement as the tangent cones at the identity of the cohomology support loci of the complement in section 4.

§1. Cohomology of the Orlik-Solomon Algebra

Let $M_n = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}$ be the configuration space of n ordered points in $\mathbb{C}$. Note that M_n may be realized as the complement of the braid arrangement $\mathcal{A}_n = \{x_i = x_j, 1 \leq i < j \leq n\}$ in $\mathbb{C}^n$. Classical work of Fadell and Neuwirth [15] shows the projection $\mathbb{C}^n \to \mathbb{C}^\ell$ defined by forgetting the last $n - \ell$ coordinates gives rise to a bundle map $p : M_n \to M_\ell$. From this it follows that M_n is a $K(P_n, 1)$-space, where $P_n = \pi_1(M_n)$ is the pure braid group on n strands.

The typical fiber of the bundle of configuration spaces $p : M_n \to M_\ell$ may be realized as the complement of an arrangement in $\mathbb{C}^{n-\ell}$, a discriminantal arrangement in the sense of Schechtman and Varchenko, see [28, 30]. The fiber over $\mathbf{z} = (z_1, \dots, z_\ell) \in M_\ell$ may be realized as the complement, $M_{n,\ell} = M(\mathcal{A}_{n,\ell})$, of the arrangement $\mathcal{A}_{n,\ell}$ consisting of the $N = \binom{n}{2} - \binom{\ell}{2}$ hyperplanes

$$H_{i,j} = \begin{cases} \ker(x_j - x_i) & \ell + 1 \leq i < j \leq n, \\ \ker(x_j - z_i) & 1 \leq i \leq \ell,\ \ell + 1 \leq j \leq n, \end{cases}$$

in $\mathbb{C}^{n-\ell}$ (with coordinates $x_{\ell+1}, \dots, x_n$). Note that $M_{n,\ell}$ is the configuration space of $n - \ell$ ordered points in $\mathbb{C} \setminus \{z_1, \dots, z_\ell\}$, and that the topology of $M_{n,\ell}$ is independent of $\mathbf{z}$, see [15, 5, 20]. We first record some known results on the cohomology of $M_{n,\ell}$.

1.1. The Orlik-Solomon Algebra

The fundamental group of the configuration space $M_{n,\ell}$ may be realized as $P_{n,\ell} = \pi_1(M_{n,\ell}) = \ker(P_n \to P_\ell)$, the kernel of the homorphism from P_n to P_ℓ defined by forgetting the last $n - \ell$ strands. From the homotopy exact sequence of the bundle $p : M_n \to M_\ell$, we see that $M_{n,\ell}$ is a $K(P_{n,\ell}, 1)$-space. The cohomology of this space, and hence of this group, may be described as follows.

Let $\mathcal{E} = \bigoplus_{q=0}^{N} \mathcal{E}^q$ be the graded exterior algebra over $\mathbb{C}$, generated by $e_{i,j}$, $\ell + 1 \leq j \leq n$, $1 \leq i < j$. Let $\mathcal{I}$ be the ideal in $\mathcal{E}$ generated, for

$1 \leq i < j < k \leq n$, by

$$e_{i,j} \wedge e_{i,k} - e_{i,j} \wedge e_{j,k} + e_{i,k} \wedge e_{j,k} \text{ if } j \geq \ell + 1, \quad \text{and} \quad e_{i,k} \wedge e_{j,k} \text{ if } j \leq \ell.$$

Note that $\mathcal{E}^q \subset \mathcal{I}$ for $q > n - \ell$. The Orlik-Solomon algebra of the discriminantal arrangement $\mathcal{A}_{n,\ell}$ is the quotient $A = \mathcal{E}/\mathcal{I}$.

Theorem 1.2. *The cohomology algebra $H^*(M_{n,\ell}; \mathbb{C}) = H^*(P_{n,\ell}; \mathbb{C})$ is isomorphic to the Orlik-Solomon algebra $A = A(\mathcal{A}_{n,\ell})$.*

The grading on $\mathcal{E}$ induces a grading $A = \bigoplus_{q=0}^{n-\ell} A^q$ on the Orlik-Solomon algebra $A = A(\mathcal{A}_{n,\ell})$. Let $a_{i,j}$ denote the image of $e_{i,j}$ in A, and note that these elements form a basis for A^1 and generate A. From the description of the ideal $\mathcal{I}$ above, it is clear that all relations among these generators are consequences of the following:

$$a_{i,k} \wedge a_{j,k} = \begin{cases} a_{i,j} \wedge (a_{j,k} - a_{i,k}) & \text{if } j \geq \ell + 1, \\ 0 & \text{if } j \leq \ell, \end{cases} \tag{1.1}$$

for $1 \leq i < j < k \leq n$.

This observation leads to a natural choice of basis for the algebra A. For $m \leq n$, write $[m, n] = \{m, m+1, \ldots, n\}$. If $I = \{i_1, \ldots, i_q\}$ and $J = \{j_1, \ldots, j_q\}$ satisfy the conditions $J \subseteq [\ell + 1, n]$ and $1 \leq i_p < j_p$ for each p, let $a_{I,J} = a_{i_1,j_1} \wedge \cdots \wedge a_{i_q,j_q}$. If $|J| = 0$, set $a_{I,J} = 1$.

Proposition 1.3. *For each q, $0 \leq q \leq n - \ell$, the forms $a_{I,J}$ with $|J| = q$ and I as above form a basis for the summand A^q of the Orlik-Solomon algebra A of the discriminantal arrangement $\mathcal{A}_{n,\ell}$. Furthermore, the summand A^q decomposes as a direct sum, $A^q = \bigoplus_{|J|=q} A_J$, where $A_J = \bigoplus_I \mathbb{C}\, a_{I,J}$.*

Remark 1.4. These results are well-known. For instance, if $\mathcal{A} = \mathcal{A}_n$ is the braid arrangement, Theorem 1.2 follows from results of Arnol'd [4] and Cohen [12], which show that $H^*(M_n; \mathbb{C})$ is generated by the forms $a_{i,j} = d\log(x_j - x_i)$, with relations (1.1) (with $\ell = 1$). For any discriminantal arrangement $\mathcal{A}_{n,\ell}$, Theorem 1.2 is a consequence of results of Brieskorn and Orlik-Solomon, see [7, 24, 25].

As mentioned in the Introduction, the discriminantal arrangements $\mathcal{A}_{n,\ell}$ are examples of (affine) fiber-type or supersolvable arrangements. The structure of the Orlik-Solomon algebra of any such arrangement $\mathcal{A}$ was determined by Terao [29]. The basis for the algebra $A(\mathcal{A}_{n,\ell})$ exhibited in Proposition 1.3 above is the **nbc**-basis (with respect to a natural ordering of the hyperplanes of $\mathcal{A}_{n,\ell}$), see [25]. The Orlik-Solomon algebra of any supersolvable arrangement admits an analogous basis,

see Björner-Ziegler [6], and see Falk-Terao [18] for affine supersolvable arrangements.

1.5. The Orlik-Solomon Algebra as a Complex

Recall that $N = \binom{n}{2} - \binom{\ell}{2}$, and consider $\mathbb{C}^N$ with coordinates $\lambda_{i,j}$, $\ell+1 \le j \le n$, $1 \le i < j$. Each point $\boldsymbol{\lambda} \in \mathbb{C}^N$ gives rise to an element $\omega = \omega_{\boldsymbol{\lambda}} = \sum \lambda_{i,j} \cdot a_{i,j}$ of A^1. Left-multiplication by ω induces a map $\mu^q(\boldsymbol{\lambda}) : A^q \to A^{q+1}$, defined by $\mu^q(\boldsymbol{\lambda})(\eta) = \omega \wedge \eta$. Clearly, $\mu^{q+1}(\boldsymbol{\lambda}) \circ \mu^q(\boldsymbol{\lambda}) = 0$, so $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a complex.

We shall obtain an inductive formula for the boundary maps of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$. The projection $\mathbb{C}^{n-\ell} \to \mathbb{C}$ onto the first coordinate gives rise to a bundle of configuration spaces, $M_{n,\ell} \to M_{\ell+1,\ell}$, with fiber $M_{n,\ell+1}$, see [15, 5, 20]. The inclusion of the fiber $M_{n,\ell+1} \hookrightarrow M_{n,\ell}$ induces a map on cohomology which is clearly surjective. This yields a surjection $\pi : A(\mathcal{A}_{n,\ell}) \to A(\mathcal{A}_{n,\ell+1})$.

Write $A = A(\mathcal{A}_{n,\ell})$ and $\widehat{A} = A(\mathcal{A}_{n,\ell+1})$, and denote the generators of both A and $\widehat{A}$ by $a_{i,j}$. In terms of these generators, the map π is given by $\pi(a_{i,\ell+1}) = 0$, and $\pi(a_{i,j}) = a_{i,j}$ otherwise. Let $\widehat{\omega} \in \widehat{A}$ denote the image of $\omega \in A^1$ under π. If we write $\omega = \sum_{k=\ell+1}^{n} \omega_k$, where $\omega_k = \sum_{i=1}^{k-1} \lambda_{i,k} \cdot a_{i,k}$, then $\widehat{\omega} = \sum_{k=\ell+2}^{n} \omega_k$. As above, left-multiplication by $\widehat{\omega}$ induces a map $\widehat{\mu}^q(\boldsymbol{\lambda}) : \widehat{A}^q \to \widehat{A}^{q+1}$, and $(\widehat{A}^\bullet, \widehat{\mu}^\bullet(\boldsymbol{\lambda}))$ is a complex. The following is straightforward.

Lemma 1.6. *The map $\pi : (A^\bullet, \mu^\bullet(\boldsymbol{\lambda})) \to (\widehat{A}^\bullet, \widehat{\mu}^\bullet(\boldsymbol{\lambda}))$ is a surjective chain map.*

Let $(B^\bullet, \mu_B^\bullet(\boldsymbol{\lambda}))$ denote the kernel of the chain map π. The terms are of the form $B^q = \bigoplus A_K^q$, where $\ell+1 \in K$ and $|K| = q$. In particular, $B^0 = 0$. We now identify the differential $\mu_B^\bullet(\boldsymbol{\lambda})$. If $k < m \le n$ and $J \subseteq [m,n]$, let $\{k, J\}$ denote the (ordered) subset $\{k\} \cup J$ of $[k,n]$. For a linear map F, write $[F]^k$ for the map $\oplus_1^k F$.

Proposition 1.7. *The complex $(B^\bullet, \mu_B^\bullet(\boldsymbol{\lambda}))$ decomposes as the direct sum of ℓ copies of the complex $\widehat{A}^\bullet$, shifted in dimension by one, with the sign of the boundary map reversed. In other words, $(B^\bullet, \mu_B^\bullet(\boldsymbol{\lambda})) \cong \big((\widehat{A}^{\bullet-1})^\ell, -[\widehat{\mu}^{\bullet-1}(\boldsymbol{\lambda})]^\ell\big)$.*

Proof. For $1 \le q \le n-\ell$, we have $B^q = \bigoplus A^q_{\{\ell+1,J\}}$, where the sum is over all $J \subseteq [\ell+2, n]$ with $|J| = q-1$. Each summand may be written as $A^q_{\{\ell+1,J\}} = \bigoplus_{i=1}^{\ell} a_{i,\ell+1} \wedge A_J^{q-1}$. Thus, $B^q = \bigoplus_{i=1}^{\ell} a_{i,\ell+1} \wedge \widehat{A}^{q-1}$ is isomorphic to the direct sum of ℓ copies of $\widehat{A}^{q-1}$ via the map $B^q \to [\widehat{A}^{q-1}]^\ell$, $a_{i,\ell+1} \wedge a_{I,J} \mapsto (0, \ldots, a_{I,J}, \ldots, 0)$.

Now consider the boundary map $\mu_B^q(\boldsymbol{\lambda}) : B^q \to B^{q+1}$ of the complex $B^\bullet$, induced by left-multiplication by $\omega = \sum_{k=\ell+1}^n \omega_k$. Let $\eta = a_{i,\ell+1} \wedge a_{I,J}$ be a generator for B^q. Since $a_{i,k} \wedge a_{j,k} = 0$ for all $i, j < k$, we have $\omega_{\ell+1} \wedge \eta = 0$. Thus,

$$\mu_B^q(\boldsymbol{\lambda})(\eta) = \omega \wedge \eta = (\omega - \omega_{\ell+1}) \wedge \eta = -a_{i,\ell+1} \wedge (\omega - \omega_{\ell+1}) \wedge a_{I,J}.$$

Write $(\omega - \omega_{\ell+1}) \wedge a_{I,J} = \xi_1 + \xi_2$ in terms of the basis for A specified in Proposition 1.3, where $\xi_1 \in \bigoplus_{\ell+1 \in K} A_K^q$ and $\xi_2 \in \bigoplus_{\ell+1 \notin K} A_K^q$. Then we have $\omega \wedge \eta = -a_{i,\ell+1} \wedge (\xi_1 + \xi_2) = -a_{i,\ell+1} \wedge \xi_2$. Checking that $\widehat{\omega} \wedge a_{I,J} = \xi_2$ in $\widehat{A}$, we have $\mu_B^q(\boldsymbol{\lambda})(a_{i,\ell+1} \wedge a_{I,J}) = -a_{i,\ell+1} \wedge \widehat{\mu}^{q-1}(\boldsymbol{\lambda})(a_{I,J})$. Thus, with the change in sign, the boundary map $\mu_B^\bullet(\boldsymbol{\lambda})$ respects the direct sum decomposition $B^\bullet \cong (\widehat{A}^{\bullet-1})^\ell$. Q.E.D.

1.8. Boundary Maps

We now study the differential of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$. The direct sum decompositions of the terms of the complexes $A^\bullet$, $\widehat{A}^\bullet$, and $B^\bullet$ exhibited above yield

$$A^q = \bigoplus\nolimits_{|J|=q} A_J^q = \left(\bigoplus\nolimits_{\ell+1 \in J} A_J^q\right) \oplus \left(\bigoplus\nolimits_{\ell+1 \notin J} A_J^q\right) = B^q \oplus \widehat{A}^q.$$

Let $\pi_B : A^q \to B^q$ denote the natural projection. With respect to the direct sum decomposition of the terms $A^q = B^q \oplus \widehat{A}^q$, the boundary map $\mu^\bullet(\boldsymbol{\lambda})$ of the complex $A^\bullet$ is given by $\mu^q(\boldsymbol{\lambda})(v_1, v_2) = (\mu_B^q(\boldsymbol{\lambda})(v_1) + \Psi^q(\boldsymbol{\lambda})(v_2), \widehat{\mu}^q(\boldsymbol{\lambda})(v_2))$, where $\Psi^q(\boldsymbol{\lambda}) = \pi_B \circ \mu^q(\boldsymbol{\lambda}) : \widehat{A}^q \to B^{q+1}$. In matrix form, we have

$$\mu^q(\boldsymbol{\lambda}) = \begin{pmatrix} \mu_B^q(\boldsymbol{\lambda}) & 0 \\ \Psi^q(\boldsymbol{\lambda}) & \widehat{\mu}^q(\boldsymbol{\lambda}) \end{pmatrix}. \tag{1.2}$$

Since $\widehat{A}^\bullet$ is the complex associated to the discriminantal arrangement $\mathcal{A}_{n,\ell+1}$ in $\mathbb{C}^{n-\ell-1}$ and $B^\bullet \cong (\widehat{A}^{\bullet-1})^\ell$ decomposes as a direct sum by Proposition 1.7, we inductively concentrate our attention on the maps $\Psi^q(\boldsymbol{\lambda})$. Fix $J \subseteq [\ell+2, n]$, and denote the restriction of $\Psi^q(\boldsymbol{\lambda})$ to the summand A_J^q of $\widehat{A}^q$ by $\Psi_J^q(\boldsymbol{\lambda})$. For $\eta \in A_J^q$, since $\pi_B(\omega_k \wedge \eta) = 0$ if $k \notin \{\ell+1, J\}$, we have $\Psi_J^q(\boldsymbol{\lambda})(\eta) = \omega_{\ell+1} \wedge \eta + \sum_{j \in J} \pi_B(\omega_j \wedge \eta)$. Thus, $\Psi_J^q(\boldsymbol{\lambda}) : A_J^q \to A_{\{\ell+1,J\}}^{q+1} = \bigoplus_{m=1}^\ell a_{m,\ell+1} \wedge A_J^q$.

For $1 \le m \le \ell$, let $\pi_{m,\ell+1} : A_{\{\ell+1,J\}}^{q+1} \to a_{m,\ell+1} \wedge A_J^q$ denote the natural projection. Then (the matrix of) $\Psi_J^q(\boldsymbol{\lambda}) : A_J^q \to \left(A_J^q\right)^\ell$ may be expressed as

$$\Psi_J^q(\boldsymbol{\lambda}) = \begin{pmatrix} \pi_{1,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda}) & \cdots & \pi_{m,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda}) & \cdots & \pi_{\ell,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda}) \end{pmatrix}, \tag{1.3}$$

and we focus our attention on one such block, that is, on the composition

$$\pi_{m,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda}) : A_J^q \longrightarrow A_{\{1,J\}}^{q+1} \longrightarrow a_{m,\ell+1} \wedge A_J^q. \tag{1.4}$$

Write $J = \{j_1, \ldots, j_q\}$ and for $1 \leq p \leq q$, let $J_p = \{j_1, \ldots, j_p\}$ and $J^p = J \setminus J_p$. If $p = 0$, set $J_0 = \emptyset$ and $J^0 = J$. Then for $a_{I,J} \in A_J^q$, it is readily checked that $\pi_{m,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda})(a_{I,J}) = \pi_{m,\ell+1} \circ \pi_B(\omega \wedge a_{I,J})$ is given by

(1.5)

$$\pi_{m,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda})(a_{I,J}) = \sum\nolimits_{p=0}^{q} \pi_{m,\ell+1} \circ \pi_B(\omega_{j_p} \wedge a_{I_p,J_p}) \wedge a_{I^p,J^p},$$

where $j_0 = \ell + 1$. In light of this, we restrict our attention to $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J})$. We describe this term using the following notion.

Definition 1.9. Fix $J = \{j_1, \ldots, j_q\} \subseteq [\ell+2, n]$ and $m \leq \ell$. If $I = \{i_1, \ldots, i_q\}$ and $1 \leq i_p < j_p$ for each p, a set $K = \{k_{s_1}, \ldots, k_{s_t}, k_{s_{t+1}}\}$ is called *I-admissible* if

1. $\{i_{s_1}, \ldots, i_{s_t}\} \subseteq I \setminus \{i_q\}$ and $i_{s_{t+1}} = i_q$;
2. $\{k_{s_1}, i_{s_1}\} = \{m, \ell + 1\}$; and
3. $\{k_{s_p}, i_{s_p}\} = \{k_{s_{p-1}}, j_{s_{p-1}}\}$ for $p = 2, \ldots, t + 1$.

Note that the last condition is vacuous if K is of cardinality one. Note also that $1 \leq k_{s_p} < j_{s_p}$ and $k_{s_p} \neq i_{s_p}$ for each p.

Lemma 1.10. *We have*

$$\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J}) = \sum\nolimits_K \lambda_{k_q,j_q} a_{m,\ell+1} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q},$$

where the sum is over all I-admissible sets $K = \{k_{s_1}, \ldots, k_{s_t}, k_{s_{t+1}} = k_q\}$, and

$$b_{j_p} = \begin{cases} a_{i_p,j_p} - a_{k_p,j_p} & \text{if } p \in \{s_1, \ldots, s_t, q\}, \\ a_{i_p,j_p} & \text{if } p \notin \{s_1, \ldots, s_t, q\}. \end{cases}$$

Proof. Let $a_{i,j}$ and $a_{k,j}$ be elements of $A_{\{j\}}^1$. Write $r = \min\{i, k\}$ and $s = \max\{i, k\}$. From (1.1), we have either $a_{i,j} \wedge a_{k,j} = a_{r,s} \wedge (a_{k,j} - a_{i,j})$ if $s \geq \ell + 1$, or $a_{i,j} \wedge a_{k,j} = 0$ if $s \leq \ell$. It follows from these considerations, and a routine exercise to check the sign, that summands $\lambda_{k_q,j_q} a_{m,\ell+1} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q}$ of $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J})$ arise only from I-admissible sets K. Q.E.D.

Now write $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J}) = \sum_R \lambda_{R,I}^J a_{m,\ell+1} \wedge a_{R,J}$, where the sum is over all $R = \{r_1, \ldots, r_q\}$, $1 \leq r_p < j_p$, $1 \leq p \leq q$, and $\lambda_{R,I}^J \in \mathbb{C}$.

Proposition 1.11. *The coefficient* $\lambda^J_{R,I}$ *of* $a_{m,\ell+1} \wedge a_{R,J}$ *in* $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J})$ *is given by*

$$\lambda^J_{R,I} = (-1)^{|R \setminus R \cap I|} \sum\nolimits_K \lambda_{k_q,j_q}$$

where the sum is over all I*-admissible sets* K *such that* $R \setminus R \cap I \subseteq K$.

Proof. Let $K = \{k_{s_1}, \ldots, k_{s_t}, k_q\}$ be an I-admissible set. Associated with K, we have the term $\lambda_{k_q,j_q} a_{m,\ell+1} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q}$ of $\pi_{m,\ell+1} \circ \pi_B(a_{I,J} \wedge \omega_{j_q})$ from Lemma 1.10. If $R \setminus R \cap I \not\subseteq K$, it is readily checked that this term contributes nothing to the coefficient $\lambda^J_{R,I}$ of $a_{m,\ell+1} \wedge a_{R,J}$. On the other hand, if $R \setminus R \cap I \subseteq K$, then the above term contributes the summand $(-1)^{|R \setminus R \cap I|} \lambda_{k_q,j_q}$ to the coefficient $\lambda^J_{R,I}$. Q.E.D.

We now obtain a complete description of the map $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda}) : A^q_J \to a_{m,\ell+1} \wedge A^q_J$ from (1.4). Write

$$\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})(a_{I,J}) = \sum\nolimits_R \Lambda^J_{R,I} a_{m,\ell+1} \wedge a_{R,J},$$

where, as above, the sum is over all $R = \{r_1, \ldots, r_q\}$, $1 \le r_p < j_p$, $1 \le p \le q$, and $\Lambda^J_{R,I} \in \mathbb{C}$. Let $\epsilon_{R,I} = 1$ if $R = I$, and $\epsilon_{R,I} = 0$ otherwise.

Theorem 1.12. *The coefficient* $\Lambda^J_{R,I}$ *of* $a_{m,\ell+1} \wedge a_{R,J}$ *in* $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})$ *is given by*

$$\Lambda^J_{R,I} = (-1)^{|R \setminus R \cap I|} \Big(\epsilon_{R,I} \lambda_{m,\ell+1} + \sum\nolimits_{j \in J} \sum\nolimits_K \lambda_{k,j} \Big),$$

where, if $j = j_p$*, the second sum is over all* I_p*-admissible sets* $K = \{k_{s_1}, \ldots, k_{s_t}, k\}$ *for which* $R \setminus R \cap I \subseteq K$.

Proof. From (1.5), we have

$$\pi_{m,\ell+1} \circ \Psi^q_J(a_{I,J}) = \sum\nolimits_{p=0}^{q} \pi_{m,\ell+1} \circ \pi_B(\omega_{j_p} \wedge a_{I_p,J_p}) \wedge a_{I^p,J^p},$$

and the summand corresponding to $p = 0$ is simply $\lambda_{m,\ell+1} a_{m,\ell+1} \wedge a_{I,J}$. For $p \ge 1$, write $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_p} \wedge a_{I_p,J_p}) = \sum_{R_p} \lambda^{J_p}_{R_p,I_p} a_{m,\ell+1} \wedge a_{R_p,J_p}$, where the sum is over all $R_p = \{r_1, \ldots, r_p\}$. For a fixed R, the coefficient of $a_{m,\ell+1} \wedge a_{R,I}$ in $\pi_{m,\ell+1} \circ \pi_B(\omega \wedge a_{I,J})$ may then be expressed as

$$\Lambda^J_{R,I} = \epsilon_{R,I} \lambda_{m,\ell+1} + \sum\nolimits_{p=1}^{q} \lambda^{J_p}_{R_p,I_p},$$

where $R = R_p \cup I^p$. Note that we have $R \setminus R \cap I = R_p \setminus R_p \cap I_p$ for such R.

By Proposition 1.11, we have $\lambda^{J_p}_{R_p,I_p} = (-1)^{|R_p \setminus R_p \cap I_p|} \sum_K \lambda_{k_p,j_p}$, where the sum is over all I_p-admissible sets K with $R_p \setminus R_p \cap I_p \subseteq K$. Thus,

$$\Lambda^J_{R,I} = \epsilon_{R,I}\lambda_{m,\ell+1} + \sum\nolimits_{p=1}^{q} (-1)^{|R_p \setminus R_p \cap I_p|} \sum\nolimits_K \lambda_{k_p,j_p},$$

and since $R = R_p \cup I^p$, we have $R \setminus R \cap I = R_p \setminus R_p \cap I_p \subseteq K$. Q.E.D.

Remark 1.13. In light of the decomposition of the boundary maps of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ given by (1.2) and (1.3), the above theorem, together with the "initial conditions"

$$\mu^0(\boldsymbol{\lambda}) : A^0 \to A^1,\ 1 \mapsto \sum\nolimits_{k=\ell+1}^{n} \omega_k = \sum\nolimits_{k=\ell+1}^{n} \sum\nolimits_{i=1}^{k-1} \lambda_{i,k} a_{i,k},$$

provides a complete description of the boundary maps $\mu^\bullet(\boldsymbol{\lambda})$.

§2. Resolutions and Local Systems

The fundamental group of the complement of a discriminantal arrangement, and more generally that of any fiber-type arrangement, may be realized as an iterated semidirect product of free groups. For any such group G, in [10] we construct a finite free $\mathbb{Z}G$-resolution $C_\bullet(G)$ of the integers. We recall the construction of this resolution in notation consonant with that of the previous section.

Denote the standard generators of the pure braid group P_n by $\gamma_{i,j}$, $1 \le i < j \le n$, and for each j, let G_j be the free group on the $j-1$ generators $\gamma_{1,j}, \ldots, \gamma_{j-1,j}$. Then the pure braid group may be realized as $P_n = G_n \rtimes \cdots \rtimes G_2$. More generally, for $1 \le \ell \le n$, the group $P_{n,\ell} = \ker(P_n \to P_\ell)$ may be realized as $P_{n,\ell} = G_n \rtimes \cdots \rtimes G_{\ell+1}$. Note that $P_n = P_{n,1}$. For $\ell < j$, the monodromy homomorphisms $P_{j-1,\ell} \to \mathrm{Aut}(G_j)$ are given by the (restriction of the) Artin representation. For $s < j$, we shall not distinguish between the braid $\gamma_{r,s}$ and the corresponding (right) automorphism $\gamma_{r,s} \in \mathrm{Aut}(G_j)$. The action of $\gamma_{r,s}$ on G_j is by conjugation: $\gamma_{r,s}(\gamma_{i,j}) = \gamma_{r,s}^{-1} \cdot \gamma_{i,j} \cdot \gamma_{r,s} = z_i \cdot \gamma_{i,j} \cdot z_i^{-1}$, where

$$z_i = \begin{cases} \gamma_{r,j}\gamma_{s,j} & \text{if } i = r \text{ or } i = s, \\ [\gamma_{r,j}, \gamma_{s,j}] & \text{if } r < i < s, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

See Birman [5] and Hansen [20] for details, and as general references on braids.

2.1. Some Fox Calculus

We first establish some notation and record some elements of the Fox Calculus [19, 5], and results from [10] necessary in the construction.

Denote the integral group ring of a (multiplicative) group G by $\mathbb{Z}G$. We regard modules over $\mathbb{Z}G$ as left modules. Elements of the free module $(\mathbb{Z}G)^n$ are viewed as row vectors, and $\mathbb{Z}G$-linear maps $(\mathbb{Z}G)^n \to (\mathbb{Z}G)^m$ are viewed as $n \times m$ matrices which act on the right. For such a map F, denote the transpose by $F^\top$, and recall that $[F]^k$ denotes the map $\oplus_1^k F$. Denote the $n \times n$ identity matrix by $\mathbb{I}_n$.

For the single free group $G_j = \langle \gamma_{i,j} \rangle$, a free $\mathbb{Z}G_j$-resolution of $\mathbb{Z}$ is given by

$$0 \to (\mathbb{Z}G_j)^{j-1} \xrightarrow{\Delta_j} \mathbb{Z}G_j \xrightarrow{\epsilon} \mathbb{Z} \to 0, \tag{2.2}$$

where $\Delta_j = \begin{pmatrix} \gamma_{1,j} - 1 & \cdots & \gamma_{j-1,j} - 1 \end{pmatrix}^\top$, and ϵ is the augmentation map, given by $\epsilon(\gamma_{i,j}) = 1$. For each element $\gamma \in P_{j-1,\ell}$, conjugation by γ induces an automorphism $\gamma : G_j \to G_j$, and a chain automorphism $\gamma_\bullet$ of (2.2), which by the "fundamental formula of Fox Calculus," can be expressed as

$$\begin{array}{ccc} (\mathbb{Z}G_j)^{j-1} & \xrightarrow{\Delta_j} & \mathbb{Z}G_j \\ \downarrow{\scriptstyle \mathcal{J}(\gamma)\circ\tilde{\gamma}} & & \downarrow{\scriptstyle \tilde{\gamma}} \\ (\mathbb{Z}G_j)^{j-1} & \xrightarrow{\Delta_j} & \mathbb{Z}G_j \end{array} \tag{2.3}$$

where $\mathcal{J}(\gamma) = \left(\frac{\partial \gamma(\gamma_{i,j})}{\partial \gamma_{k,j}}\right)$ is the $(j-1) \times (j-1)$ Jacobian matrix of Fox derivatives of γ, and $\tilde{\gamma}$ denotes the extension of γ to the group ring $\mathbb{Z}G_j$, resp., to $(\mathbb{Z}G_j)^{j-1}$. For a second element β of $P_{j-1,\ell}$, we have $(\gamma \cdot \beta)_\bullet = (\beta \circ \gamma)_\bullet = \beta_\bullet \circ \gamma_\bullet$ by the "chain rule of Fox Calculus": $\mathcal{J}(\beta \circ \gamma) = \tilde{\beta}(\mathcal{J}(\gamma)) \cdot \mathcal{J}(\beta)$. In particular, $\mathcal{J}(\gamma^{-1}) = \tilde{\gamma}^{-1}(\mathcal{J}(\gamma)^{-1})$.

Now fix ℓ, $1 \leq \ell \leq n$, and consider the group $P_{n,\ell} = G_n \rtimes \cdots \rtimes G_{\ell+1}$. Let $\mathcal{R} = \mathbb{Z}P_{n,\ell}$ denote the integral group ring of $P_{n,\ell}$, For $\gamma \in P_{j-1,\ell}$ as above, define $m_\gamma : \mathcal{R} \to \mathcal{R}$ by $m_\gamma(r) = \gamma \cdot r$. From (2.3) and extension of scalars, we obtain

$$\begin{array}{ccc} \mathcal{R} \otimes_{\mathbb{Z}G_j} (\mathbb{Z}G_j)^{j-1} & \xrightarrow{\mathrm{id}\otimes\Delta_j} & \mathcal{R} \otimes_{\mathbb{Z}G_j} \mathbb{Z}G_j \\ \downarrow{\scriptstyle m_\gamma \otimes \mathcal{J}(\gamma)\circ\tilde{\gamma}} & & \downarrow{\scriptstyle m_\gamma \otimes \tilde{\gamma}} \\ \mathcal{R} \otimes_{\mathbb{Z}G_j} (\mathbb{Z}G_j)^{j-1} & \xrightarrow{\mathrm{id}\otimes\Delta_j} & \mathcal{R} \otimes_{\mathbb{Z}G_j} \mathbb{Z}G_j \end{array}$$

The map $m_\gamma \otimes \mathcal{J}(\gamma) \circ \tilde{\gamma}$ and the canonical isomorphism $\mathcal{R} \otimes_{\mathbb{Z}G_j} (\mathbb{Z}G_j)^{j-1} \cong \mathcal{R}^{j-1}$ define an $\mathcal{R}$-linear automorphism $\rho_j(\gamma) : \mathcal{R}^{j-1} \to \mathcal{R}^{j-1}$, whose matrix is $\gamma \cdot \mathcal{J}(\gamma)$, see [10, Lemma 2.4]. Furthermore, we have the following.

Lemma 2.2 ([10, Lemma 2.6]). *For each j, $2 \leq j \leq n$, the action of the group $P_{j-1,\ell}$ on the free group G_j gives rise to a representation $\rho_j : P_{j-1,\ell} \to \mathrm{Aut}_{\mathcal{R}}(\mathcal{R}^{j-1})$ with the property that $\rho_j(\gamma) = m_\gamma \otimes \mathcal{J}(\gamma) \circ \tilde{\gamma}$ for every $\gamma \in P_{j-1,\ell}$.*

Remark 2.3. Via the convention $\rho_j(\gamma_{p,q}) = \mathbb{I}_{j-1}$ for $q \geq j$, the above extends to a representation $\rho_j : P_{n,\ell} \to \mathrm{Aut}_{\mathcal{R}}(\mathcal{R}^{j-1})$ of the entire group $P_{n,\ell}$. We denote by $\tilde{\rho}_j : \mathcal{R} \to \mathrm{End}_{\mathcal{R}}(\mathcal{R}^{j-1})$ the extension of ρ_j to the group ring $\mathcal{R}$. We also use $\tilde{\rho}_j$ to denote the homomorphism $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^m, \mathcal{R}^n) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^{m(j-1)}, \mathcal{R}^{n(j-1)})$ defined by replacing each entry x of an $m \times n$ matrix by $\tilde{\rho}_j(x)$.

2.4. The Resolution

We now recall the construction of the free resolution $\epsilon : C_\bullet = C_\bullet(G) \to \mathbb{Z}$ over the ring $\mathcal{R} = \mathbb{Z}G$ from [10], in the case where $G = P_{n,\ell}$ is the fundamental group of the complement of the discriminantal arrangement $\mathcal{A}_{n,\ell}$. If $J = \{j_1, \ldots, j_q\} \subseteq [\ell+1, n]$, recall that for $p < q$, $J_p = \{j_1, \ldots, j_p\}$ and $J^p = J \setminus J_p$. For such a set J, let C_q^J be a free $\mathcal{R}$-module of rank $(j_1 - 1) \cdots (j_q - 1)$.

Let $C_0 = \mathcal{R}$, and, for $1 \leq q \leq n - \ell$, let $C_q = \bigoplus_{|J|=q} C_q^J$, where the sum is over all $J \subseteq [\ell+1, n]$. The augmentation map, $\epsilon : C_0 \to \mathbb{Z}$, is the usual augmentation of the group ring, given by $\epsilon(\gamma) = 1$, for $\gamma \in P_{n,\ell}$. We define the boundary maps of $C_\bullet$ by recursively specifying their restrictions Δ^J to the summands C_q^J as follows:

If $J = \{j\}$, we define $\Delta_J : C_1^J = \mathcal{R}^{j-1} \to \mathcal{R} = C_0$ as in the resolution (2.2), by $\Delta_J = \begin{pmatrix} \gamma_{1,j} - 1 & \cdots & \gamma_{j-1,j} - 1 \end{pmatrix}^\top$.

In general, if $J = \{j_1, \ldots, j_q\}$, then $J^1 = \{j_2, \ldots, j_q\}$ and $J_{q-1} = \{j_1, \ldots, j_{q-1}\}$, and we define $\Delta_J : C_q^J \to C_{q-1}^{J^1}$ by $\Delta_J = -\tilde{\rho}_{j_q}(\Delta_{J_{q-1}})$

Now define $\Delta^J : C_q^J \to \bigoplus_{p=1}^q C_{q-1}^{J \setminus \{j_p\}}$ by

$$\Delta^J = \left(\Delta_J, [\Delta_{J^1}]^{d_1}, \ldots, [\Delta_{J^p}]^{d_p}, \ldots, [\Delta_{J^{q-1}}]^{d_{q-1}}\right),$$

where $d_p = (j_1 - 1) \cdots (j_p - 1)$.

Finally, define $\partial_q : C_q \to C_{q-1}$ by $\partial_q = \sum_{|J|=q} \Delta^J$.

Theorem 2.5 ([10, Theorem 2.10]). *Let $\mathcal{R} = \mathbb{Z}P_{n,\ell}$ be the integral group ring of the group $P_{n,\ell}$. Then the system of $\mathcal{R}$-modules and homomorphisms $(C_\bullet, \partial_\bullet)$ is a finite, free resolution of $\mathbb{Z}$ over $\mathcal{R}$.*

Remark 2.6. The proof of this result in [10] makes use of a mapping cone decomposition of the complex $(C_\bullet, \partial_\bullet)$. This decomposition may be described as follows. Let $(\widehat{C}_\bullet, \widehat{\partial}_\bullet)$ denote the subcomplex of $(C_\bullet, \partial_\bullet)$ with terms $\widehat{C}_q = \bigoplus_{\ell+1 \notin J} C_q^J$, and boundary maps $\widehat{\partial}_q = \partial_q|_{\widehat{C}_q}$ given by restriction. The complex $\widehat{C}_\bullet$ may be realized as $\widehat{C}_\bullet = C_\bullet(P_{n,\ell+1}) \otimes_{P_{n,\ell}} \mathcal{R}$, where $\epsilon : C_\bullet(P_{n,\ell+1}) \to \mathbb{Z}$ is the resolution over $\mathbb{Z}P_{n,\ell+1}$ obtained by applying the above construction to the group $P_{n,\ell+1} < P_{n,\ell}$.

Let $(D_\bullet, \partial_\bullet^D)$ denote the direct sum of ℓ copies of the complex $\widehat{C}_\bullet$, with the sign of the boundary map reversed. That is, $D_q = (\widehat{C}_q)^\ell$ and $\partial_q^D = -[\widehat{\partial}_q]^\ell$. The terms of this complex may be expressed as $D_q = \bigoplus_{\ell+1 \in K} C_{q+1}^K$, where $|K| = q$. Using this description, define a map $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$ by setting the restriction of Ξ_q to the summand C_{q+1}^K of D_q to be equal to $\Delta_K : C_{q+1}^K \to C_q^J \subset \widehat{C}_q$, where $K = \{\ell+1\} \cup J$.

As shown in [10], the map $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$ is a chain map, and the original complex $(C_\bullet, \partial_\bullet)$ may be realized as the mapping cone of $\Xi_\bullet$. Explicitly, the terms of $C_\bullet$ decompose as $C_q = D_{q-1} \oplus \widehat{C}_q$. With respect to this decomposition, the boundary map $\partial_{q+1} : C_{q+1} \to C_q$ is given by $\partial_{q+1}(u, v) = (-\partial_q^D(u), \Xi_q(u) + \widehat{\partial}_{q+1}(v))$.

2.7. Rank One Local Systems

The abelianization of the group $P_{n,\ell}$ is free abelian of rank $N = \binom{n}{2} - \binom{\ell}{2}$. Let $(\mathbb{C}^*)^N$ denote the complex torus, with coordinates $t_{i,j}$, $\ell + 1 \leq j \leq n$, $1 \leq i < j$. Each point $\mathbf{t} \in (\mathbb{C}^*)^N$ gives rise to a rank one representation $\nu_\mathbf{t} : P_{n,\ell} \to \mathbb{C}^*$, $\gamma_{i,j} \mapsto t_{i,j}$, an associated $P_{n,\ell}$-module $L = L_\mathbf{t}$, and a rank one local system $\mathcal{L} = \mathcal{L}_\mathbf{t}$ on the configuration space $M_{n,\ell}$. The homology and cohomology of $P_{n,\ell}$ with coefficients in L (resp., that of $M_{n,\ell}$ with coefficients in $\mathcal{L}$) are isomorphic to the homology and cohomology of the complexes $\mathbf{C}_\bullet := C_\bullet \otimes_{P_{n,\ell}} L$ and $\mathbf{C}^\bullet := \mathcal{H}om_{P_{n,\ell}}(C_\bullet, L)$ respectively, see [8].

The terms, $\mathbf{C}_q = C_q \otimes_\mathcal{R} \mathbb{C}$ and $\mathbf{C}^q = \mathrm{Hom}_{P_{n,\ell}}(C_q, L)$, of these complexes are finite dimensional complex vector spaces. Notice that $\dim \mathbf{C}_q = \dim \mathbf{C}^q = \dim A^q = \sum_{|J|=q} (j_1 - 1) \cdots (j_q - 1)$, where the sum is over all $J \subseteq [\ell + 1, n]$. Denote the boundary maps of $\mathbf{C}_\bullet$ and $\mathbf{C}^\bullet$ by $\boldsymbol{\partial}_q(\mathbf{t}) : \mathbf{C}_q \to \mathbf{C}_{q-1}$ and $\boldsymbol{\delta}^q(\mathbf{t}) : \mathbf{C}^q \to \mathbf{C}^{q+1}$. As we follow [8] in our

definition of $\mathbf{C}^\bullet$, these maps are related by

$$\boldsymbol{\delta}^q(\mathbf{t})(u)(x) = (-1)^q u(\boldsymbol{\partial}_{q+1}(\mathbf{t})(x)) \tag{2.4}$$

for $u \in \mathbf{C}^q$ and $x \in C_{q+1}$. To describe these maps further, we require some notation.

Consider the evaluation map $\mathcal{R} \times (\mathbb{C}^*)^N \to \mathbb{C}$, which takes an element f of the group ring, and a point $\mathbf{t}$ in $(\mathbb{C}^*)^N$ and yields $f(\mathbf{t}) := \tilde{\nu}_{\mathbf{t}}(f)$, the evaluation of f at $\mathbf{t}$. Fixing $f \in \mathcal{R}$ and allowing $\mathbf{t} \in (\mathbb{C}^*)^N$ to vary, we get a holomorphic map $\mathbf{f} : (\mathbb{C}^*)^N \to \mathbb{C}$. More generally, we have the map $\mathrm{Mat}_{r\times s}(\mathcal{R}) \times (\mathbb{C}^*)^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$, $(F, \mathbf{t}) \mapsto F(\mathbf{t}) := \tilde{\nu}_{\mathbf{t}}(F)$. For fixed $F \in \mathrm{Mat}_{p\times q}(\mathcal{R})$, we get a map $\mathbf{F} : (\mathbb{C}^*)^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$. With these conventions, if $\dim \mathbf{C}_q = r$ and $\dim \mathbf{C}_{q+1} = s$, the boundary maps of the complexes $\mathbf{C}_\bullet$ and $\mathbf{C}^\bullet$ may be viewed as evaluations, $\boldsymbol{\partial}_q(\mathbf{t})$ and $\boldsymbol{\delta}^q(\mathbf{t})$, of maps $\boldsymbol{\partial}_q : (\mathbb{C}^*)^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$ and $\boldsymbol{\delta}^q : (\mathbb{C}^*)^N \to \mathrm{Mat}_{s\times r}(\mathbb{C})$.

We shall subsequently be concerned with the derivatives of these maps at the identity element $\mathbf{1} = (1, \ldots, 1)$ of $(\mathbb{C}^*)^N$. The (holomorphic) tangent space of $H^1(M_{n,\ell}; \mathbb{C}^*) = (\mathbb{C}^*)^N$ at $\mathbf{1}$ is $H^1(M_{n,\ell}; \mathbb{C}) = \mathbb{C}^N$, with coordinates $\lambda_{i,j}$. The exponential map $\mathrm{T}_{\mathbf{1}}(\mathbb{C}^*)^N \to (\mathbb{C}^*)^N$ is the coefficient map $H^1(M_{n,\ell}; \mathbb{C}) \to H^1(M_{n,\ell}; \mathbb{C}^*)$ induced by $\exp : \mathbb{C} \to \mathbb{C}^*$, $\lambda_{i,j} \mapsto e^{\lambda_{i,j}} = t_{i,j}$. For an element f of $\mathcal{R}$, the derivative of the corresponding map $\mathbf{f} : (\mathbb{C}^*)^N \to \mathbb{C}$ at $\mathbf{1}$ is given by $\mathbf{f}_* : \mathbb{C}^N \to \mathbb{C}$, $\mathbf{f}_*(\boldsymbol{\lambda}) = \frac{d}{dx}\big|_{x=0} f(\ldots e^{x\lambda_{i,j}} \ldots)$. More generally, for $F \in \mathrm{Mat}_{r\times s}(\mathcal{R})$, we have $\mathbf{F}_* : \mathbb{C}^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$.

§3. A Complex of Derivatives

We now relate the cohomology theories $H^*(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ and $H^*(M_n; \mathcal{L})$ by relating the complexes $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ and $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$. As above, let $(\boldsymbol{\partial}_q)_*$ and $\boldsymbol{\delta}^q_*$ denote the derivatives of the maps $\boldsymbol{\partial}_q$ and $\boldsymbol{\delta}^q$ at $\mathbf{1} \in (\mathbb{C}^*)^N$.

Theorem 3.1. *The complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a linear approximation of the complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$. For each $\boldsymbol{\lambda} \in \mathbb{C}^N$, the system of complex vector spaces and linear maps $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet_*(\boldsymbol{\lambda}))$ is a complex. For each q, we have $A^q \cong \mathbf{C}^q$, and, under this identification, $\mu^q(\boldsymbol{\lambda}) = \boldsymbol{\delta}^q_*(\boldsymbol{\lambda})$.*

From the discussions in sections 1.1 and 2.7, it is clear that $A^q \cong \mathbf{C}^q$. In light of the sign conventions (2.4) used in the construction of the complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$ and the fact that $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a complex, to show that $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet_*(\boldsymbol{\lambda}))$ is a complex, and to prove the theorem, it suffices to establish the following.

Proposition 3.2. *For each q, we have $\mu^q(\boldsymbol{\lambda}) = (-1)^q \big[(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$.*

The maps $\mu^q(\boldsymbol{\lambda})$ were analyzed in section 1.5. We now carry out a similar analysis of the maps $(\partial_{q+1})_*(\boldsymbol{\lambda})$.

3.3. Some Calculus

We first record some facts necessary for this analysis. Recall that $\mathcal{R}$ denotes the integral group ring of the group $P_{n,\ell}$. For $f, g \in \mathcal{R}$, the Product Rule yields $(\mathbf{f} \cdot \mathbf{g})_*(\boldsymbol{\lambda}) = \mathbf{f}_*(\boldsymbol{\lambda}) \cdot \mathbf{g}(\mathbf{1}) + \mathbf{f}(\mathbf{1}) \cdot \mathbf{g}_*(\boldsymbol{\lambda})$. Similarly, for $F \in \mathrm{Mat}_{p\times q}(\mathcal{R})$ and $G \in \mathrm{Mat}_{q\times r}(\mathcal{R})$, matrix multiplication and the differentiation rules yield

$$(\mathbf{F} \cdot \mathbf{G})_*(\boldsymbol{\lambda}) = \mathbf{F}_*(\boldsymbol{\lambda}) \cdot \mathbf{G}(\mathbf{1}) + \mathbf{F}(\mathbf{1}) \cdot \mathbf{G}_*(\boldsymbol{\lambda}). \tag{3.1}$$

As an immediate consequence of the Product Rule, for $\gamma, \zeta \in P_{n,\ell}$ and $\tau = [\zeta, \gamma]$ a commutator, we have $(\gamma^{-1})_* = -\gamma_*$, and $\tau_* = 0$. Consequently, $(\zeta \cdot \gamma \cdot \zeta^{-1})_* = \gamma_*$.

Now recall the representations ρ_j defined in Lemma 2.2, and used in the construction of the resolution $C_\bullet$. Associated to each $\gamma \in P_{j-1,\ell}$, we have a map $\boldsymbol{\rho}_j(\gamma) : (\mathbb{C}^*)^N \to \mathrm{Aut}(\mathbb{C}^{j-1})$. Since γ acts on the free group by conjugation, we have $\boldsymbol{\rho}_j(\gamma)(\mathbf{1}) = \mathbb{I}_{j-1}$. Identify $\mathrm{End}(\mathbb{C}^{j-1})$ as the tangent space to $\mathrm{Aut}(\mathbb{C}^{j-1})$ at the identity, and denote the derivative of the map $\boldsymbol{\rho}_j(\gamma)$ at $\mathbf{1}$ by $\boldsymbol{\rho}_j(\gamma)_* : \mathbb{C}^N \to \mathrm{End}(\mathbb{C}^{j-1})$.

Define $(\boldsymbol{\rho}_j)_* : P_{j-1,\ell} \to \mathrm{Hom}(\mathbb{C}^N, \mathrm{End}(\mathbb{C}^{j-1}))$ by $(\boldsymbol{\rho}_j)_*(\gamma) = \boldsymbol{\rho}_j(\gamma)_*$. The chain rule of Fox Calculus and a brief computation reveal that $(\boldsymbol{\rho}_j)_*$ is a homomorphism, and is trivial on the commutator subgroup $P'_{n,\ell}$. This yields a map $\mathbb{C}^N \to \mathrm{Hom}(\mathbb{C}^N, \mathrm{End}(\mathbb{C}^{j-1}))$, $\lambda_{r,s} \mapsto \boldsymbol{\rho}_j(\gamma_{r,s})_*$, which we continue to denote by $(\boldsymbol{\rho}_j)_*$. For $\gamma \in P_{n,\ell}$, view the derivative, $\gamma_*(\boldsymbol{\lambda}) = \sum c_{r,s}\lambda_{r,s}$, of the corresponding map $\boldsymbol{\gamma}$ as a linear form in the $\lambda_{r,s}$. Then we have the following "chain rule":

$$\boldsymbol{\rho}_j(\gamma)_*(\boldsymbol{\lambda}) = \sum c_{r,s}(\boldsymbol{\rho}_j)_*(\lambda_{r,s}) = (\boldsymbol{\rho}_j)_*(\gamma_*(\boldsymbol{\lambda})). \tag{3.2}$$

In particular, $\boldsymbol{\rho}(\gamma_{r,s})_* = \boldsymbol{\rho}_*(\lambda_{r,s})$, which we now compute.

Lemma 3.4. *For $r < s < j$, the derivative of the map $\boldsymbol{\rho}_j(\gamma_{r,s})$ is given by $\boldsymbol{\rho}_j(\gamma_{r,s})_*(\boldsymbol{\lambda}) =$*

$$\begin{pmatrix} \lambda_{r,s}\cdot\mathbb{I}_{r-1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{r,s}+\lambda_{s,j} & 0 & -\lambda_{r,j} & 0 \\ 0 & 0 & \lambda_{r,s}\cdot\mathbb{I}_{s-r-1} & 0 & 0 \\ 0 & -\lambda_{s,j} & 0 & \lambda_{r,s}+\lambda_{r,j} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{r,s}\cdot\mathbb{I}_{j-s-1} \end{pmatrix}.$$

Proof. The matrix of $\rho_j(\gamma_{r,s})$ is $\gamma_{r,s} \cdot \mathcal{J}(\gamma_{r,s})$, where $\mathcal{J}(\gamma_{r,s})$ is the Fox Jacobian. Thus, $\boldsymbol{\rho}_j(\gamma_{r,s})(\mathbf{t}) = t_{r,s} \cdot \boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t}) = (t_{r,s} \cdot \mathbb{I}_{j-1}) \cdot \boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t})$, where $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t})$ is the map induced by the Fox Jacobian. By the Product Rule (3.1), we have

$$\boldsymbol{\rho}_j(\gamma_{r,s})_*(\boldsymbol{\lambda}) = (\lambda_{r,s} \cdot \mathbb{I}_{j-1}) \cdot \boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{1}) + \boldsymbol{\mathcal{J}}(\gamma_{r,s})_*(\boldsymbol{\lambda}).$$

The action of $\gamma_{r,s}$ on the free group $G_j = \langle \gamma_{i,j} \rangle$ is recorded in (2.1). Computing Fox derivatives and evaluating at $\mathbf{t}$ yields the familiar Gassner matrix of $\gamma_{r,s}$ (see [5]), $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t}) =$

$$\begin{pmatrix} \mathbb{I}_{r-1} & 0 & 0 & 0 & 0 \\ 0 & 1 - t_{r,j} + t_{r,j}t_{s,j} & 0 & t_{r,j}(1 - t_{r,j}) & 0 \\ 0 & \vec{u} & \mathbb{I}_{s-r-1} & -\vec{u} & 0 \\ 0 & 1 - t_{s,j} & 0 & t_{r,j} & 0 \\ 0 & 0 & 0 & 0 & \mathbb{I}_{j-s-1} \end{pmatrix},$$

where $\vec{u} = \begin{pmatrix}(1 - t_{r+1,j})(1 - t_{r,j}) & \cdots & (1 - t_{s-1,j})(1 - t_{r,j})\end{pmatrix}^\top$. Since $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{1}) = \mathbb{I}_{j-1}$, the result follows upon differentiating $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t})$. Q.E.D.

3.5. Boundary Map Derivatives

We now obtain an inductive formula for the derivatives of the boundary maps of the complex $(\mathbf{C}_\bullet, \boldsymbol{\partial}_\bullet(\mathbf{t}))$. The mapping cone decomposition of the resolution $(C_\bullet, \partial_\bullet)$ discussed in Remark 2.6 gives rise to an analogous decomposition of the complex $(\mathbf{C}_\bullet, \boldsymbol{\partial}(\mathbf{t}))$. Specifically, the terms decompose as $\mathbf{C}_q = \mathbf{D}_{q-1} \oplus \widehat{\mathbf{C}}_q$, and with respect to this decomposition, the matrix of the boundary map $\boldsymbol{\partial}_{q+1}(\mathbf{t}) : \mathbf{C}_{q+1} \to \mathbf{C}_q$ is given by

$$\boldsymbol{\partial}_{q+1}(\mathbf{t}) = \begin{pmatrix} -\boldsymbol{\partial}_q^D(\mathbf{t}) & \boldsymbol{\Xi}_q(\mathbf{t}) \\ 0 & \widehat{\boldsymbol{\partial}}_{q+1}(\mathbf{t}) \end{pmatrix}. \tag{3.3}$$

Up to sign, the complex $(\mathbf{D}_\bullet, \boldsymbol{\partial}_\bullet^D(\mathbf{t}))$ is a direct sum of ℓ copies of the complex $(\widehat{\mathbf{C}}_\bullet, \widehat{\boldsymbol{\partial}}_\bullet(\mathbf{t}))$, which arises from the group $P_{n,\ell+1} < P_{n,\ell}$. In light of this, we restrict our attention to the chain map $\boldsymbol{\Xi}_\bullet$ and its components $\Delta_{\{\ell+1,J\}}$, their evaluations $\boldsymbol{\Xi}_\bullet(\mathbf{t})$ and $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t})$, and the derivatives of these evaluations at $\mathbf{1}$.

For $J = \{j_1, \ldots, j_q\} \subseteq [\ell + 2, n]$, let $\rho_J = \tilde{\rho}_{j_q} \circ \cdots \circ \tilde{\rho}_{j_1}$ and $d_J = (j_1 - 1) \cdots (j_q - 1)$. Then $\Delta_{\{\ell+1,J\}} = (-1)^q \rho_J(\Delta_{\ell+1})$, where $\Delta_{\ell+1} = \begin{pmatrix}\gamma_{1,\ell+1} - 1 & \cdots & \gamma_{\ell,\ell+1} - 1\end{pmatrix}^\top$, and the matrix of $\Delta_{\{\ell+1,J\}}$ is $\ell \cdot d_J \times d_J$ with $d_J \times d_J$ blocks $(-1)^q \rho_J(\gamma_{m,\ell+1} - 1)$, $1 \leq m \leq \ell$. We concentrate our attention on one such block.

Fix m, $1 \leq m \leq \ell$, and let M denote the matrix of $\rho_J(\gamma_{m,\ell+1} - 1)$. Similarly, let M' denote the matrix of $\rho_{J_{q-1}}(\gamma_{m,\ell+1} - 1)$. Then M is the matrix of $\tilde{\rho}_{j_q}(M')$. Since M is $d_J \times d_J$, its rows and columns are naturally indexed by sets $R = \{r_1, \ldots, r_q\}$ and $I = \{i_1, \ldots, i_q\}$, $1 \leq r_p, i_p \leq j_p - 1$. We thus denote the entries of M by $M_{R,I}$. With these conventions, we have

$$M_{R,I} = \left[\tilde{\rho}_{j_q}(M'_{R',I'})\right]_{r_q,i_q} \tag{3.4}$$

where, for instance, $I' = I_{q-1} = I \setminus \{i_q\}$.

Now consider the block $\mathbf{M}(\mathbf{t})$ of $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t})$ arising from the block M of the matrix of $\Delta_{\{\ell+1,J\}}$ above. Recall the notion of an I-admissible set from Definition 1.9, and recall that $\epsilon_{R,I} = 1$ if $R = I$, and $\epsilon_{R,I} = 0$ otherwise.

Theorem 3.6. *Let $J \subseteq [\ell+2, n]$ and let M denote the matrix of $\rho_J(\gamma_{m,\ell+1} - 1)$. Then the entries of the derivative, $\mathbf{M}_*(\boldsymbol{\lambda})$, of the evaluation $\mathbf{M}(\mathbf{t})$ are given by*

$$[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I} = (-1)^{|R \setminus R \cap I|}\Big(\epsilon_{R,I}\lambda_{m,\ell+1} + \sum\nolimits_{j \in J}\sum\nolimits_{K}\lambda_{k,j}\Big),$$

where, if $j = j_p$, the second sum is over all I_p-admissible sets $K = \{k_{s_1}, \ldots, k_{s_t}, k\}$ for which $R \setminus R \cap I \subseteq K$.

Proof. The proof is by induction on $|J|$.

If $J = \{j\}$, then $M = \gamma_{m,\ell+1} \cdot \mathcal{J}(\gamma_{m,\ell+1}) - \mathbb{I}_{j-1}$ is the matrix of $\tilde{\rho}_j(\gamma_{m,\ell+1} - 1)$, so $\mathbf{M}(\mathbf{t}) = \boldsymbol{\gamma}_{m,\ell+1}(\mathbf{t}) \cdot \boldsymbol{\mathcal{J}}(\gamma_{m,\ell+1})(\mathbf{t}) - \mathbb{I}_{j-1}$. Since the derivative of the constant $\mathbb{I}_{j-1}$ is zero, the entries of $\mathbf{M}_*(\boldsymbol{\lambda})$ are given by Lemma 3.4 (with $r = m$ and $s = \ell+1$). In this instance, we have $I = \{i\}$, and a set $K = \{k\}$ is I-admissible if $k \neq i$ and $\{k, i\} = \{m, \ell+1\}$. It follows that the case $|J| = 1$ is a restatement of Lemma 3.4.

In general, let $J = \{j_1, \ldots, j_{q-1}, j_q\}$ and, as in (3.4) above, write $M_{R,I} = \left[\tilde{\rho}_{j_q}(M'_{R',I'})\right]_{r_q,i_q}$. Then we have

$$[\mathbf{M}(\mathbf{t})]_{R,I} = \left[\tilde{\boldsymbol{\rho}}_{j_q}(M'_{R',I'})(\mathbf{t})\right]_{r_q,i_q} \quad \text{and}$$
$$[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I} = \left[\tilde{\boldsymbol{\rho}}_{j_q}(M'_{R',I'})_*(\boldsymbol{\lambda})\right]_{r_q,i_q}.$$

By induction, the entries of the matrix $\mathbf{M}'_*(\boldsymbol{\lambda})$ are given by

$$[\mathbf{M}'_*(\boldsymbol{\lambda})]_{R',I'} = (-1)^{|R' \setminus R' \cap I'|}\Big(\epsilon_{R',I'}\lambda_{m,\ell+1} + \sum\nolimits_{j_p \in J'}\sum\nolimits_{K}\lambda_{k_p,j_p}\Big),$$

where $J' = J_{q-1} = J \setminus \{j_q\}$, and for $j_p \in J'$, the second sum is over all I_p-admissible sets K for which $R' \setminus R' \cap I' \subseteq K$.

By the chain rule (3.2), the entries of $\mathbf{M}_*(\boldsymbol{\lambda})$ are given by

(3.5)

$$[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I} = \left[\tilde{\boldsymbol{\rho}}_{j_q}(M'_{R',I'})_*(\boldsymbol{\lambda})\right]_{r_q,i_q} = \left[(\boldsymbol{\rho}_{j_q})_*\big((\mathbf{M}'_{R',I'})_*(\boldsymbol{\lambda})\big)\right]_{r_q,i_q}$$
$$= \left[S\Big(\epsilon_{R',I'}(\boldsymbol{\rho}_{j_q})_*(\lambda_{m,\ell+1}) + \sum\nolimits_{j_p \in J'} \sum\nolimits_K (\boldsymbol{\rho}_{j_q})_*(\lambda_{k_p,j_p})\Big)\right]_{r_q,i_q},$$

where $S = (-1)^{|R' \setminus R' \cap I'|}$. By Lemma 3.4, for $r < s < j_q$, we have

(3.6)

$$\left[(\boldsymbol{\rho}_{j_q})_*(\lambda_{r,s})\right]_{r_q,i_q} = \begin{cases} \lambda_{r,s} + \lambda_{k_q,j_q} & \text{if } i_q = r_q \text{ and } \{k_q, i_q\} = \{r, s\}, \\ -\lambda_{r_q,j_q} & \text{if } i_q \neq r_q \text{ and } \{r_q, i_q\} = \{r, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

The entries of $\mathbf{M}_*(\boldsymbol{\lambda})$ may be calculated from (3.5) using (3.6), yielding the formula in the statement of the theorem. We conclude the proof by making several observations which elucidate this calculation.

First consider the case $R' = I'$. Then $S = 1$ and $\epsilon_{R',I'} = 1$. If $r_q = i_q$, then the first case of (3.6) yields a contribution of $\lambda_{m,\ell+1} + \lambda_{k_q,j_q}$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{I,I}$, provided that $\{k_q, i_q\} = \{m, \ell+1\}$. Note that this condition implies that the set $K = \{k_q\}$ is I-admissible (and that $k_q \neq i_q$). Note also that in this instance we have $R = I$, $R \setminus R \cap I = \emptyset \subset K$, $\epsilon_{R,I} = 1$, and $|R \setminus R \cap I| = 0$.

If $R' = I'$ and $r_q \neq i_q$, then the second case of (3.6) contributes $-\lambda_{r_q,j_q}$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I}$ if $\{r_q, i_q\} = \{m, \ell+1\}$. In this instance, the set $\{r_q\}$ is I-admissible. Since $R' = I'$ and $r_q \neq i_q$, we have $|R \setminus R \cap I| = 1$.

For general R' and I', suppose that $S \cdot \lambda_{k_p,j_p}$ is a summand of $[\mathbf{M}'_*(\boldsymbol{\lambda})]_{R',I'}$ for some $p \leq q-1$. Then, by the inductive hypothesis, this summand arises from an I_p-admissible set $K = \{k_{s_1}, \ldots, k_{s_t}, k_p\}$ with $R' \setminus R' \cap I' \subseteq K$. If $r_q = i_q$, then the first case of (3.6) yields a contribution of $S \cdot (\lambda_{k_p,j_p} + \lambda_{k_q,j_q})$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I}$, provided that $\{k_q, i_q\} = \{k_p, j_p\}$. For such k_q, it is readily checked that the set $K \cup \{k_q\}$ is I-admissible. Also, since $r_q = i_q$, we have $R \setminus R \cap I = R' \setminus R' \cap I' \subseteq K$.

If, as above, $S \cdot \lambda_{k_p,j_p}$ is a summand of $[\mathbf{M}'_*(\boldsymbol{\lambda})]_{R',I'}$ and $r_q \neq i_q$, then the second case of (3.6) contributes $-S \cdot \lambda_{r_q,j_q}$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I}$ provided $\{r_q, i_q\} = \{k_p . j_p\}$. In this instance, the set $K \cup \{r_q\}$ is I-admissible, and since $r_q \neq i_q$, we have $R \setminus R \cap I = (R' \setminus R' \cap I') \cup \{r_q\} \subseteq K \cup \{r_q\}$, and $|R \setminus R \cap I| = |R' \setminus R' \cap I'| + 1$.

Applying these observations to (3.5) above completes the proof. Q.E.D.

3.7. Proof of Proposition 3.2

We now use Theorems 1.12 and 3.6 to show that the differential of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is given by $\mu^q(\boldsymbol{\lambda}) = (-1)^q\big[(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$, where $(\boldsymbol{\partial}_q)_*(\boldsymbol{\lambda})$ is the derivative of the boundary map of the complex $(\mathbf{C}_\bullet, \boldsymbol{\partial}_\bullet(\mathbf{t}))$, thereby proving Proposition 3.2 and hence Theorem 3.1 as well.

The proof is by induction on $d = n-\ell$, the cohomological dimension of the group $P_{n,\ell}$, (resp., the rank of the discriminantal arrangement $\mathcal{A}_{n,\ell}$).

In the case $d = 1$, the complexes $A^\bullet$ and $\mathbf{C}_\bullet$ are given by

$$A^0 \xrightarrow{\ \mu^0(\boldsymbol{\lambda})\ } A^1 \quad \text{and} \quad \mathbf{C}_1 \xrightarrow{\ \boldsymbol{\partial}_1(\mathbf{t})\ } \mathbf{C}_0$$

respectively, where $A^0 = \mathbf{C}_0 = \mathbb{C}$, $A^1 = \oplus_{i<n}\mathbb{C}a_{i,n}$, and $\mathbf{C}_1 = \mathbb{C}^{n-1}$. The boundary maps are $\mu^0(\boldsymbol{\lambda}) : 1 \mapsto \sum_{i<n}\lambda_{i,n}\cdot a_{i,n}$ and $\boldsymbol{\partial}_1(\mathbf{t}) = \begin{pmatrix} t_{1,n}-1 & \cdots & t_{n-1,n}-1\end{pmatrix}^\top$. Identifying A^1 and $\mathbf{C}_1$ in the obvious manner, we have $\mu^0(\boldsymbol{\lambda}) = (-1)^0\big[(\boldsymbol{\partial}_1)_*(\boldsymbol{\lambda})\big]^\top$.

In the general case, we identify A^q and $\mathbf{C}_q$ in an analogous manner. In particular, the rows and columns of the matrix of the boundary map $\boldsymbol{\partial}_{q+1}(\mathbf{t}) : \mathbf{C}_{q+1} \to \mathbf{C}_q$ are indexed by basis elements $a_{I,J}$ of A^{q+1} and A^q, or simply by the underlying sets I and J, respectively. To show that $\mu^q(\boldsymbol{\lambda}) = (-1)^q\big[(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$, we make use of the decomposition of the complex $A^\bullet$ established in Proposition 1.7, and that of $\mathbf{C}_\bullet$ stemming from the mapping cone decomposition of the resolution $C_\bullet$ described in Remark 2.6. Recall from (1.2) and (3.3) that with respect to these decompositions, the boundary maps may be expressed as

$$\mu^q(\boldsymbol{\lambda}) = \begin{pmatrix} \mu_B^q(\boldsymbol{\lambda}) & 0 \\ \Psi^q(\boldsymbol{\lambda}) & \widehat{\mu}^q(\boldsymbol{\lambda})\end{pmatrix} \quad \text{and} \quad \boldsymbol{\partial}_{q+1}(\mathbf{t}) = \begin{pmatrix} -\boldsymbol{\partial}_q^{\mathbf{D}}(\mathbf{t}) & \boldsymbol{\Xi}_q(\mathbf{t}) \\ 0 & \widehat{\boldsymbol{\partial}}_{q+1}(\mathbf{t})\end{pmatrix}.$$

The maps $\widehat{\mu}^q(\boldsymbol{\lambda})$ and $\widehat{\boldsymbol{\partial}}_{q+1}(\mathbf{t})$ are the boundary maps of the complexes $\widehat{A}^\bullet$ and $\widehat{\mathbf{C}}_\bullet$ arising from the cohomology algebra $A(\mathcal{A}_{n,\ell+1})$ and fundamental group $P_{n,\ell+1}$ of the complement of the discriminantal arrangement $\mathcal{A}_{n,\ell+1}$. So by induction, we have $\widehat{\mu}^q(\boldsymbol{\lambda}) = (-1)^q\big[(\widehat{\boldsymbol{\partial}}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$ for each q. Since the complexes $B^\bullet \cong (\widehat{A}^\bullet)^\ell$ and $\mathbf{D}_\bullet \cong (\widehat{\mathbf{C}}_\bullet)^\ell$ decompose as direct sums, with boundary maps $\mu_B^q(\boldsymbol{\lambda}) = -[\widehat{\mu}^{q-1}(\boldsymbol{\lambda})]^\ell$ and $\boldsymbol{\partial}_q^{\mathbf{D}}(\mathbf{t}) = -[\widehat{\boldsymbol{\partial}}_q(\mathbf{t})]^\ell$ the inductive hypothesis also implies that

$$\begin{aligned}\mu_B^q(\boldsymbol{\lambda}) &= -\Big[(-1)^{q-1}\big[(\widehat{\boldsymbol{\partial}}_q)_*(\boldsymbol{\lambda})\big]^\top\Big]^\ell \\ &= (-1)^q\Big[\big[(\widehat{\boldsymbol{\partial}}_q)_*(\boldsymbol{\lambda})\big]^\ell\Big]^\top = (-1)^q\big[-(\boldsymbol{\partial}_q^{\mathbf{D}})_*(\boldsymbol{\lambda})\big]^\top.\end{aligned}$$

Thus it remains to show that $\Psi^q(\boldsymbol{\lambda}) = (-1)^q\big[(\Xi_q)_*(\boldsymbol{\lambda})\big]^\top$. For this, it suffices to show that the restriction $\Psi^q_J(\boldsymbol{\lambda}) : A^q_J \to A^{q+1}_{\{\ell+1,J\}}$ of $\Psi^q(\boldsymbol{\lambda})$ is dual to the derivative of the summand $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t}) : \mathbf{C}^{\{\ell+1,J\}}_{q+1} \to \mathbf{C}^J_q$ of $\Xi_q(\mathbf{t})$ for each $J = \{j_1, \dots, j_q\} \subseteq [\ell+2, n]$. As noted in (1.3), the matrix of $\Psi^q_J(\boldsymbol{\lambda})$ is $d_J \times \ell \cdot d_J$ with $d_J \times d_J$ blocks $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})$, where $d_J = (j_1 - 1)\cdots(j_q - 1)$. Similarly, from the discussion in section 3.5, we have that the matrix of $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t})$ is $\ell \cdot d_J \times d_J$ with $d_J \times d_J$ blocks $(-1)^q\big(\boldsymbol{\rho}_J(\gamma_{m,\ell+1})(\mathbf{t}) - \mathbb{I}_{d_J}\big)$. Comparing the formulas obtained in Theorem 1.12 and Theorem 3.6, we see that $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda}) = \big[\boldsymbol{\rho}_J(\gamma_{m,\ell+1})_*(\boldsymbol{\lambda})\big]^\top$. It follows readily that

$$\Psi^q_J(\boldsymbol{\lambda}) = (-1)^q\big[\big(\boldsymbol{\Delta}_{\{\ell+1,J\}}\big)_*(\boldsymbol{\lambda})\big]^\top,$$

completing the proof.

§4. Cohomology Support Loci and Resonant Varieties

In an immediate application of Theorem 3.1, we establish the relationship between the cohomology support loci of the complement of the discriminantal arrangement $\mathcal{A}_{n,\ell}$ and the resonant varieties of its Orlik-Solomon algebra.

Recall that each point $\mathbf{t} \in (\mathbb{C}^*)^N$ gives rise to a local system $\mathcal{L} = \mathcal{L}_\mathbf{t}$ on the complement $M_{n,\ell}$ of the arrangement $\mathcal{A}_{n,\ell}$. For sufficiently generic $\mathbf{t}$, the cohomology $H^k(M_{n,\ell}, \mathcal{L}_\mathbf{t})$ vanishes (for $k < n - \ell$), see for instance [21, 10]. Those $\mathbf{t}$ for which $H^k(M_{n,\ell}; \mathcal{L}_\mathbf{t})$ does not vanish comprise the cohomology support loci

$$\Sigma^k_m(M_{n,\ell}) = \{\mathbf{t} \in (\mathbb{C}^*)^N \mid \dim H^k(M_{n,\ell}; \mathcal{L}_\mathbf{t}) \geq m\}.$$

These loci are algebraic subvarieties of $(\mathbb{C}^*)^N$, which, since $M_{n,\ell}$ is a $K(P_{n,\ell}, 1)$-space, are invariants of the group $P_{n,\ell}$.

Similarly, each point $\boldsymbol{\lambda} \in \mathbb{C}^N$ gives rise to an element $\omega = \omega_\boldsymbol{\lambda} \in A^1$ of the Orlik-Solomon algebra of the arrangement $\mathcal{A}_{n,\ell}$. For sufficiently generic $\boldsymbol{\lambda}$, the cohomology $H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ vanishes (for $k < n - \ell$), see [31, 16]. Those $\boldsymbol{\lambda}$ for which $H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ does not vanish comprise the resonant varieties

$$\mathcal{R}^m_k(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^N \mid \dim H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda})) \geq m\}.$$

These subvarieties of $\mathbb{C}^N$ are invariants of the Orlik-Solomon algebra A of $\mathcal{A}_{n,\ell}$.

Recall that $\mathbf{1} = (1, \dots, 1)$ denotes the identity element of $(\mathbb{C}^*)^N$.

Theorem 4.1. *Let $\mathcal{A}_{n,\ell}$ be a discriminantal arrangement with complement $M_{n,\ell}$ and Orlik-Solomon algebra A. Then for each k and each m, the resonant variety $\mathcal{R}^m_k(A)$ coincides with the tangent cone of the cohomology support locus $\Sigma^k_m(M_{n,\ell})$ at the point* $\mathbf{1}$.

Proof. For each $\mathbf{t} \in (\mathbb{C}^*)^N$, the cohomology of $M_{n,\ell}$ with coefficients in the local system $\mathcal{L}_\mathbf{t}$ is isomorphic to that of the complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$. So $\mathbf{t} \in \Sigma^k_m(M_{n,\ell})$ if and only if $\dim H^k(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}) \geq m$. An exercise in linear algebra shows that

$$\Sigma^k_m(M_{n,\ell}) = \{\mathbf{t} \in (\mathbb{C}^*)^N \mid \operatorname{rank} \boldsymbol{\delta}^{k-1}(\mathbf{t}) + \operatorname{rank} \boldsymbol{\delta}^k(\mathbf{t}) \leq \dim \mathbf{C}^k - m\}.$$

For $\boldsymbol{\lambda} \in \mathbb{C}^N$, we have $\boldsymbol{\lambda} \in \mathcal{R}^m_k$ if $\dim H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda})) \geq m$. So, as above,

$$\mathcal{R}^m_k(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^N \mid \operatorname{rank} \mu^{k-1}(\boldsymbol{\lambda}) + \operatorname{rank} \mu^k(\boldsymbol{\lambda}) \leq \dim A^k - m\}.$$

By Theorem 3.1, $\dim A^k = \dim \mathbf{C}^k$ and $\mu^k(\boldsymbol{\lambda}) = \boldsymbol{\delta}^k_*(\boldsymbol{\lambda})$ for each k. Thus,

$$\mathcal{R}^m_k(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^N \mid \operatorname{rank} \boldsymbol{\delta}^{k-1}_*(\boldsymbol{\lambda}) + \operatorname{rank} \boldsymbol{\delta}^k_*(\boldsymbol{\lambda}) \leq \dim \mathbf{C}^k - m\},$$

and the result follows. Q.E.D.

The cohomology support loci are known to be unions of torsion-translated subtori of $(\mathbb{C}^*)^N$, see [3]. In particular, all irreducible components of $\Sigma^k_m(M_{n,\ell})$ passing through $\mathbf{1}$ are subtori of $(\mathbb{C}^*)^N$. Consequently, all irreducible components of the tangent cone are linear subspaces of $\mathbb{C}^N$. So we have the following.

Corollary 4.2. *For each k and each m, the resonant variety $\mathcal{R}^m_k(A)$ is the union of an arrangement of subspaces in $\mathbb{C}^N$.*

Remark 4.3. For $k = 1$, Theorem 4.1 and Corollary 4.2 hold for any arrangement $\mathcal{A}$, see [11, 22, 23]. In particular, as conjectured by Falk [16, Conjecture 4.7], the resonant variety $\mathcal{R}^m_1(A(\mathcal{A}))$ is the union of a subspace arrangement. Thus, Corollary 4.2 above may be viewed as resolving positively a strong form of this conjecture in the case where $\mathcal{A} = \mathcal{A}_{n,\ell}$ is a discriminantal arrangement.

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Department of Mathematics,
Louisiana State University,
Baton Rouge, LA 70803
U. S. A.
cohen@math.lsu.edu
http://math.lsu.edu/~cohen