XVIII. More on Proper Forcing

§0. Introduction

From the last eight chapters you may have gotten the impression that we are done with properness, but this is not so. First, we turn to the problem of not adding reals; remember that by V §7, VIII §4, for CS iterations of proper forcing notions not adding reals, the limit does not add reals, provided that two additional conditions hold: one is $\mathbb{D}$-completeness (for, say, a simple 2-completeness system) and the second is $(< \omega_1)$-properness (see V §2). Now, the first restriction is justified by the weak diamond (see V 5.1, 5.1A and AP §1); that is not to say that we have to demand exactly $\mathbb{D}$-completeness, but certainly we have to demand something in this direction. However, there was nothing there to justify the second demand: α-proper for every α. In the first section, (following [Sh:177]), we show that we cannot just omit it, even if we use an $\aleph_1$-completeness system. It is natural to hope that this counterexample will lead to a principle like the weak diamond (so provable from CH). Thus the construction of this counterexample leads to questions like: Assuming CH, can we find $\langle C_\delta : \delta < \omega_1 \rangle$, C_δ an unbounded subset of δ, say of order type ω, such that for every club E of ω_1, for stationarily many limit $\delta < \omega_1$, $C_\delta \subseteq E$ or $\delta = \sup(C_\delta \cap E)$ or $(\forall \alpha \in \delta)[\min(E \setminus \alpha) < \min(C_\delta \setminus (\alpha + 1))]$? (They are kin to "the guessing clubs", the existence of which for, e.g. $\aleph_2$, follows from ZFC, see [Sh:g].) It interests us as the theorems (and proofs) from V, VIII §4 do not give