NEGATIVE DEPENDENCE

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1. Introduction

Various concepts of positive dependence have been considered in the literature. Many of these concepts have been developed to obtain conditions on a random vector $\underline{T} = (T_1,\ldots,T_n)$ such that

$$(1) \qquad P\{T_1 > t_1,\ldots,T_n > t_n\} \geq \prod_{i=1}^{n} P\{T_i > t_i\} \text{ for all } t_1,\ldots,t_n \quad .$$

See Lehmann (1966) for a discussion of the bivariate case and Barlow and Proschan (1975) and Block and Ting (1981) for details in the multivariate case.

There are many distributions such as the multinomial and the Dirichlet for which the reverse inequality holds (see Mallows (1968) and Jogdeo and Patil (1975)). However no systematic study of negative dependence concepts in the $n \geq 3$ case was attempted until recently (the bivariate case was considered by Lehmann (1966)).

In this paper we discuss various multivariate concepts of negative dependence. Many of these concepts arose out of discussions which were begun at the NSF/CBMS conference held in Columbia, Missouri in June 1979 at which Frank

Proschan was the principal speaker. Sections 2 and 3 contain results concerning positive dependence. Results on the positive dependence of the multivariate normal are given in Section 4. Section 5 deals with some basic negative dependence results in the bivariate case. Various concepts of multivariate negative dependence are compared in Section 6. These are based on topics considered in Block, Savits and Shaked (1982), Ebrahimi and Ghosh (1981) and Karlin and Rinott (1980). One of the fundamental results discussed is that if a distribution satisfies an intuitive structural condition called Condition N, then it satisfies all of the other conditions mentioned including the fundamental negative dependence inequality given by (3). Condition N is satisfied by the multinomial, hypergeometric and Dirichlet distribution as well as several others. Section 7 concludes with two other negative dependence conditions. One is a concept of negative association introduced by Jogdeo and Proschan (1981) and a second deals with a condition of stochastic ordering due to Block, Savits and Shaked (1981).

2. Positive Dependence - Bivariate Case

For the bivariate random vector (S,T) where $\mathrm{cov}(S,T) \geq 0$ we often have or want to show that

$$P\{S > s,\ T > t\} \geq P\{S > s\}\ P\{T > t\} \text{ for all } s,t \quad . \tag{2}$$

If this inequality holds we have a lower bound for the joint survival function in terms of the marginal survival functions. We say that (S,T) is positive quadrant dependent (PQD) if (2) holds. Condition (2) is often difficult to verify directly for specific bivariate distributions, so that a condition which implies (2) and is easy to check is useful. One such condition is that (S,T) has a joint density $f(s,t)$ which is TP_2, i.e.,

$$f(s,t)\ f(s',t') \geq f(s,t')\ f(s',t) \text{ for all } t < t',\ s < s' \quad .$$

This has been also called positive likelihood ratio dependent by Lehmann (1966) who establishes the following relationships. He shows that if (S,T) has a TP_2 density then $P\{T > t | S = s\}$ increases in s (called positive regression dependence). This last condition implies that (S,T) is PQD which in turn implies that $\mathrm{cov}(S,T) \geq 0$.

3. Positive Dependence - Multivariate Case

A parallel multivariate theory has been developed (see Barlow and Proschan (1975)). The various notions are:

1) $\underset{\sim}{T} = (T_1, \ldots, T_n)$ has a joint density which is TP_2 (totally positive of order 2) in pairs, i.e., the joint density $f(t_1, \ldots, t_n)$ is TP_2 in t_i and t_j for any $i \neq j$ in $\{1,2,\ldots,n\}$ when the remaining variables are fixed (there are also some conditions on the support of f, see Kempermann (1977));

2) $P\{T_i > t | T_1 = t_1, \ldots, T_{i-1} = t_{i-1}\}$ increases in $t_1, \ldots, t_{i-1}$ for $i = 2, \ldots, n$ and for all t;

3) $\underset{\sim}{T}$ is associated which means that $\mathrm{cov}(f(\underset{\sim}{T}), g(\underset{\sim}{T})) \geq 0$ for all functions f and g which increase componentwise;

4) $P\{T_1 > t_1, \ldots, T_n > t_n\} \geq \prod_{i=1}^{n} P\{T_i > t_i\}$ for all $t_1, t_2, \ldots, t_n$;

5) $\mathrm{cov}(T_i, T_j) \geq 0$ for all $i \neq j$ in $\{1,2,\ldots,n\}$.

Each of these conditions implies the following condition. The most important result is that 1) implies 4). For recent developments concerning multivariate positive dependence conditions see Block and Ting (1981). A new concept of positive dependence by stochastic ordering has been given by Block, Savits and Shaked (1981).

4. Positive Dependence of the Multivariate Normal

The results of the previous section have been applied to the multivariate normal distribution. Conditions on the covariance matrix have been given so that the various positive dependence conditions are satisfied. These are summarized below.

Let $\underline{X} = (X_1,\ldots,X_n)$ be a random vector having a multivariate normal distribution with mean vector $\underline{0}$ and covariance matrix $\underline{\Sigma}$. The strongest condition, TP_2 in pairs, is satisfied if and only if $\underline{\Lambda} = \underline{\Sigma}^{-1}$ exists and has nonpositive off diagonal elements. A weaker condition is that $\underline{\Sigma}$ has nonnegative elements and this is necessary and sufficient for $\underline{X}$ to be associated as was recently shown by Pitt (1982) (see also Jogdeo and Perlman (1981)). Since condition 4) implies condition 5), it is also clear, because of the previous result, that the nonnegativity of the elements of $\underline{\Sigma}$ is also necessary and sufficient for 4) to hold.

Summarizing, for the multivariate normal the main result is that if $\operatorname{cov}(X_i,X_j) \geq 0$ for all i,j in $\{1,\ldots,n\}$ then

$$P\{X_1 > x_1,\ldots,X_n > x_n\} \geq \prod_{i=1}^{n} P\{X_i > x_i\} \text{ for all } x_1,\ldots,x_n \quad .$$

This result is not true for most multivariate distributions. For related results concerning the dependence of $(|X_1|,|X_2|,\ldots,|X_n|)$ and the comparison of two normal vectors see Block and Sampson (1982).

5. Negative Dependence - Bivariate Case

The results of this section are practically identical to those of the bivariate positive dependence case and can be obtained by reversing inequalities and monotonicities (or by considering (S,-T) where (S,T) satisfies one of the positive dependence conditions). See Lehmann (1966) for details.

The negative dependence analog of TP_2 is RR_2 (or reverse rule of order 2). The vector (S,T) is said to be RR_2 if (S,T) has density f(s,t) which satisfies

$$f(s,t)\ f(s',t') \leq f(s,t')\ f(s',t) \text{ for all } t < t',\ s < s' \ .$$

This condition implies that $P\{T > t | S = s\}$ decreases in s for all t. The latter condition implies that $P\{S > s,\ T > t\} \leq P\{S > s\}\ P\{T > t\}$ for all s and t from which it follows that $\operatorname{cov}(S,T) \leq 0$.

6. Negative Dependence - Multivariate Case

Until recently negative dependence concepts in the $n \geq 3$ case had not been widely studied. One reason for this was that the structure of negative dependence was not generally understood. Recent studies have shown that negative dependence is quite different from positive dependence.

Ebrahimi and Ghosh (1981) have reversed inequalities and directions of monotonicities for the multivariate positive dependence concepts and compared resulting concepts. These analogs of the positive dependence concepts do not in general imply the condition

$$\text{(3)} \qquad P\{T_1 > t_1, \ldots, T_n > t_n\} \leq \prod_{i=1}^{n} P\{T_i > t_i\} \text{ for all } t_1, \ldots, t_n \quad .$$

The condition that $P\{T_i > t_i | T_1 > t_1, \ldots, T_{i-1} > t_{i-1}\}$ is decreasing in $t_1, \ldots, t_{i-1}$ for all t_i, $i = 2, \ldots, n$ does imply (3), but it is no easier to check than (3) itself.

We now illustrate some of the problems in obtaining a condition which implies (3). Since TP_2 in pairs implies (1), it might seem reasonable to hope that the negative dependence analog given by Definition 1 below would imply (3).

Definition 1: The random vector $\underline{T} = (T_1, \ldots, T_n)$ is said to be RR_2 in pairs if it has joint density $f(t_1, \ldots, t_n)$ which is RR_2 in any two variables when the others are fixed.

This definition does not imply (3). Moreover not even the marginals of $\underline{T}$ need to be RR_2 in pairs. A modified defintion was given by Ebrahimi and Ghosh (1981).

Definition 2: The random vector $\underline{T} = (T_1, \ldots, T_n)$ is said to be completely RR_2 in pairs if the joint density and all of its marginals are RR_2 in pairs.

It remains an unsolved problem as to whether this implies (3). A further attempt to modify this definition was made by Block, Savits and Shaked (1982). This still stronger definition follows.

Definition 3: The random vector $\underline{T} = (T_1,\ldots,T_n)$ has an RR_2 in pairs measure if $\mu(I_1,I_2,\ldots,I_n) = \int_{I_1} \cdots \int_{I_n} d\,P\{T_1 \le t_1,\ldots,T_n \le t_n\}$ is RR_2 in pairs, e.g., $\mu(I_1,I_2,\ldots,I_n)$ is RR_2 in I_1 and I_2 if

$$\mu(I_1,I_2,I_3,\ldots,I_n)\ \mu(I_1',\ I_2',\ I_3,\ldots,I_n)$$
$$\le \mu(I_1,I_2',I_3,\ldots,I_n)\ \mu(I_1',I_2,I_3,\ldots,I_n)$$

for all intervals $I_1 < I_1'$ and $I_2 < I_2'$ where $I_i < I_i'$ means every point in I_i is less than every point in I_i'.

Block, Savits and Shaked (1982) prove that if $\underline{T}$ satisfies Definition 3 then $\underline{T}$ satisfies (3). The only problem is that Definition 3 is not easy to check. In the same paper these authors propose a structural condition, which many standard multivariate distributions satisfy. This is Condition N below.

Condition N: The random vector $\underline{T} = (T_1,\ldots,T_n)$ is such that there exist independent r.v.'s $S_0,S_1,\ldots,S_n$ having PF_2 densities (i.e., the densities $f_i(s_i)$, $i=0,\ldots,n$ satisfy $f_i(t_1-t_2)$ is TP_2 in (t_1,t_2)) and there exists a real number s such that

$$(T_1,\ldots,T_n) \overset{st}{=} [(S_1,\ldots,S_n) \mid S_0 + S_1 + \cdots + S_n = s] .$$

Note - For a discussion of PF_2 (Polya frequency of order 2) densities see Barlow and Proschan (1975). Densities with these properties have increasing failure rates.

Essentially a random vector $\underline{T}$ satisfying this condition is like the multinomial distribution, i.e., the sum of the components of the random vector is fixed. Remarkably it turns out that many of the standard multivariate distributions which have negatively correlated univariate marginals satisfy this condition. It can be shown that the multinomial, the symmetric normal with negative correlations, the multivariate hypergeometric and the Dirichlet all satisfy this definition.

Example: The Dirichlet distribution has the same distribution as the conditional distribution of a sum of certain independent gamma distributions given that their sum is equal to 1 and it is easy to show that those gammas have PF_2 densities. Thus, the Dirichlet satisfies Condition N.

Furthermore, Condition N can be seen to imply Definition 3 and so (3) is satisfied. Thus Condition N is an intuitive condition, which is easy to check, and which implies the inequality (3).

Karlin and Rinott (1980) have also proposed a definition which implies the basic inequality (3). Their definition is also a strengthening of the RR_2 condition.

Definition 4: The random vector $\underline{T} = (T_1, \ldots, T_n)$ having joint density f is called strongly multivariate RR_2 (S-MRR_2) if for any set of PF_2 functions $\phi_1, \ldots, \phi_{n-k}$, $2 \leq k \leq n$ the function

$$g(x_{v_1}, \ldots, x_{v_k}) = \int \cdots \int f(x_1, \ldots, x_n)\, \phi_1(x_{j_1}) \ldots \phi_{n-k}(x_{j_{n-k}})\, dx_{j_1}, \ldots, dx_{j_{n-k}}$$

is RR_2 in pairs of the unintegrated variables $x_{v_1}, \ldots, x_{v_k}$.

Note - The case $k = n$ in the above definition corresponds to assuming the density is strictly positive and RR_2 in pairs on a product set in $\mathbb{R}^n$.

As mentioned above if $\underline{T}$ is S-MRR_2 then $\underline{T}$ satisfies (3), but as can be seen from the computations in Karlin and Rinott (1980) this definition is cumbersome to check. Fortunately, if $\underline{T}$ satisfies Condition N, then it is S-MRR_2 (see Block, Savits and Shaked (1982)).

Summarizing the relationships, Condition N is stronger than Definition i which is stronger than Definition $i - 1$ for $i = 2,3,4$ and Definition 3 implies (3).

7. Other Negative Dependence Conditions

One of the problems of negative dependence was that there did not seem to be a natural analog of the positive dependence concept of association. Simply reversing inequalities or monotonicities in the definition of association leads to concepts which have certain anomalies. For example, assume

$$\text{cov}(f(\underline{T}),\ g(\underline{T})) \leq 0 \text{ for all increasing } f,g\ .$$

Then for f and g which are functions only of the i^{th} variables it must be that $\text{cov}(f(\underline{T}), g(\underline{T})) = 0$. A similar circumstance occurs when f and g are taken to be the same function.

Jogdeo and Proschan (1981) and Alam and Saxena (1981) have proposed a definition which avoids these difficulties. They define $\underline{T} = (T_1,\ldots,T_n)$ to be negatively associated if

$$\text{cov}(f(T_i,\ i \in \Lambda),\ g(T_i,\ i \in \bar{\Lambda})) \leq 0$$

where $\Lambda \subset \{1,2,\ldots,n\}$ and $\bar{\Lambda} = \{1,2,\ldots,n\}\setminus\Lambda$ and f and g are any nondecreasing functions of the appropriate number of variables.

Jogdeo and Proschan (1981) have shown that if $\underline{T}$ is negatively associated then random variables defined as increasing functions on disjoint subsets of $\underline{T}$ are negatively associated. The negative association condition can be seen to imply (3), but the condition is not easy to check. However, these authors have essentially shown that negative association is implied by Condition N. Thus, all of the distributions which satisfy Condition N are negatively associated.

One other condition has been proposed by Block, Savits and Shaked (1981). One of the motivations was to obtain a condition satisfied by a wider class of multivariate normals than satisfies Condition N. Although Condition N is a natural condition for distributions like the multinomial and the Dirichlet it is not natural for multivariate normal distributions. The new condition resembles, but is more general than, a condition used by Mallows (1968) and by Jogdeo and Patil (1975) to show (3) for specific distributions. The condition used by these authors was $P\{X_2 > x_2,\ldots,X_n > x_n | X_1 > x_1\}$ is decreasing in x_1 for all $x_2,\ldots,x_n$. The new condition called negatively dependent by stochastic ordering is that $E(f(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)|X_i = x_i)$ is decreasing in x_i for all nondecreasing functions f and all $i = 1,\ldots,n$.

Several natural models and distributions which satisfy this condition are:

1) Condition N (actually a slightly stronger version);

2) $\underline{T} = (T_1,\ldots,T_n)$ such that there exists independent, identically distributed, and continuous $X_1,\ldots,X_n$ and a real number z such that

$$\underline{T} \overset{st}{=} [(X_1,\ldots,X_n) \mid \min(X_1,\ldots,X_n) = z] ;$$

3) $\underline{T}$ is such that there exist independent and identically normally distributed $X_1,\ldots,X_n$ such that

$$\underline{T} \overset{st}{=} (X_1 - \overline{X},\ldots,X_n - \overline{X}) ;$$

4) $\underline{T}$ is multivariate normal with nonpositive correlations;

5) All of the distributions mentioned in Section 6.

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