CHAPTER 2

ASYMPTOTIC PROPERTIES OF WEIGHTED EMPIRICALS

2.1. INTRODUCTION.

Let, for each $n \geq 1$, $\eta_{n1}, \ldots, \eta_{nn}$ be independent r.v.'s taking values in [0, 1] with respective d.f.'s $G_{n1}, \ldots, G_{nn}$ and $d_{n1}, \ldots, d_{nn}$ be real numbers. Define

$$W_d(t) = \sum_{i=1}^{n} d_{ni}\{I(\eta_{ni} \leq t) - G_{ni}(t)\}, \qquad 0 \leq t \leq 1. \tag{1}$$

Observe that both V_h of (1.4.1) and W_d belong to $\mathbb{D}[0, 1]$ for each n and for any triangular arrays $\{h_{ni}, 1 \leq i \leq n\}$ and $\{d_{ni}, 1 \leq i \leq n\}$.

In this chapter we first prove certain weak convergence results about suitably standardized W_d and V_h processes. This is done in Sections 2.2a and 2.2b, respectively. Sections 2.3.1 uses the asymptotic continuity of a certain W_d-process to obtain the asymptotic uniform linearity result about $\mathbf{V}(\cdot, \mathbf{u})$ of (1.1.2) in $\mathbf{u}$. Analogous result for $\mathbf{T}(\cdot, \mathbf{u})$ of (1.4.3) uses the asymptotic continuity of a certain V_h-process and is proved in Section 7.2.

A proof of an exponential inequality for a stopped martingale with bounded differences due to Johnson, Schechtman and Zinn (1985) and Levental (1989) is included in Section 2.2b. This inequality is of general interest and an important tool needed to carry out a chaining argument pertaining to the weak convergence of V_h.

Section 2.4 treats laws of iterated logarithm pertaining to W_d, the weak convergence of W_d when $\{\eta_i\}$ are in $[0, 1]^p$, the weak convergence of W_d w.r.t. some other metrics when $\{\eta_i\}$ are in [0, 1], an embedding result for W_d when $\{\eta_i\}$ are i.i.d. uniform [0, 1] r.v.'s, and a proof of its martingale property. It also includes an exponential inequality for the tail probabilities of w.e.p.'s of independent r.v.'s. This inequality is an extension of the well celebrated Dvoretzky, Kiefer and Wolfowitz (1956) inequality for the ordinary empirical process. These results are stated for the sake of completeness, without proofs. They are not used in the subsequent sections.

2.2. WEAK CONVERGENCE

2.2a. W_d – Processes.

In this section we give two proofs of the weak convergence of suitably standardized $\{W_d\}$ to a limit in $\mathbb{C}[0, 1]$. Accordingly, let

$$(1) \qquad G_d(t) := \sum_{i=1}^{n} d_{ni}^2\, G_{ni}(t), \qquad 0 \le t \le 1,$$

and

$$(2) \qquad \mathcal{C}_d(s, t) := \sum_{i=1}^{n} d_{ni}^2\, [G_{ni}(s \wedge t) - G_{ni}(s)\, G_{ni}(t)], \qquad 0 \le s, t \le 1.$$

Let $\mathscr{d}$ denote the *supremum metric.*

Theorem 2.2a.1. *Let* $\{\eta_{ni}\}$, $\{d_{ni}\}$ *and* $\{G_{ni}\}$ *be as in Section* 2.1. *In addition assume that the following hold:*

$$\text{(N1)} \qquad \tau_d^2 := \sum_{i=1}^{n} d_{ni}^2 = 1, \ \textit{for all } n \ge 1.$$

$$\text{(N2)} \qquad \max_{1 \le i \le n} d_{ni}^2 \longrightarrow 0.$$

$$\text{(C)} \qquad \lim_{\delta \to 0} \limsup_n \sup_{0 \le t \le 1-\delta} [G_d(t + \delta) - G_d(t)] = 0.$$

Then, for every $\epsilon > 0$,

$$\text{(i)} \qquad \lim_{\delta \to 0} \limsup_n P\Big(\sup_{|t-s| < \delta} |W_d(t) - W_d(s)| > \epsilon\Big) = 0.$$

(ii) *Moreover,* $W_d \Rightarrow$ *some* W *on* $(\mathbb{D}[0, 1], \mathscr{d})$ *if and only if for every* $0 \le s, t \le 1$, $\mathcal{C}_d(s, t)$ *converges to some covariance function* $\mathcal{C}(s, t)$.

In this case W *is necessarily a continuous Gaussian process with covariance function* $\mathcal{C}$ *and* $W(0) = 0 = W(1)$.

Remark 2.2a.1. Perhaps a remark about the labeling of the conditions is in order. The letter **N** in **(N1)** and **(N2)** stands for **Noether** who was the first person to use these conditions to obtain the asymptotic normality of certain weighted sums of r.v.'s. See Noether (1949).

The letter **C** in the condition **(C)** stands for the specified *continuity* of the sequence $\{G_d\}$. Observe that the d.f.'s $\{G_i\}$ need not be continuous for each i and n; only $\{G_d\}$ needs to be equicontinuous in the sense of **(C)**. Of course if $\{\eta_i\}$ are i.i.d. G then, because of **(N1)**, **(C)** is equivalent to the continuity of G. □

The proof of the theorem will follow from the following *two* lemmas.

Lemma 2.2a.1. *For any* $0 \le s \le t \le u \le 1$ *and each* $n \ge 1$

$$E|W_d(t) - W_d(s)|^2 |W_d(u) - W_d(t)|^2$$

$$(3a) \qquad \le 3\, [G_d(u) - G_d(t)][G_d(t) - G_d(s)].$$

(3b) $$\le 3[G_d(u) - G_d(s)]^2.$$

Proof. Fix $0 \le s, t, u \le 1$ and let

$$p_i = G_i(t) - G_i(s), \qquad q_i = G_i(u) - G_i(t),$$

$$\alpha_i = I(s < \eta_i \le t) - p_i, \qquad \beta_i = I(t < \eta_i \le u) - q_i, \quad 1 \le i \le n.$$

Observe that $E\,\alpha_i = 0 = E\,\beta_j$ for all $1 \le i,\ j \le n$, $\{\alpha_i\}$ are independent as are $\{\beta_j\}$ and that α_i is independent of β_j for $i \neq j$. Moreover,

$$W_d(t) - W_d(s) = \Sigma_i\, d_i\alpha_i\,, \qquad W_d(u) - W_d(t) = \Sigma_i\, d_i\beta_i.$$

Now expand and multiply the quadratics and use the above facts to obtain

(4) $$E|W_d(t) - W_d(s)|^2|W_d(u) - W_d(t)|^2$$

$$= \sum_i d_i^4\, E\alpha_i^2\beta_i^2 + \sum_{i\neq j}\sum d_i^2\, d_j^2\, E\alpha_i^2\, E\beta_j^2 + 2\sum_{i\neq j}\sum d_i^2\, d_j^2\, E(\alpha_i\beta_i)\, E(\alpha_j\,\beta_j).$$

But

$$E\alpha_i^2 = p_i(1-p_i), \qquad E\beta_j^2 = q_j(1-q_j),$$

$$\begin{aligned} E\alpha_i^2\beta_i^2 &= (1-p_i)^2\, p_iq_i^2 + (1-q_i)^2\, q_ip_i^2 + p_iq_i(1-q_i-p_i) \\ &\le \{(1-p_i) + (1-q_i) + (1-q_i-p_i)\}\, p_iq_i \\ &\le 3p_iq_i, \end{aligned}$$

$$\begin{aligned} E(\alpha_i\beta_i) &= -(1-p_i)\, p_iq_i - (1-q_i)\, q_ip_i + p_iq_i\,(1-q_i-p_i) \\ &= -p_iq_i\,, \qquad 1 \le i \le n, \quad 1 \le j \le n. \end{aligned}$$

Therefore,

(5) $$\text{LHS (4)} \le 3\{\Sigma_i\, d_i^4\, p_iq_i + \sum_{i\neq j}\sum d_i^2d_j^2\, p_iq_j\} = 3[\Sigma_i\, d_i^2p_i]\,[\Sigma_j\, d_j^2\, q_j].$$

This completes the proof of (3a), in view of the definition of $\{p_i, q_j\}$. That of (3b) follows from (3a), (1) and the monotonicity of the G_i, $1 \le i \le n$. □

Lemma 2.2a.2. *For every* $\epsilon > 0$ *and* $s \le u$,

(6) $$P[\sup_{s\le t\le u} |W_d(t) - W_d(s)| \ge \epsilon]$$
$$\le \kappa\,\epsilon^{-4}\,[G_d(u) - G_d(s)]^2 + P[|W_d(u) - W_d(s)| \ge \epsilon/2]$$

where κ *does not depend on* ϵ, n *or on any underlying quantity.*

Proof. Let $\delta = u - s$, $m \geq 1$ be an integer,

$$\xi_j = W_d((j/m)\delta + s) - W_d(((j-1)/m)\delta + s), \quad 1 \leq j \leq m, \tag{7}$$
$$S_k = \sum_{j=1}^{k} \xi_j, \qquad M_m = \max_{1 \leq k \leq m} |S_k|.$$

The right continuity of W_d implies that for each n and each sample path, $M_m \longrightarrow \sup\{|W_d(t) - W_d(s)|;\ s \leq t \leq u\}$ as $m \longrightarrow \infty$, w.p.1. In view of Lemma 2.2a.1, Lemma A.1 in the Appendix is applicable to the above r.v.'s $\{\xi_j\}$ with $\gamma = 2$, $\alpha = 1$ and

$$u_j = 3^{1/2}\{G_d((j/m)\delta + s) - G_d(((j-1)/m)\delta + s)\}, \quad 1 \leq j \leq m.$$

Hence (6) follows from that lemma and the right continuity of W_d. □

Proof of Theorem 2.2a.1. For a $\delta > 0$, let $r = [\delta^{-1}]$, the greatest integer less than or equal to $1/\delta$. Define $t_j = j\delta$, $1 \leq j \leq r$ and $t_0 = 0$. Let $\Gamma_j = W_d(t_j) - W_d(t_{j-1})$, $1 \leq j \leq r$. Then

$$\begin{aligned}
&P(\sup_{|t-s|<\delta} |W_d(s) - W_d(s)| \geq \epsilon) \\
&\leq \sum_{j=1}^{r} P[\sup_{t_{j-1} \leq s \leq t_j} |W_d(s) - W_d(t_{j-1})| \geq \epsilon/3] \\
&\leq \kappa\epsilon^{-2} \sum_{j=1}^{r} [G_d(t_j) - G_d(t_{j-1})]^2 + \sum_{j=1}^{r} P[|\Gamma_j| \geq \epsilon/6] \\
&\leq \kappa\epsilon^{-2} \sup_{0 \leq t \leq 1-\delta} [G_d(t+\delta) - G_d(t)] + \sum_{j=1}^{r} P[|\Gamma_j| \geq \epsilon/6] \\
&= I_n(\delta) + II_n(\delta), \qquad \text{(say)}.
\end{aligned} \tag{8}$$

In the above the first inequality follows from Lemma A.2 of the Appendix, the second inequality follows from Lemma 2.2a.2 above and the last inequality follows because, by (N1),

$$\sum_{j=1}^{r} [G_d(t_j) - G_d(t_{j-1})] \leq G_d(1) = 1. \tag{9}$$

Next, observe that

$$\begin{aligned}
\sigma_j^2 &:= \mathrm{Var}(\Gamma_j) = \Sigma_i\, d_i^2\{G_i(t_j) - G_i(t_{j-1})\}\, \{1 - G_i(t_j) + G_i(t_{j-1})\}, \\
&\leq G_d(t_j) - G_d(t_{j-1}), \qquad 1 \leq j \leq r,
\end{aligned} \tag{10}$$

and, by (9), that

(11) $$\sum_{j=1}^{r} \sigma_j^4 \le \sup_{0\le t\le 1-\delta} [G_d(t+\delta) - G_d(t)], \quad \text{all } r \text{ and all } n.$$

Furthermore, (N1) and (N2) enable one to apply the Lindeberg–Feller Central Limit Theorem (L–F CLT) to conclude that $\sigma_j^{-1}\Gamma_j \xrightarrow[d]{} Z$, Z a N(0, 1) r.v. Therefore, for every $\delta > 0$ (or $r < \infty$)

(12) $$|II_n(\delta) - \sum_{j=1}^{r} P(|Z| \ge (\epsilon/6)\sigma_j^{-1})| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

By the Markov Inequality applied to the summands in the second term of (12) and by (11),

(13) $$\begin{aligned} \limsup_n II_n(\delta) &\le 3 \limsup_n \sum_{j=1}^{r} (6\sigma_j/\epsilon)^4 \qquad (EZ^4 = 3) \\ &\le \kappa\epsilon^{-4} \limsup_n \sup_{0\le t\le 1-\delta} [G_d(t+\delta) - G_d(t)]. \end{aligned}$$

The result (i) now follows from (13), (8) and the assumption (C).

Proof of (ii). Suppose $\mathcal{C}_d \longrightarrow \mathcal{C}$. Let m be a positive integer, $0 \le t_1, \ldots, t_m \le 1$ and $a_1, \ldots, a_m$ be arbitrary but fixed numbers. Consider

(14) $$T_n := \sum_{j=1}^{m} a_j W_d(t_j) = \sum_{i=1}^{n} d_i \nu_i$$

where

$$\nu_i := \sum_{j=1}^{m} a_j\{I(\eta_i \le t_j) - G_i(t_j)\}, \qquad 1 \le i \le n.$$

Note that

(15) $$|\nu_i| \le \sum_{j=1}^{m} |a_j| < \infty, \qquad 1 \le i \le n.$$

Also, $\mathrm{Var}(T_n) \longrightarrow \sigma^2 := \sum_{j=1}^{m}\sum_{r=1}^{m} a_j a_r \mathcal{C}(t_j, t_r)$. In view of (N1) and (N2), the L–F CLT yields that $T_n \xrightarrow[d]{} N(0, 1)$. Hence all finite dimensional distributions of W_d converge weakly to those of a Gaussian process W with the covariance function $\mathcal{C}$ and $W(0) = 0 = W(1)$. In view of (i), this implies that $W_d \Rightarrow W$ in $(\mathbb{D}[0, 1], d)$ with W denoting a continuous Gaussian process tied down at 0 and 1.

Conversely, suppose $W_d \Rightarrow W$. By (i), W is in $\mathbb{C}[0, 1]$. In particular the T_n of (14) converges in distribution to $T := \sum_{j=1}^{m} a_j W(t_j)$. Moreover, (15) and (N1) imply that, for all $n \ge 1$,

$$ET_n^4 = E(\Sigma_i\, d_i \nu_i)^4 = \Sigma_i\, d_i^4\, E\nu_i^4 + 3 \sum_{i \neq j}\sum d_i^2\, d_j^2\, E\nu_i^2\, E\nu_j^2 \leq 3(\sum_{j=1}^{m} |a_j|)^4,$$

Therefore $\{T_n^2, n \geq 1\}$ is uniformly integrable and hence

$$ET_n^2 = \sum_{j=1}^{m} \sum_{k=1}^{m} a_j\, a_k\, \mathcal{C}_d(t_j, t_k) \longrightarrow \sum_{j=1}^{m} \sum_{k=1}^{m} a_j\, a_k\, \mathrm{Cov}[W(t_j), W(t_k)]$$

for any set of numbers $0 \leq \{t_j\} \leq 1$ and any finite real numbers $a_1, \ldots, a_m$. Hence

$$\mathcal{C}_d(s, t) \longrightarrow \mathrm{Cov}[W(s), W(t)] = \mathcal{C}(s, t) \text{ for all } 0 \leq s, t \leq 1.$$

Now repeat the above argument of the "only if" part to conclude that W must be a tied down Gaussian process in $\mathbb{C}[0, 1]$. □

Another set of sufficient conditions for the weak convergence of $\{W_d\}$ is given in the following

Theorem 2.2a.2. *Under the notation of Theorem* 2.2a.1, *suppose that* (N1) *holds. In addition, assume that the following hold:*

(B) $$n \max_{1 \leq i \leq n} d_{ni}^2 = O(1).$$

and

(D) $n^{-1}\, \Sigma_i\, G_{ni}(t) - t$ *is nonincreasing in* t, $0 \leq t \leq 1$, $n \geq 1$.

Then also (i) *and* (ii) *of Theorem* 2.2a.1 *hold.*

Remark 2.2a.1. Clearly (B) implies (N2). Moreover

$$\begin{aligned}
[G_d(t + \delta) - G_d(t)] &\leq n \max_i d_i^2\, [n^{-1}\Sigma_i\, \{G_i(t + \delta) - G_i(t)\}] \\
&= n \max_i d_i^2\, [n^{-1}\Sigma_i\, \{G_i(t + \delta) - (t + \delta)\} \\
&\qquad - n^{-1}\Sigma_i\, \{G_i(t) - t\} + \delta] \\
&\leq n \max_i d_i^2\, \delta, \qquad 0 \leq t \leq 1 - \delta, \qquad \text{by (D)}.
\end{aligned}$$

Thus (B) and (D) together imply (N2) and (C). Hence Theorem 2.2a.2 follows from Theorem 2.2a.1. However, we can also give a different proof of Theorem 2.2a.2 which is direct and quite interesting (see (19) below). This proof will be based on the following *three* lemmas.

Lemma 2.2a.3. *Under* (D), *for all* $n \geq 1$,

(16) $$E|W_d(t) - W_d(s)|^4 \le k_d^2 \{3(t-s)^2 + (t-s)n^{-1}\}, \qquad 0 \le s, t \le 1$$

where $k_d^2 := n \max_{1 \le i \le n} d_{ni}^2$.

Proof. Suppose $0 \le s \le t \le 1$. Let α_i and p_i be as in the proof of Lemma 2.2a.1. Using the independence of $\{\alpha_i\}$ and the fact that $E\alpha_i = 0$ for all $1 \le i \le n$, one obtains

$$\begin{aligned} E|W_d(t) - W_d(s)|^4 &= E(\Sigma d_i\, \alpha_i)^4 \\ &= \Sigma_i d_i^4 E\alpha_i^4 + 3 \mathop{\Sigma\Sigma}_{i \ne j} d_i^2 d_j^2 E\alpha_i^2 E\alpha_j^2 \\ &= \Sigma_i d_i^4 \{E\alpha_i^4 - 3E^2(\alpha_i^2)\} + 3(\Sigma_i d_i^2 E\alpha_i^2)^2 \\ &= \Sigma_i d_i^4 p_i(1-p_i)(1 - 6p_i(1-p_i)) + \\ &\qquad + 3[\Sigma_i d_i^2 p_i(1-p_i)]^2 \end{aligned}$$

(17) $$\le k_d^2 \{n^{-2}\Sigma_i p_i + 3(n^{-1}\Sigma_i p_i)^2\}.$$

But $s \le t$ and (D) imply

$$0 \le n^{-1}\Sigma_i p_i = n^{-1}\Sigma_i [G_i(t) - G_i(s)] \le (t-s).$$

Hence,

$$\text{l.h.s. } (16) \le k_d^2 \{n^{-1}(t-s) + 3(t-s)^2\}, \qquad 0 \le s \le t \le 1.$$

The proof is completed by interchanging the role of s and t in the above argument in the case $t \le s$. □

Next, define, for $(i-1)/n \le t \le i/n$, $1 \le i \le n$,

(18) $$Z_d(t) = W_d((i-1)/n) + \{nt - (i-1)\}[W_d(i/n - W_d(i-1)/n].$$

Lemma 2.2a.4. *The assumption* (D) *implies that*

(19) $$E|Z_d(t) - Z_d(s)|^4 \le k_d^2\, 144|t-s|^2, \quad 0 \le s, t \le 1, \ n \ge 1.$$

If, in addition, (N1) *and* (B) *hold, then,*

(20) $$\sup_t |W_d(t) - Z_d(t)| = o_p(1).$$

Proof. Let $n \ge 1$ and $0 \le s, t \le 1$ be arbitrary but fixed. Choose integers $1 \le i, j \le n$ such that

(21) $$(i-1)/n \le s \le i/n \text{ and } (j-1)/n \le t \le j/n.$$

For the sake of convenience, let

$$\delta_{k,m} := |Z_d(m/n) - Z_d(k/n)| = |W_d(m/n) - W_d(k/n)|,$$

$$b_{k,m} := 4k_d^2\,[(m-k)/n]^2, \qquad m,\ k \text{ integers;}$$

$$\Delta_{u,v} := |Z_d(u) - Z_d(v)|, \qquad 0 \le u,\ v \le 1.$$

From (16),

(22) $$E\delta_{k,m}^4 \le k_d^2\,\{3(m-k)^2/n^2 + n^{-2}\cdot|m-k|\} \le 4k_d^2\,[(m-k)/n]^2 = b_{k,m}.$$

The proof of (19) will be completed by considering the following three cases.

Case 1. $i < j-1$. Then because of (18) and (21),

$$\Delta_{s,t} \le \max\{\delta_{i,j-1},\ \delta_{i,j},\ \delta_{i-1,j-1},\ \delta_{i-1,j}\}$$

which entails that

(23) $$\begin{aligned} E\,\Delta_{s,t}^4 &\le E\{\delta_{i,j-1}^4 + \delta_{i,j}^4 + \delta_{i-1,j-1}^4 + \delta_{i-1,j}^4\} \\ &\le b_{i,j-1} + b_{i,j} + b_{i-1,j-1} + b_{i-1,j} \qquad \text{(by (21))} \\ &\le 4\,b_{i-1,j} = 16\,k_d^2\,[(j-(i-1))/n]^2 \end{aligned}$$

where the last inequality follows from $0 \le j-i-1 < j-i < j-(i-1)$.

Note that (21), $i < j-1$ and i, j integers imply that

(24) $$3(t-s) \ge [j-(i-1)]/n.$$

From (23) and (24) one obtains

(25) $$E\Delta_{s,t}^4 \le 144\,k_d^2\,(t-s)^2.$$

Case 2. $i = j$. In this case $(i-1)/n \le s,\ t \le i/n$. From (18) one has

$$\Delta_{s,t} = n|t-s|\ \delta_{i-1,i}$$

so that from (22)

(26) $$E\Delta_{s,t}^4 < n^4(t-s)^4\cdot 4k_d^2\cdot n^{-2} \le 4k_d^2\,(t-s)^2.$$

The last inequality follows because $n(t-s) \le 1$.

Case 3. $i = j-1$. By the triangle inequality

$$\Delta_{s,t} \le 2\max(\Delta_{s,i/n},\ \Delta_{i/n,t}).$$

Thus by Case 2, applied once with s and i/n and once with i/n and t, one obtains

$$E\,\Delta_{s,t}^4 \le 2^4\{E\Delta_{s,i/n}^4 + E\Delta_{i/n,t}^4\}$$

(27) $$\le 2^6\,k_d^2\,\{(i/n-s)^2 + (t-i/n)^2\} \le 2^7\,k_d^2\,(t-s)^2.$$

In view of (27), (26) and (25), the proof of (19) is complete.

To **prove** (20), let $d_{i+} = \max(0, d_i)$, $d_{i-} = \max(0, -d_i)$. Then one has $d_i = d_{i+} - d_{i-}$. Decompose W_d and Z_d accordingly. Note that $\max(d_{i+}^2, d_{i-}^2) = d_{i+}^2 + d_{i-}^2 = d_i^2$, $1 \le i \le n$. This and (N1) imply that $\tau_{d+} \le 1$, $\tau_{d-} \le 1$. It also implies that if (N2) is satisfied by the $\{d_i\}$ then it is also satisfied by $\{d_{i+}, d_{i-}\}$. By the triangle inequality,

(28) $$\|W_d - Z_d\|_\infty \le \|W_{d+} - Z_{d+}\|_\infty + \|W_{d-} - Z_{d-}\|_\infty.$$

Moreover $d_{i+}\wedge d_{i-} \ge 0$, for all i. Therefore, it is enough to prove (20) for $d_i \ge 0$, $1 \le i \le n$. Accordingly suppose that is the case. Then

(29) $$\|W_d - Z_d\|_\infty \le \mathcal{U}_1 + \mathcal{U}_2,$$

where

(30) $$\mathcal{U}_1 = \max_{1\le i\le n}\ \sup_{(i-1)/n\le t\le i/n} |W_d(t) - W_d((i-1)/n)|,$$

$$\mathcal{U}_2 = \max_{1\le i\le n}\ \sup_{(i-1)/n\le t\le i/n} |W_d(t) - W_d(i/n)|.$$

For $(i-1)/n \le t \le i/n$, and $d_i \ge 0$, $1 \le i \le n$,

$$\begin{aligned}|W_d(t) - W_d(i/n)| &\le |\Sigma_j\, d_j\, I(t < \eta_j \le i/n)| + \Sigma_j\, d_j\, [G_j(i/n) - G_j(t)]\\ &\le |W_d(i/n) - W_d((i-1)/n)| +\\ &\qquad + 2\,\Sigma_j\, d_j\, [G_j(i/n) - G_j((i-1)/n)]\end{aligned}$$

(31) $$\le \delta_{i-1,i} + 2\max_j d_j, \qquad \text{by (D).}$$

Therefore, by (22), (30), (31) and the Markov inequality, for every $\epsilon > 0$ and for n sufficiently large such that $2 \max_j d_j < \epsilon$, the existence of which is guarenteed by (B),

$$\begin{aligned} P(\mathcal{U}_2 \geq \epsilon) &\leq P(\max_i \delta_{i-1,i} \geq \epsilon - 2 \max_i d_i) \\ &\leq (\epsilon - 2 \max_i d_i)^{-4} \sum_{i=1}^n E\delta_{i-1,i}^4 \\ &\leq (\epsilon - 2 \max_i d_i)^{-4} \cdot 4\, k_d^2\, n^{-1} \longrightarrow 0. \end{aligned} \tag{32}$$

Exactly similar calculations show that $\mathcal{U}_1 = o_p(1)$. □

Proof of Theorem 2.2a.2. Observe that $Z_d(0) = 0 = Z_d(1)$ and that $Z_d \in \mathbb{C}[0, 1]$ for every $n \geq 1$ and each sequence $\{d_i\}$. Hence by (19) and Theorem A.2 of the Appendix, $\{Z_d\}$ is tight in $\mathbb{C}[0, 1]$. Thus claim (i) follows from (20). To prove (ii) just argue as in the proof of (ii) of Theorem 2.2a.1 above. □

The following corollary will be useful later on. To state it we need some more notation. Let $F_{n1}, \ldots, F_{nn}$ be d.f.'s on $\mathbb{R}$ and X_{ni} be a r.v. with d.f. F_{ni}, $1 \leq i \leq n$. Define

$$(33)\quad H(x) := n^{-1} \Sigma_i F_{ni}(x),\ x \in \mathbb{R}; \qquad H^{-1}(t) := \inf\{x; H(x) \geq t\},\ 0 \leq t \leq 1;$$

$$L_{ni}(t) := F_{ni}(H^{-1}(t)),\quad 1 \leq i \leq n; \qquad L_d(t) := \Sigma_i d_{ni}^2 L_{ni}(t),$$

$$W_d^*(t) := \Sigma_i d_{ni}\{I(X_{ni} \leq H^{-1}(t)) - L_{ni}(t)\}, \qquad 0 \leq t \leq 1.$$

Corollary 2.2a.1. *Assume that*

(34) $X_{n1}, \ldots, X_{nn}$ *are independent r.v.'s with respective d.f.'s* $F_{n1}, \ldots, F_{nn}$ *on* $\mathbb{R}$.

In addition, suppose that $\{d_{ni}\}$, $\{F_{ni}\}$ *satisfy* (N1), (N2) *and*

$$(C^*)\qquad \lim_{\delta\to 0} \limsup_n \sup_{0 \leq t \leq 1-\delta} [L_d(t + \delta) - L_d(t)] = 0.$$

Then, for every $\epsilon > 0$,

$$(35)\qquad \lim_{\delta\to 0} \limsup_n P\Big(\sup_{|t-s|<\delta} |W_d^*(t) - W_d^*(s)| \geq \epsilon\Big) = 0.$$

Proof. Follows from Theorem 2.2a.1(i) applied to $\eta_i \equiv H(X_i)$, $G_i \equiv L_i$, $1 \leq i \leq n$. □

Remark 2.2a.3. Note that if H is continuous then $n^{-1} \Sigma_i L_{ni}(t) \equiv t$. Therefore,

$$(36) \qquad \sup_{0 \le t \le 1-\delta} [L_d(t+\delta) - L_d(t)] \le n \max_i d_{ni}^2 \, \delta.$$

Thus, if we strengthen (N2) to require (B) then (C*) is *a priori* satisfied. That is, the conditions of Theorem 2.2a.2(i) are satisfied.

If $F_{ni} \equiv F$, F a continuous d.f. then $L_{ni}(t) \equiv t$. Therefore, in view of (N1), (C*) is *a priori* satisfied. Moreover $\mathcal{C}_d^*(s, t) := \mathrm{Cov}(W_d^*(s), W_d^*(t)) = s(1-t)$, $0 \le s \le t \le 1$. Therefore we obtain

Corollary 2.2a.2. *Suppose that* $X_{n1}, \ldots, X_{nn}$ *are i.i.d.* F, F *a continuous d.f.. Suppose that* $\{d_{ni}\}$ *satisfy* (N1) *and* (N2). *Then* $W_d^* \Rightarrow B$ *in* $(\mathbb{D}[0, 1], d)$ *with* B *a Brownian bridge in* $\mathbb{C}[0, 1]$. □

Observe that $d_{ni} \equiv n^{-1/2}$ satisfy (N1) and (N2). In other words the above corollary includes the well celebrated result, v.i.z., the weak convergence of the sequence of the ordinary empirical processes.

Note. A variant of Theorem 2.2a.1 was first proved in Koul (1970). The above formulation and proof is based on this work and that of Withers (1975). Theorem 2.2a.2 is motivated by the work of Shorack (1973) which deals only with the weak convergence of the W_1-process, the process W_d with $d_{ni} \equiv n^{-1/2}$. The sufficiency of condition (D) for (16) was observed by Eyster (1977). □

2.2b. V_h-processes

In this subsection we shall investigate the weak convergence of the r.w.e.p.'s $\{V_h(x), \ x \in \mathbb{R}\}$ of (1.4.1). To state the general result we need some more structure on the underlying r.v.'s.

Accordingly, let $(\Omega, \mathcal{A}, P)$ be a probability space and G be a d.f. on $\mathbb{R}$. For each integer $n \ge 1$, let $(\zeta_{ni}, h_{ni}, \delta_{ni})$, $1 \le i \le n$, be an array of trivariate r.v.'s defined on $(\Omega, \mathcal{A})$ such that $\{\zeta_{ni}, 1 \le i \le n\}$ are i.i.d. G r.v.'s and ζ_{ni} is independent of (h_{ni}, δ_{ni}) for each $1 \le i \le n$. Furthermore, let $\{\mathcal{A}_{ni}\}$ be an array of sub σ–fields such that $\mathcal{A}_{ni} \subset \mathcal{A}_{n,i+1}$, $\mathcal{A}_{ni} \subset \mathcal{A}_{n+1,i}$, $1 \le i \le n$, $n \ge 1$; (h_{n1}, δ_{n1}) is $\mathcal{A}_{n1}$-measurable; the r.v.'s $\{\zeta_{n1},\ldots,\zeta_{n,j-1}; (h_{ni}, \delta_{ni}), 1 \le i \le j\}$ are $\mathcal{A}_{nj}$-measurable, $2 \le j \le n$; and ζ_{nj} is independent of $\mathcal{A}_{nj}$, $1 \le j \le n$. Define

$$(1) \qquad V_h(x) := n^{-1} \sum_{i=1}^n h_{ni} I(\zeta_{ni} \le x + \delta_{ni}), \qquad V_h^*(x) := n^{-1} \sum_{i=1}^n h_{ni} I(\zeta_{ni} \le x),$$

$$J_h(x) := n^{-1} \sum_{i=1}^n E[\{h_{ni} I(\zeta_{ni} \le x + \delta_{ni})\} \mid \mathcal{A}_{ni}] = n^{-1} \sum_{i=1}^n h_{ni} G(x + \delta_{ni}),$$

$$J_h^*(x) := n^{-1} \sum_{i=1}^n h_{ni} G(x),$$

$$U_h(x) := n^{1/2}[V_h(x) - J_h(x)], \quad U_h^*(x) = n^{1/2}[V_h^*(x) - J_h^*(x)], \quad x \in \mathbb{R}.$$

We are now ready to state the following

Theorem 2.2b.1. *In addition to the above, assume that the following conditions hold:*

(A1) $\sup_{n\geq 1} \max_i |h_{ni}| \leq c$, a.s., *for some constant* $c < \infty$.

(A2) $\max_i |\delta_{ni}| = o_p(1)$.

(A3) $n^{-1/2} \Sigma_i |h_{ni}\, \delta_{ni}| = O_p(1)$.

(A4) G *has a uniformly continuous density* g *w.r.t.* λ, *and* $g > 0$, *a.e.*

Then

(2) $$\|U_h - U_h^*\|_\infty = o_p(1).$$

If, in addition,

(A5) $n^{-1} \Sigma_i |h_{ni}|^2 \longrightarrow \alpha^2$ *in probability,* α *a r.v.,*

then

(3) $$U_h \Rightarrow \alpha \cdot B(G), \qquad U_h^* \Rightarrow \alpha \cdot B(G)$$

where B *is a Brownian Bridge in* $\mathbb{C}[0, 1]$, *independent of* α.

The proof of (2) uses a restricted chaining argument and an exponential inequality for martingales with bounded differences. It will be a consequence of the following *two* lemmas.

Lemma 2.2b.1. *Under* (A1) – (A4), $\forall\, \epsilon > 0$ *and for* $r = 1, 2$,

$$\lim_n P\Big(\sup_{x,y} n^{-1/2} \sum_{i=1}^{n} |h_{ni}|^r \, |G(y + \delta_{ni}) - G(x + \delta_{ni})| \leq 2c^r \epsilon\Big) = 1,$$

where the supremum is taken over the set $\{x, y \in \mathbb{R};\ n^{1/2}|G(x) - G(y)| \leq \epsilon\}$.

Proof. Let $\epsilon > 0$, $q(u) := g(G^{-1}(u))$, $0 \leq u \leq 1$; $\quad \gamma_n := \max_i |\delta_i|$,

$$\omega_n := \sup\{|q(u) - q(v)|;\ |u - v| \leq \epsilon n^{-1/2}\}$$
$$= \sup\{|g(x) - g(y)|;\ |G(x) - G(y)| \leq \epsilon n^{-1/2}\},$$
$$\Delta_n := \sup\{|g(x) - g(y)|;\ |y - x| \leq \gamma_n\}.$$

By (A4), q is uniformly continuous on $[0, 1]$. Hence, by (A2),

$$\Delta_n = o_p(1), \qquad \omega_n = o(1). \tag{4}$$

But

$$\sup_{x,y} n^{-1/2} \sum_{i=1}^{n} |h_i|^r \, |G(y + \delta_i) - G(x + \delta_i)|$$
$$\le \sup_{x,y} n^{-1/2} \sum_{i=1}^{n} |h_i|^r \, |G(y) - G(x)| +$$
$$+ n^{-1/2} \sum_{i=1}^{n} |h_i|^r |\delta_i| \, [\omega_n + 2\Delta_n]$$
$$\le c^r \epsilon + O_p(1) \cdot o_p(1), \qquad \text{by (A3) and (4).}$$

This completes the proof of the Lemma. □

Lemma 2.2b.2. *Let* $\{\mathcal{F}_i,\ i \ge 0\}$ *be an increasing sequence of* σ*-fields,* m *be a positive integer,* $\tau \le m$ *be a stopping time relative to* $\{\mathcal{F}_i\}$ *and* $\{\xi_i, 1 \le i \le m\}$ *be a sequence of real valued martingale differences w.r.t.* $\{\mathcal{F}_i\}$. *In addition, suppose that*

(i) $|\xi_i| \le M < \infty$, *for some constant* $M < \infty$, $1 \le i \le m$,

and

(ii) $\sum_{i=1}^{\tau} E(\xi_i^2 | \mathcal{F}_{i-1}) \le L$, *for a constant* $L < \infty$.

Then, for every $a > 0$,

$$P\left(\left|\sum_{i=1}^{\tau} \xi_i\right| > a\right) \le 2 \exp\{-(a/2M) \operatorname{arcsinh}(Ma/2L)\}. \tag{5}$$

Proof. Write $\sigma_i^2 = E(\xi_i^2 | \mathcal{F}_{i-1})$, $i \ge 1$. First, consider the

case $\tau = m$:

Recall the following elementary facts: For all $x \in \mathbb{R}$,

(6a) $\exp(x) - x - 1 \le 2(\cosh x - 1) \le x \sinh x$,

(6b) $(\sinh x)/x$ is increasing in $|x|$,

(6c) $x \le \exp(x - 1)$.

Because $E(\xi_i | \mathcal{F}_{i-1}) \equiv 0$ and by (i), for a $\delta > 0$ and for all $1 \le i \le m$,

$$E\{[\exp(\delta\xi_i) - 1] \,|\, \mathcal{F}_{i-1}\} \le E\{\delta\xi_i \sinh(\delta\xi_i) \,|\, \mathcal{F}_{i-1}\}, \qquad \text{by (6a)}$$

$$\le \sigma_i^2\, \delta \sinh(\delta M)/M, \qquad \text{by (6b).} \tag{7}$$

Use a conditioning argument to obtain

$$\begin{aligned}
&E\exp\{\delta \sum_{i=1}^{m} \xi_i\} \\
&= E[\exp(\delta \sum_{i=1}^{m-1} \xi_i)\, E\{\exp(\delta\, \xi_m) \,|\, \mathcal{F}_{m-1}\}] \\
&\le E[\exp(\delta \sum_{i=1}^{m-1} \xi_i)\, \exp(E\{\exp(\delta\, \xi_m) \,|\, \mathcal{F}_{m-1}\} - 1)], && \text{by (6c)} \\
&\le E[\exp(\delta \sum_{i=1}^{m-1} \xi_i)\, \exp(\sigma_m^2 \cdot \delta/M \cdot \sinh(\delta M))], && \text{by (7)} \\
&\le E[\exp(\delta \sum_{i=1}^{m-1} \xi_i)\, \exp\{(L - \sum_{i=1}^{m-1} \sigma_i^2) \cdot \delta/M \cdot \sinh(\delta M)\}].
\end{aligned}$$

Observe that $L - \sum_{i=1}^{j-1} \sigma_i^2$ is $\mathcal{F}_{j-2}$ measurable, for all $j \ge 2$. Hence, iterating the above argument $m-1$ times will give

$$E\exp\{\delta \sum_{i=1}^{m} \xi_i\} \le \exp\{L \cdot \delta/M \cdot \sinh(\delta M)\}.$$

Now, by the Markov inequality, $\forall\, a > 0$,

$$P(\sum_{i=1}^{m} \xi_i \ge a) \le E\exp\{\delta(\sum_{i=1}^{m} \xi_i - a)\} \le \exp\{\delta\, [L/M \cdot \sinh(\delta M) - a]\}.$$

The choice of $\delta = (1/M)\operatorname{arcsinh}(Ma/2L)$ in this leads to the inequality

$$P(\sum_{i=1}^{m} \xi_i \ge a) \le \exp\{(-a/2M)\operatorname{arcsinh}(Ma/2L)\}.$$

An application of this inequality to $\{-\xi_i\}$ will yield the same bound for $P(\sum_{i=1}^{m} \xi_i \le -a)$, thereby completing the proof of (5) in the case $\tau = m$. Now consider the

general case $\tau \le m$:

Let $\chi_j = \xi_j I(j \le \tau)$. Because the event $[j \le \tau] \in \mathcal{F}_{j-1}$, it follows that $\{\chi_j, \mathcal{F}_j\}$ satisfy the conditions of the previous case. Hence,

$$P(|\sum_{i=1}^{\tau} \xi_i| \ge a) = P(|\sum_{i=1}^{m} \chi_i| \ge a) \le \exp\{(-a/2M) \operatorname{arcsinh}(Ma/2L)\}. \quad \square$$

Proof of Theorem 2.2b.1. For the clarity of the proof it is important to emphasize the dependence of various underlying processes on n. Accordingly, we shall write V_n, U_n etc. for V_h, U_h etc. in the proof.

On $\mathbb{R}$ define the metric $d(x,y) := |G(x) - G(y)|^{1/2}$. This metric makes $\mathbb{R}$ totally bounded. Thus, to prove the theorem, it suffices to prove

(a) $\forall\ y \in \mathbb{R}, \qquad |U_n(y) - U_n^*(y)| = o_p(1),$

(b) $\forall\ \epsilon > 0\ \exists\ \delta > 0\ \ni$

(i) $\limsup_n P(\sup_{d(x,y) \le \delta} |U_n(y) - U_n(x)| > \epsilon) < \epsilon,$

(ii) $\limsup_n P(\sup_{d(x,y) \le \delta} |U_n^*(y) - U_n^*(x)| > \epsilon) < \epsilon.$

Proof of (a). The fact that $U_n - U_n^*$ is a sum of conditionally centered bounded r.v.'s yields that

$$\operatorname{Var}(U_n(y) - U_n^*(y)) \le E\, n^{-1} \Sigma_i h_i^2 |G(y + \delta_i) - G(y)| = o(1),$$

by (A1), (A2), (A4) and the Dominated Convergence Theorem.

Proof of (b)(i). The following proof of (b)(i) uses a restricted chaining argument as discussed in Pollard (1984: p. 160–162), and the exponential inequality of Lemma 2.2b.2 above.

Fix an $\epsilon > 0$. Let $a_n := [n^{1/2}/\epsilon]$, the greatest integer less than or equal to $n^{1/2}/\epsilon$, and define the grid

$$\mathcal{H}_n := \{y_j;\ G(y_j) = j\epsilon n^{-1/2},\ 1 \le j \le a_n\}, \qquad n \ge 1.$$

Also let

$$Z_i(x) := I(\zeta_i \le x + \delta_i) - G(x + \delta_i), \quad x \in \mathbb{R}, \quad 1 \le i \le n.$$

Write $h_i \equiv h_{i+} - h_{i-}$, $h_{i+} \equiv \max(0, h_i)$, so that

$$U_n(x) = n^{-1/2}\sum_{i=1}^{n} h_{i+} Z_i(x) - n^{-1/2}\sum_{i=1}^{n} h_{i-} Z_i(x) = U_n^+(x) - U_n^-(x), \quad \text{say.}$$

Thus to prove (b)(i), by the triangle inequality, it suffices to prove it for $U_n^{\pm}$ processes. The details of the proof shall be given for the U_n^+ process only; those for the U_n^- being similar.

Next, we need to define the sequence of stopping times

$$\tau_n^+ := n \wedge \max\{k \geq 1;\ \max_{x,\, y \in \mathscr{H}_n} \frac{\sum_{i=1}^{k} (h_{i+})^2 E\{[Z_i(x) - Z_i(y)]^2 \mid \mathcal{A}_i\}}{d^2(x,y)} < 3\epsilon c^2 n\}.$$

Observe that $\tau_n^+ \leq n$. To adapt the present situation to that of the Pollard, we first prove that $P(\tau_n^+ < n) \longrightarrow 0$ (see (8) below). This allows one to work with $n^{-1/2}(\tau_n^+)^{1/2}\, U^+_{\tau_n^+}$ instead of U_n^+. By Lemma 2.2b.2 and the fact that $\operatorname{arcsinh}(x)$ is increasing and concave in x, one obtains that if x, y in $\mathscr{H}_n$ are such that $d^2(x, y) \geq t\epsilon n^{-1/2}$ then

$$P(n^{-1/2}(\tau_n^+)^{1/2} \mid U^+_{\tau_n^+}(x) - U^+_{\tau_n^+}(y) \mid \geq t)$$

$$\leq 2 \exp\{-\frac{t^2}{2c\,d^2(x,y)} \cdot \epsilon \operatorname{arcsinh}(1/(6\epsilon^2 c))\}, \qquad \text{for all } t > 0.$$

This enables one to carry out the chaining argument as in Pollard. What remains to be done is to connect between the points in $\mathbb{R}$ and a point in $\mathscr{H}_n$ which will be done in (9) below. We shall now prove

(8) $$P(\tau_n^+ < n) \longrightarrow 0.$$

Proof of (8). For y_j, y_k in $\mathscr{H}_n$ with $y_j < y_k$, $d^2(y_j, y_k) \geq (k-j)\epsilon n^{-1/2}$. Hence, using the fact that $(h_{i+})^2 \leq h_i^2$,

$$\sum_{i=1}^{n} (h_{i+})^2\, E\{[Z_i(y_k) - Z_i(y_j)]^2 \mid \mathcal{A}_i\} / d^2(y_i, y_k)$$

$$\leq \sum_{i=1}^{n} h_i^2\, [G(y_k + \delta_i) - G(y_j + \delta_i)]\, \{(k-j)\epsilon\}^{-1}\, n^{1/2}$$

$$\leq \{(k-j)\epsilon\}^{-1}\, n^{1/2} \sum_{i=1}^{n} h_i^2 \sum_{r=j}^{k-1} [G(y_{r+1} + \delta_i) - G(y_r + \delta_i)]$$

$$\leq \epsilon^{-1}\, n^{1/2} \max_{1 \leq r \leq a_n} \sum_{i=1}^{n} h_i^2 \,[G(y_{r+1} + \delta_i) - G(y_r + \delta_i)].$$

Now apply Lemma 2.2b.1 with $r = 2$ to obtain

$$P\Big(\max_{1 \leq k, j \leq a_n} \Big[\sum_{i=1}^{n} (h_{i+})^2\, E\{[Z_i(y_k) - Z_i(y_j)]^2 | \mathcal{A}_i\} / d^2(y_j, y_k)\Big] < 3\epsilon c^2 n\Big) \longrightarrow 1.$$

This completes the proof of (8).

Next, for each $x \in \mathbb{R}$, let y_{j_x} denote the point in $\mathcal{H}_n$ that is the closest to x in d-metric from the points in $\mathcal{H}_n$ that satisfy $y_{j_x} \leq x$. We shall now prove: $\forall\, \epsilon > 0$,

$$\text{(9)} \qquad P(\sup_x | \mathcal{U}_n^+(x) - \mathcal{U}_n^+(y_{j_x})| > 8c\epsilon) \longrightarrow 0.$$

Proof of (9). Now write $\mathcal{V}_n^+$, $\mathcal{J}_n^+$ for $\mathcal{V}_n$, $\mathcal{J}_n$ when $\{h_i\}$ in these quantities is replaced by $\{h_{i+}\}$.

The definition of y_{j_x}, G increasing, and the fact that $h_{i+} \leq |h_i|$ for all i, imply that

$$\sup_x |n^{1/2}[\mathcal{J}_n^+(x) - \mathcal{J}_n^+(y_{j_x})]|$$

$$\leq \max_{1 \leq j \leq a_n} n^{-1/2} \sum_{i=1}^{n} |h_i| \,[G(y_{j+1} + \delta_i) - G(y_j + \delta_i)].$$

An application of Lemma 2.2b.1 with $r = 1$ now yields that

$$\text{(10)} \qquad P(\sup_x |n^{1/2}[\mathcal{J}_n^+(x) - \mathcal{J}_n^+(y_{j_x})]| > 4c\epsilon) \longrightarrow 0.$$

But $h_{i+} \geq 0$, $1 \leq i \leq n$, implies that $\mathcal{V}_n^+$ is nondecreasing in x. Therefore, using the definition of y_{j_x},

$$n^{-1/2}[\mathcal{U}_n^+(y_{j_x-1}) - \mathcal{U}_n^+(y_{j_x})] + \mathcal{J}_n^+(y_{j_x-1}) - \mathcal{J}_n^+(y_{j_x}) =$$

$$= \mathcal{V}_n^+(y_{j_x-1}) - \mathcal{V}_n^+(y_{j_x}) \leq \mathcal{V}_n^+(x) - \mathcal{V}_n^+(y_{j_x}) \leq \mathcal{V}_n^+(y_{j_x+1}) - \mathcal{V}_n^+(y_{j_x})$$

$$= n^{-1/2}[\mathcal{U}_n^+(y_{j_x+1}) - \mathcal{U}_n^+(y_{j_x})] + \mathcal{J}_n^+(y_{j_x+1}) - \mathcal{J}_n^+(y_{j_x}).$$

Hence,

(11) $$\sup_x |n^{1/2}[V_n^+(x) - V_n^+(y_{j_x})]| \le 2 \max_{1\le j\le a_n} |U_n^+(y_{j+1}) - U_n^+(y_j)| + \\ + 2 \max_{1\le j\le a_n} |n^{1/2}[J_n^+(y_{j+1}) - J_n^+(y_j)]|.$$

Thus, (9) will follow from (10), (11) and

(12) $$P(\max_{1\le j\le a_n} |U_n^+(y_{j+1}) - U_n^+(y_j)| > c\,\epsilon) \longrightarrow 0.$$

In view of (8), to prove (12), it suffices to show that

(13) $$P(\max_{1\le j\le a_n} n^{-1/2}(\tau_n^+)^{1/2} |U_{\tau_n^+}(y_{j+1}) - U_{\tau_n^+}(y_j)| > c\,\epsilon) \longrightarrow 0.$$

But,

$$\text{l.h.s.}(13) \le \sum_{j=1}^{a_n} P(|\sum_{i=1}^{\tau_n^+} h_{i*}[Z_i(y_{j+1}) - Z_i(y_j)]| > c\epsilon n^{1/2}).$$

Now apply Lemma 2.2b.2 with $\xi_i \equiv h_{i*}[Z_i(y_{j+1}) - Z_i(y_j)]$, $\mathcal{F}_{i-1} \equiv \mathcal{A}_i$, $\tau \equiv \tau_n^+$, $M = c$, $a = c\epsilon n^{1/2}$, $m = n$. By the definition of τ_n^+, $L = 3c^2\epsilon^2 n^{1/2}$. Hence by Lemma 2.2b.2,

$$P(|\sum_{i=1}^{\tau_n^+} h_{i*}[Z_i(y_{j+1}) - Z_i(y_j)]| > c\epsilon n^{1/2}) \le 2\exp[-\frac{n^{1/2}\epsilon}{2}\operatorname{arcsinh}(1/6\epsilon)].$$

Since this bound does not depend on j, it follows that

$$\text{l.h.s.}(13) \le 2\epsilon^{-1} n^{1/2} \exp[-\frac{n^{1/2}\epsilon}{2}\operatorname{arcsinh}(1/6\epsilon)] \longrightarrow 0 .$$

This completes the proof of (9) for U_n^+. As mentioned earlier the proof of (9) for U_n^- is exactly similar, thereby completing the proof of (b)(i).

Adapt the above proof of (b)(i) with $\delta_i \equiv 0$ to conclude (b)(ii). *Note that* (b)(ii) *holds solely under* (A1) *and the assumption that* G *is continuous and strictly increasing; the other assumptions are not required here.* The proof of (2) is now complete.

The claim (3) follows from (1), (b)(ii) above, Lemma A.3 of the Appendix and the Cramer–Wold device. □

As noted in the proof of the above theorem, the weak convergence of U_h^* holds only under (A1), (A5) and the assumption that G is continuous and strictly increasing. For an easy reference later on we state this result as

Corollary 2.2b.1. *Let the setup of Theorem* 2.2b.1 *hold. Assume that* G *is continuous and strictly increasing and that* (A1), (A5) *hold. Then,* $U_h^* \Rightarrow \alpha \cdot B(G)$, *where B is a Brownian bridge in* $\mathbb{C}[0, 1]$, *independent of* α. □

Remark 2.2b.1. Consider the process $\mathcal{U}_h(t) := U_h^*(G^{-1}(t))$, $0 \le t \le 1$. Now work with the metric $|t-s|^{1/2}$ on $[0, 1]$. Upon repeating the arguments in the proof of the above theorem, modified appropriately, one can readily conclude the following

Corollary 2.2b.2. *Let the setup of Theorem* 2.2b.1 *hold. Assume that* G *is continuous and that* (A1), (A5) *hold. Then* $\{\mathcal{U}_h\} \Rightarrow \alpha \cdot B$, *where B is a Brownian Bridge in* $\mathbb{C}[0, 1]$, *independent of* α. □

Remark 2.2b.2. Suppose that in Theorem 2.2a.2 the r.v.'s $\eta_{n1}, \ldots, \eta_{nn}$ are i.i.d Uniform [0, 1]. Then, upon choosing $h_{ni} \equiv n^{1/2} d_{ni}$, $\zeta_{ni} \equiv G^{-1}(\eta_{ni})$, one sees that $U_h \equiv W_d(G)$, provided G is continuous. Moreover, the condition (D) is *a priori* satisfied, (B) is equivalent to (A1) and (N1) implies (A5) trivially. Consequently, for this special setup, Theorem 2.2a.2 is a special case of Corollary 2.2b.2. But in general these two results serve different purposes. Theorem 2.2a.1 is the most general for the independent setup given there and cannot be deduced from Theorem 2.2b.1. □

Note: The inequality (5) and its proof appears in Levental (1989). See also Proposition 3.1 in Johnson, Schechtman and Zinn (1985). The proof of Theorem 2.2b.1 has its roots in Levental and Koul (1989) and Koul (1991). It was recently generalized by Koul and Ossiander (1992) to include unbounded weights. □

2.3. ASYMPTOTIC UNIFORM LINEARITY (A.U.L.) OF RESIDUAL W.E.P.'s.

In this section we shall obtain the asymptotic uniform linearity (a.u.l.) of residual w.e.p.'s. It will be observed that the asymptotic continuity property of the type specified in Theorem 2.2a.1(i) is the basic tool to obtain this result. Accordingly let $\{X_{ni}\}$, $\{F_{ni}\}$, $\{H\}$ and $\{L_{ni}\}$ be as in (2.2a.33) and define

$$(1)\qquad \begin{aligned} S_d(t, \mathbf{u}) &:= \Sigma_i\, d_{ni}\, I(X_{ni} \le H^{-1}(t) + \mathbf{c}_{ni}'\mathbf{u}), \\ \mu_d(t, \mathbf{u}) &:= \Sigma_i\, d_{ni}\, F_{ni}(H^{-1}(t) + \mathbf{c}_{ni}'\mathbf{u}), \\ Y_d(t, \mathbf{u}) &:= S_d(t, \mathbf{u}) - \mu_d(t, \mathbf{u}), \qquad 0 \le t \le 1, \ \mathbf{u} \in \mathbb{R}^p, \end{aligned}$$

where $\{\mathbf{c}_{ni},\ 1 \le i \le n\}$ are $p \times 1$ vectors of real numbers. We also need

(2a) $$S_d^o(x, \mathbf{u}) := \Sigma_i\, d_{ni}\, I(X_{ni} \le x + \mathbf{c}_{ni}'\mathbf{u}),$$
$$\mu_d^o(x, \mathbf{u}) := \Sigma_i\, d_{ni}\, F_{ni}(x + \mathbf{c}_{ni}'\mathbf{u}),$$
$$Y_d^o(x, \mathbf{u}) := S_d^o(x, \mathbf{u}) - \mu_d^o(x, \mathbf{u}), \qquad -\infty \le x \le \infty, \ \mathbf{u} \in \mathbb{R}^p.$$

Clearly, if H is strictly increasing then $S_d^o(x, \mathbf{u}) \equiv S_d(H(x), \mathbf{u})$. Similar remark applies to other functions.

Throughout the text, any w.e.p. with weights $d_{ni} \equiv n^{-1/2}$ *will be indicated by the subscript* 1. Thus, e.g., $\forall\ -\infty \le x \le \infty,\ \mathbf{u} \in \mathbb{R}^p$,

(2b) $$S_1^o(x, \mathbf{u}) = n^{-1/2}\, \Sigma_i\, I(X_{ni} \le x + \mathbf{c}_{ni}'\mathbf{u}),$$
$$Y_1^o(x, \mathbf{u}) = n^{-1/2}\, \Sigma_i\, \{I(X_{ni} \le x + \mathbf{c}_{ni}'\mathbf{u}) - F_{ni}(x + \mathbf{c}_{ni}'\mathbf{u})\}.$$

Theorem 2.3.1. *In addition to* (2.2a.34), (N1), (N2), *and* (C*) *assume that d.f.'s* $\{F_{ni},\ 1 \le i \le n\}$ *have densities* $\{f_{ni},\ 1 \le i \le n\}$ *w.r.t.* λ *such that the following hold:*

(3a) $$\lim_{\delta\to 0} \limsup_n \max_{1\le i\le n} \sup_{|x-y|\le\delta} |f_{ni}(x) - f_{ni}(y)| = 0,$$

(3b) $$\max_{i,n} \|f_{ni}\|_\infty \le k < \infty.$$

In addition, assume that

(4) $$\max_{1\le i\le n} \|\mathbf{c}_{ni}\| = o(1)$$

and

(5) $$\Sigma_i\, \|d_{ni}\, \mathbf{c}_{ni}\| = O(1).$$

Then, for every $0 < B < \infty$,

(6) $$\sup |S_d(t, \mathbf{u}) - S_d(t, \mathbf{0}) - \mathbf{u}'\Sigma_i\, d_{ni}\, \mathbf{c}_{ni}\, q_{ni}(t)| = o_p(1),$$

where $q_{ni} := f_{ni}H^{-1}$, $1 \le i \le n$, *and the supremum is taken over* $0 \le t \le 1$, $\|\mathbf{u}\| \le B$.

Consequently, if H *is strictly increasing on* $\mathbb{R}$, *then*

(7) $$\sup |S_d^o(x, \mathbf{u}) - S_d^o(x, \mathbf{0}) - \mathbf{u}'\Sigma_i\, d_{ni}\, \mathbf{c}_{ni}\, f_{ni}(x)| = o_p(1).$$

where the supremum is taken over $-\infty \le x \le \infty$, $\|\mathbf{u}\| \le B$.

Theorem 2.3.1 is a consequence of the following *four* lemmas. In these lemmas the setup is as in the theorem.

In what follows, $\mathcal{N}(B) = \{\mathbf{u}\in\mathbb{R}^p;\ \|\mathbf{u}\| \le B\}$; $\sup_{t,\mathbf{u}}$ stands for the supremum over $0 \le t \le 1$ and $\mathbf{u} \in \mathcal{N}(B)$, unless mentioned otherwise. Let

$$(8)\qquad \nu_d(t) := \Sigma_i\, d_{ni}\, \mathbf{c}_{ni}\, q_{ni}(t),$$

$$R_d(t, \mathbf{u}) := S_d(t, \mathbf{u}) - S_d(t, \mathbf{0}) - \mathbf{u}'\nu_d(t), \qquad 0 \le t \le 1,\ \mathbf{u} \in \mathbb{R}^p.$$

Lemma 2.3.1. *Under* (3), (4) *and* (5),

$$(9)\qquad \sup_{t,\mathbf{u}} |\mu_d(t, \mathbf{u}) - \mu_d(t, \mathbf{0}) - \mathbf{u}'\nu_d(t)\,| = o(1).$$

Proof. Let $\delta_n = B \max_i \|\mathbf{c}_i\|$. By (3), $\{F_i\}$ are uniformly differentiable for sufficiently large n, uniformly in $1 \le i \le n$. Hence,

$$\text{l.h.s. } (9) \le (\Sigma_i \|d_i\mathbf{c}_i\|) \max_i \sup_{|x-y|\le\delta_n} |f_i(x) - f_i(y)| = o(1),$$

by (3), (4) and (5). □

Lemma 2.3.2. *Under* (N1), (N2), (C*), (3), (4) *and* (5), $\forall\ \mathbf{u} \in \mathcal{N}(B)$,

$$(10)\qquad \sup_{0\le t\le 1} |\,Y_d(t, \mathbf{u}) - Y_d(t, \mathbf{0}| = o_p(1).$$

Proof. Fix a $\mathbf{u} \in \mathcal{N}(B)$. The lemma will follow if we show

(i) $\quad Y_d(t, \mathbf{u}) - Y_d(t, \mathbf{0}) = o_p(1)$ for each $0 \le t \le 1$,

and

(ii) $\quad \forall\ \epsilon > 0$, and for $\mathbf{b} = \mathbf{u}$ or $\mathbf{b} = \mathbf{0}$,

$$\lim_{\delta\to 0} \limsup_n P(\sup_{|t-s|<\delta} |\,Y_d(t, \mathbf{b}) - Y_d(s, \mathbf{b})| \ge \epsilon) = 0.$$

Since $Y_d(., \mathbf{0}) = W_d^*(.)$ of (2.2a.33), for $\mathbf{b} = \mathbf{0}$, (ii) follows from (2.2a.35) of Corollary 2.2a.1.

To verify (ii) for $\mathbf{b} = \mathbf{u}$, take $\eta_i = H(X_i - \mathbf{c}_i'\mathbf{u})$, $1\le i\le n$, in (2.2a.1). Then $Y_d(\cdot, \mathbf{u}) \equiv W_d(\cdot)$ of (2.2a.1) and $G_i(\cdot) = F_i(H^{-1}(\cdot) + \mathbf{c}_i'\mathbf{u})$, $1\le i\le n$. Moreover,

$$\text{(11)}\quad \sup_{0\le t\le 1-\delta} \Sigma_i\, d_i^2\, [F_i(H^{-1}(t+\delta) + c_i'u) - F_i(H^{-1}(t) + c_i'u)]$$
$$\le 2Bk \max_i \|c_i\| + \sup_{0\le t\le 1-\delta} [L_d(t+\delta) - L_d(t)], \quad \text{by (3)}$$
$$= o(1) \text{ as } n \longrightarrow \infty, \text{ then } \delta \longrightarrow 0, \text{ by (4) and (C*)}.$$

Hence (C) is satisfied by the above $\{G_i\}$. The other conditions being (N1) and (N2) which are also assumed here, it follows that the above $\{\eta_i\}$ and $\{W_d\}$ satisfy the conditions of Theorem 2.2a.1.(i). Thus (ii) for $b = u$ follows from Theorem 2.2a.1(i). Hence (ii) is proved.

To obtain (i), note that the

$$\mathrm{Var}[Y_d(t, u) - Y_d(t, 0)] \le \Sigma_i\, d_i^2\, |F_i(H^{-1}(t) + c_i'u) - F_i(H^{-1}(t))|$$
$$\le Bk \max_i \|c_i\|, \qquad \text{by (3)},$$
$$= o(1), \qquad \text{by (4)}.$$

This together with the Chebychev inequality yields (i) and hence (10). □

To state and prove the next lemma we need some more notation. Let $\kappa_{ni} = \|c_{ni}\|$, $1 \le i \le n$, and define

$$\text{(12)}\quad S_d^*(t, u, b) = \Sigma\, d_{ni}\, I(X_{ni} \le H^{-1}(t)+c_{ni}'u+b\, \kappa_{ni}),$$
$$\mu_d^*(t, u, b) = E\, S_d^*(t, u, b),$$
$$Y_d^*(t, u, b) = S_d^*(t, u, b) - \mu_d^*(t, u, b), \qquad 0 \le t \le 1,\ u \in \mathbb{R}^p,\ b \in \mathbb{R}.$$

Lemma 2.3.3. *Under* (N1), (N2), (C*), (3), (4) *and* (5), $\forall\ \epsilon > 0$, $|b| < \infty$ *and* $u \in \mathcal{N}(B)$,

$$\text{(13)}\quad \lim_{\delta\to 0} \limsup_n P\Big(\sup_{|t-s|<\delta} |Y_d^*(t, u, b) - Y_d^*(s, u, b)| \ge \epsilon\Big) = 0.$$

Proof. In Theorem 2.2a.1(i), take $\eta_i = H(X_i - c_i'u - b\kappa_i)$, $1\le i\le n$. Then $W_d(\cdot) = Y_d^*(\cdot, u, b)$ and $G_i(\cdot) = F_i(H^{-1}(\cdot)+c_i'u+b\kappa_i)$, $1\le i\le n$. Again, similar to (11),

$$\sup_{0\le t\le 1-\delta} [G_d(t+\delta) - G_d(t)] \le 2k(B+b) \max_i \|c_i\| + \sup_{0\le t\le 1-\delta} [L_d(t+\delta) - L_d(t)]$$
$$= o(1), \qquad \text{by (4) and (C*)}.$$

Hence (13) follows from Theorem 2.2a.1(i). □

Lemma 2.3.4. *Under* (N1), (N2), (C*), (3), (4) *and* (5), $\forall\ \epsilon > 0$ *there is a* $\delta > 0$ *such that for every* $\mathbf{v} \in \mathcal{N}(B)$,

$$\limsup_n P\Big(\sup_{t,\|\mathbf{u}-\mathbf{v}\|\le\delta} |R_d(t,\mathbf{u}) - R_d(t,\mathbf{v})| \ge \epsilon\Big) = 0, \tag{14}$$

where R_d *is defined at* (8).

Proof. Assume, without loss of generality, that $d_i \ge 0$, $1 \le i \le n$. For, otherwise write $d_i = d_{i+} - d_{i-}$, $1 \le i \le n$, where $\{d_{i+}, d_{i-}\}$ are as in the proof of Lemma 2.2a.4. Then $S_d = \tau_{d+}S_{d+} - \tau_{d-}S_{d-}$, $R_d = \tau_{d+}R_{d+} - \tau_{d-}R_{d-}$, where $\tau_{d+}^2 = \Sigma_i (d_{i+})^2$, $\tau_{d-}^2 = \Sigma_i (d_{i-})^2$. In view of (N1), $\tau_{d+} \le 1$, $\tau_{d-} \le 1$. Moreover, if $\{d_i\}$ satisfy (N2) and (5) above, so do $\{d_{i+}, d_{i-}\}$ because $d_{i+}^2 \vee d_{i-}^2 = d_{i+}^2 + d_{i-}^2 = d_i^2$, $1 \le i \le n$. Hence the triangle inequality will yield (14), if proved for R_{d+} and R_{d-}. But note that $d_{i+} \wedge d_{i-} \ge 0$ for all i.

Now, $\|\mathbf{u} - \mathbf{v}\| \le \delta$ implies

$$-\delta\kappa_i + \mathbf{c}_i'\mathbf{v} \le \mathbf{c}_i'\mathbf{u} \le \delta\kappa_i + \mathbf{c}_i'\mathbf{v}, \qquad \kappa_i = \|\mathbf{c}_i\|, \quad 1 \le i \le n. \tag{15}$$

Therefore, because $d_i \ge 0$ for all i,

$$\overset{*}{S}_d(t,\mathbf{v},-\delta) \le S_d(t,\mathbf{u}) \le \overset{*}{S}_d(t,\mathbf{v},\delta) \text{ for all } t, \tag{16}$$

yielding

$$\begin{aligned} L_1(t,\mathbf{u},\mathbf{v}) := \overset{*}{S}_d(t,\mathbf{v},-\delta) - S_d(t,\mathbf{v}) - (\mathbf{u}-\mathbf{v})'\nu_d(t) \\ \le R_d(t,\mathbf{u}) - R_d(t,\mathbf{v}) \\ \le \overset{*}{S}_d(t,\mathbf{v},\delta) - S_d(t,\mathbf{v}) - (\mathbf{u}-\mathbf{v})'\nu_d(t) =: L_2(t,\mathbf{u},\mathbf{v}). \end{aligned} \tag{17}$$

We shall show that there is a $\delta > 0$ such that for every $\mathbf{v} \in \mathcal{N}(B)$,

$$P\Big(\sup_{t,\|\mathbf{u}-\mathbf{v}\|\le\delta} |L_j(t,\mathbf{u},\mathbf{v})| \ge \epsilon\Big) = o(1), \qquad j = 1, 2. \tag{18}$$

We shall first prove (18) for L_2. Observe that

$$\begin{aligned} |L_2(t,\mathbf{u},\mathbf{v})| \le |\overset{*}{Y}_d(t,\mathbf{v},\delta) - \overset{*}{Y}_d(t,\mathbf{v},0)| \\ + |\overset{*}{\mu}_d(t,\mathbf{v},\delta) - \overset{*}{\mu}_d(t,\mathbf{v},0)| + |(\mathbf{u}-\mathbf{v})'\nu_d(t)| \end{aligned} \tag{19}$$

The Mean Value Theorem, (3), and $\|\mathbf{u} - \mathbf{v}\| \le \delta$ imply

$$\text{(20)} \qquad \sup|\overset{*}{\mu}_d(t, \mathbf{v}, \delta) - \overset{*}{\mu}_d(t, \mathbf{v}, 0)| \le \delta\, k\, \Sigma_i \|d_i\mathbf{c}_i\|,$$

$$\sup_t |(\mathbf{u}-\mathbf{v})' \nu_d(t)| \le k\, \delta\, \Sigma_i \|d_i\mathbf{c}_i\|.$$

Let $M(t)$ denote the first term on the r.h.s. of (19). I.e.,

$$M(t) = \overset{*}{Y}_d(t, \mathbf{v}, \delta) - \overset{*}{Y}_d(t, \mathbf{v}, 0), \qquad 0 \le t \le 1.$$

(21) **Claim:** $\sup_t |M(t)| = o_p(1)$.

To begin with,

$$\begin{aligned} \mathrm{Var}\,(M(t)) &\le \Sigma_i\, d_i^2\, [F_i(H^{-1}(t)+\mathbf{c}_i'\mathbf{v}+\delta\kappa_i) - F_i(H^{-1}(t)+\mathbf{c}_i'\mathbf{v})] \\ &\le \delta k \max_i \kappa_i, && \text{by (3a), (3b),} \\ &= o(1), && \text{by (5).} \end{aligned}$$

Hence

$$\text{(22)} \qquad M(t) = o_p(1) \quad \text{for every } 0 \le t \le 1.$$

Next, note that, for a $\gamma > 0$,

$$\begin{aligned} \sup_{|t-s|<\gamma} |M(t) - M(s)| \le \sup_{|t-s|<\gamma} &|\overset{*}{Y}_d(t, \mathbf{v}, \delta) - \overset{*}{Y}_d(s, \mathbf{v}, \delta)| + \\ &+ \sup_{|t-s|<\gamma} |\overset{*}{Y}_d(t, \mathbf{v}, 0) - \overset{*}{Y}_d(s, \mathbf{v}, 0)|. \end{aligned}$$

Apply Lemma 2.3.3 twice, once with $b = \delta$ and once with $b = 0$, to obtain that $\forall\ \epsilon > 0$,

$$\text{(23)} \qquad \lim_{\gamma\to 0} \limsup_n P\Big(\sup_{|t-s|<\gamma} |M(t) - M(s)| \ge \epsilon\Big) = 0.$$

But (23) and (22) imply the Claim (21).

Now choose $\delta > 0$ so that

$$\text{(24)} \qquad \limsup_n \delta\, k\, \Sigma_i \|d_i\mathbf{c}_i\| \le \epsilon/3. \qquad \text{(use (5) here).}$$

From (19), (20) and (21) one readily obtains

$$\limsup_n P\Big(\sup_{t,\|u-v\|\le\delta} |L_2(t,u,v)| \ge \epsilon\Big) \le \limsup_n P\big(\sup_t |M(t)| \ge \epsilon/3\big) = 0.$$

This prove (18) for L_2. A similar argument proves (18) for L_1 with the same δ as in (24), thereby completing the proof of the Lemma. □

Proof of Theorem 2.3.1. Fix an $\epsilon > 0$ and choose a $\delta > 0$ satisfying (24). By the compactness of $\mathcal{N}(B)$ there exist points $v_1, \ldots, v_r$ in $\mathcal{N}(B)$ such that for any $u \in \mathcal{N}(B)$, $\|u - v_j\| \le \delta$ for some $j = 1, 2, \ldots, r$. Thus

$$\begin{aligned}\limsup_n &P\big(\sup_{t,u} |R_d(t, u)| \ge \epsilon\big)\\ &\le \sum_{j=1}^{r} \limsup_n P\Big(\sup_{t,\|u-v_j\|\le\delta} |R_d(t, u) - R_d(t, v_j)| \ge \epsilon/2\Big)\\ &\quad + \sum_{j=1}^{r} \limsup_n P\big(\sup_t |R_d(t, v_j)| \ge \epsilon/2\big) = 0\end{aligned}$$

by Lemmas 2.3.2 and 2.3.4. □

Remark 2.3.1. Upon a reexamination of the above proof one finds that Theorem 2.3.1 is a sole consequence of the continuity of certain w.e.p.'s and the smoothness of $\{F_{ni}\}$. Note that the above proof does not use the full force of the weak convergence of these w.e.p.'s. □

Remark 2.3.2. By the relationship

$$R_d(t, u) = \mathcal{Y}_d(t, u) - \mathcal{Y}_d(t, 0) + \mu_d(t, u) - \mu_d(t, 0) - u'\nu_d(t)$$

and by Lemma 2.3.1, (6) of Theorem 2.3.1 is equivalent to

$$\sup_{0\le t\le 1,\, u\in\mathcal{N}(B)} |\mathcal{Y}_d(t, u) - \mathcal{Y}_d(t, 0)| = o_p(1). \tag{25}$$

This will be useful when dealing with w.e.p.'s based on ranks in Chapter 3. □

The above theorem needs to be extended and reformulated when dealing with a linear regression model with an unknown scale parameter or with M-estimators in the presence of a preliminary scale estimator. To that end, define, for $x, s \in \mathbb{R}$, $0 \le t \le 1$, $u \in \mathbb{R}^p$,

$$\begin{aligned} S_d(s, t, u) &:= \Sigma_i\, d_{ni} I(X_{ni} \le (1+sn^{-1/2})H^{-1}(t) + c_{ni}'u), \\ S_d^o(s, x, u) &:= \Sigma_i\, d_{ni} I(X_{ni} \le (1+sn^{-1/2})x + c_{ni}'u), \end{aligned} \tag{26}$$

and define $\mathcal{Y}_d(s, t, u)$, $\mu_d(s, t, u)$ similarly. We are now ready to prove

Theorem 2.3.2. *In addition to the assumptions of Theorem 2.3.1, assume that*

$$\max_{i,n} \sup_x |x\, f_{ni}(x)| \le k < \infty. \tag{27}$$

Then

$$\sup |S_d(s, t, \mathbf{u}) - S_d(0, t, 0) - \Sigma_i\, d_{ni}\{s n^{-1/2} H^{-1}(t) + \mathbf{c}_{ni}'\mathbf{u}\} q_{ni}(t)| = o_p(1). \tag{28}$$

where the supremum is taken over $|s| \le b$, $\mathbf{u} \in \mathcal{N}(B)$, $0 \le t \le 1$.

Consequently, if H *is strictly increasing for all* $n \ge 1$, *then*

$$\sup |S_d^o(s, x, \mathbf{u}) - S_d^o(0, x, 0) - \Sigma d_{ni}\, \{s n^{-1/2} x + \mathbf{c}_{ni}'\mathbf{u}\} f_{ni}(x)| = o_p(1). \tag{29}$$

where the supremum is taken over $|s| \le b$, $\mathbf{u} \in \mathcal{N}(B)$ *and* $x \in \mathbb{R}$.

Sketch of proof. The argument is quite similar to that of Theorem 2.3.1. We briefly indicate the modifications of the previous proof.

An analogue of Lemma 2.3.1 will now assert

$$\sup |\mu_d(s, t, \mathbf{u}) - \mu_d(1, t, 0) - \{(n^{-1/2}\Sigma d_i q_i(t) H^{-1}(t))s + \mathbf{u}'\nu_d(t)\}| = o(1).$$

This uses (3), (4), (5), (27) and (N1).

An analogue of Lemma 2.3.2 is obtained by applying Theorem 2.2a.1(i) to $\eta_i := H((X_i - \mathbf{c}_i'\mathbf{u})\sigma_n^{-1})$, $1 \le i \le n$, $\sigma_n := (1 + s n^{-1/2})$. This states that for every $|s| \le b$ and every $\mathbf{u} \in \mathcal{N}(B)$,

$$\sup_{0 \le t \le 1} |Y_d(s, t, \mathbf{u}) - Y_d(s, t, 0)| = o_p(1). \tag{30}$$

In verifying (C) for these $\{\eta_i\}$, one has an analogue of (11):

$$\sup_{0 \le t \le 1-\delta} [G_d(t + \delta) - G_d(t)]$$
$$\le 2k\{B \max_i \|\mathbf{c}_i\| + b n^{-1/2}\} + \sup_{0 \le t \le 1-\delta} [L_d(t+\delta) - L_d(t)].$$

Note that here $G_d(t) \equiv \Sigma_i\, d_i^2 F_i(\sigma_n H^{-1}(t) + \mathbf{c}_i'\mathbf{u})$.

One similarly has an analogue of Lemma 2.3.3. Consequently, from Theorem 2.3.1 one can conclude that for each fixed $s \in [-b, b]$,

$$\sup_{0 \le t \le 1,\, \|\mathbf{u}\| \le B} |R_d(s, t, \mathbf{u})| = o_p(1), \tag{31}$$

where $R_d(s, t, u)$ equals the l.h.s. of (28) without the supremum. To complete the proof, once again exploit the compactness of $[-b, b]$ and the monotonic structure that is present in S_d and μ_d. Details are left for interested readers. □

Consider now the specialization of Theorems 2.3.1 and 2.3.2 to the case when $F_{ni} \equiv F$, F a d.f.. Note that in this case **(N1)** implies that $L_d(t) \equiv t$ so that (C^*) is *a priori* satisfied. To state these specializations we need the following assumptions:

(F1) F *has uniformly continuous density* f *w.r.t.* λ.

(F2) $f > 0$, a.e. λ.

(F3) $\sup_{x \in \mathbb{R}} |xf(x)| \le k < \infty$.

Note that **(F1)** implies that f is bounded and that **(F2)** implies that F is strictly increasing.

Corollary 2.3.1. *Let* $X_{n1}, \ldots, X_{nn}$ *be i.i.d.* F. *In addition, suppose that* **(N1)**, **(N2)**, (4), (5) *and* **(F1)** *hold. Then* (6) *holds with* $q_{ni} = f(F^{-1})$.

If, in addition, **(F2)** *holds, then* (7) *holds with* $f_{ni} \equiv f$. □

Corollary 2.3.2. *Let* $X_{n1}, \ldots, X_{nn}$ *be i.i.d.* F. *In addition, suppose that* **(N1)**, **(N2)**, (4), (5), **(F1)** *and* **(F3)** *hold. Then* (28) *holds with* $H \equiv F$ *and* $q_{ni} \equiv f(F^{-1})$.

If, in addition, **(F2)** *holds, then* (29) *holds with* $f_{ni} \equiv f$. □

We shall now apply the above results to the model (1.1.1) and the $\{V_j\}$-processes of (1.1.2). The results thus obtained are useful in studying the asymptotic distributions of certain goodness-of-fit tests and a class of M-estimators of $\boldsymbol{\beta}$ of (1.1.1) when there is an unknown scale parameter also.

We need the following assumption about the design matrix $\mathbf{X}$.

(NX) $(\mathbf{X}'\mathbf{X})^{-1}$ *exists*, $n \ge p$; $\max_i \mathbf{x}_{ni}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_{ni} = o(1)$.

This is Noether's condition for the design matrix $\mathbf{X}$. Now, let

$$
\begin{aligned}
(32)\qquad & \mathbf{A} = (\mathbf{X}'\mathbf{X})^{-1/2}, && \mathbf{D} := \mathbf{XA}, \\
& \mathbf{q}'(t) := (q_{n1}(t), \ldots, q_{nn}(t)), && \mathbf{\Lambda}(t) := \operatorname{diag}(\mathbf{q}(t)), \\
& \mathbf{\Gamma}_1(t) := \mathbf{AX}'\mathbf{\Lambda}(t)\,\mathbf{XA}, \quad \mathbf{\Gamma}_2(t) := n^{-1/2}H^{-1}(t)\,\mathbf{D}'\mathbf{q}(t), && 0 \le t \le 1.
\end{aligned}
$$

Write $\mathbf{D} = ((d_{ij}))$, $1 \le i \le n$, $1 \le j \le p$, and let $\mathbf{d}_{(j)}$ denote the j^{th} column of $\mathbf{D}$. Note that $\mathbf{D}'\mathbf{D} = \mathbf{I}_{pxp}$. This in turn implies that

(33) $\qquad$ (N1) is satisfied by $\mathbf{d}_{(j)}$ for all $1 \le j \le p$.

Moreover, with $\mathbf{a}_{(j)}$ denoting the j^{th} column of $\mathbf{A}$,

$$(34) \quad \max_i d_{ij}^2 = \max_i (\mathbf{x}_i'\mathbf{a}_{(j)})^2 \le \max_i \sum_{j=1}^{p} (\mathbf{x}_i'\mathbf{a}_{(j)})^2$$
$$= \max_i \mathbf{x}_i' \Big(\sum_{j=1}^{p} \mathbf{a}_{(j)}\mathbf{a}_{(j)}'\Big) \mathbf{x}_i$$
$$= \max_i \mathbf{x}_i' (\mathbf{X}'\mathbf{X})^{-1} \mathbf{x}_i = o(1), \quad \text{by (NX)}.$$

Let

$$(35) \qquad L_j(t) := \sum_{i=1}^{n} d_{ij}^2 \, F_i(H^{-1}(t)), \qquad 0 \le t \le 1, \ 1 \le j \le p.$$

We are now ready to state

Theorem 2.3.3. *Let* $\{(\mathbf{x}_{ni}', Y_{ni}),\ 1 \le i \le n\}$, β, $\{F_{ni},\ 1 \le i \le n\}$ *be as in the model* (1.1.1). *In addition, assume that* $\{F_{ni}\}$ *satisfy* (3a), (3b) *and that* (C*) *is satisfied by each* L_j *of* (35), $1 \le j \le p$.

Then, for every $0 < B < \infty$,

$$(36) \qquad \sup \|\mathbf{A}\{\mathbf{V}(H^{-1}(t), \beta + \mathbf{A}\mathbf{u}) - \mathbf{V}(H^{-1}(t), \beta)\} - \Gamma_1(t)\mathbf{u}\| = o_p(1).$$

where the supremum is over $0 \le t \le 1$, $\mathbf{u} \in \mathcal{N}(B)$.

If, in addition, H *is strictly increasing for all* $n \ge 1$, *then, for every* $0 < B < \infty$,

$$(37) \qquad \sup \|\mathbf{A}\{\mathbf{V}(x, \beta + \mathbf{A}\mathbf{u}) - \mathbf{V}(x, \beta)\} - \Gamma_1(H(x))\mathbf{u}\| = o_p(1).$$

where the supremum is over $-\infty \le x \le \infty$, $\mathbf{u} \in \mathcal{N}(B)$.

Theroem 2.3.4. *Suppose that* $\{(\mathbf{x}_{ni}', Y_{ni}),\ 1 \le i \le n\}$ *and* $\beta \in \mathbb{R}^p$ *obey the model*

$$(38) \qquad Y_{ni} = \mathbf{x}_{ni}'\beta + \gamma\, \epsilon_{ni}, \qquad 1 \le i \le n, \ \gamma > 0,$$

with $\{\epsilon_{ni}\}$ *independent r.v.'s having d.f.'s* $\{F_{ni}\}$. *Assume that* (NX) *holds. In addition, assume that* $\{F_{ni}\}$ *satisfy* (3a), (3b), (27) *and that* (C*) *is satisfied by each* L_j *of* (35), $1 \le j \le p$.

Then for every $0 < b, B < \infty$,

(39) $\sup \|\mathbf{A}\{\mathbf{V}(\alpha H^{-1}(t), \boldsymbol{\beta}+\mathbf{A}\mathbf{u}\gamma)-\mathbf{V}(\gamma H^{-1}(t), \boldsymbol{\beta})\}-\boldsymbol{\Gamma}_1(t)\mathbf{u}-\boldsymbol{\Gamma}_2(t)v\| = o_p(1),$

where $v := n^{1/2}(\alpha-\gamma)\gamma^{-1}$, $\alpha > 0$, *and the supremum is over* $0 \le t \le 1$, $\mathbf{u} \in \mathcal{N}(B)$ *and* $|v| \le b$.

If, in addition, H *is strictly increasing for every* $n \ge 1$, *then*

(40) $\sup\|\mathbf{A}\{\mathbf{V}(\alpha x, \boldsymbol{\beta}+\mathbf{A}\mathbf{u}\gamma)-\mathbf{V}(\gamma x, \boldsymbol{\beta})\} - \boldsymbol{\Gamma}_1(H(x))\mathbf{u} - \boldsymbol{\Gamma}_2(H(x))v\| = o_p(1).$

where $v := n^{1/2}(\alpha-\gamma)\gamma^{-1}$, $\alpha > 0$, *and the supremum is over* $-\infty \le x \le \infty$, $\mathbf{u} \in \mathcal{N}(B)$ *and* $|v| \le b$.

Proof of Theorem 2.3.3. Apply Theorem 2.3.1 to $X_i = Y_i - \mathbf{x}_i'\boldsymbol{\beta}$, $c_i = \mathbf{x}_i'A$, $1 \le i \le n$. Then F_i is the d.f. of X_i and the j^th^ components of $\mathbf{AV}(H^{-1}(t), \boldsymbol{\beta}+\mathbf{Au})$ and $\mathbf{AV}(H^{-1}(t), \boldsymbol{\beta})$ are $S_d(t, \mathbf{u})$, $S_d(t, \mathbf{0})$ of (1), respectively, with $d_i = d_{ij}$, $1 \le i \le n$, $1 \le j \le p$. Therefore (36) will follows by p applications of (6), one for each $\mathbf{d}_{(j)}$, provided the assumptions of Theorem 2.3.1 are satisfied. But in view of (33) and (34), the assumption (NX) implies (N1), (N2) for $\mathbf{d}_{(j)}$, $1 \le j \le p$. Also, (4) for the specified $\{c_i\}$ is equivalent to (NX). Finally, the C–S inequality and (33) verifies (5) in the present case. This makes Theorem 2.3.1 applicable and hence (36) follows. □

Proof of Theorem 2.3.4. Follows from Theorem 2.3.2 when applied to $X_i = (Y_i - \mathbf{x}_i'\boldsymbol{\beta})\gamma^{-1}$, $c_i' = \mathbf{x}_i'A$, $1 \le i \le n$, in a fashion similar to the proof of Theorem 2.3.3 above. □

The following corollaries follow from Corollaries 2.3.1 and 2.3.2 in the same way as the above Theorems 2.3.3 and 2.3.4 follow from Theorems 2.3.1 and 2.3.2. These are stated for an easy reference later on.

Corollary 2.3.3. *Suppose that the model* (1.1.1) *with* $F_{ni} \equiv F$ *holds. Assume that the design matrix* X *and the d.f.* F *satisfying* **(NX)** *and* **(F1)**. *Then,* $\forall\ 0 < B < \infty$,

(41) $\sup \|\mathbf{A}\{\mathbf{V}(F^{-1}(t), \mathbf{s}) - \mathbf{V}(F^{-1}(t), \boldsymbol{\beta})\} - f(F^{-1}(t))\mathbf{A}^{-1}(\mathbf{s}-\boldsymbol{\beta})\| = o_p(1).$

where the supremum is over $0 \le t \le 1$; $\mathbf{s} \in \mathbb{R}^p$, $\|\mathbf{A}^{-1}(\mathbf{s}-\boldsymbol{\beta})\| \le B$.

If, in addition, F *satisfies* **(F2)**, *then*

(42) $\sup \|\mathbf{A}\{\mathbf{V}(x, \mathbf{s}) - \mathbf{V}(x, \boldsymbol{\beta})\} - f(x)\, \mathbf{A}^{-1}(\mathbf{s}-\boldsymbol{\beta})\| = o_p(1).$

where the supremum is over $-\infty \le x \le \infty$; $\mathbf{s} \in \mathbb{R}^p$, $\|\mathbf{A}^{-1}(\mathbf{s}-\boldsymbol{\beta})\| \le B$. □

Corollary 2.3.4. *Suppose that the model* (38) *with* $F_{ni} \equiv F$ *holds and that the design matrix* X *and the d.f.* F *satisfy* (NX), (F1) *and* (F3). *Then* (39) *holds with*

$$(43) \quad \Gamma_1(t) = f(F^{-1}(t))I_{pxp}, \qquad \Gamma_2(t) = F^{-1}(t)\, f(F^{-1}(t))AX'1, \quad 0 \le t \le 1.$$

If, in addition, F *satisfies* (F2), *then* (40) *holds with* $\Gamma_j(H) \equiv \Gamma_j(F)$, j = 1, 2. I.e.,

$$(44) \qquad \sup \|A\{V(\alpha x, \beta + Au\gamma) - V(\gamma x, \beta)\} - f(x)u - xf(x)v\| = o_p(1),$$

where the supremum is over $-\infty \le x \le \infty$; $u \in \mathcal{N}(B)$ *and* $|v| \le b$, *with* v *as in* (39). □

We end this section by stating an a.u.l. result about the ordinary residual empirical processes H_n of (1.2.1) for an easy reference later on.

Corollary 2.3.5. *Suppose that the model* (1.1.1) *with* $F_{ni} \equiv F$ *holds. Assume that the design matrix* X *and the d.f.* F *satisfying* (NX) *and* (F1). *Then,* $\forall\ 0 < B < \infty$,

$$(45) \quad \sup|n^{1/2}\{H_n(F^{-1}(t), s) - H_n(F^{-1}(t), \beta)\} - $$
$$- f(F^{-1}(t)) \cdot n^{-1/2}\Sigma_i\, x_{ni}'\, A \cdot A^{-1}(s - \beta)| = o_p(1),$$

where the supremum is over $0 \le t \le 1$; $s \in \mathbb{R}^p$, $\|A^{-1}(s - \beta)\| \le B$.

If, in addition, F *satisfies* (F2), *then,* $\forall\ 0 < B < \infty$,

$$(46) \quad \sup|n^{1/2}\{H_n(x, s) - H_n(x, \beta)\} - f(x) \cdot n^{-1/2}\Sigma_i\, x_{ni}'A \cdot A^{-1}(s - \beta)| = o_p(1).$$

where the supremum is over $-\infty \le x \le \infty$; $s \in \mathbb{R}^p$, $\|A^{-1}(s - \beta)\| \le B$.

Proof. The proof follows from Theorem 2.3.1 by specializing it to the case where $d_{ni} \equiv n^{-1/2}$ and the rest of the entities as in the proof of Theorem 2.3.3. □

Note: Ghosh and Sen (1971) and Koul and Zhu (1991) have proved an almost sure version of (42) in the case p = 1 and p > 1, respectively. □

2.4. SOME FURTHER PROBABILISTIC RESULTS FOR W.E.P.'S.

For the sake of general interest, here we state some further results about w.e.p.'s. To begin with, we have

2.4.1. Laws of the iterated logarithm:

In this subsection, we assume that

(1) $$d_{ni} \equiv d_i, \quad \eta_{ni} \equiv \eta_i, \quad G_{ni} \equiv G_i, \quad 1 \le i \le n.$$

Define

(2) $$\mathscr{U}_n(t) := \sum_{i=1}^{n} d_i \{I(\eta_i \le t) - G_i(t)\}, \qquad \sigma_n^2 := \sum_{i=1}^{n} d_i^2,$$

$$\xi_n(t) := \mathscr{U}_n(t)/\{2\sigma_n^2 \,\ell n\, \ell n\, \sigma_n^2\}^{1/2}, \qquad n \ge 1, \ 0 \le t \le 1.$$

Let $r(s,t) := s \wedge t - st$, $0 \le s, t \le 1$, and $H(r)$ be the reproducing kernel Hilbert space generated by the kernel r with $\|\cdot\|_r$ denoting the associated norm on $H(r)$. Let

(3) $$K = \{f \in H(r); \ \|f\|_r \le 1\}.$$

Theorem 2.4.1. *If* $\eta_1, \eta_2, \ldots$ *are i.i.d. uniform on* $[0, 1]$ *and* $d_1, d_2, \ldots$ *are any real numbers satisfying*

(a) $$\lim_n \sigma_n^2 = \infty, \qquad \lim_n \left(\max_{1 \le i \le n} d_i^2\right) \frac{\ell n\ \ell n\ \sigma_n^2}{\sigma_n^2} = 0,$$

then

$$P(\mathscr{d}(\xi_n, K) \longrightarrow 0 \textit{ and the set of limit points of } \{\xi_n\} \textit{ is } K) = 1. \quad \square$$

Theorem 2.4.1 was proved by Vanderzanden (1980, 1984) using some of the results of Kuelbs (1976) and certain martingale properties of ξ_n.

Theorem 2.4.2. *Let* $\eta_1, \eta_2, \ldots.$ *be independent nonnegative r.v.'s. Let* $\{d_i\}$ *be any real numbers. Then*

$$\limsup_n \sup_{t \ge 0} \sigma_n^{-1} |\,\mathscr{U}_n(t-)| < \infty, \text{ a.s.}. \quad \square$$

A proof of this appears in Marcus and Zinn (1984). Actually they prove some other interesting results about w.e.p.'s with weights which are r.v.'s and functions of t. Most of their results, however, are concerned with the bounded law of the iterated logarithm. They also proved the following inequality that is similar to, yet a generalization of, the classical Dvoretzky-Kiefer-Wolfowitz exponential inequality for the ordinary empirical process. Their proof is valid for triangular arrays and real r.v.'s.

Exponential inequality. *Let* $X_{n1}, X_{n2}, \ldots, X_{nn}$ *be independent r.v.'s with respective d.f.'s* $F_{n1}, \ldots, F_{nn}$ *and* $\{d_{ni}\}$ *be any real numbers satisfying* (N1). *Then,* $\forall\ \lambda > 0, \forall\ n \ge 1,$

$$(4)\quad P\Big(\sup_{|x|<\infty} \Big|\sum_{i=1}^{n} d_{ni}\{I(X_{ni} \le x) - F_{ni}(x)\}\Big| \ge \lambda\Big) \le [1+(8\pi)^{1/2}\lambda]\exp(-\lambda^2/8). \quad \square$$

The above two theorems immediately suggest some interesting probabilistic questions. For example, is Vanderzanden's result valid for nonidentical r.v.'s $\{\eta_i\}$? Or can one remove the assumption of nonnegative $\{\eta_i\}$ in Theorem 2.4.1?

2.4.2. Weak convergence of w.e.p.'s in $\mathbb{D}[0, 1]^p$, in d_q-metric and an embedding result.

Next, we state a weak convergence result for multivariate r.v.'s. For this we revert back to triangular arrays. Now suppose that $\boldsymbol{\eta}_{ni} \in [0, 1]^p$, $1 \le i \le n$, are independent r.v.'s of dimension p. Define

$$(5)\qquad W_d(\mathbf{t}) := \sum_{i=1}^{n} d_{ni}\{I(\boldsymbol{\eta}_{ni} \le \mathbf{t}) - G_{ni}(\mathbf{t})\}, \quad \mathbf{t} \in [0, 1]^p.$$

Let G_{nij} be the j^{th} marginal of G_{ni}, $1 \le i \le n$, $1 \le j \le p$.

Theorem 2.4.3. *Let* $\{\boldsymbol{\eta}_{ni}, 1 \le i \le n\}$ *be independent* p*-variate r.v.'s and* $\{d_{ni}\}$ *satisfy* (N1) *and* (N2). *Moreover suppose that for each* $1 \le j \le p$,

$$\lim_{\delta\to 0} \limsup_n \sup_{0\le t\le 1-\delta} \sum_{i=1}^{n} d_{ni}^2\{G_{nij}(t+\delta) - G_{nij}(t)\} = 0.$$

Then, for every $\epsilon > 0$

(i) $$\lim_{\delta\to 0} \limsup_n P\Big(\sup_{|\mathbf{s}-\mathbf{t}|<\delta} |W_d(\mathbf{t}) - W_d(\mathbf{s})| > \epsilon\Big) = 0.$$

(ii) *Moreover,* $W_d \Rightarrow$ *some* W *on* $(\mathbb{D}[0,1]^p, d)$ *if, and only if, for each* $\mathbf{s}, \mathbf{t} \in [0, 1]^p$, $\mathrm{Cov}(W_d(\mathbf{s}), W_d(\mathbf{t})) \longrightarrow \mathrm{Cov}(W(\mathbf{s}), W(\mathbf{t})) =: \mathcal{C}(\mathbf{s}, \mathbf{t})$. *In this case* W *is necessarily a Gaussian process,* $P(W \in \mathbb{C}[0, 1]^p) = 1$, $W(0) = 0 = W(1)$. $\square$

Theorem 2.4.3 is essentially proved in Vanderzanden (1980), using results of Bickel and Wichura (1971).

Mehra and Rao (1975), Withers (1975), and Koul (1977), among others, obtain the weak convergence results for $\{W_d\}$-processes when $\{\eta_{ni}\}$ are weakly dependent. See Dehling and Taqqu (1989) and Koul and Mukherjee (1992) for similar results when $\{\eta_{ni}\}$ are long range dependent.

Shorack (1979) proved the weak convergence of W_d/q-process in the d-metric, where $q \in \mathcal{Q}$, with

$$\mathcal{Q} := \{q;\ q \text{ a continuous function on } [0,1],\ q \geq 0,\ q(t) = q(1-t),\ q(t)\uparrow \text{ and } t^{-1/2}q(t)\downarrow \text{ for } 0 \leq t \leq 1/2,\ \int_0^1 q^{-2}(t)dt < \infty\}.$$

Theorem 2.4.4. *Suppose that* $\eta_{n1}, \ldots, \eta_{nn}$ *are independent r.v.'s in* $[0, 1]$ *with respective d.f.'s* $G_{n1}, \ldots, G_{nn}$ *such that*

$$n^{-1}\sum_{i=1}^{n} G_{ni}(t) = t, \quad 0 \leq t \leq 1.$$

In addition, suppose that $\{d_{ni}\}$ *satisfy* (**N1**) *and* (**B**). *Then,*

(i) $\forall\ \epsilon > 0,\ \forall\ q \in \mathcal{Q},$

$$\lim_{\delta\to 0} \limsup_n P\Big(\sup_{|t-s|<\delta} \Big|\frac{W_d(t)}{q(t)} - \frac{W_d(s)}{q(s)}\Big| > \epsilon\Big) = 0.$$

(ii) $q^{-1}W_d \Rightarrow q^{-1}W$, W *a continuous Gaussian process with covariance function* C *if, and only if* $C_d \longrightarrow C$. □

Shorack (1991) and Einmahl and Mason (1991) proved the following embedding result.

Theorem 2.4.5. *Suppose that* $\eta_{n1}, \ldots, \eta_{nn}$ *are i.i.d. Uniform* $[0, 1]$ *r.v.'s. In addition, suppose that* $\{d_{ni}\}$ *satisfy* (**N1**) *and that*

$$\sum_{i=1}^{n} d_{ni} = 0, \qquad n\sum_{i=1}^{n} d_{ni}^4 = O(1).$$

Then on a rich enough probability space there exist a sequence of versions $\mathcal{W}_d$ *of the processes* W_d *and a fixed Brownian bridge* B *on* $[0, 1]$ *such that*

$$\sup_{1/n\leq t\leq 1-1/n} n^{\nu}\frac{|\mathcal{W}_d(t) - B(t)|}{\{t(1-t)\}^{1/2-\nu}} = O_p(1), \qquad \textit{for all } 0 \leq \nu < 1.$$

The closed interval $1/n \leq t \leq 1-1/n$ *may be replaced by the open interval* $\min\{\eta_{nj};\ 1 \leq j \leq n\} < t < \max\{\eta_{nj};\ 1 \leq j \leq n\}$. □

2.4.3. A martingale property.

In this subsection we shall prove a martingale property of w.e.p.'s. Let $X_{n1}, X_{n2}, \ldots, X_{nn}$ be independent real r.v.'s with respective d.f.'s $F_{n1}, \ldots, F_{nn}$; $d_{n1}, \ldots, d_{nn}$ be real numbers. Let $a \leq b$ be fixed real numbers. Define,

$$M_n(t) := \sum_{i=1}^{n} d_{ni}\{I(X_{ni} \in (a, t] - p_{ni}(a, t]\}\,\{1 - p_{ni}(a, t]\}^{-1},$$

$$R_n(t) := \sum_{i=1}^{n} d_{ni}\{I(X_{ni} \in (t, b] - p_{ni}(t, b]\} \{1 - p_{ni}(t, b]\}^{-1}, \qquad t \in \mathbb{R},$$

where

$$p_{ni}(s,t] := F_{ni}(t) - F_{ni}(s), \quad 0 \le s \le t \le 1, \ 1 \le i \le n.$$

Let $T_1 \subset [a, \infty)$, $T_2 \subset (-\infty, b]$ be such that $M_n(t)$ $[R_n(t)]$ is well–defined for $t \in T_1$ $[t \in T_2]$. Let

$$\mathcal{F}_{1n}(t) := \sigma\text{–field } \{I(X_{ni} \in (a, s]), \ a \le s \le t, \ i = 1, ..., n\}, \quad t \in T_1,$$

$$\mathcal{F}_{2n}(t) := \sigma\text{–field } \{I(X_{ni} \in (s, b]), \ t \le s \le b, \ i = 1, ..., n\}, \quad t \in T_2.$$

Martingale Lemma. *Under the above set up, for each* $n \ge 1$, $\{M_n(t), \mathcal{F}_{1n}(t), t \in T_1\}$ *is a martingale and* $\{R_n(t), \mathcal{F}_{2n}(t), t \in T_2\}$ *is a reverse martingale.*

Proof. Write $q_i(a, s] = 1 - p_i(a, s]$. Because $\{X_i\}$ are independent, for $a \le s \le t$

$$\begin{aligned} &E\{M_n(t) \mid \mathcal{F}_{1n}(s)\} \\ &= \sum_{i=1}^{n} d_i\{q_i(a,t]\}^{-1}[I(X_i \in (a, s])\, E\{(I(X_i \in (a, t]) - p_i(a, t]) \mid X_i \in (a, s]\} \\ &\qquad + I(X_i \notin (a, s])\, E\{(I(X_i \in (a, t]) - p[_i(a, t]) \mid X_i \notin (a, s]\}] \\ &= \sum_{i=1}^{n} d_i \{q_i(a, t)\}^{-1}[I(X_i \in (a, s]q_i(a, t] + \\ &\qquad + I(X_i \notin (a, s] \{\frac{p_i(s, t]}{q_i(a, s]} - p_i(a, t]\}] \\ &= \sum_{i=1}^{n} d_i\{q_i(a, s)\}^{-1}(I(X_i \in (a, s]) - q_i(a, s]\} = M_n(s). \end{aligned}$$

A similar argument yields the result about R_n. □

Note. In the case $\{X_{ni}\}$ are i.i.d. and $d_{ni} \equiv n^{-1/2}$, this Lemma is well known. In the case $\{X_{ni}\}$ are i.i.d. and $\{d_{ni}\}$ are arbitrary, the observation about $\{M_n\}$ being a martingale first appeared in Sinha and Sen (1979). The above Martingale Lemma appears in Vanderzanden (1980, 1984).

Theorem 2.4.1 above generalizes a result of Finkelstein (1971) for the ordinary empirical process to w.e.p.'s of i.i.d. r.v.'s.. In fact, the set K is the same as the set K of Finkelstein. □□